NRLS: Structural Estimation of Directional Dynamic Games With Multiple Equilibria

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ROAD MAP

1. Solving directional dynamic games (DDGs):
   - Bertrand pricing and investment game
   - Solving once: State recursion algorithm
   - Solving for all MPE: Recursive lexicographical search (RLS) algorithm
2. Structural estimation of DDGs: nested MLE RLS
   - Is it even feasible computationally?
3. Monte Carlo to compare to exising estimators in dynamic games (two-step CCP, NPL, EPL, MPEC)
   - One equilibrium in the model and data
   - Multiplicity of equilibria at true parameter
   - Multiple equilibria in the data

 Iskhakov, Rust and Schjerning, 2016, ReStud

 Iskhakov, Rust and Schjerning, 2018, IER

 Currently: estimation + Monte Carlo exercises

Dynamic Bertrand price competition

Directional stochastic dynamic game

- Two Bertrand competitors, $n = 2$, no entry or exit
- Discrete time, infinite horizon ($t = 1, 2, \ldots, \infty$)
- Firms maximize expected discounted profits
- Each firm has two choices in each period:
  1. Price for the product — simultaneous
  2. Whether or not to buy the state of the art technology
     - Simultaneous moves
     - Alternating moves

Static Bertrand price competition in each period

- Continuum of consumers make static purchase decision
- No switching costs: buy from the lower price supplier
- Per period profits (c_i is the marginal cost)

$$r_i(c_1, c_2) = \begin{cases} 0 & \text{if } c_i \geq c_j \\ c_j - c_i & \text{if } c_i < c_j \end{cases}$$

Cost-reducing investments

State-of-the-art production cost c process

- Initial value c_0, lowest value 0: $0 \leq c \leq c_0$
- Discretized with n points
- Follows exogenous Markov process and only improves
- Markov transition probability $\pi(c_{t+1}|c_t)$
  $\pi(c_{t+1}|c_t) = 0$ if $c_{t+1} > c_t$

State space of the problem

- State of the game: cost structure (c_1, c_2, c)
- State space is $S = (c_1, c_2, c) \subset R^3$: $c_1 \geq c$, $c_2 \geq c$
- Actions are observable
- Private information EV(1) i.i.d. shocks $\eta\epsilon_{i,I}$ and $\eta\epsilon_{i,N}$

Bellman equations, firm 1, simultaneous moves

$$V_1(c_1, c_2, c, \epsilon_1) = \max\left[v_1(I, c_1, c_2, c) + \eta\epsilon_1(I), v_1(N, c_1, c_2, c) + \eta\epsilon_1(N)\right]$$

$$\begin{aligned} v_1(N, c_1, c_2, c) &= r_1(c_1, c_2) + \beta EV_1(c_1, c_2, c, N) \\ v_1(I, c_1, c_2, c) &= r_1(c_1, c_2) - K(c) + \beta EV_1(c_1, c_2, c, I) \end{aligned}$$

With extreme value shocks, the investment probability (CCP) is

$$P_1(I|c_1, c_2, c) = \frac{\exp\{v_1(I, c_1, c_2, c)/\eta\}}{\exp\{v_1(I, c_1, c_2, c)/\eta\} + \exp\{v_1(N, c_1, c_2, c)/\eta\}}$$

- There is a separate Bellman equation for player 2, with "outputs" V_2 and P_2, where $P_2(I|c_1, c_2, c)$ is firm 2's probability of investing in state (c_1, c_2, c).

Bellman equations, firm 1, simultaneous moves

The expected values are given by

$$EV_1(c_1, c_2, c, N) = \int_0^c \left[P_2(I|c_1, c_2, c) H_1(c_1, c, c') + [1 - P_2(I|c_1, c_2, c)] H_1(c_1, c_2, c') \right] \pi(dc'|c)$$

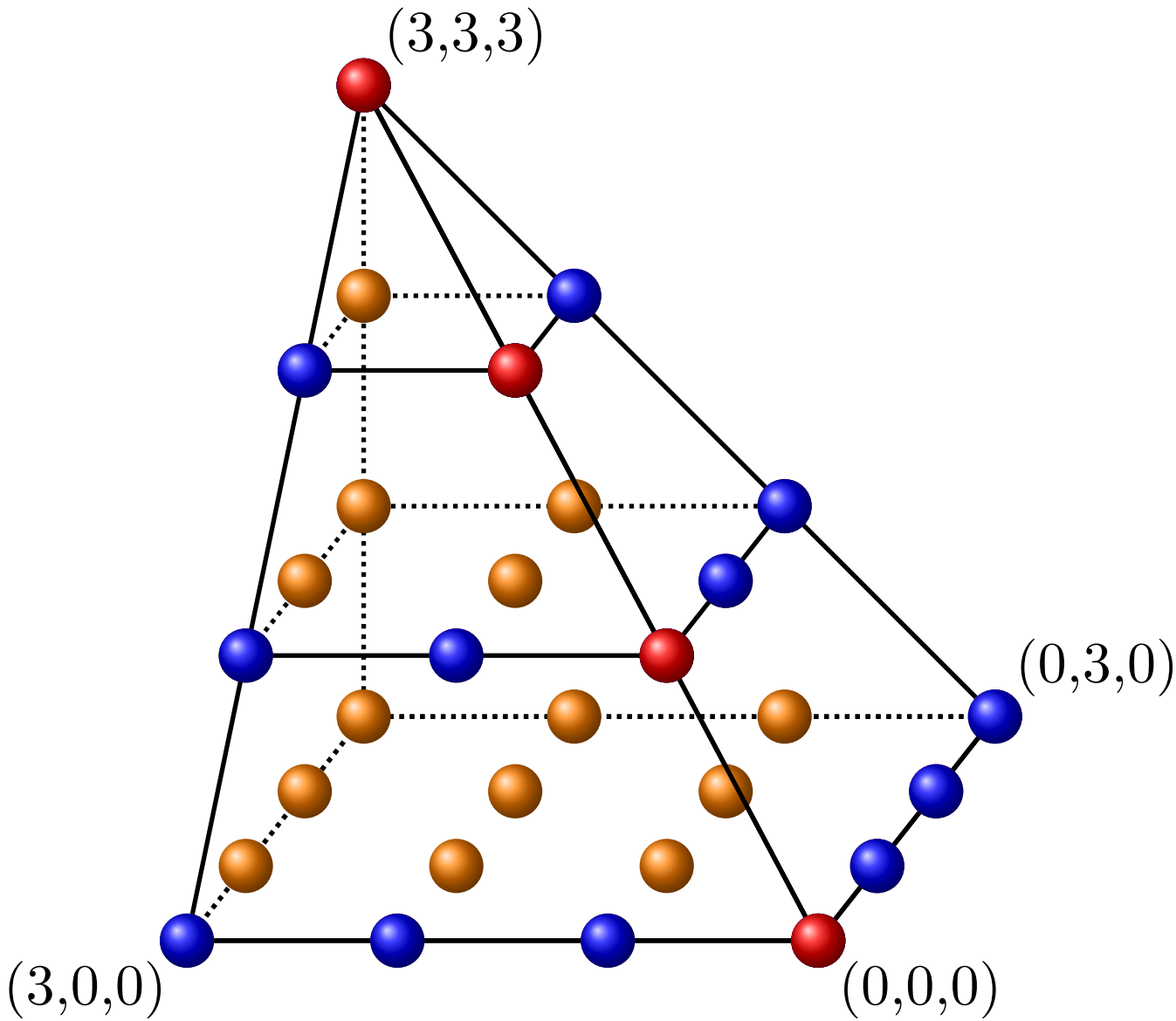
$$EV_1(c_1, c_2, c, I) = \int_0^c \left[P_2(I|c_1, c_2, c) H_1(c, c, c') + [1 - P_2(I|c_1, c_2, c)] H_1(c, c_2, c') \right] \pi(dc'|c)$$

$$H_1(c_1, c_2, c) = \eta \log \left[\exp\left(v_1^N(c_1, c_2, c)/\eta\right) + \exp\left(v_1^I(c_1, c_2, c)/\eta\right) \right].$$

is the "smoothed max" or logsum function

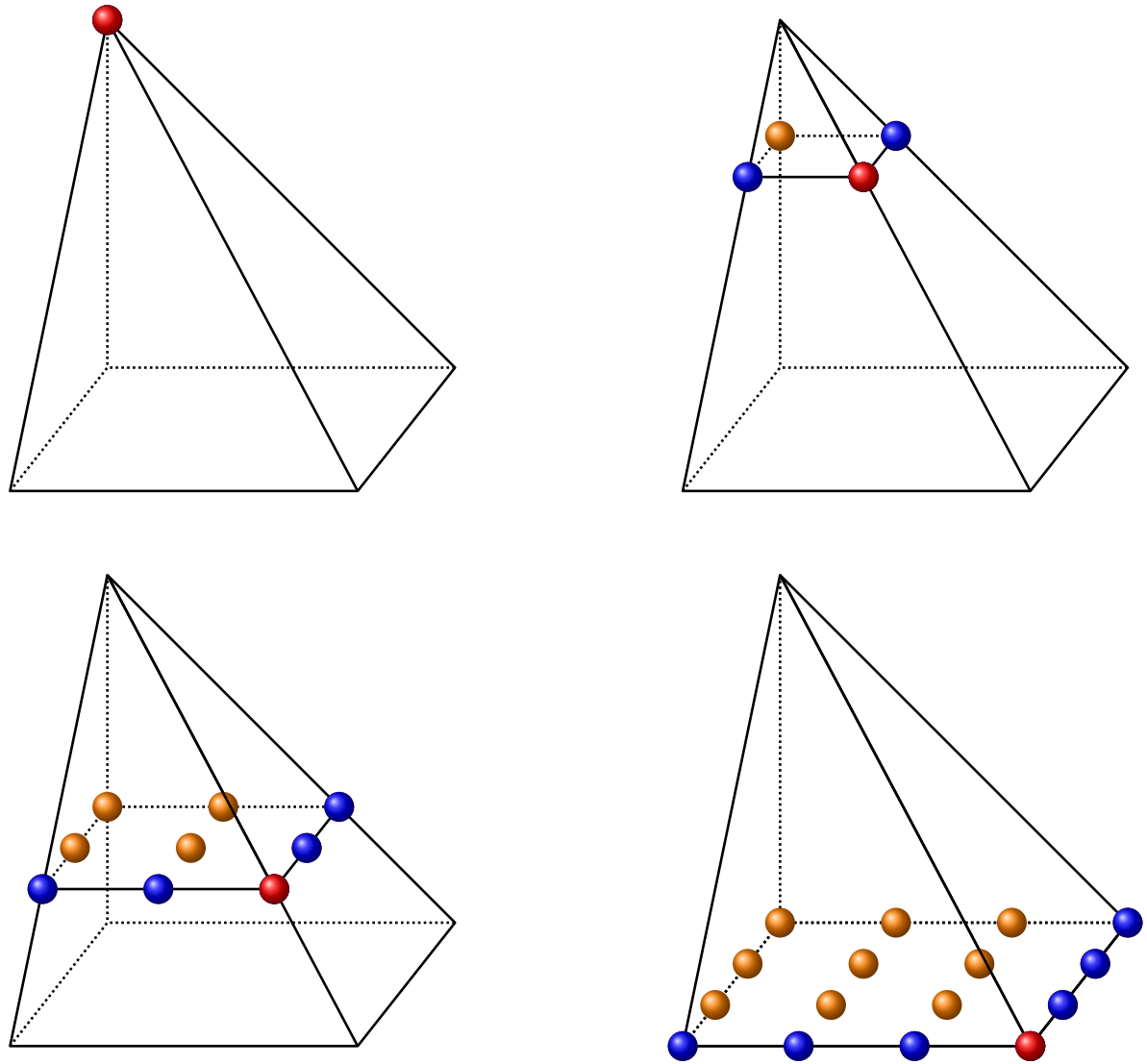
Discretized state space = a "quarter pyramid"

$S = \{(c_1, c_2, c) | c_1 \geq c,\ c_2 \geq c, c \in [0, 3]\},\ n = 4$



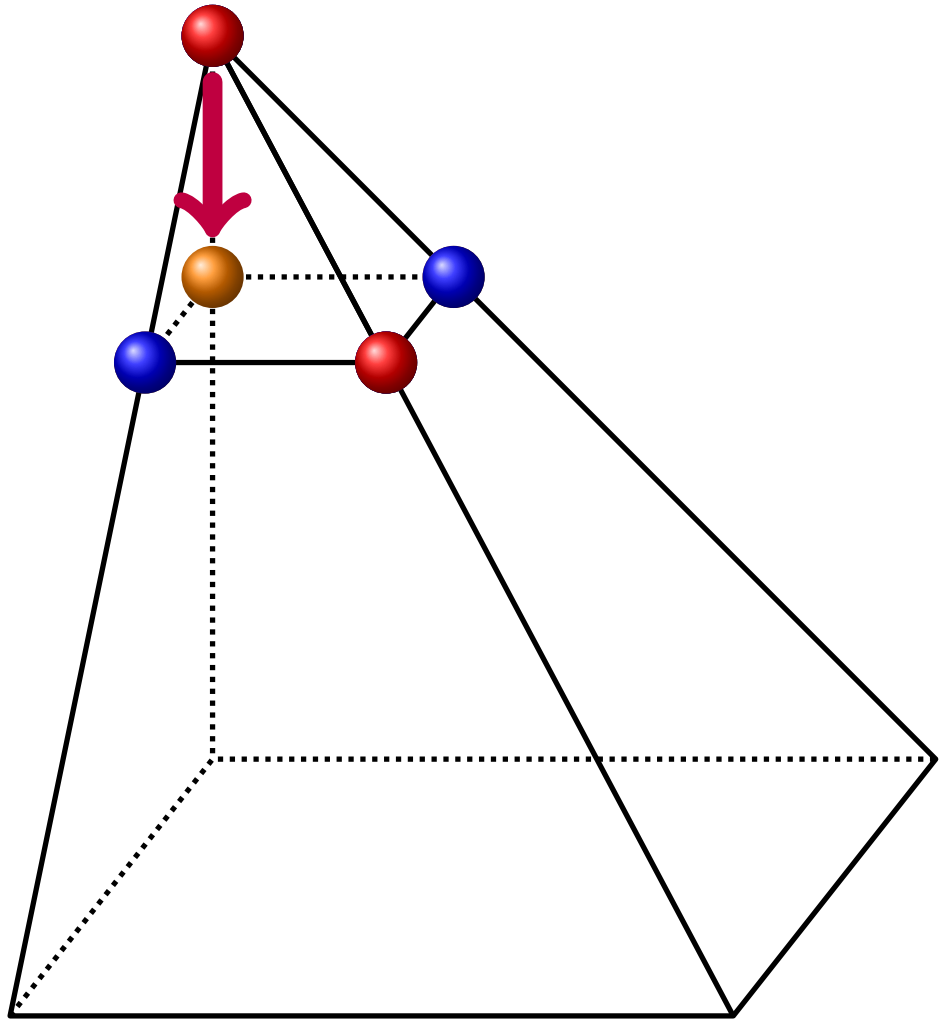
Transitions due to technological progress

As c decreases, the game falls through the layers of the pyramid



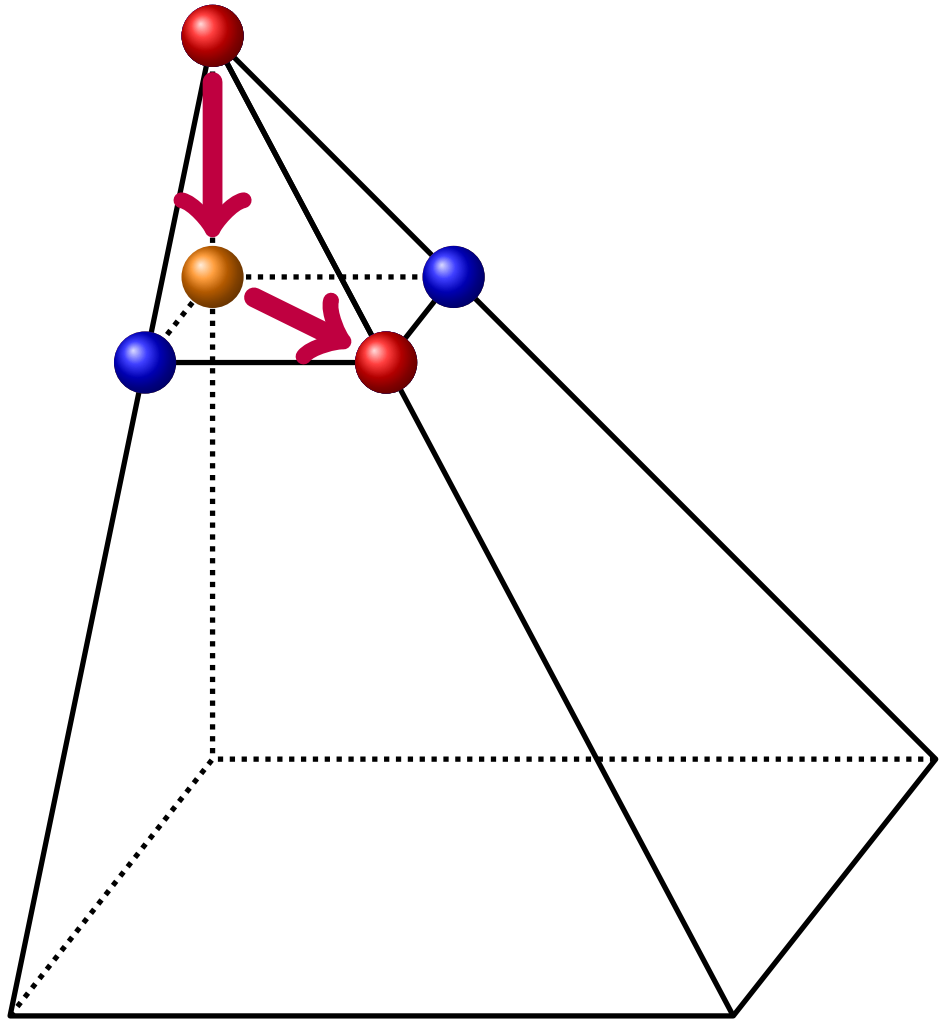
Game dynamics: example

The game starts at the apex, as some point technology improves



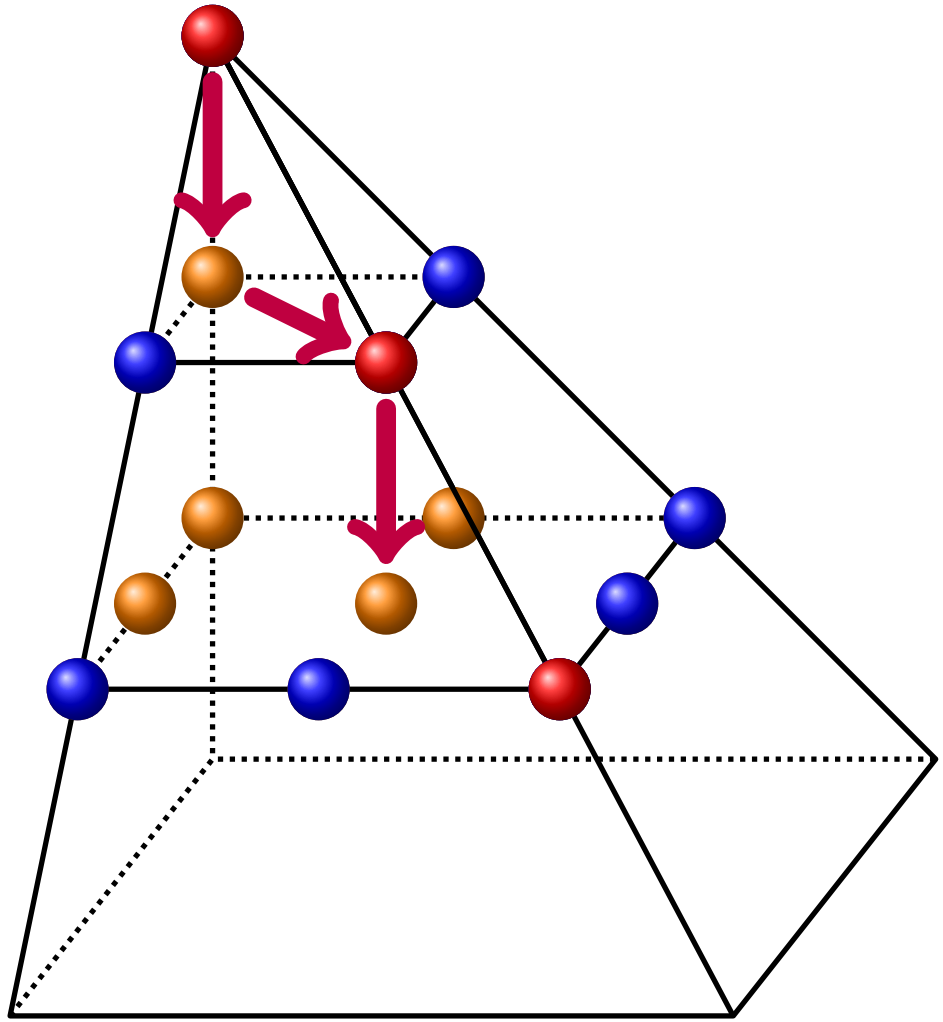
Game dynamics: example

Both firms buy new technology $c = 2 \rightsquigarrow (c_1, c_2, c) = (2, 2, 2)$



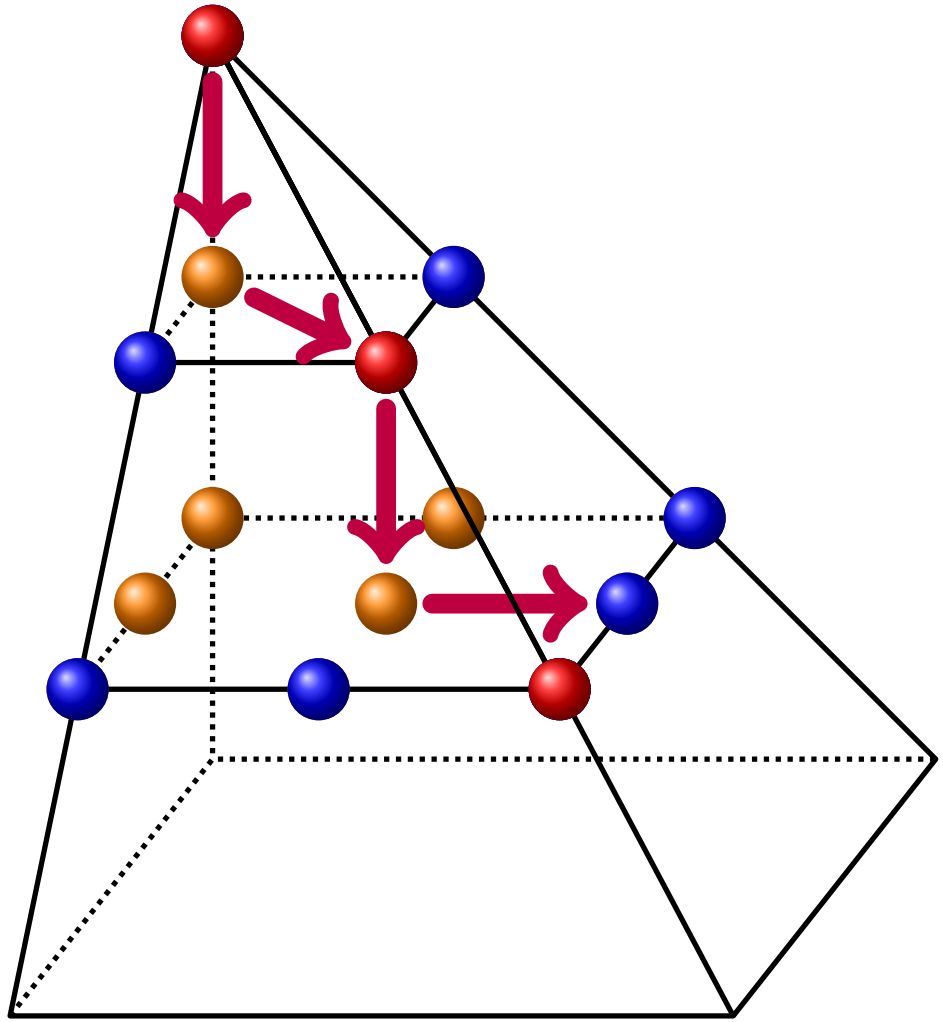
Game dynamics: example

State-of-the-art technology becomes $c = 1 \rightsquigarrow (c_1, c_2, c) = (2, 2, 1)$



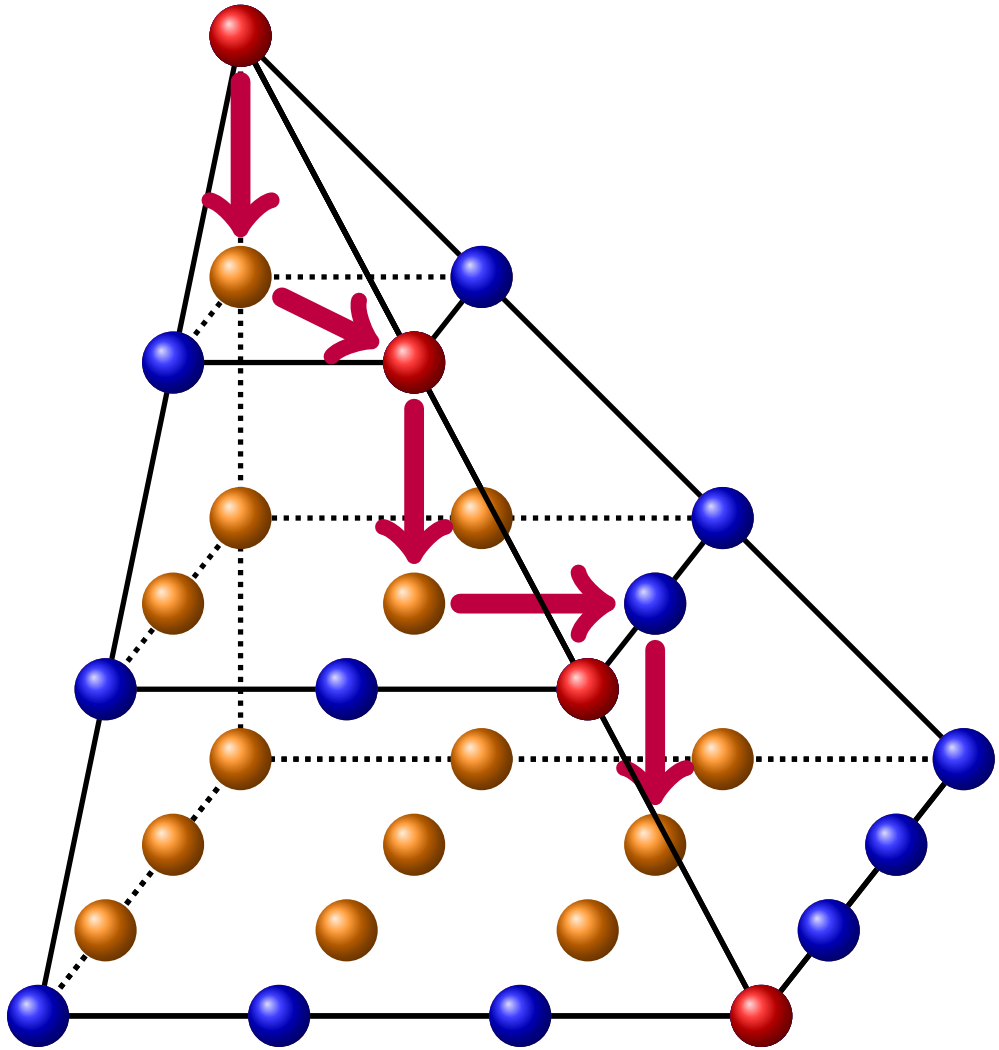
Game dynamics: example

Firm 1 invests and becomes cost leader $\rightsquigarrow (c_1, c_2, c) = (1, 2, 1)$



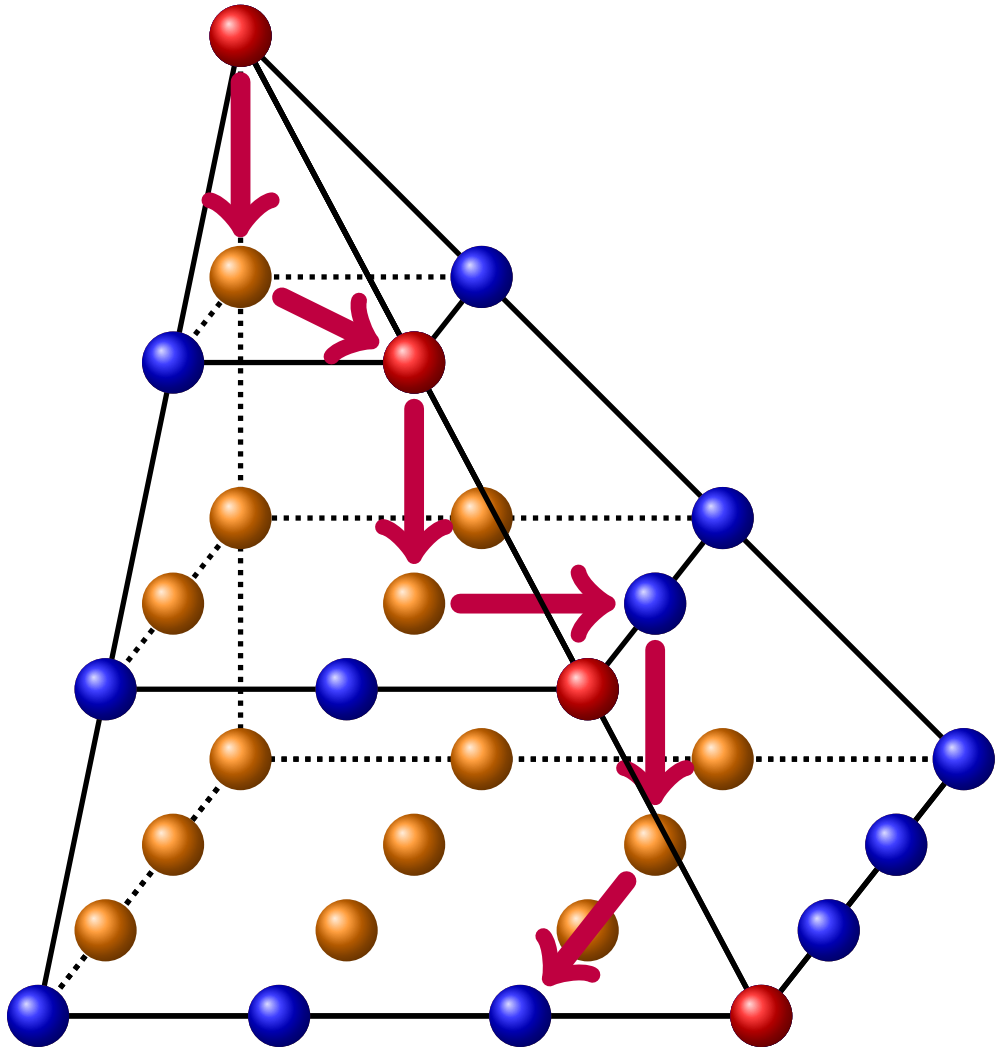
Game dynamics: example

State-of-the-art technology becomes $c = 0 \rightsquigarrow (c_1, c_2, c) = (1, 2, 0)$



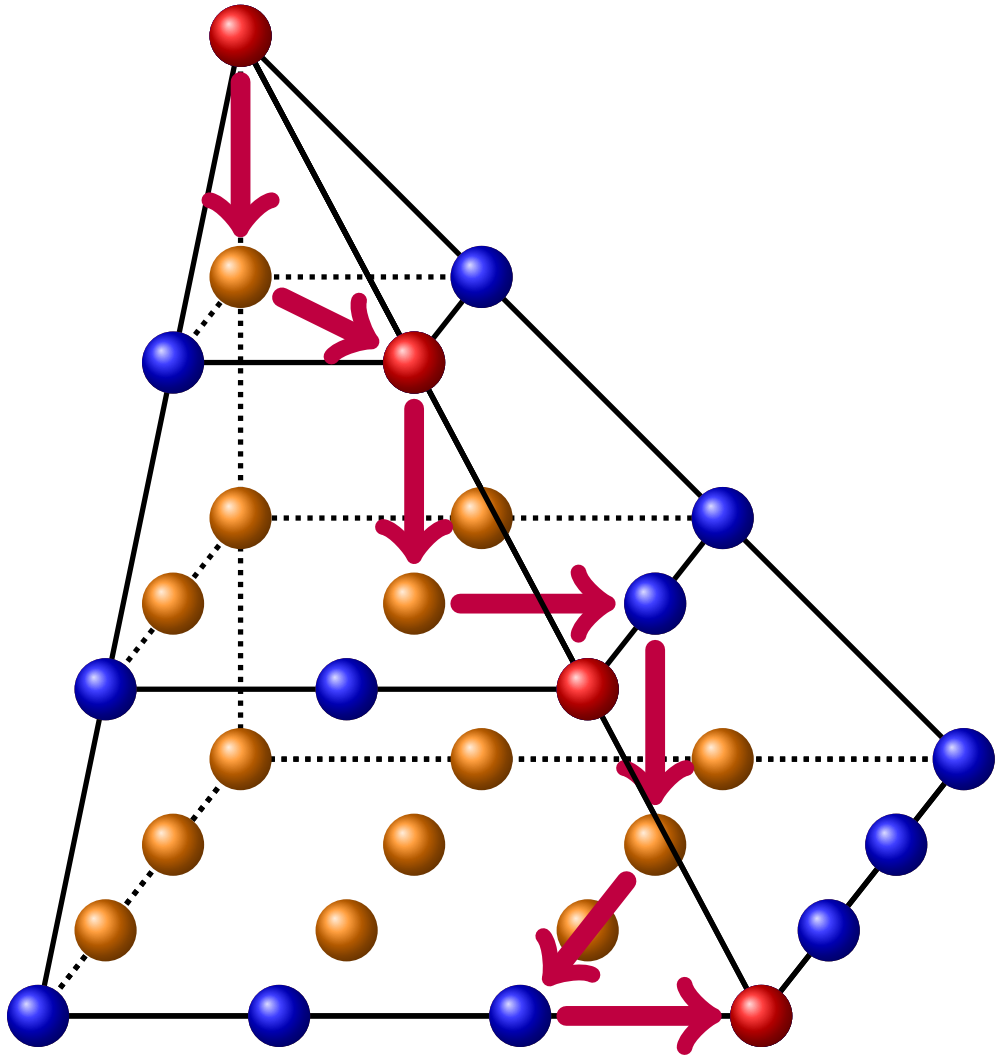
Game dynamics: example

Firm 2 leapfrogs firm 1 to become new cost leader $\rightsquigarrow (c_1, c_2, c) = (1, 0, 0)$



Game dynamics: example

A particular sequence of investment decisions along technological progress pass



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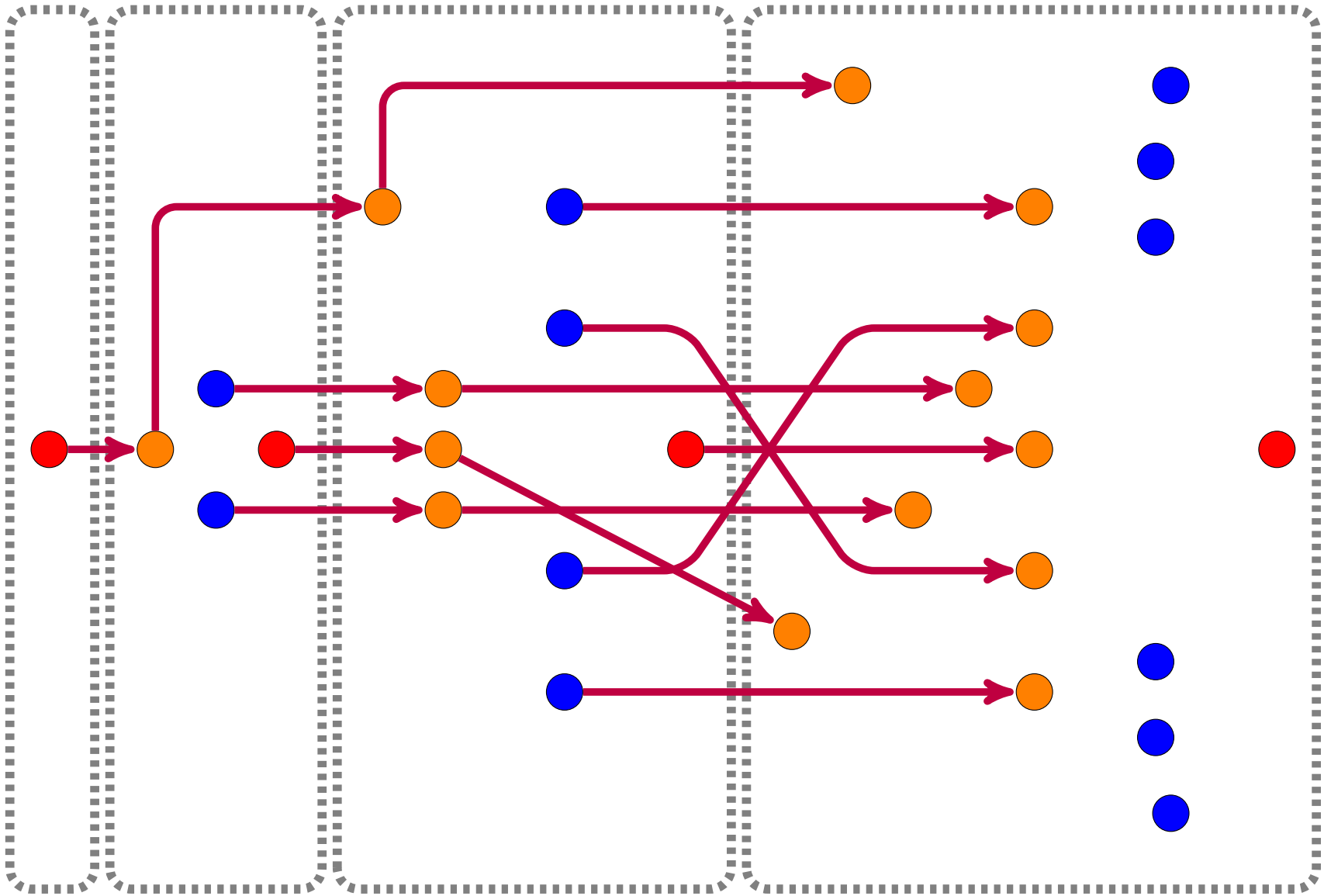
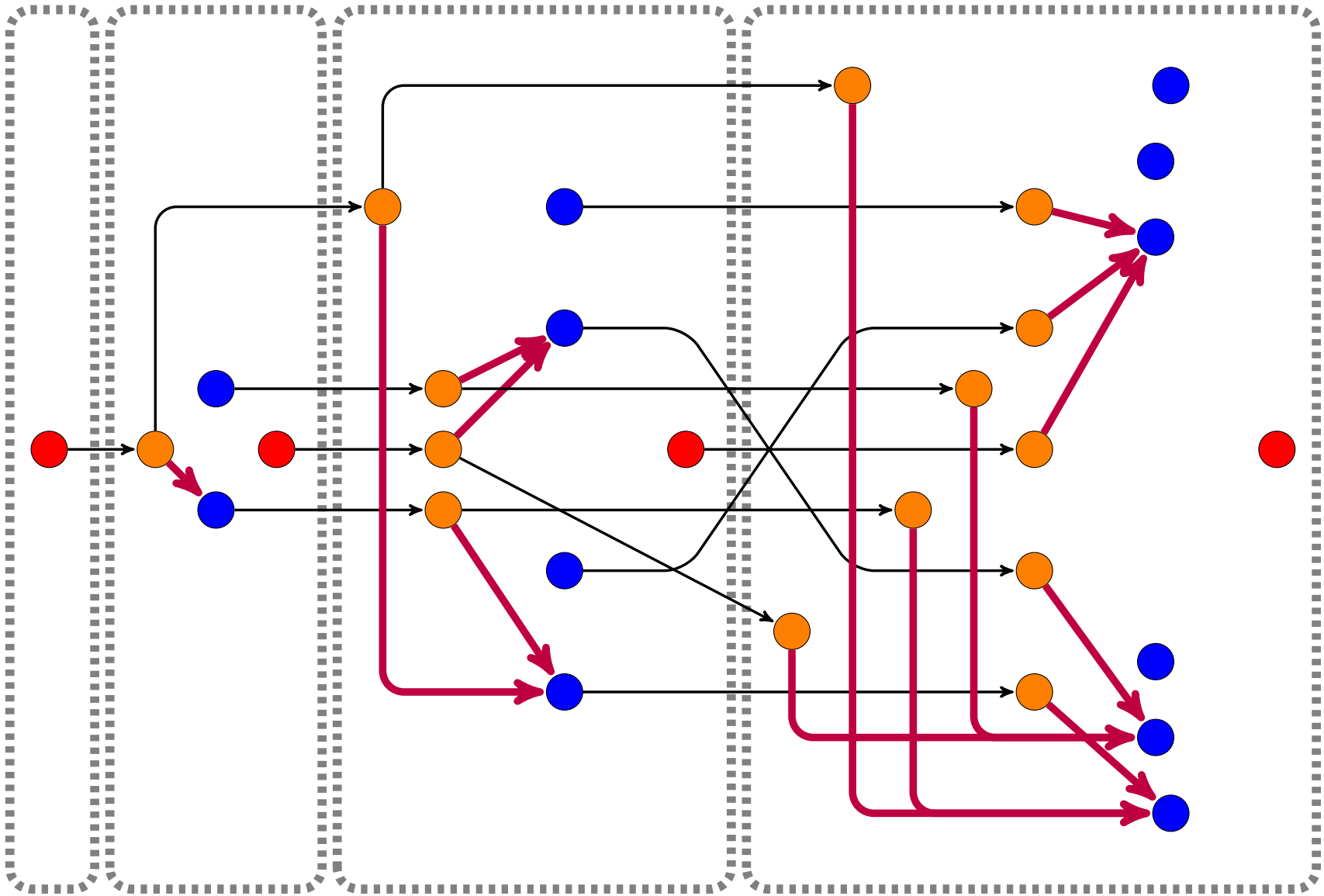


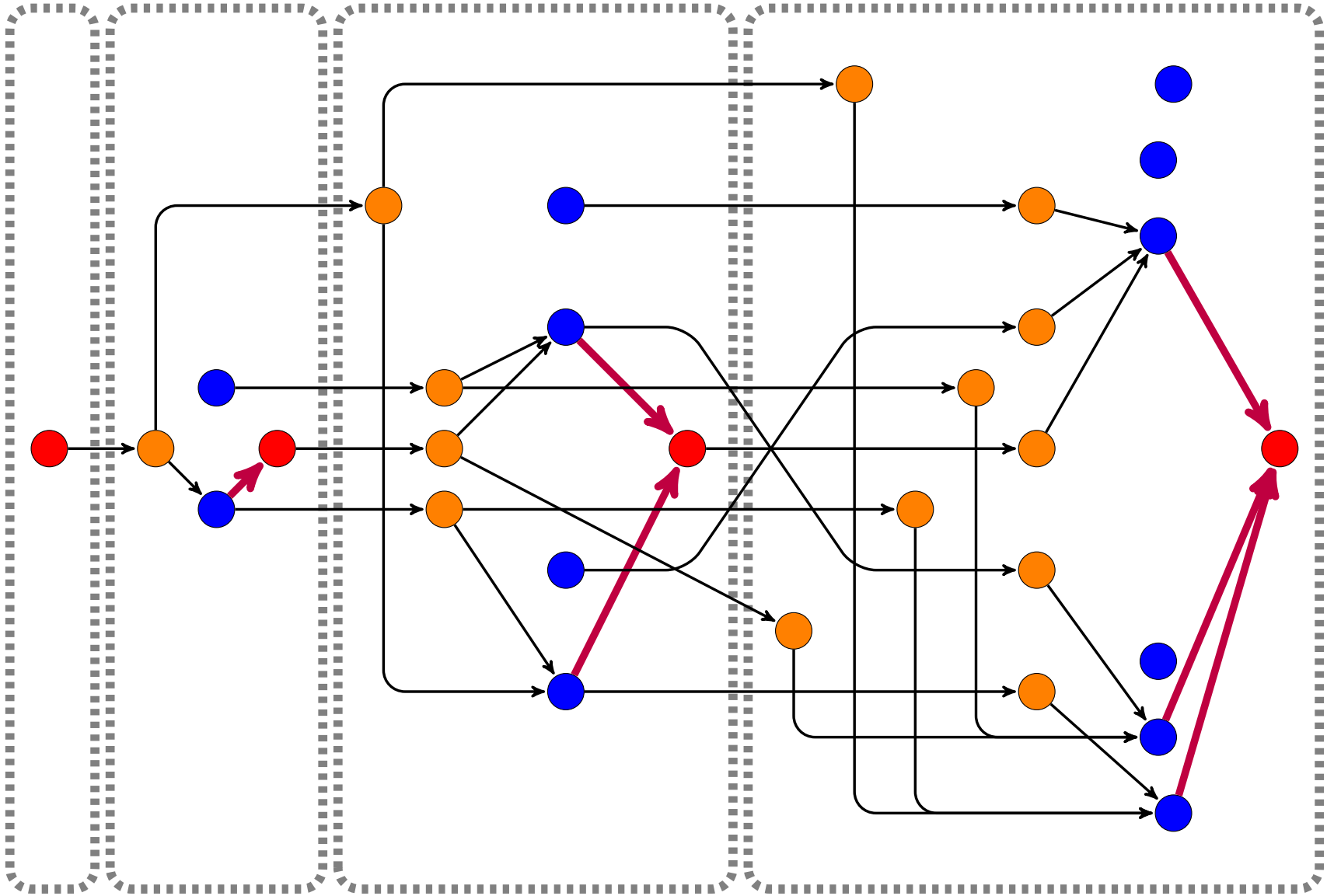
Strategy-specific partial order on S

Strategy σ_1 of firm 1: invest at all interior points



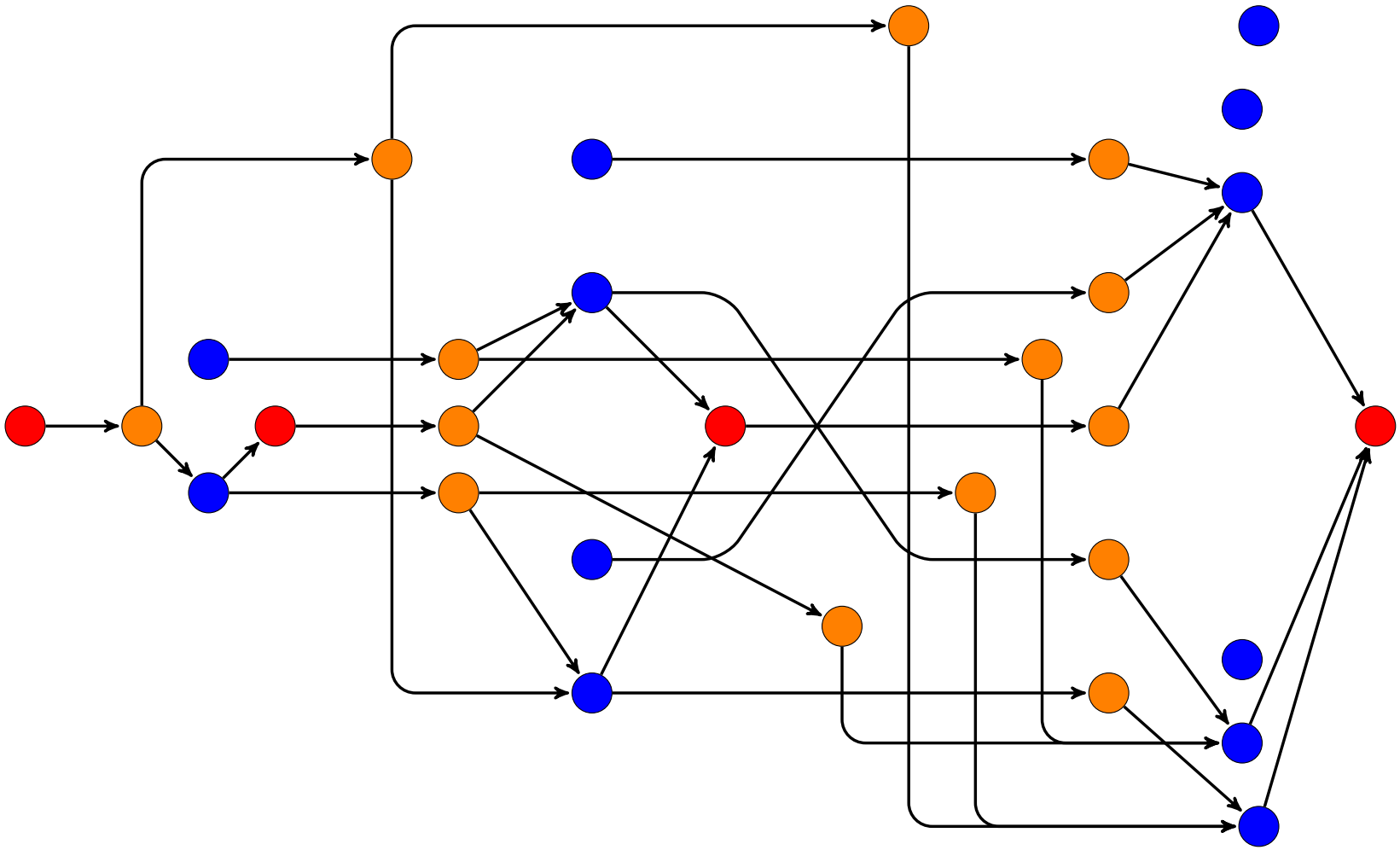
Strategy-specific partial order on S

Strategy σ_2 of firm 2: invest at all edge points



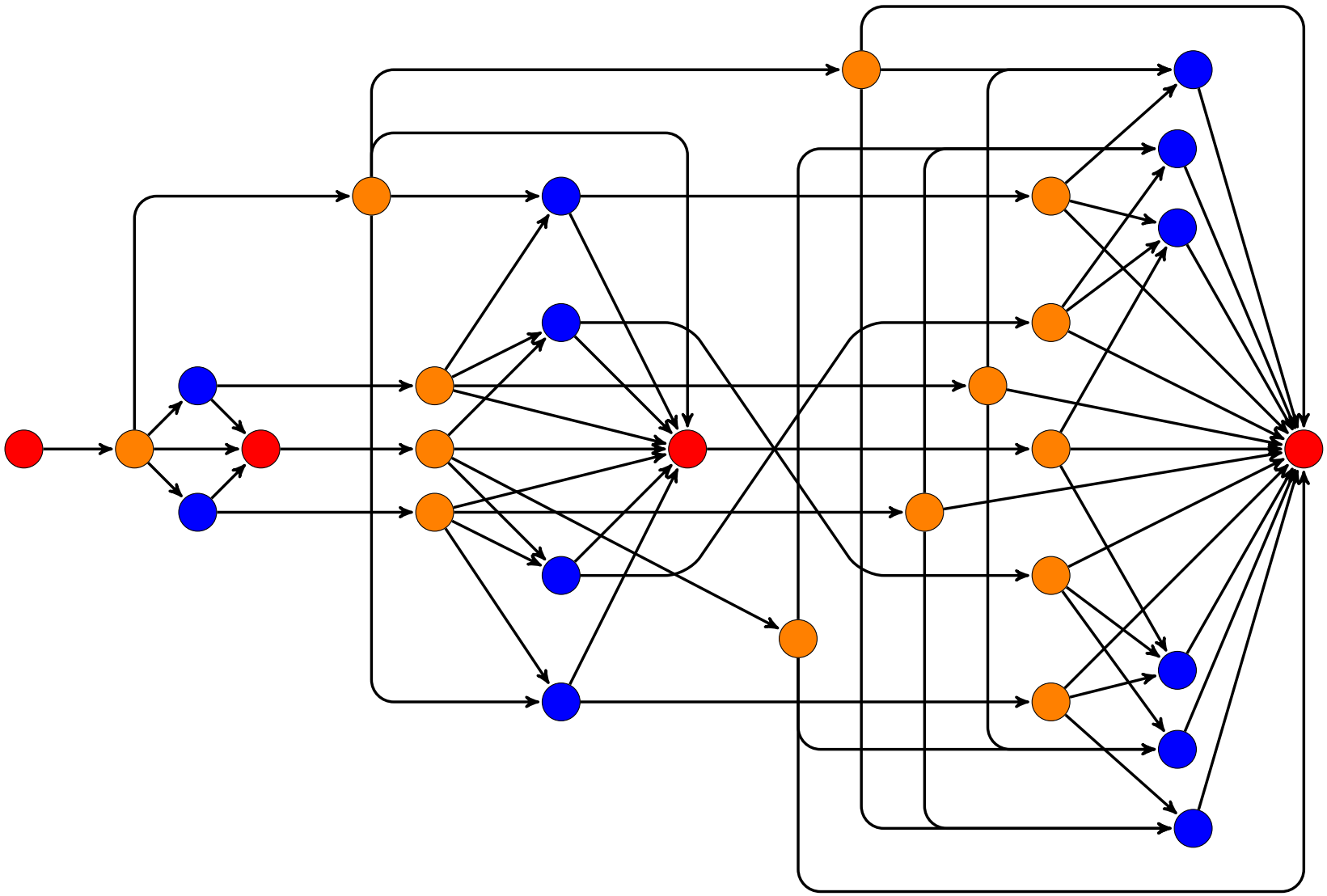
Strategy-specific partial order on S

Strategy $\sigma = (\sigma_1, \sigma_2)$ of both firms



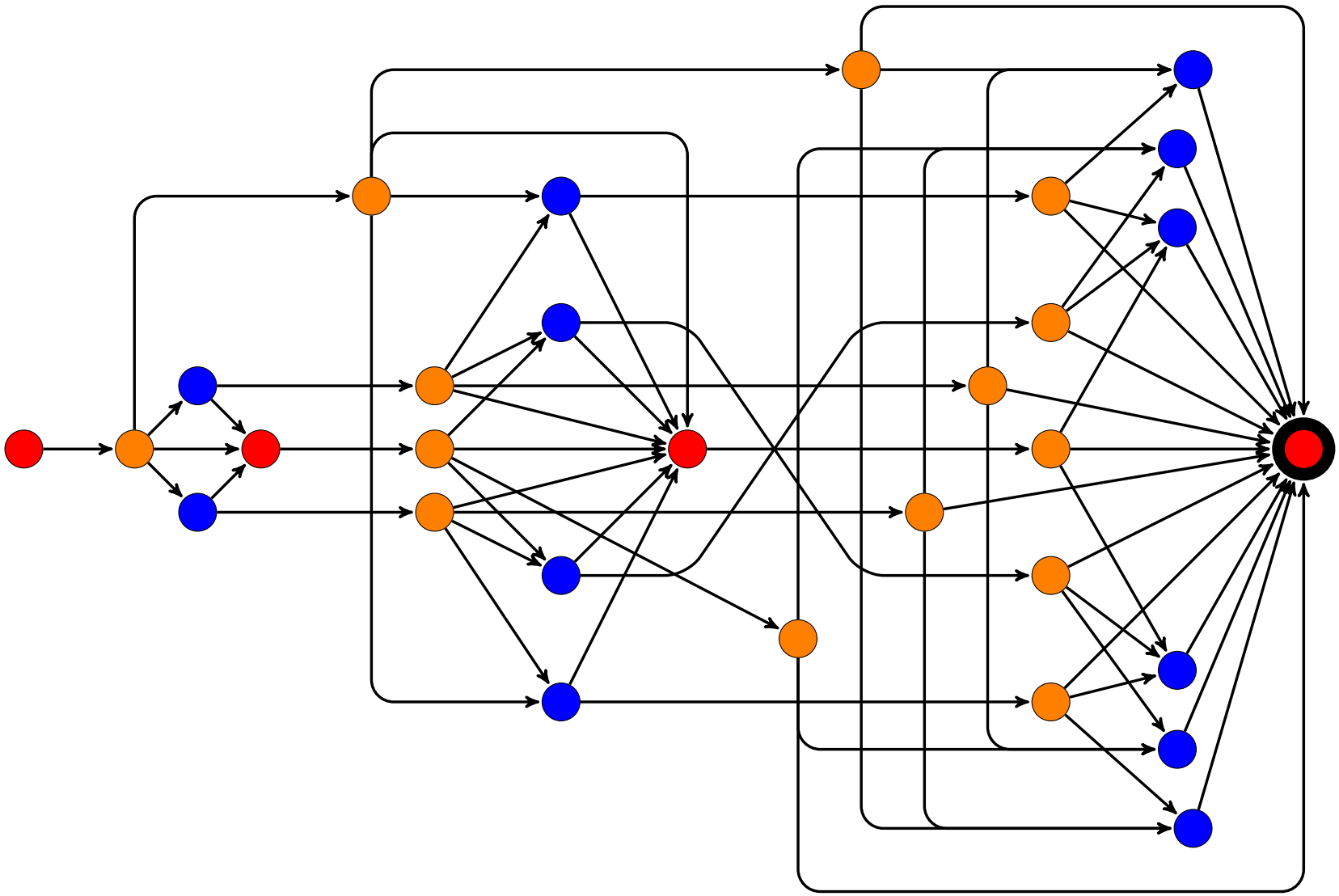
Digraph of strategy independent partial order on S

Over all feasible strategies



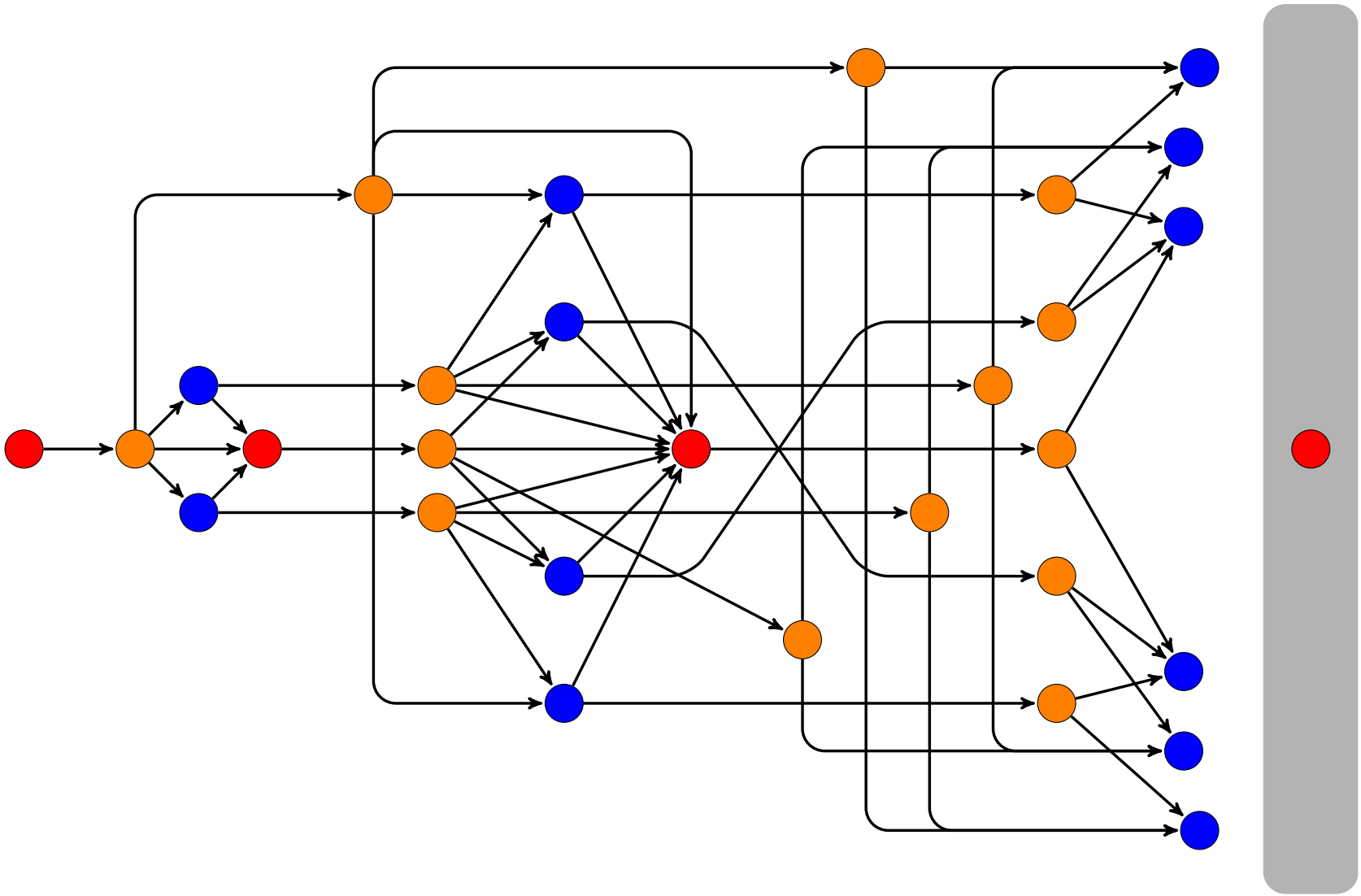
Topological sort of game graph

Identify terminal states



Topological sort of game graph

Remove terminal states



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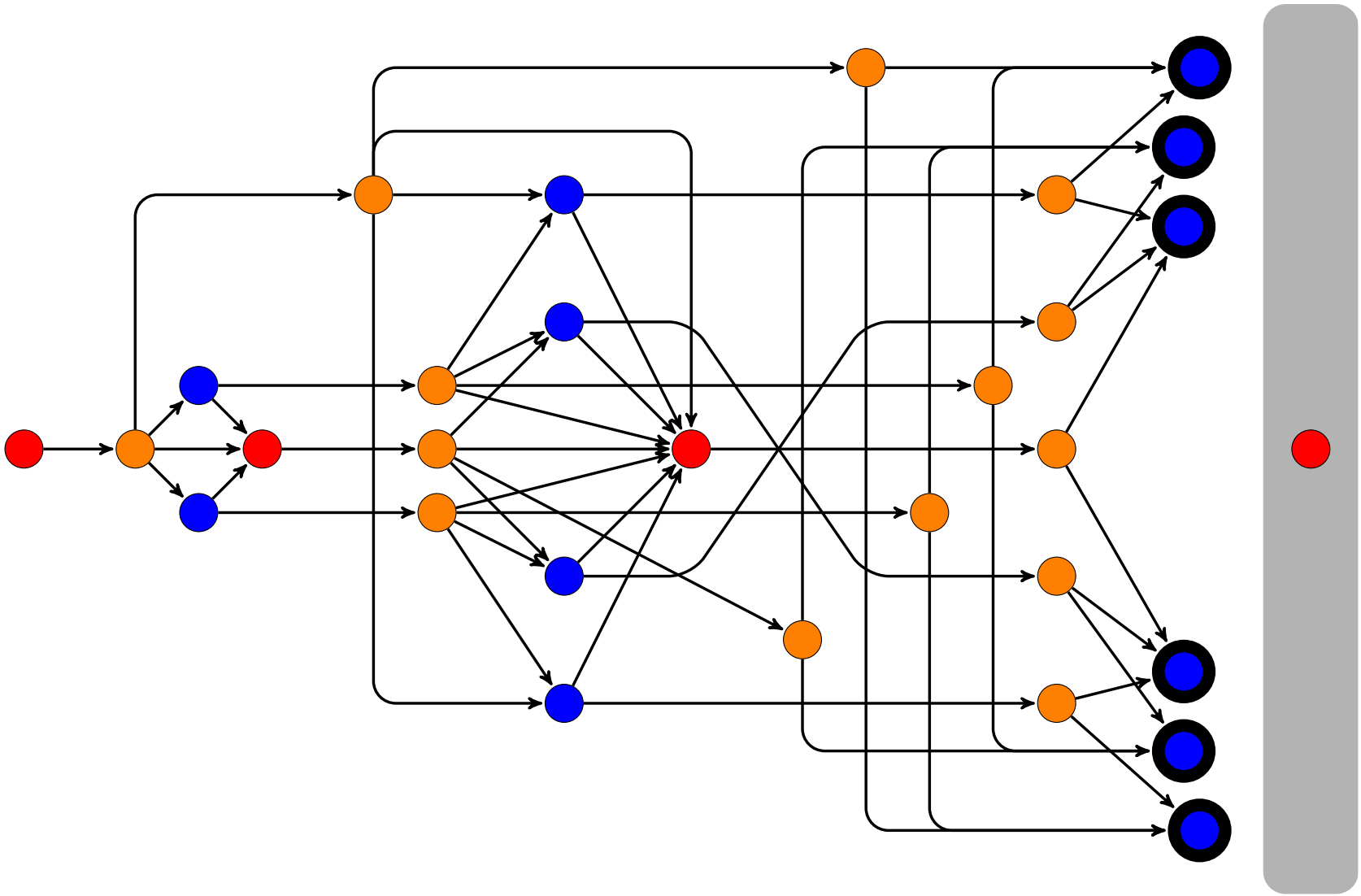

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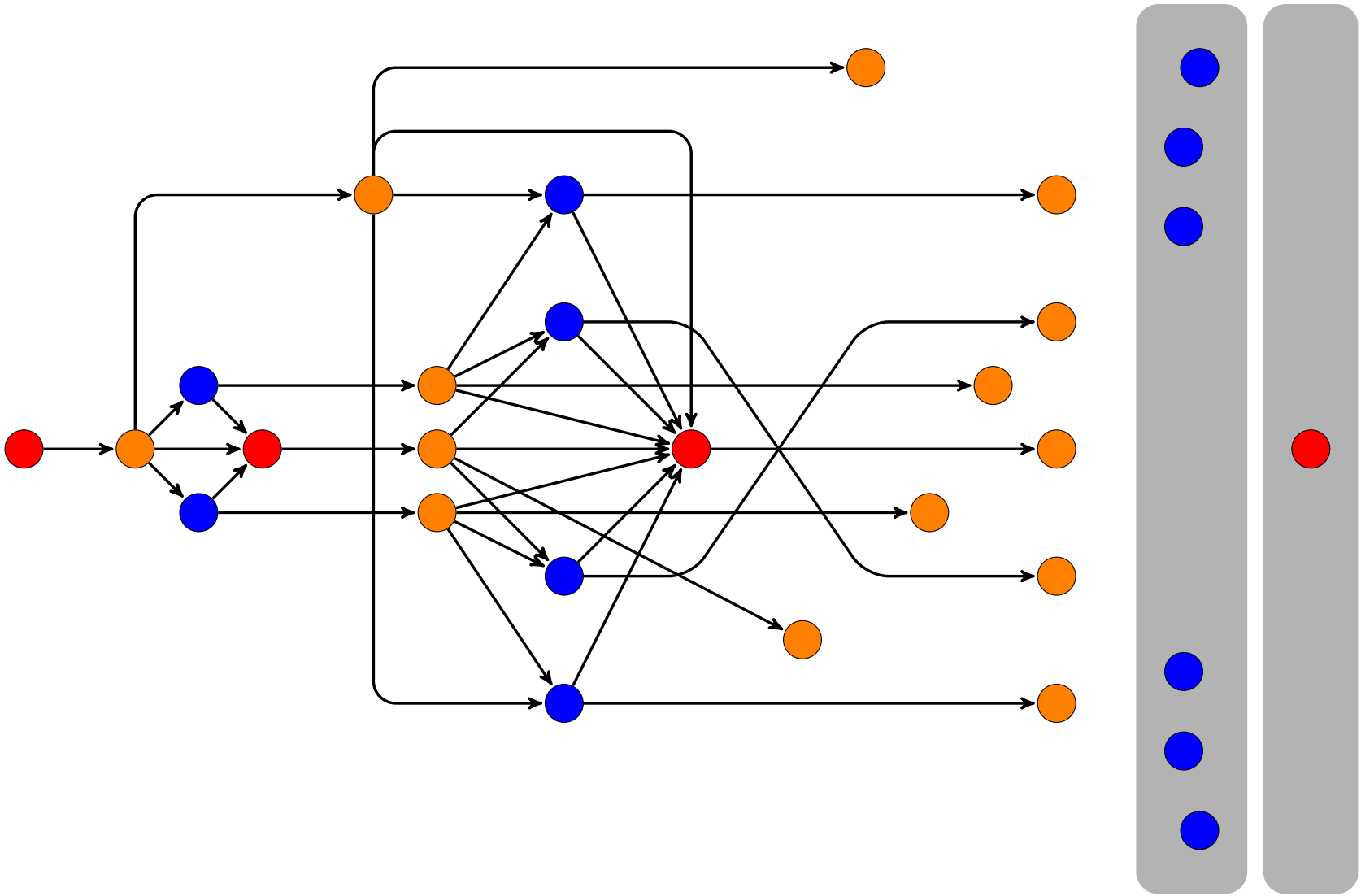

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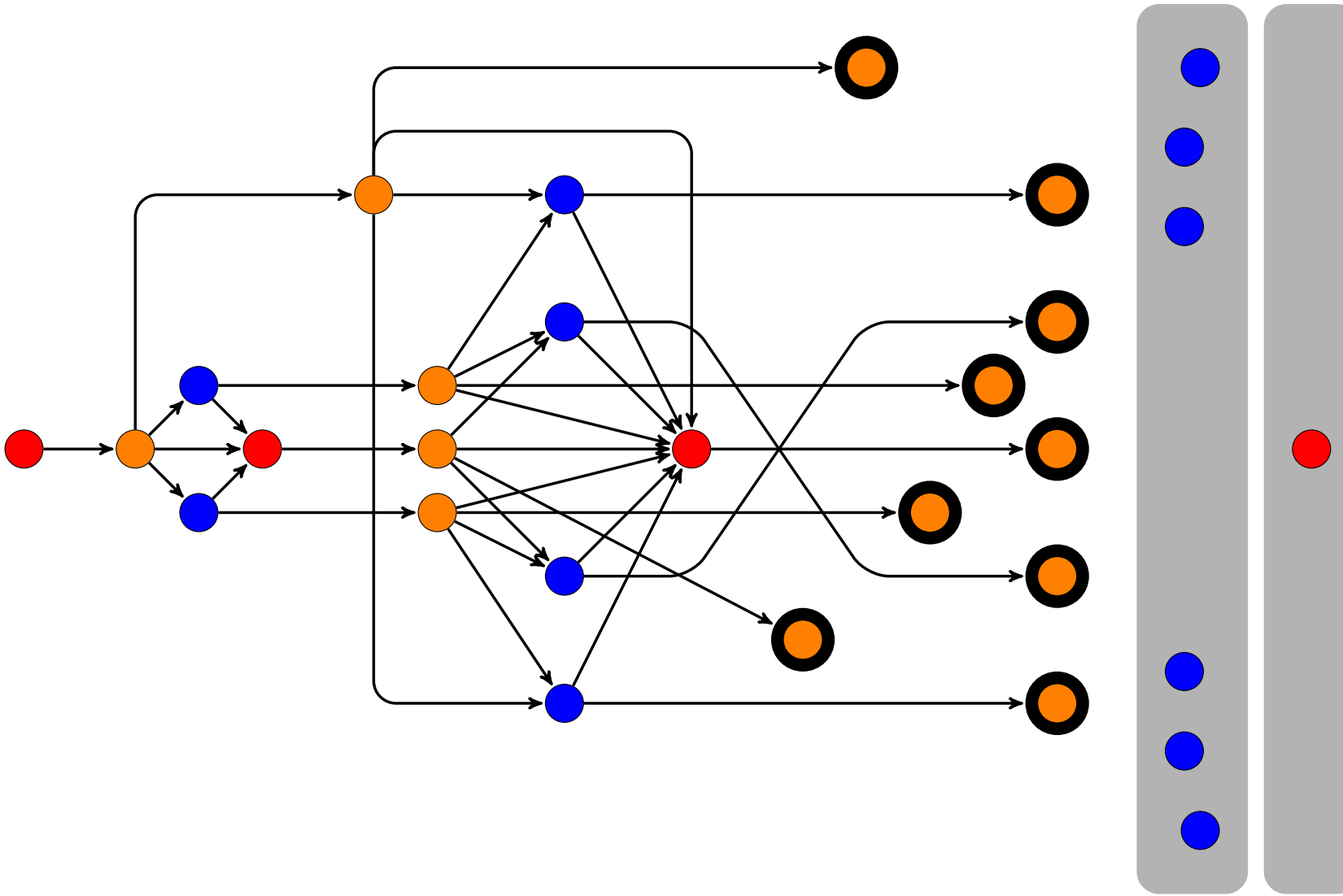

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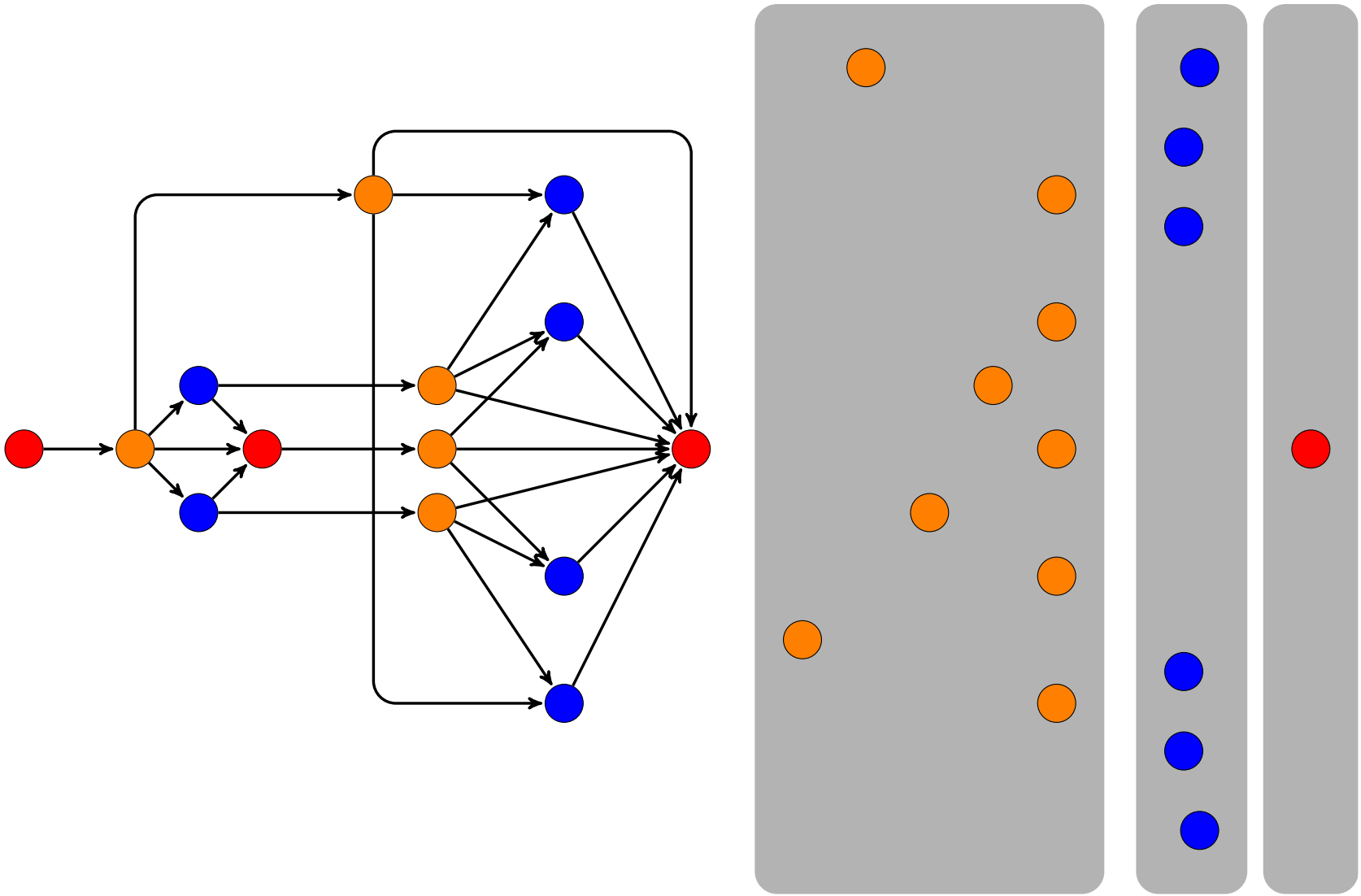

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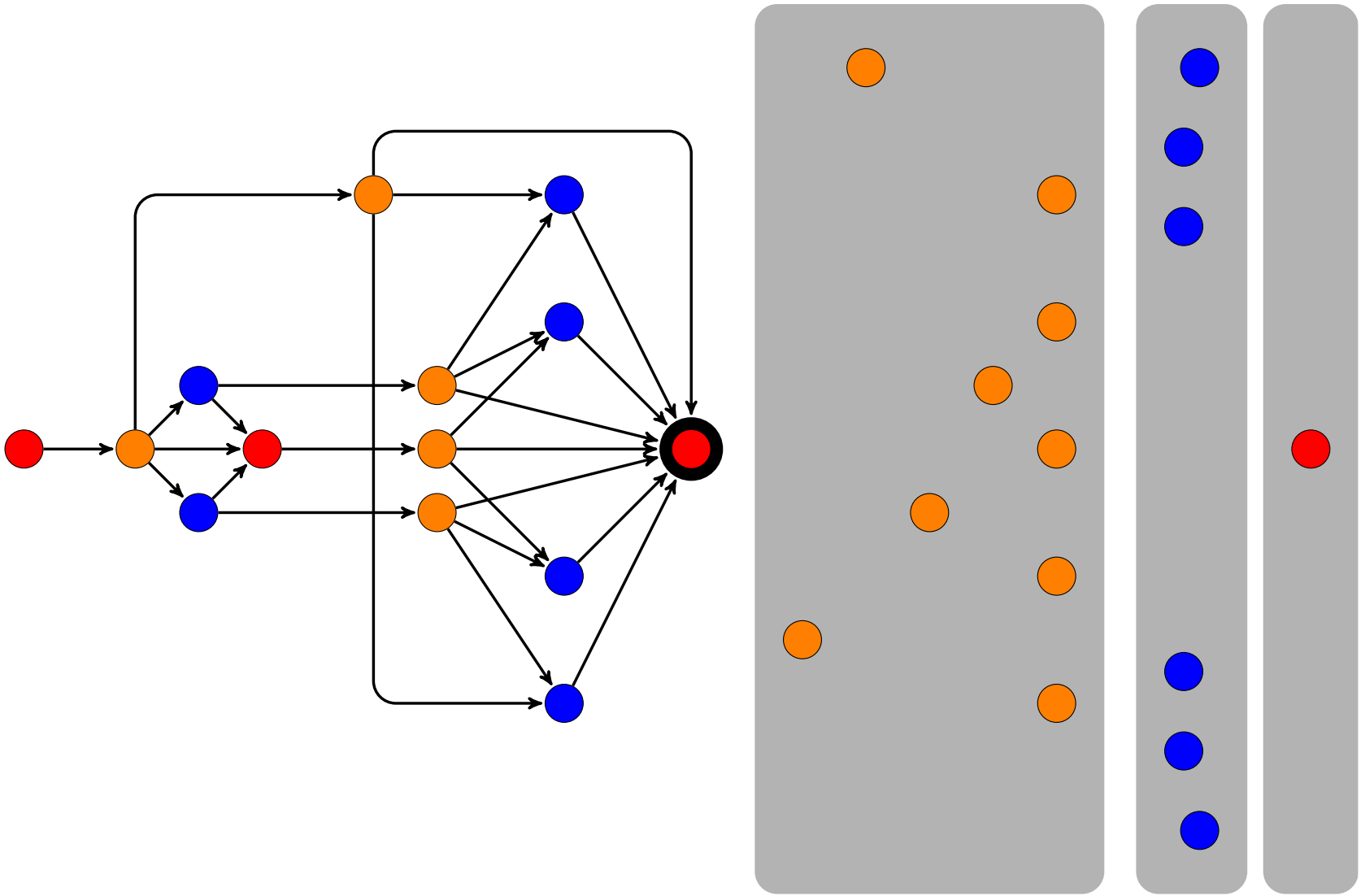

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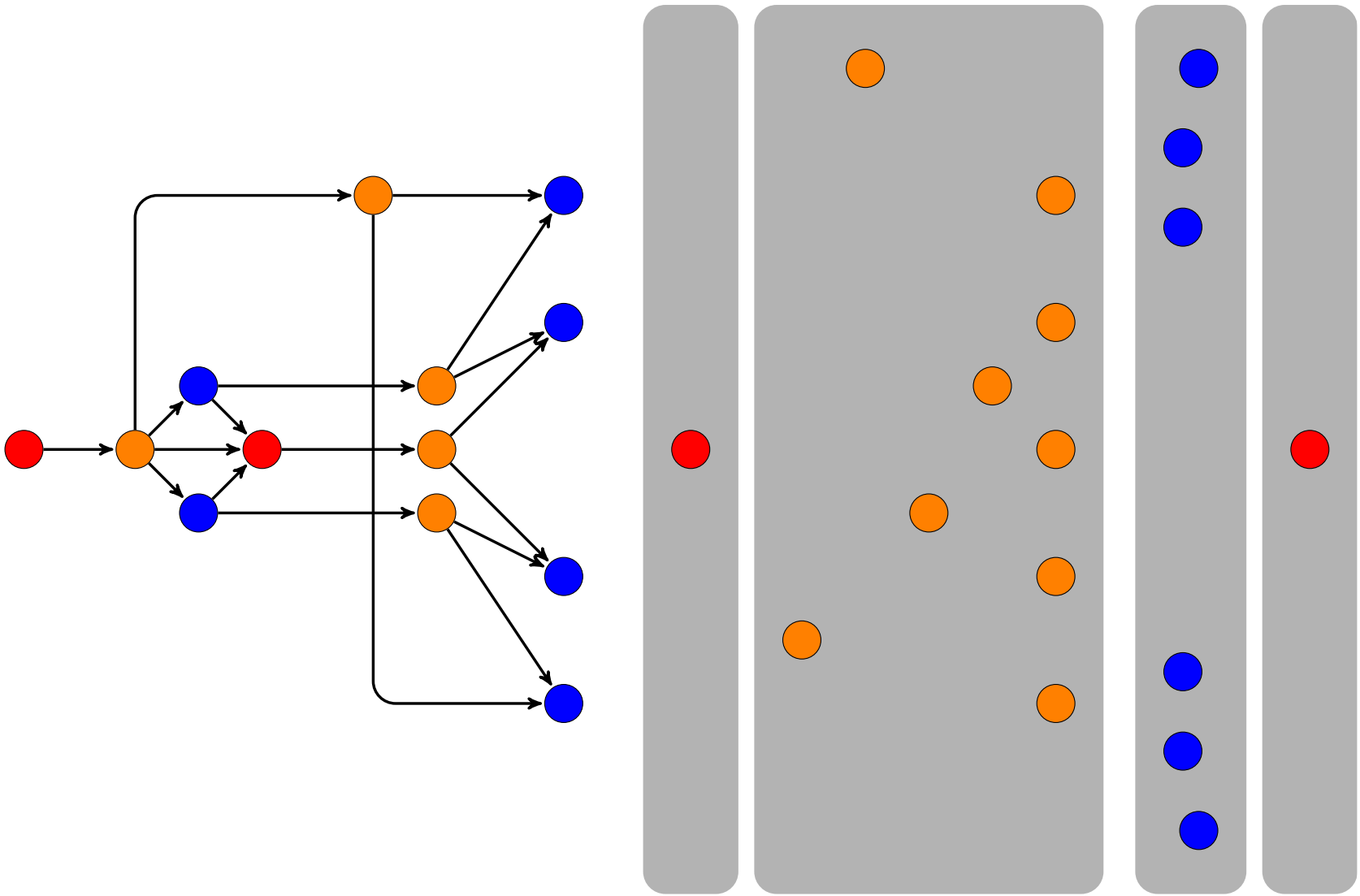

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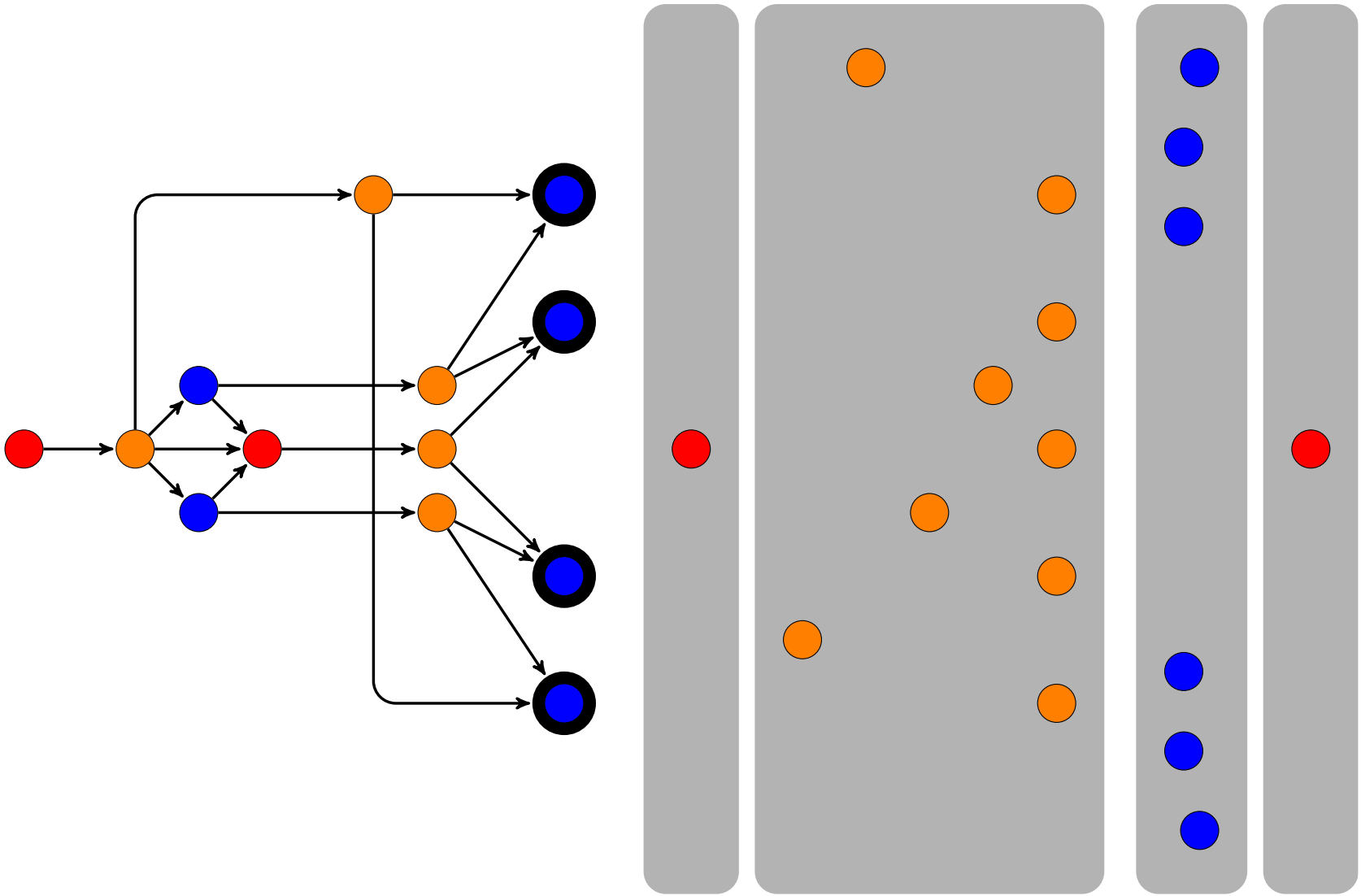

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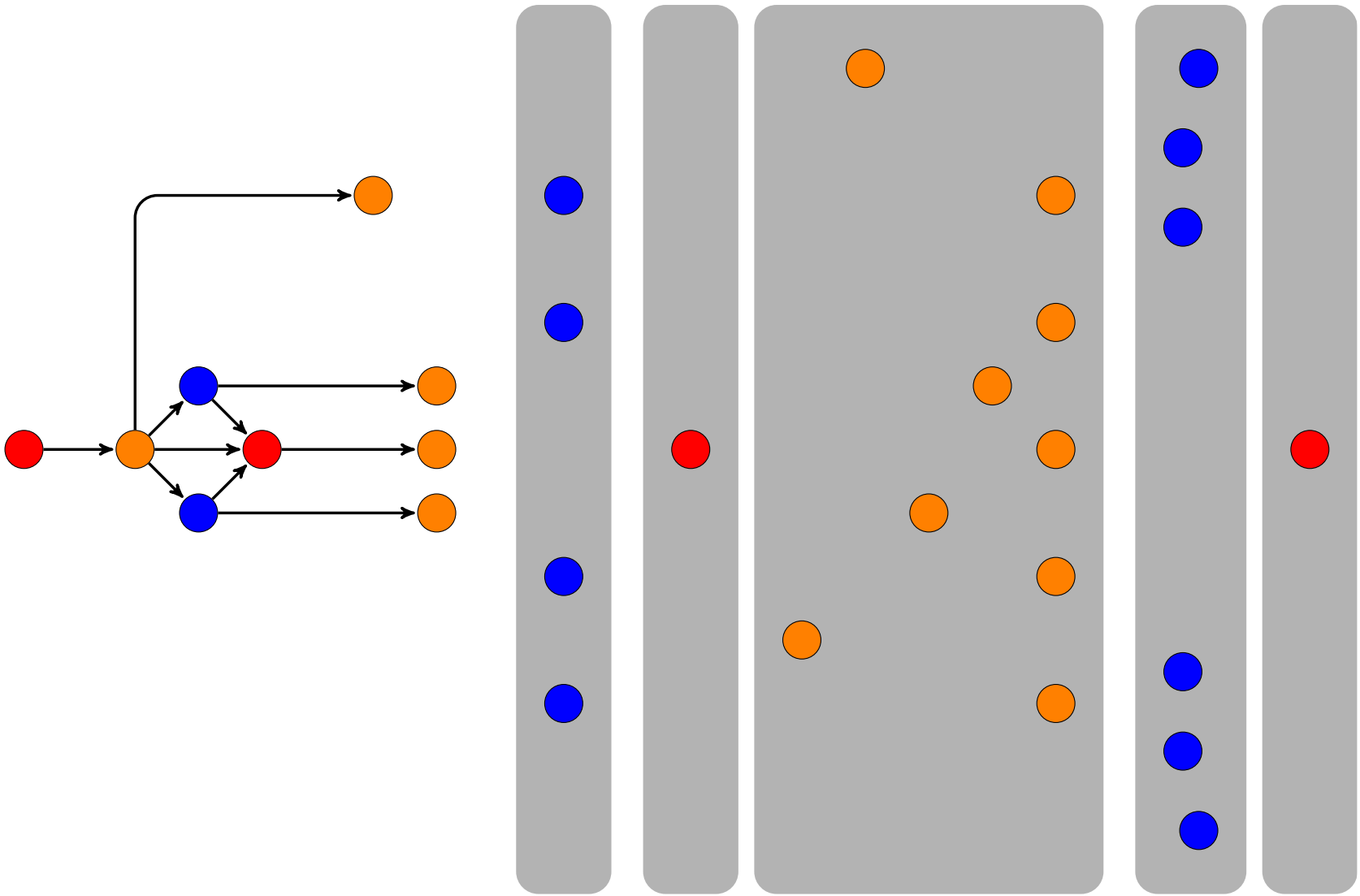

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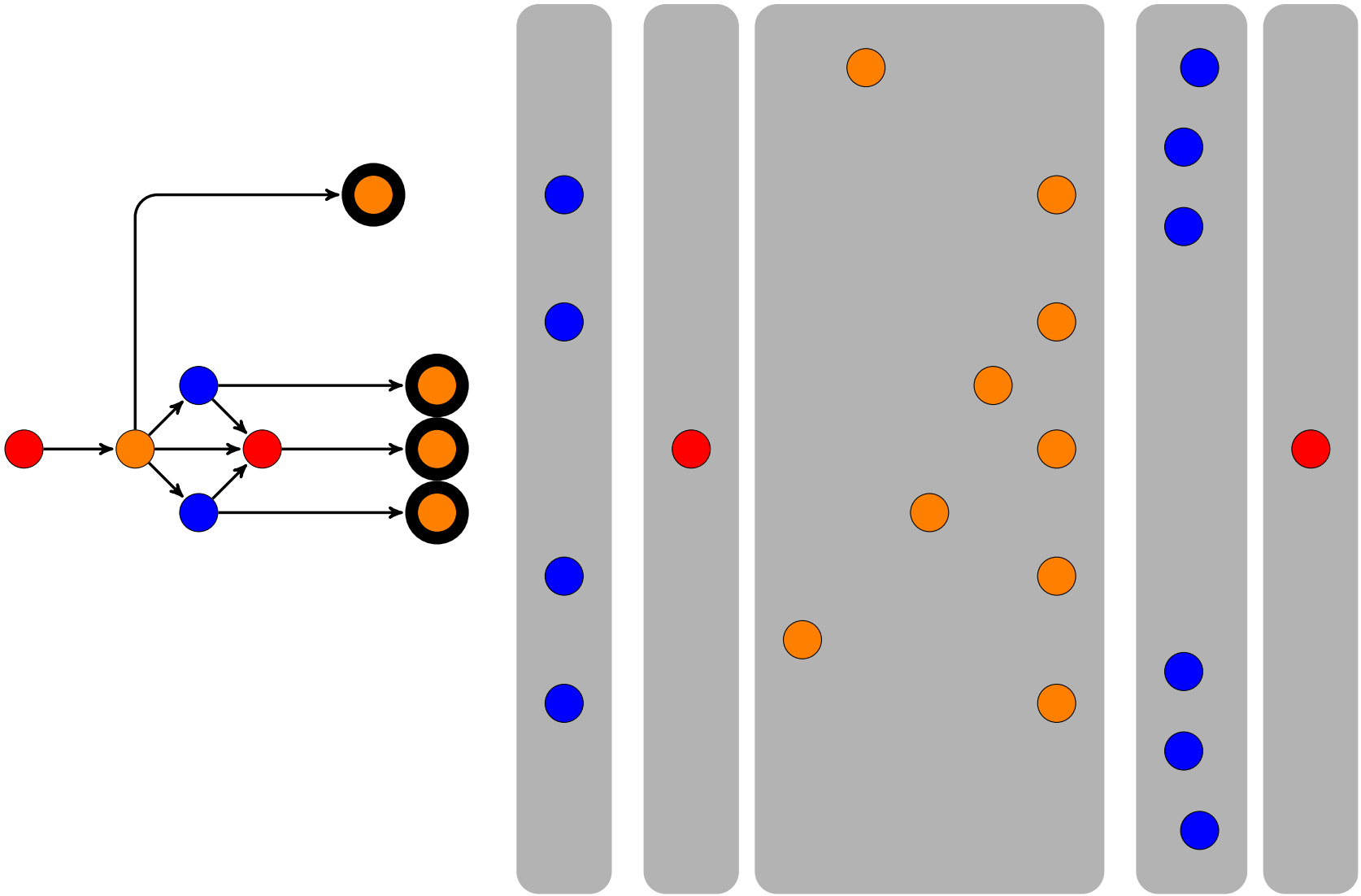

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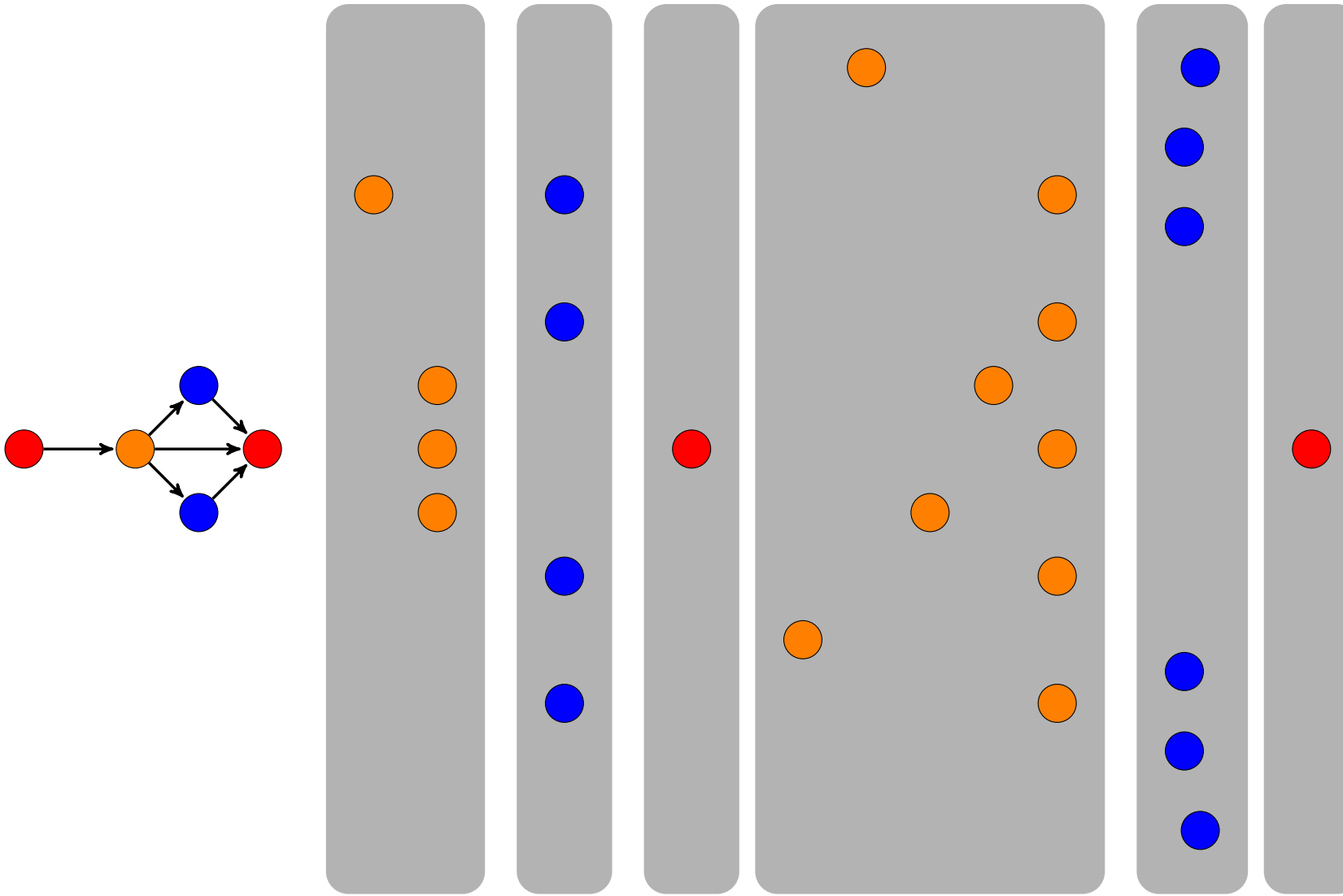

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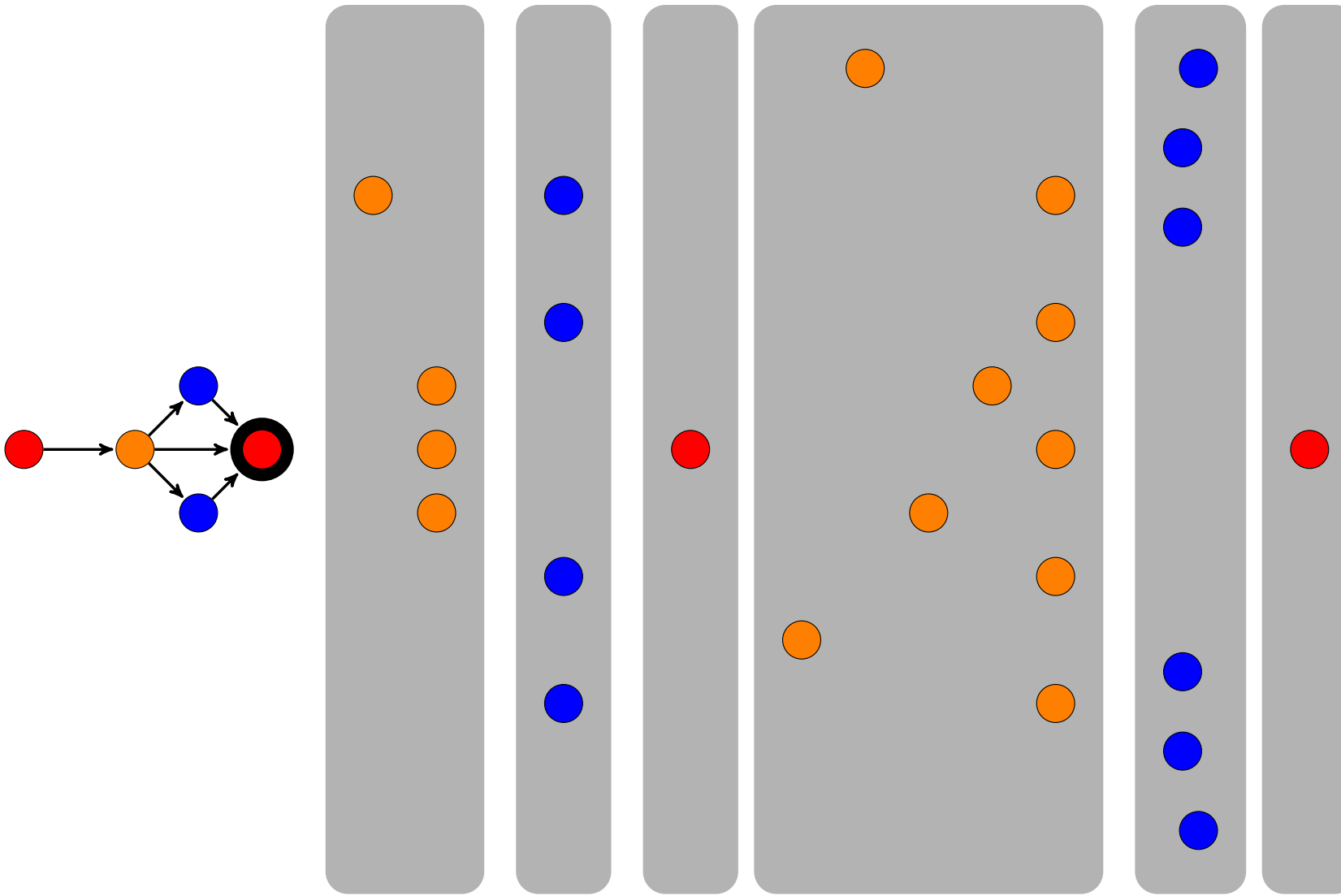

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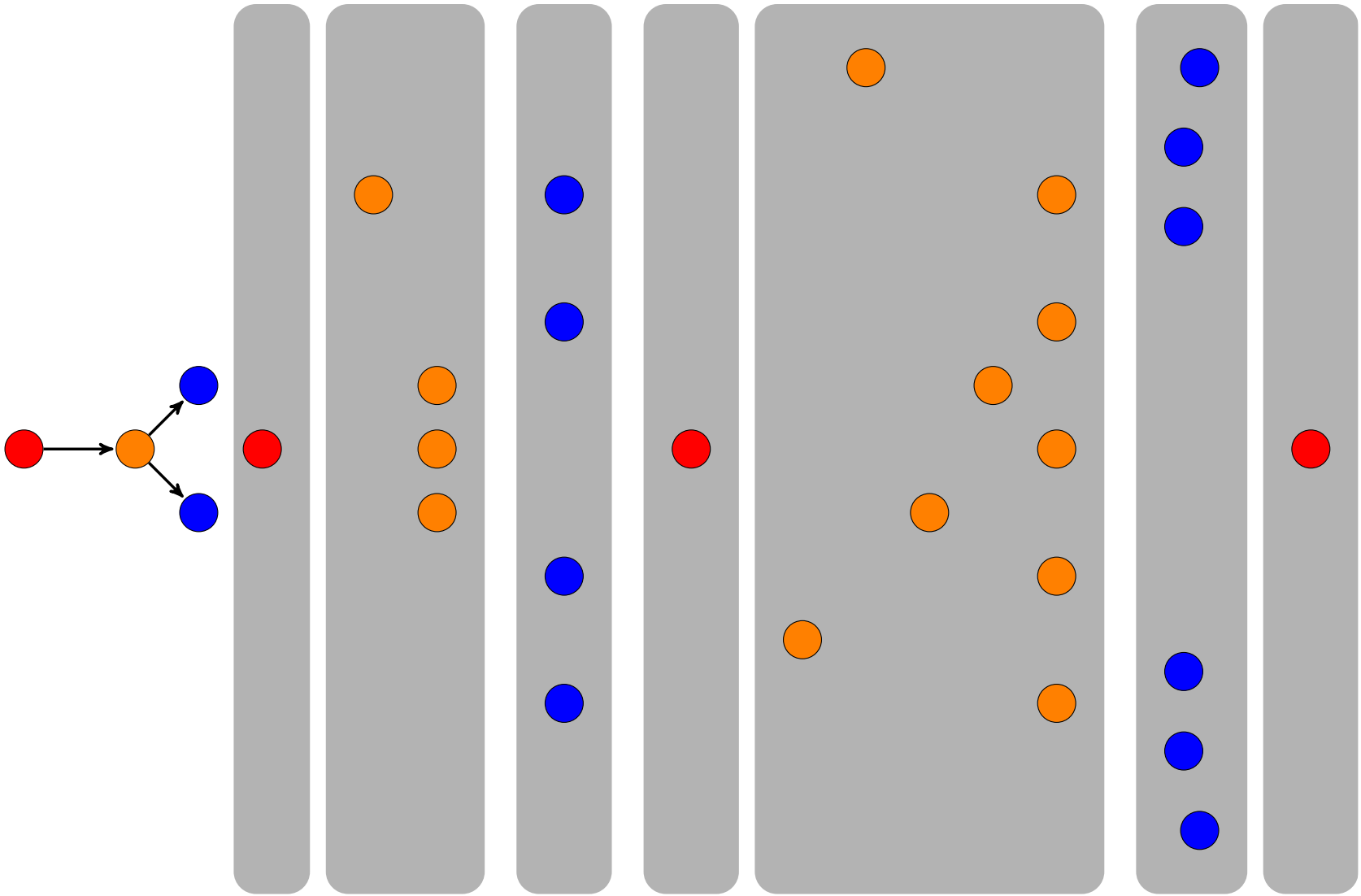

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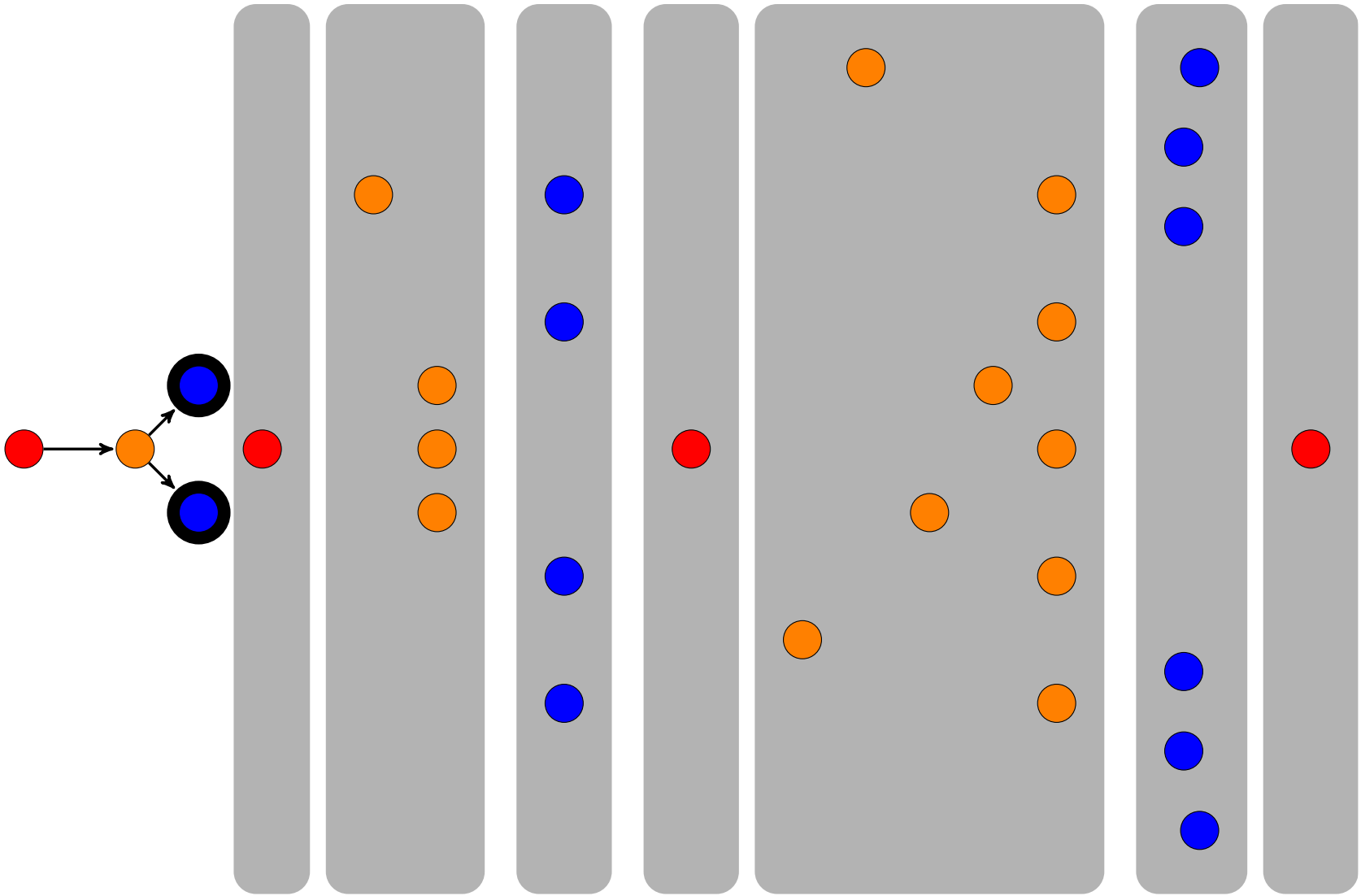

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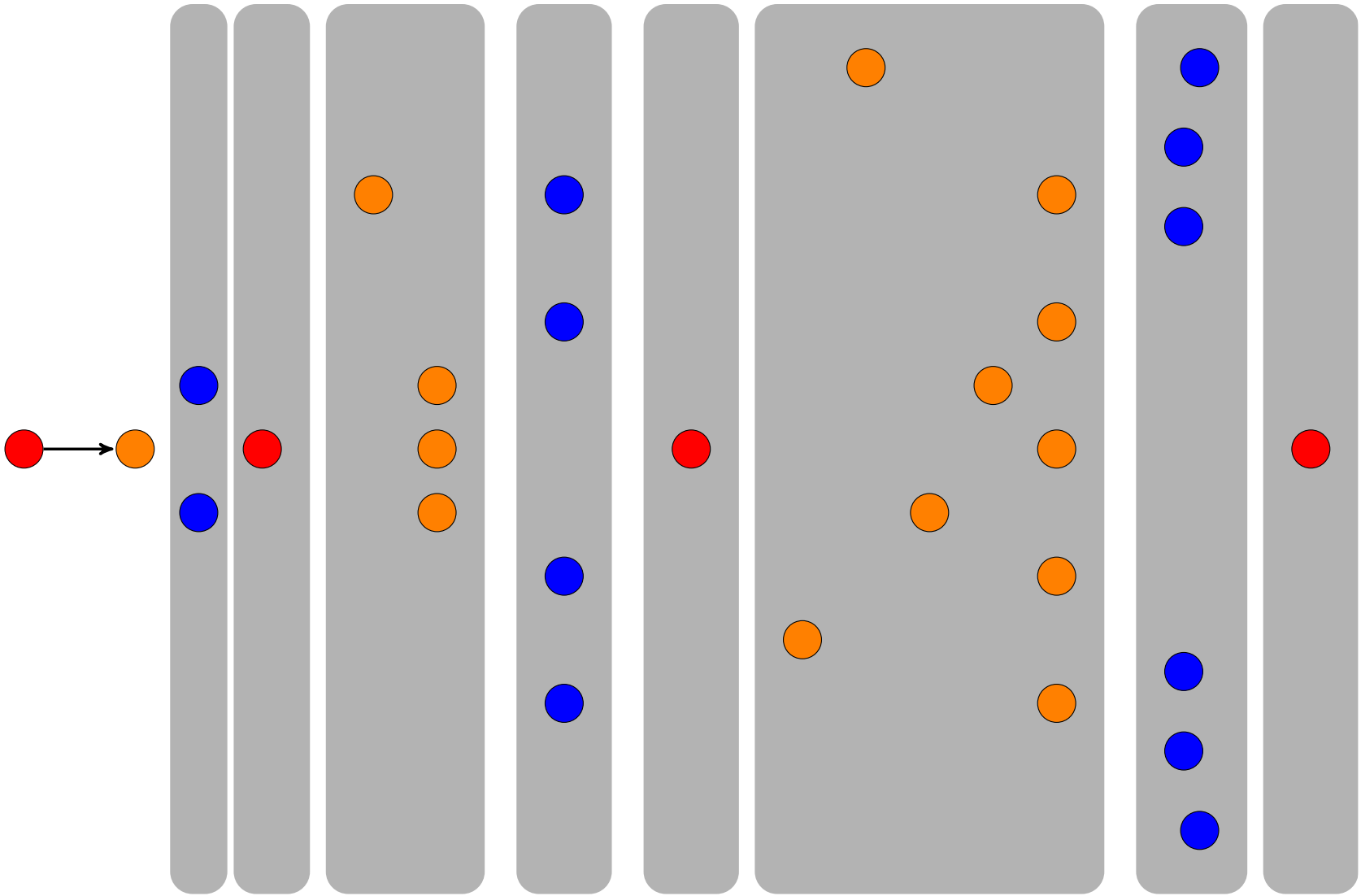

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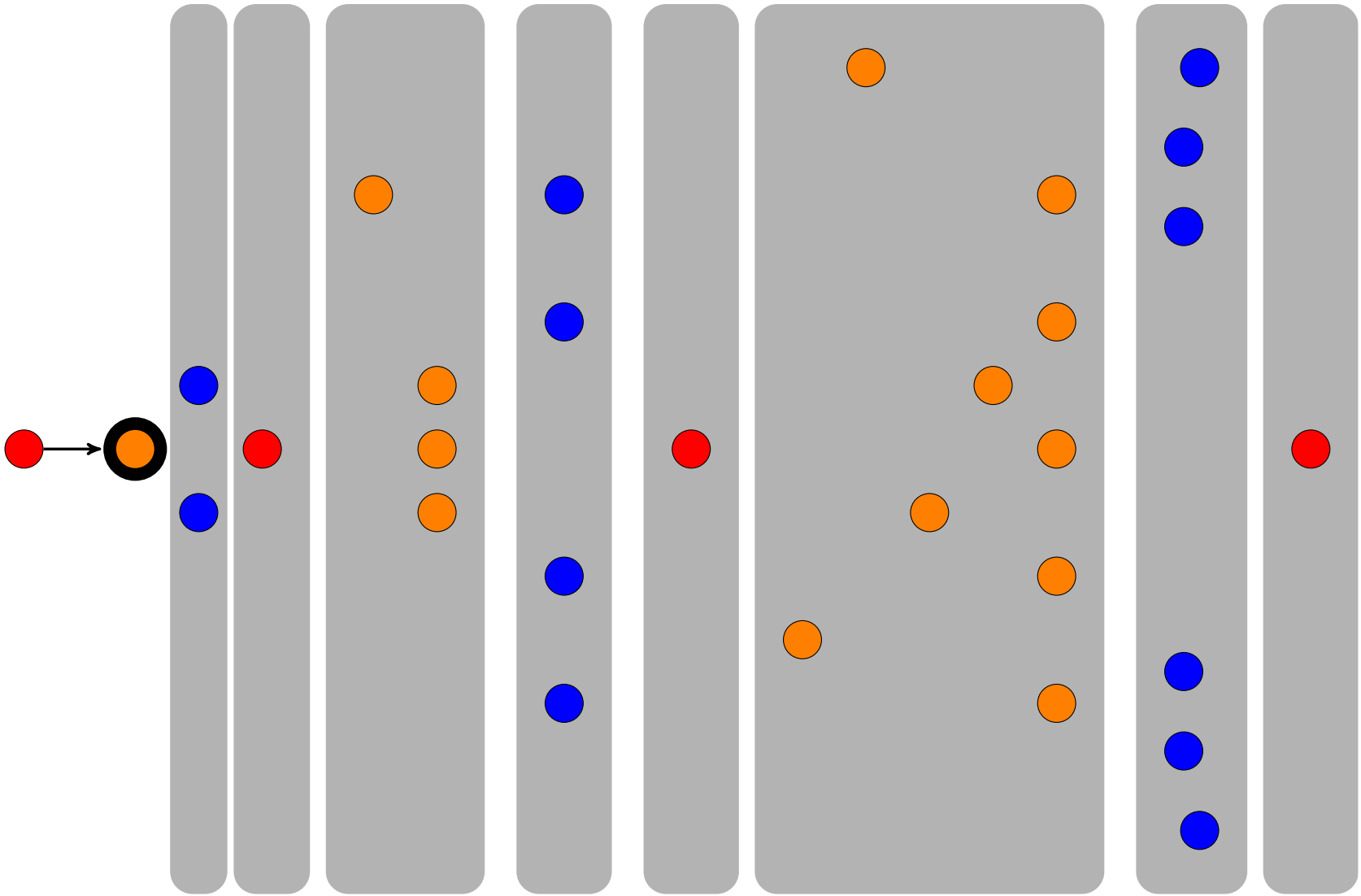

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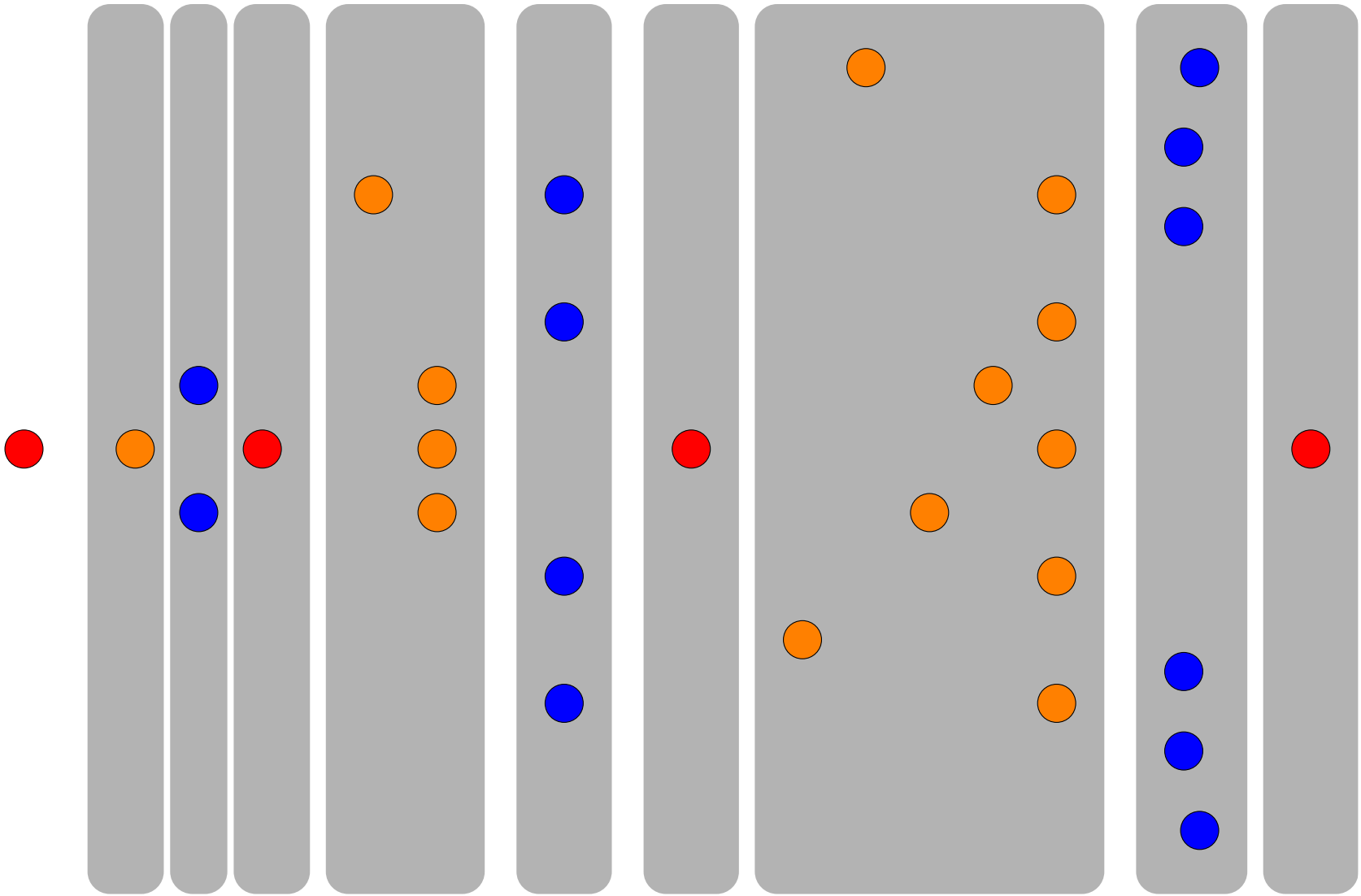


Directional dynamic games and DAG

- S is finite $\implies$ Topological sort completes in finite number of steps
- If null-graph is reached after K iterations, then the game graph is a DAG and S is partitioned into K stages
- If the algorithm stops and there are still edges in the remaining graph, then there are cycles in the game graph

Game is DDG

$\Updownarrow$

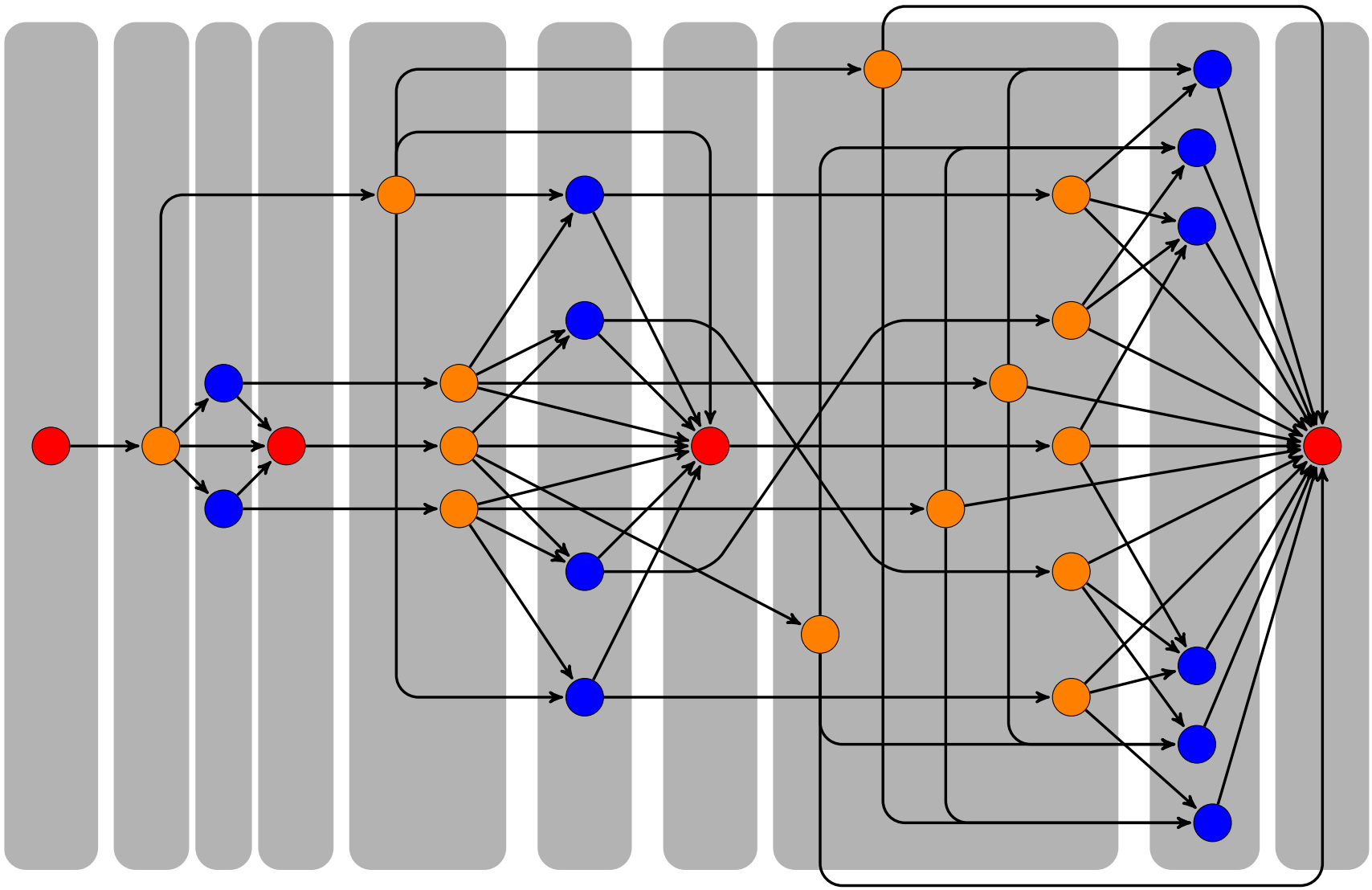
Topological sort produces a full partition of S into stages

$\Updownarrow$

The game graph is a DAG

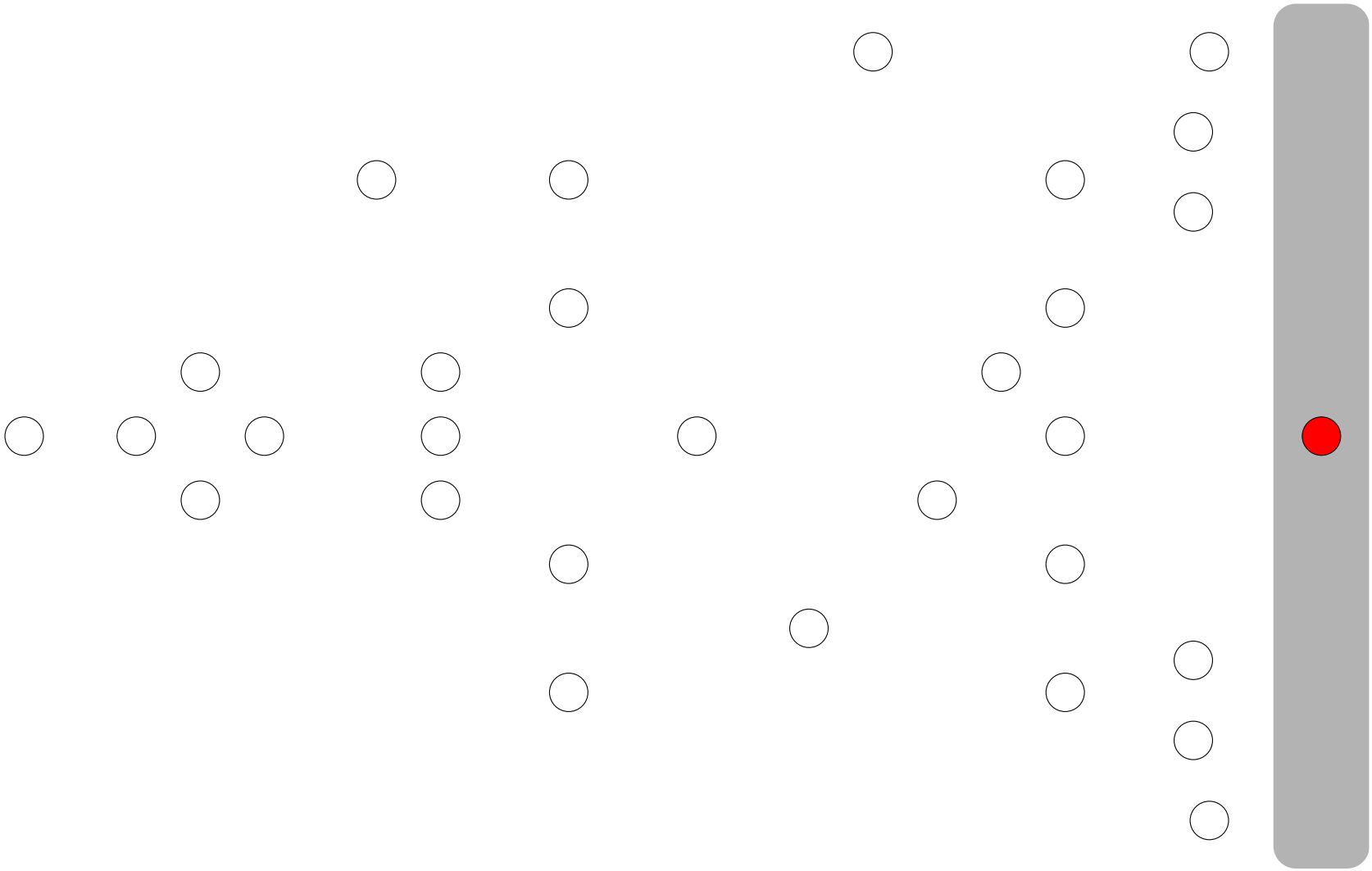
Total order on the set of stages

After running a topoligical sort algorithm on the DAG



State recursion algorithm

Backward induction on stages of DDG



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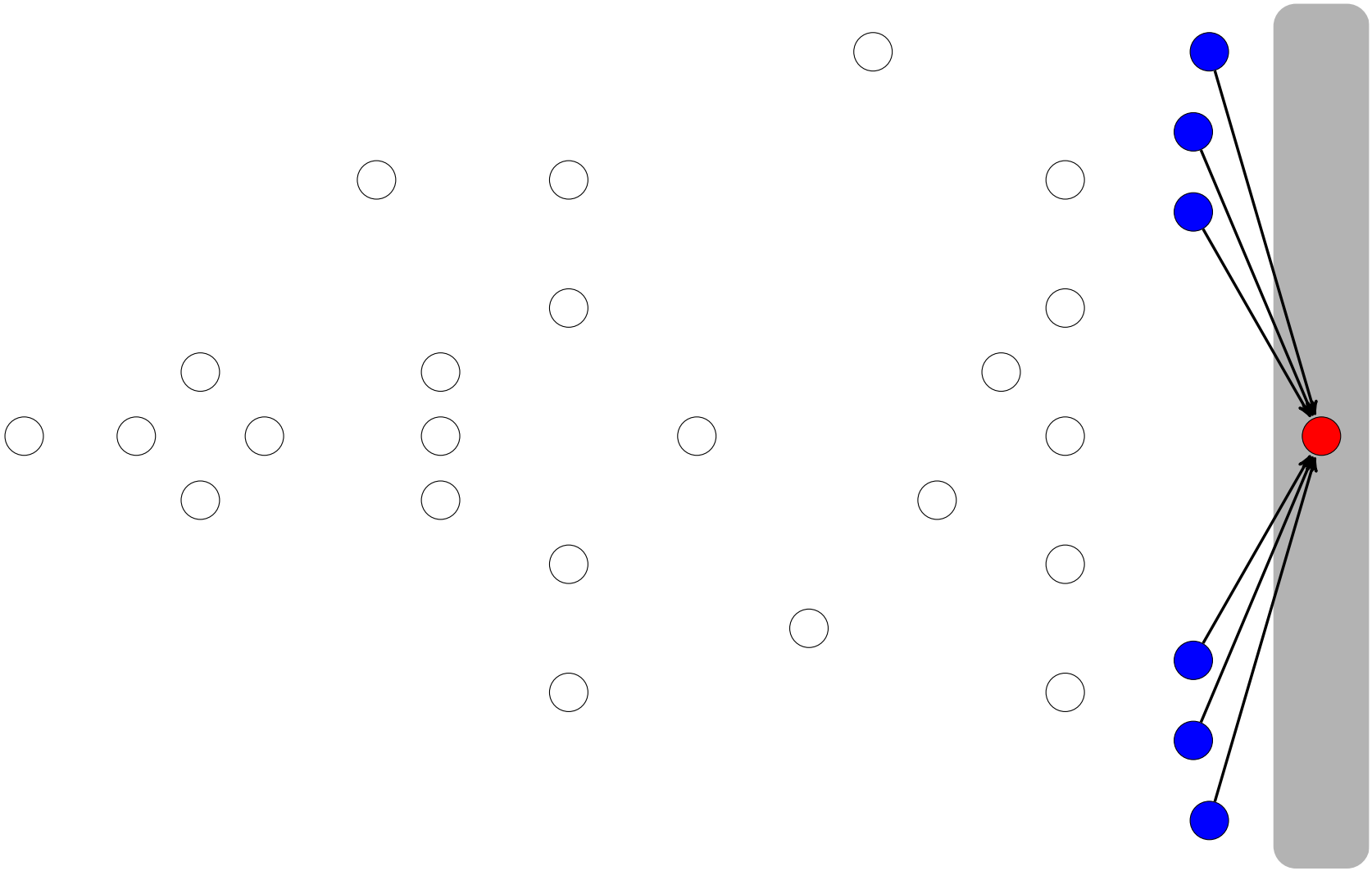

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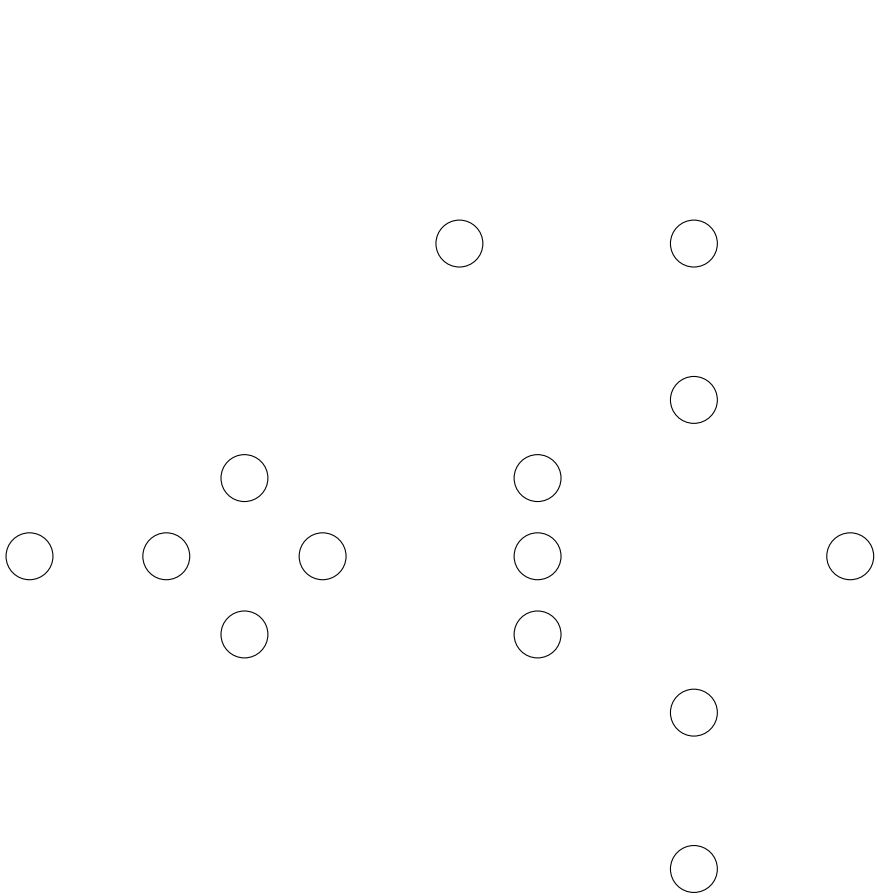

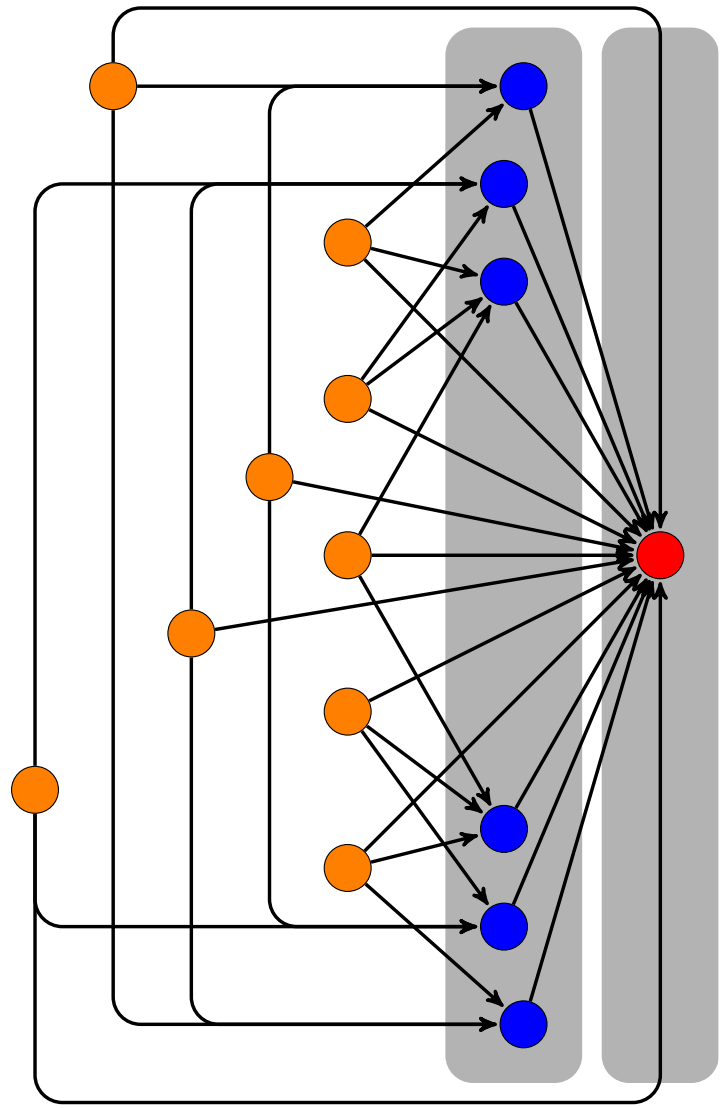

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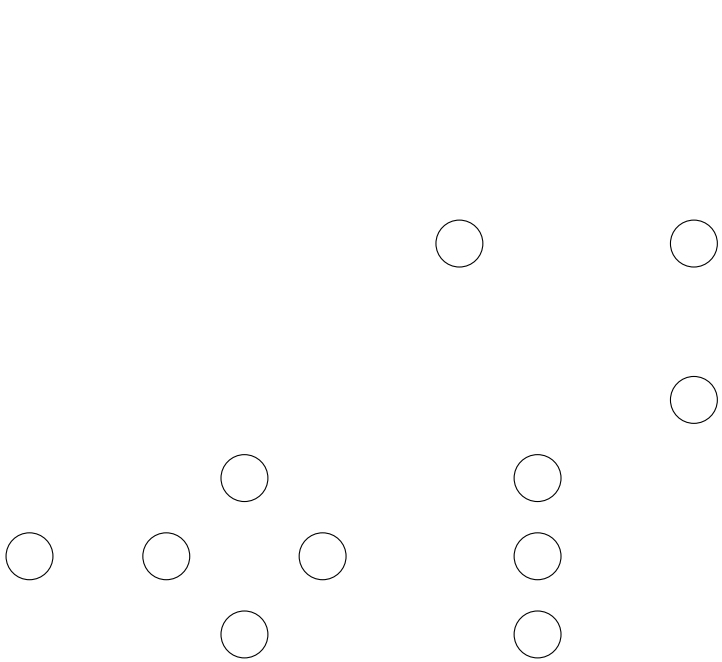
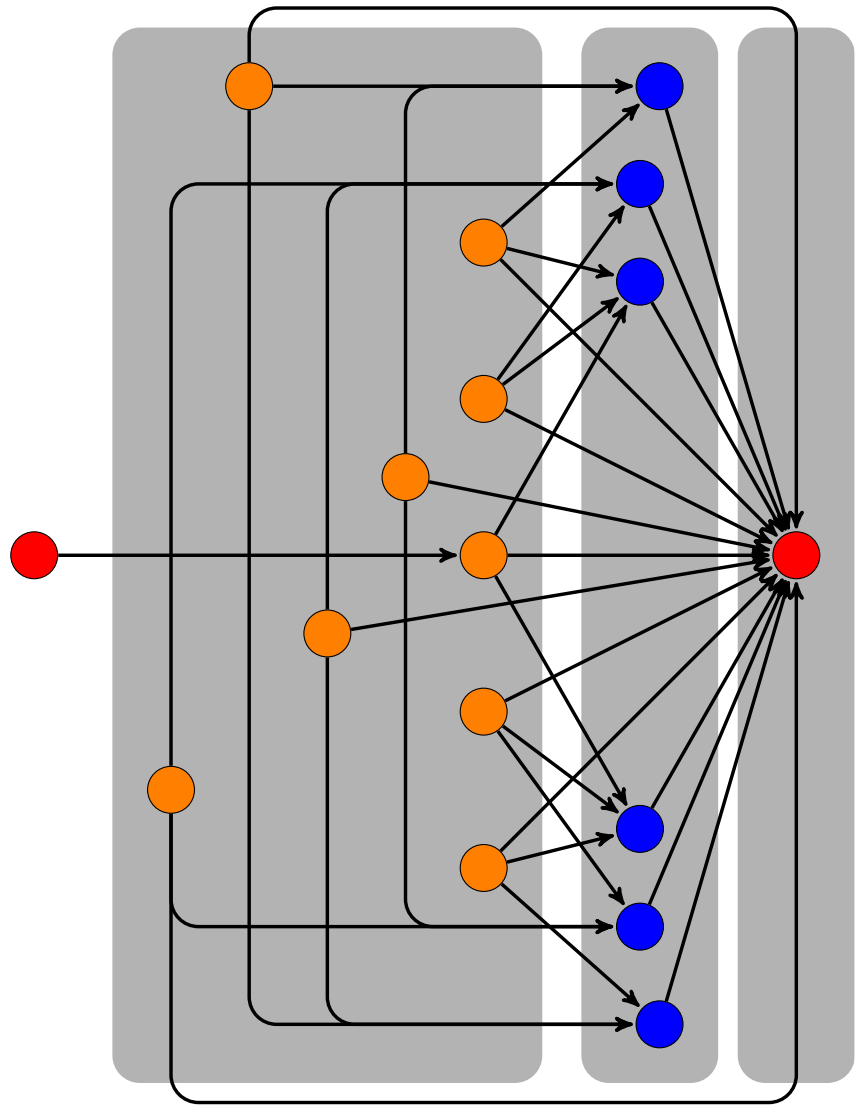
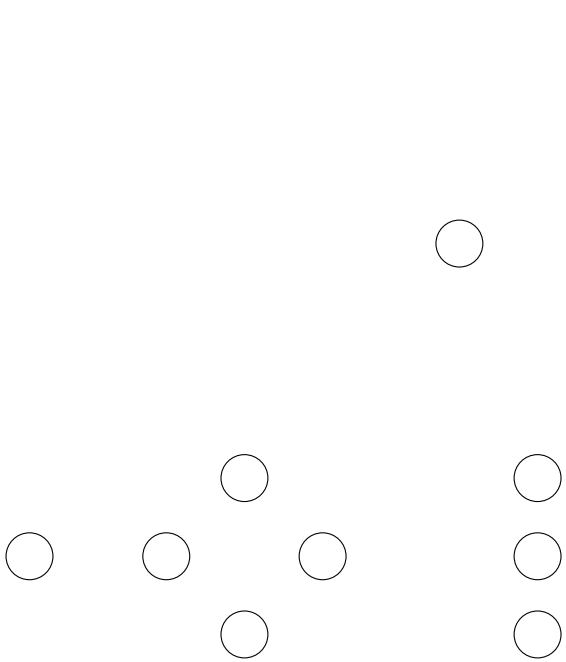
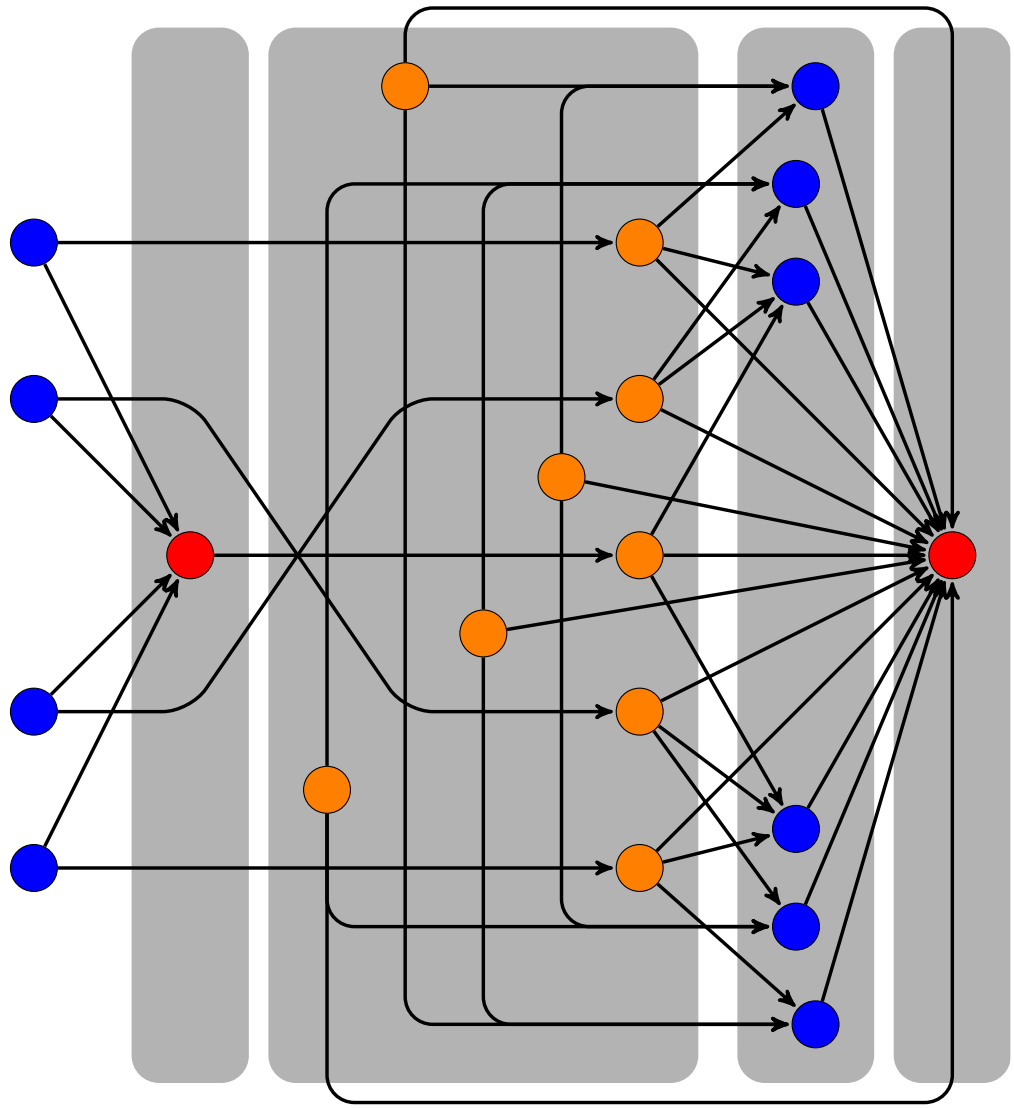




State recursion algorithm

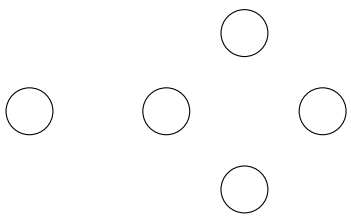
Backward induction on stages of DDG

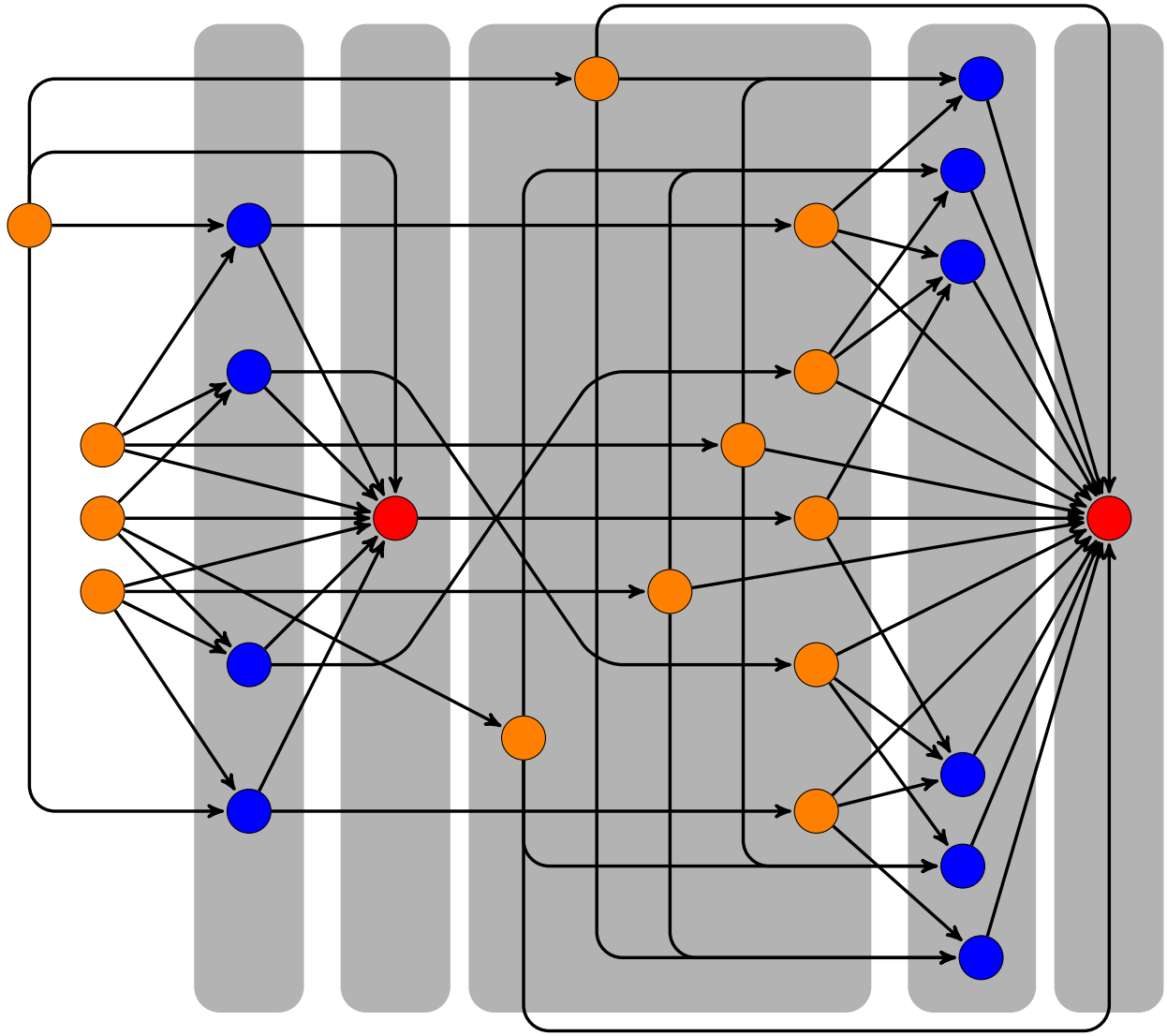




State recursion algorithm

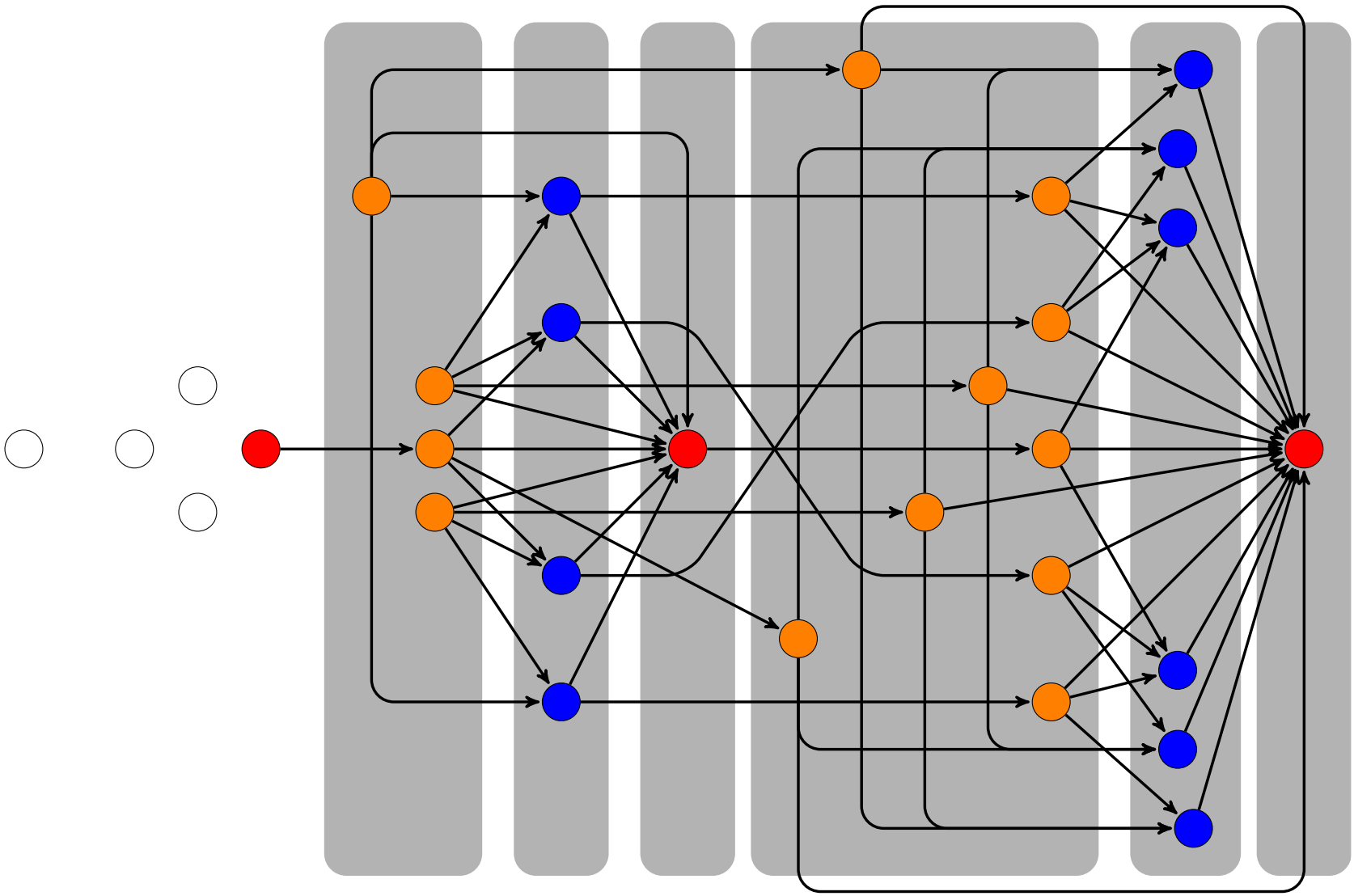
Backward induction on stages of DDG





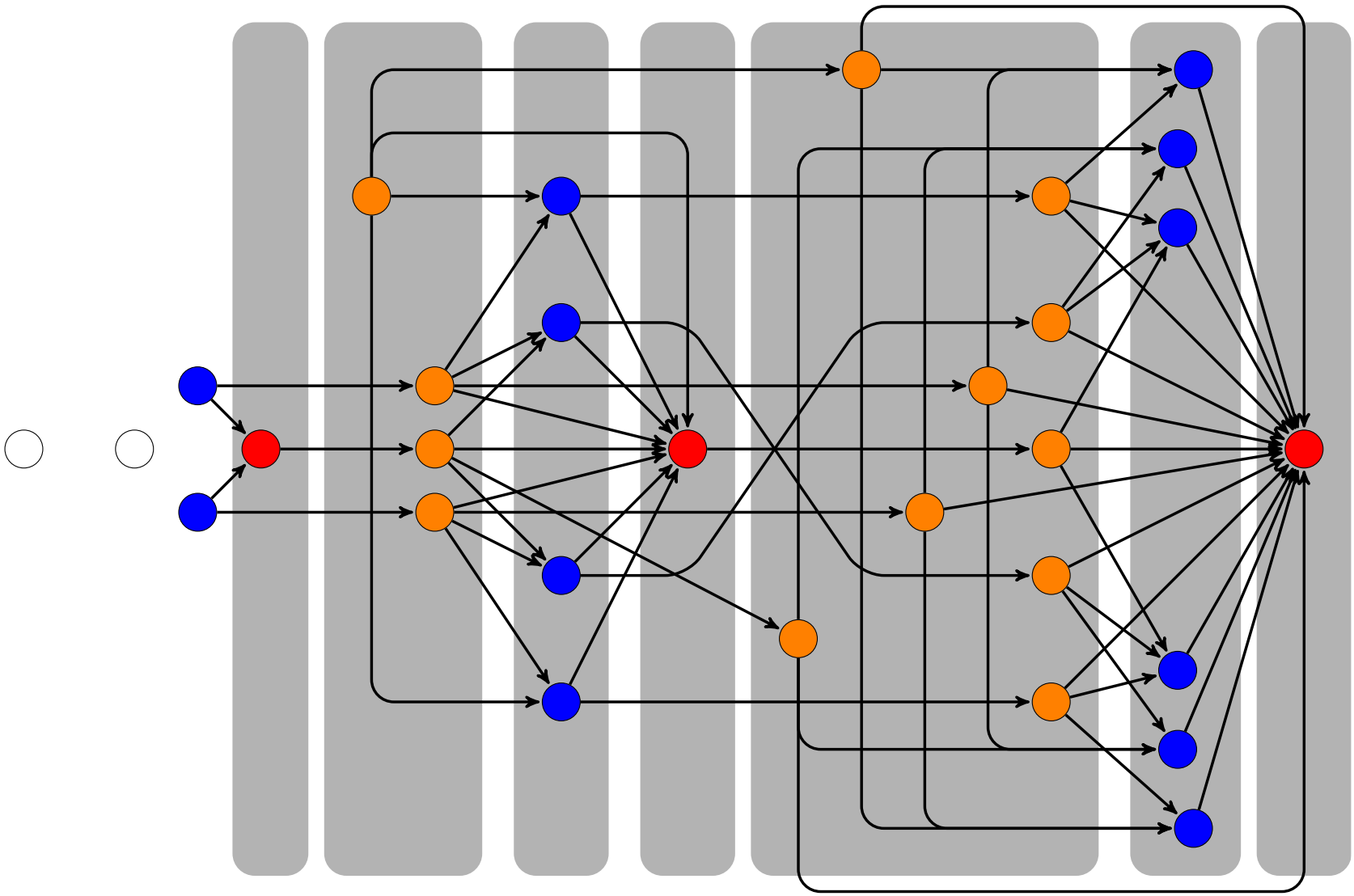
State recursion algorithm

Backward induction on stages of DDG



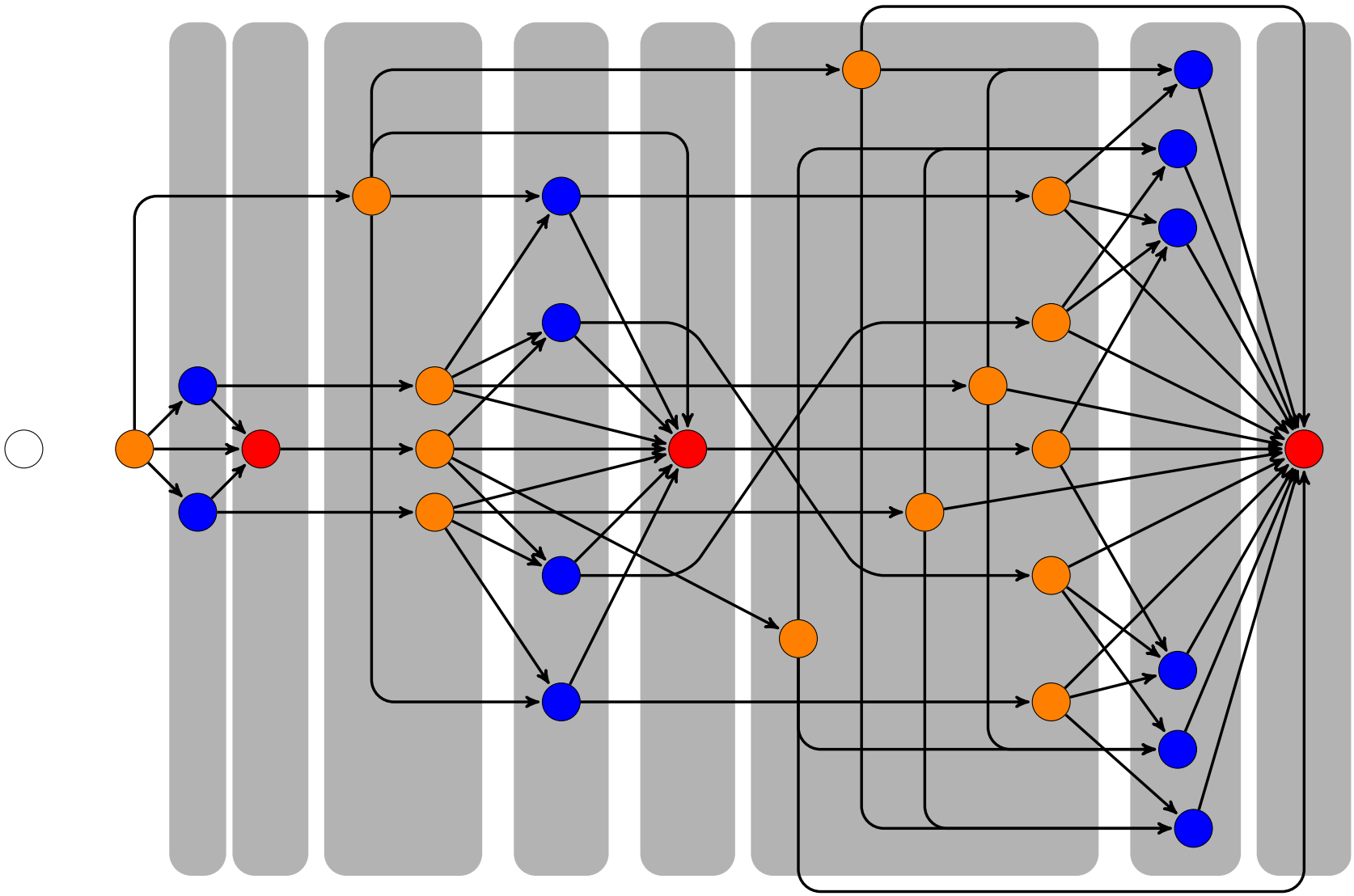
State recursion algorithm

Backward induction on stages of DDG



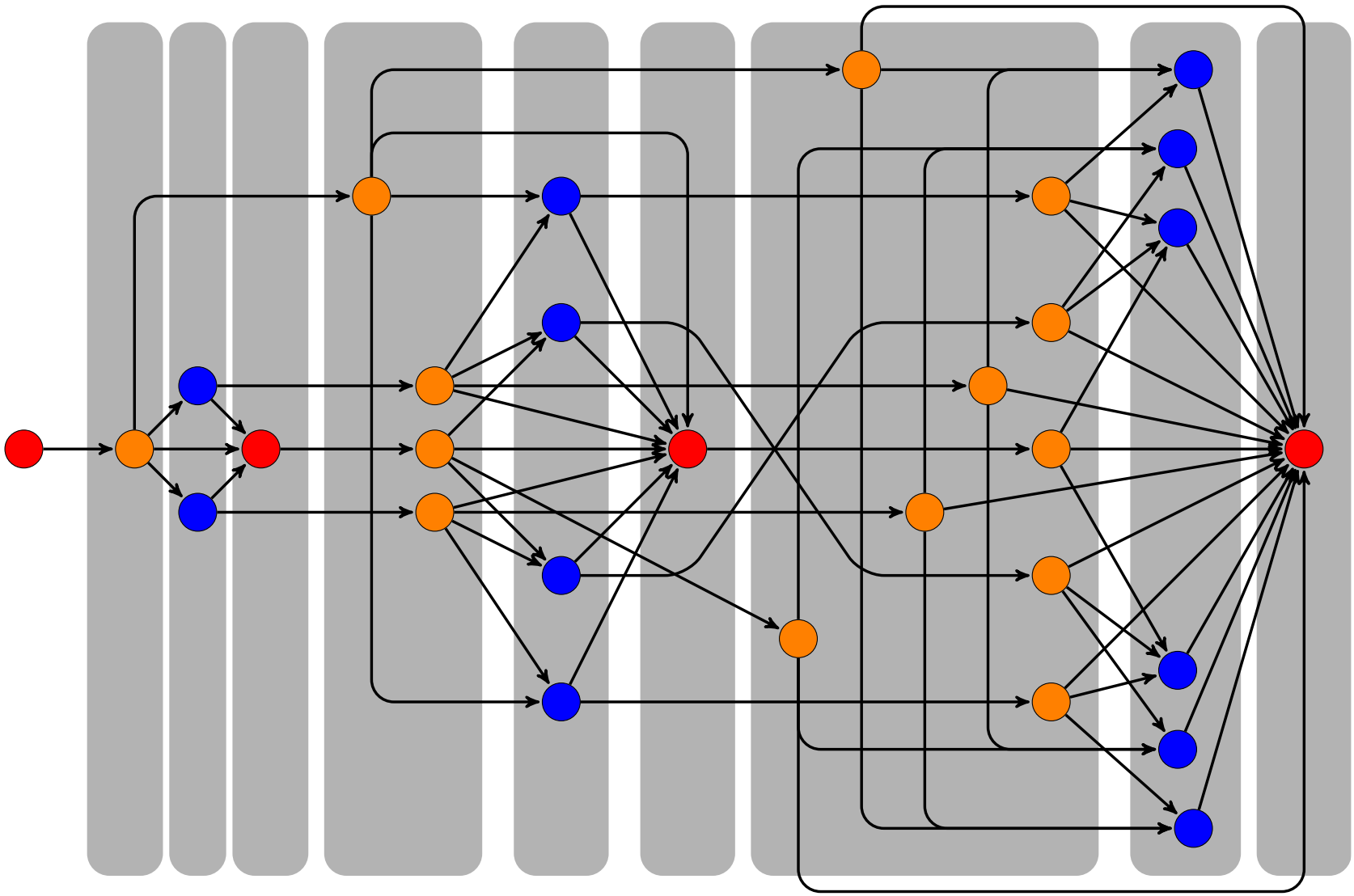
State recursion algorithm

Backward induction on stages of DDG



State recursion algorithm

Backward induction on stages of DDG



State Recursion versus Backward Induction

- State recursion – generalization of backward induction
- Runs on state space instead of time periods
- Time (t) evolves as $t \to t+1$ with probability 1
- For stages of state space (τ) transitions are stochastic and not necessarily sequential
- Yet, probability of going $\tau \to \tau'$ is zero when $\tau' < \tau$
- With multiplicity, state recursion is performed conditional of a particular equilibrium selection rule (ESR)

State Recursion and Model Transitions

- Bellman equation in integrated/expected value function form

$$V_i^\sigma(x) = \int \max_{a_i \in A} \{v_i^\sigma(a_i, x) + \eta \varepsilon_i[a_i]\} g_i(d\varepsilon_i) = \eta \log \sum_{a_i \in A} \exp\left(\frac{v_i^\sigma(a_i, x)}{\eta}\right) + \eta\gamma$$

  - $V_i^\sigma(x)$ – value function integrated over taste shocks ε_i of player i
  - $v_i^\sigma(x)$ – choice-specific value function for action a_i

- Taking expectation over choices with choice probabilities $P_i^\sigma(a_i|x)$

$$V_i^\sigma(x) = \sum_{a_i \in A} P_i^\sigma(a_i|x) \left[\pi_i^\sigma(a_i, x) + e_i^\sigma(a_i, x) + \beta \sum_{x' \in X} V_i^\sigma(x') f_i^\sigma(x'|a_i, x)\right]$$

  - $e_i^\sigma(x)$ – expectation of taste shock ε_i conditional on action a_i chosen
  - $f_i^\sigma(x'|a_i, x)$ – transition probability under σ and action a_i by player i

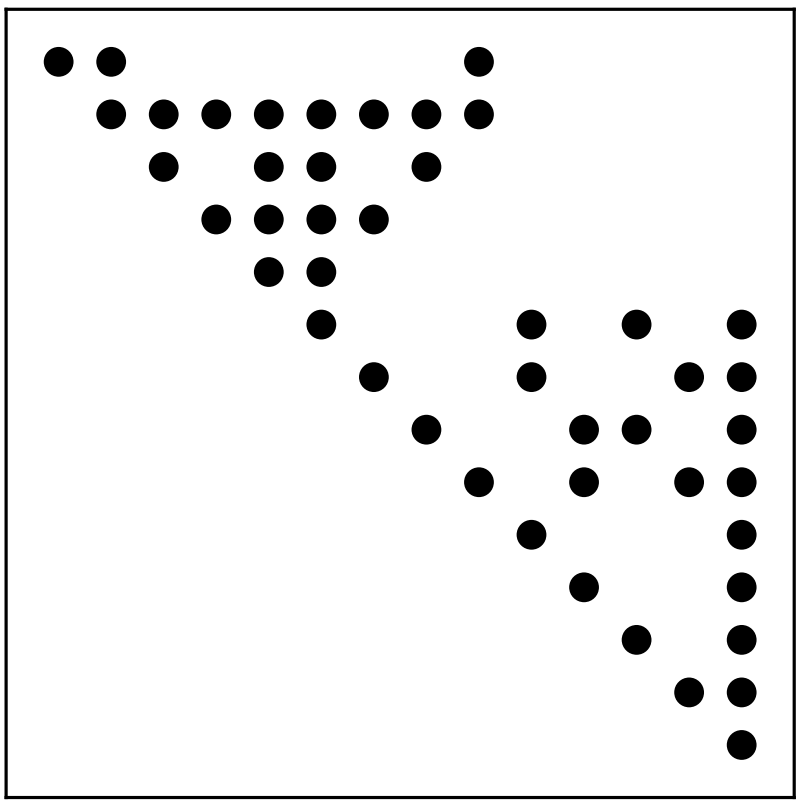
- Stacking over the points of state space, in matrix form

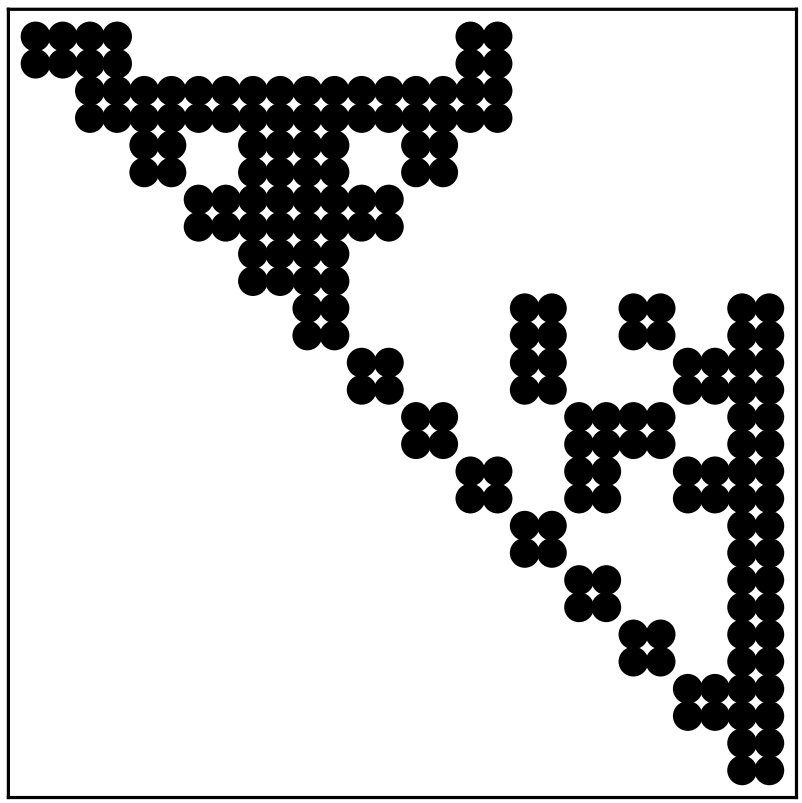
$$V_i^\sigma = (I - \beta F)^{-1} \sum_{a_i \in A} P_i^\sigma(a_i) * [\pi_i^\sigma(a_i) + e_i^\sigma(a_i)],$$

Upper block-triangularity of the transision matrix F is directionality

F in simultaneous and alternating move leapfrogging

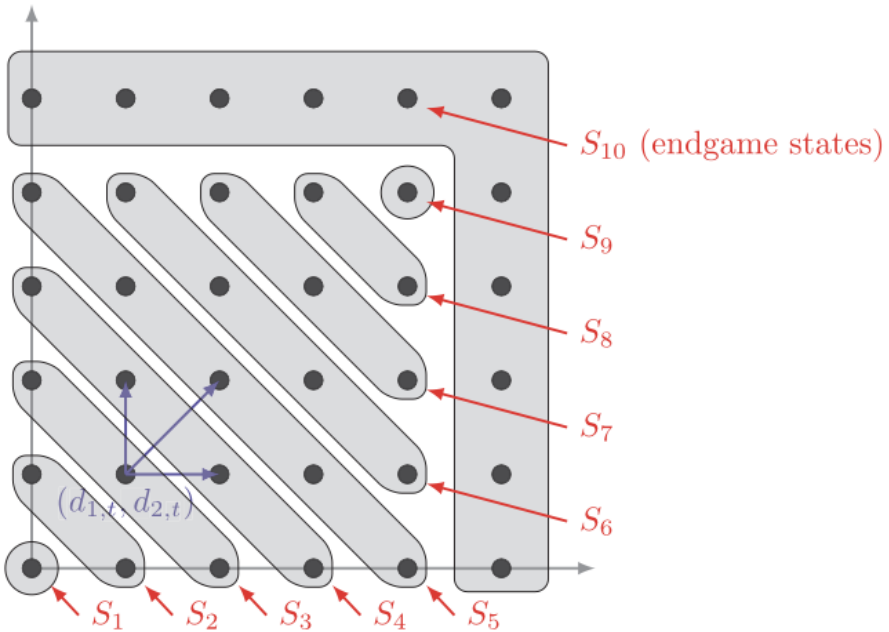
- Simultaneous move: 14 points in the state space for (c_1, c_2, c_3)
- Alternating move: 28 points with $m \in \{1, 2\} \to 2 \times 2$ blocks

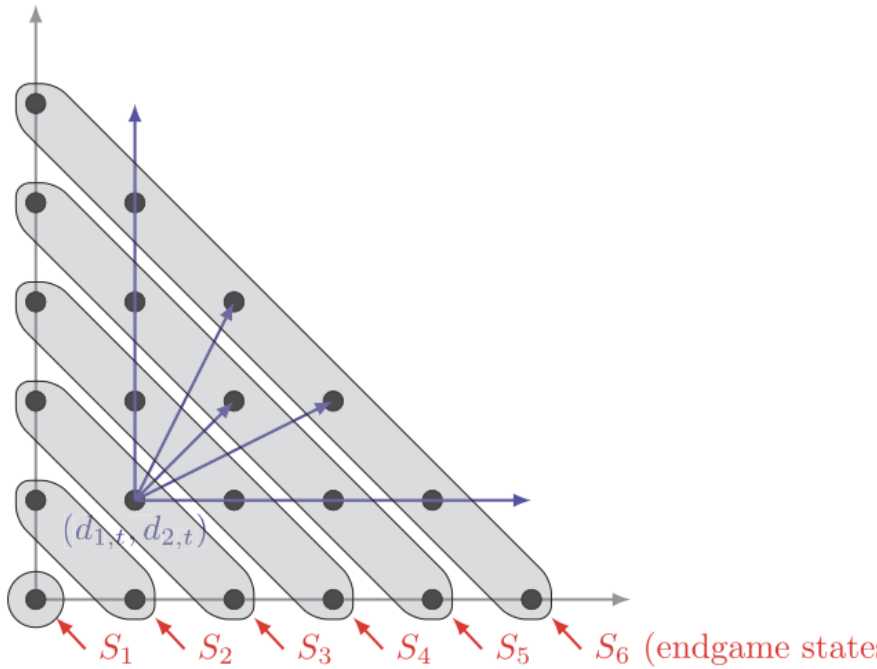




Examples of Directional Dynamic Games

Many games have state dynamic evolutions described by a DAGs





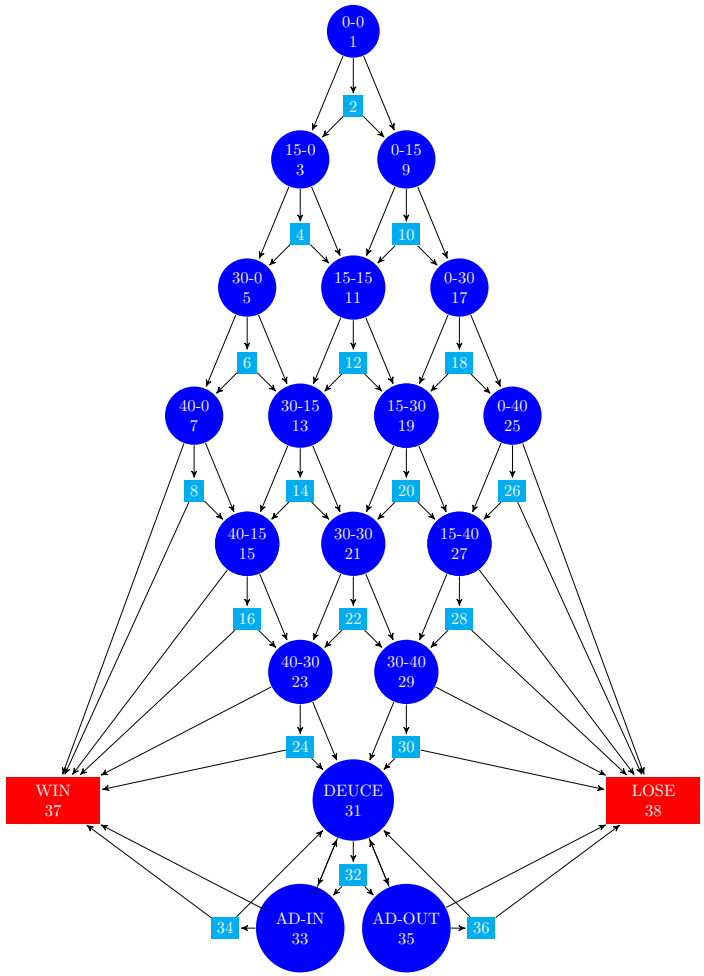
Judd, Schmedders, Yeltekin (2012), *IER*

"Optimal rules for patent researchers"

Dube, Hitsch, Chintagunta (2010), *Marketing Science*

"Tipping and concentration in markets with indirect network effects"

Tennis is a Directional Dynamic Game

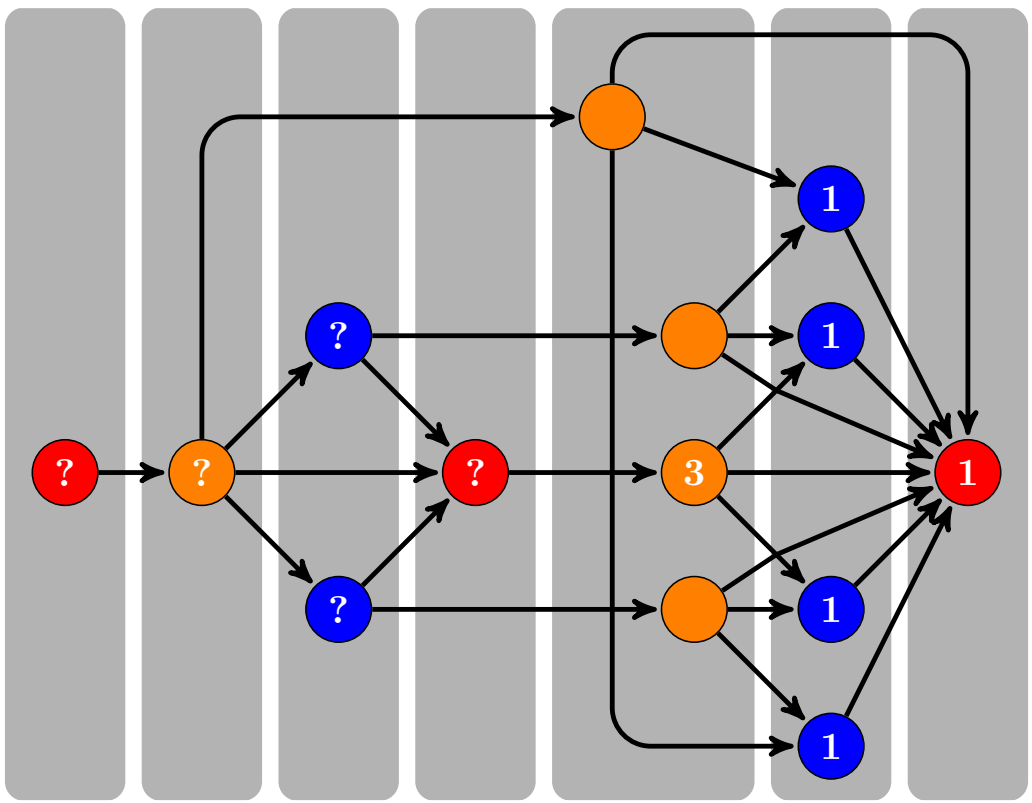


Anderson, Rosen, Rust, Wong (2024) *JPE* (forthcoming)
"Disequilibrium Play in Tennis"

Multiplicity of stage equibiria

Number of equilibria in the higher stages depends on the selected equilibria

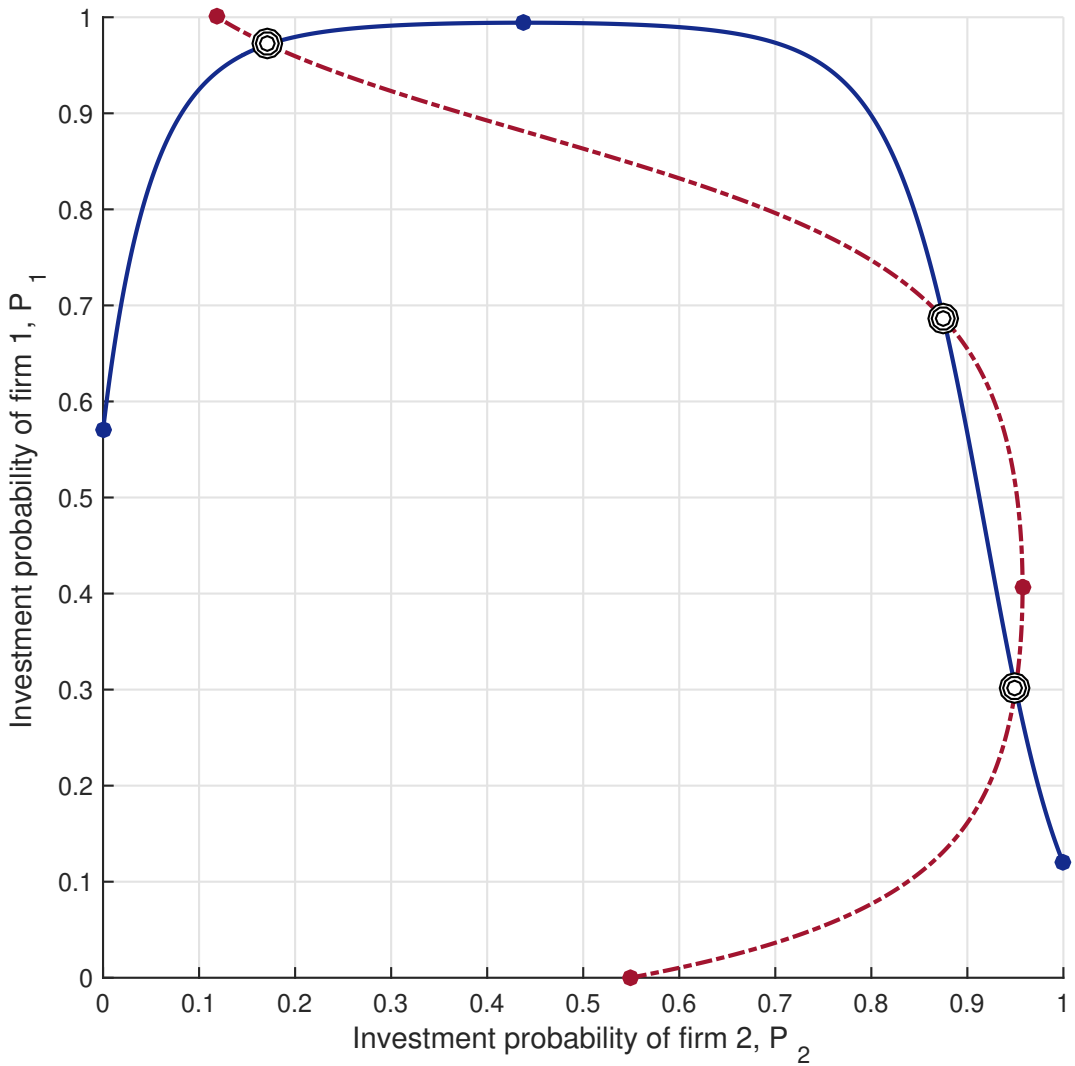
- State recursion proceeds conditional on equilibrium selection rule
- Multiplicity of stage equilibria $\Leftrightarrow$ multiplicity
- Can systematically combine different stage equilibria



Best response functions

Typically one or three stage equilibria, but may be 5

- Smooth best response function with $\eta > 0$



Recursive Lexicographic Search Algorithm

Building blocks of RLS algorithm:

1. State recursion algorithm solves the game conditional on equilibrium selection rule (ESR)
2. RLS algorithm efficiently cycles through all feasible ESRs

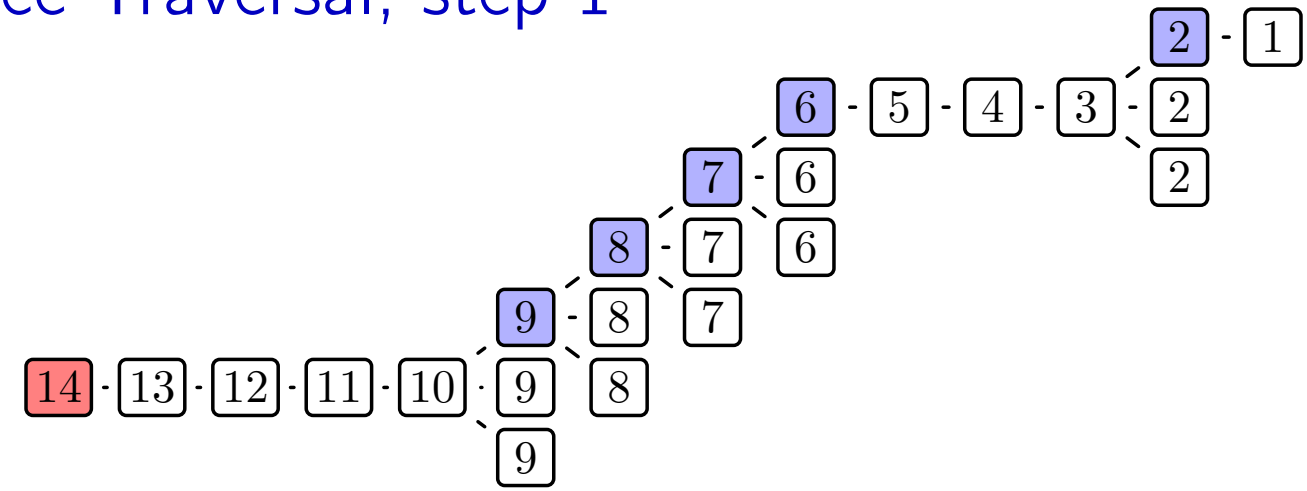
Challenge:

- Choice of a particular MPE for any stage game at any stage
- may alter the set and even the number of stage equilibria at earlier stages

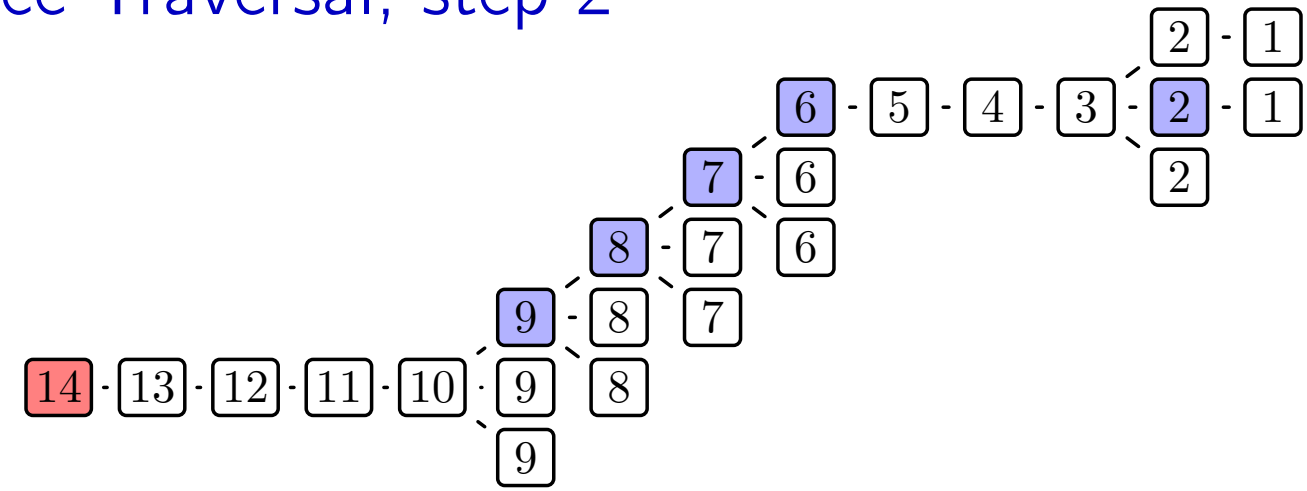
Solution: RLS = depth-first tree traversal (illustration coming)

- Root of the tree is one of the absorbing states
- Levels of the tree correspond to the state points
- Branching happens when stages have multiple equilibria
- MPE of the game is given by a path from root to a leaf

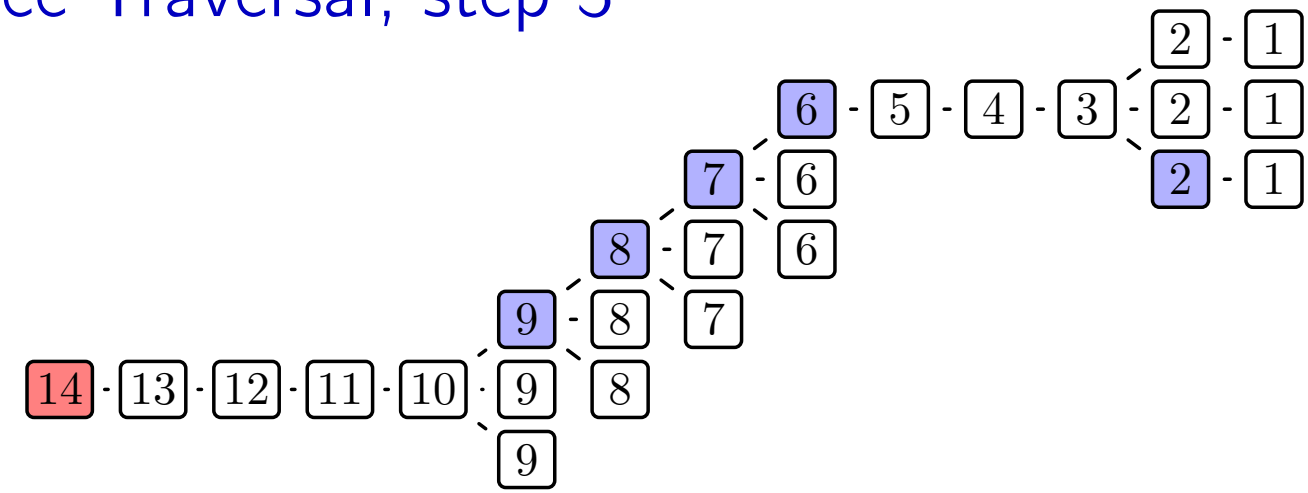
RLS Tree Traversal, step 1



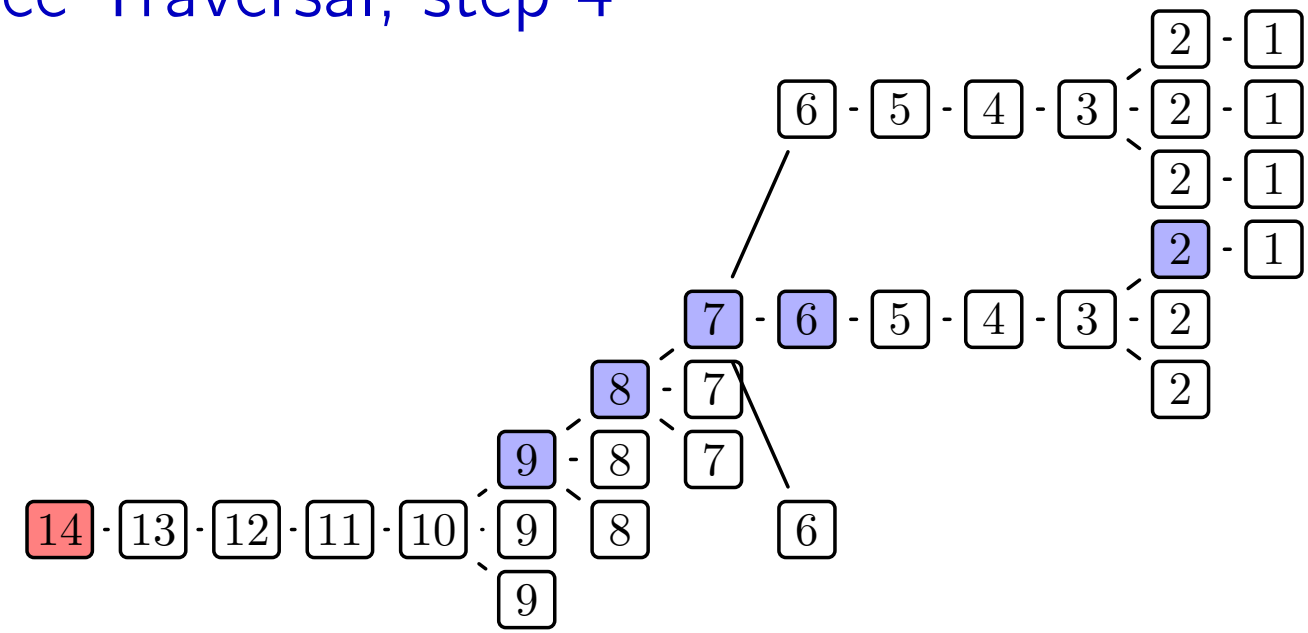
RLS Tree Traversal, step 2



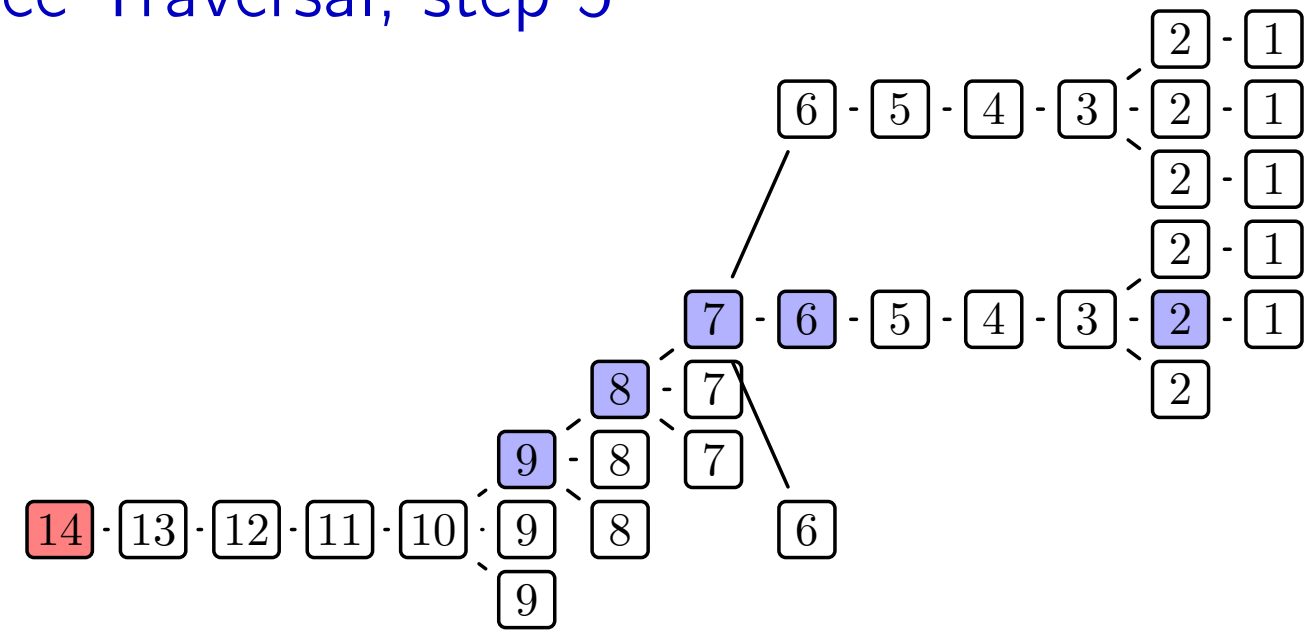
RLS Tree Traversal, step 3



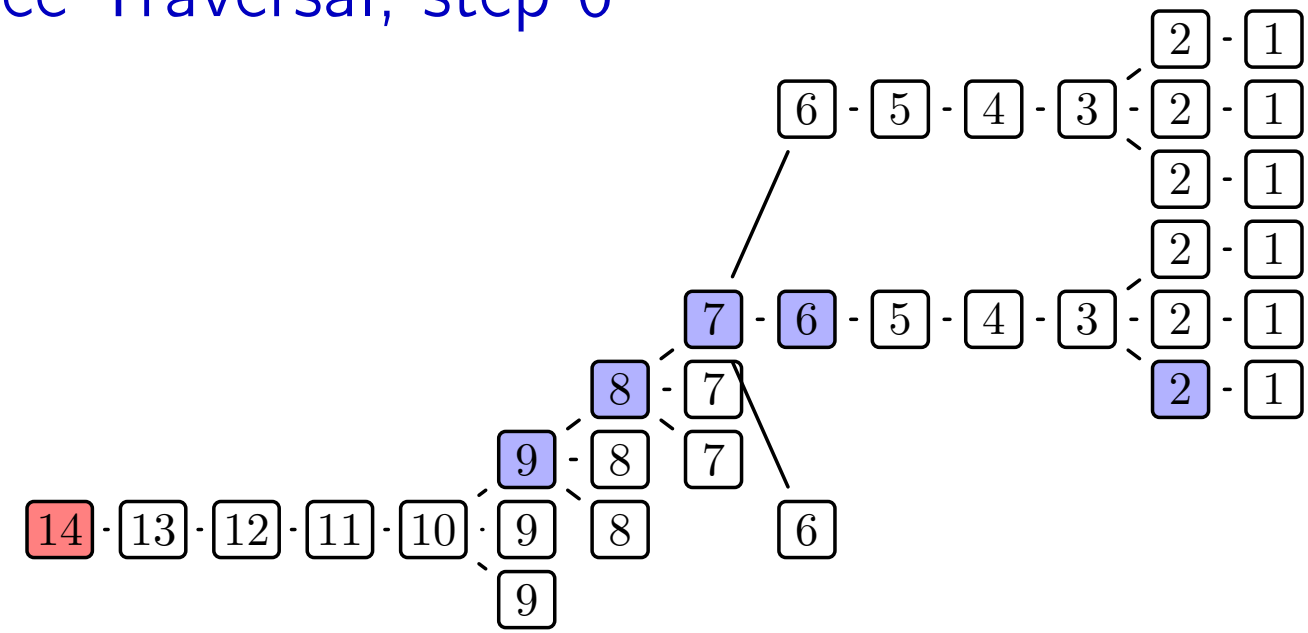
RLS Tree Traversal, step 4



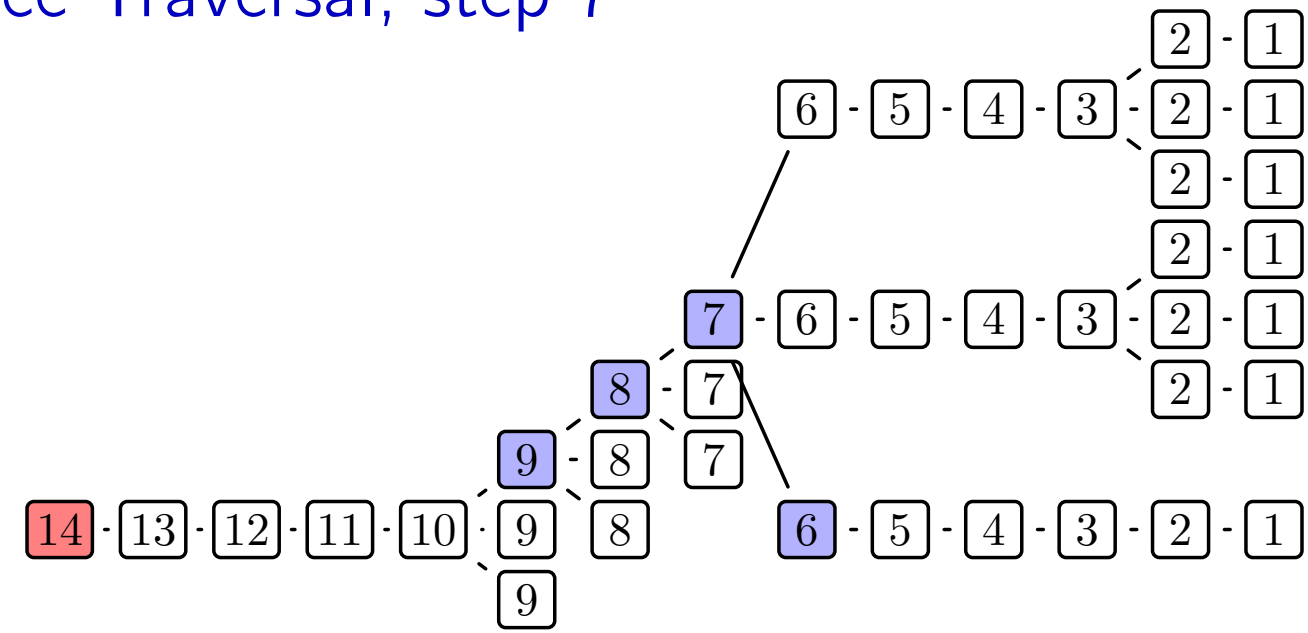
RLS Tree Traversal, step 5



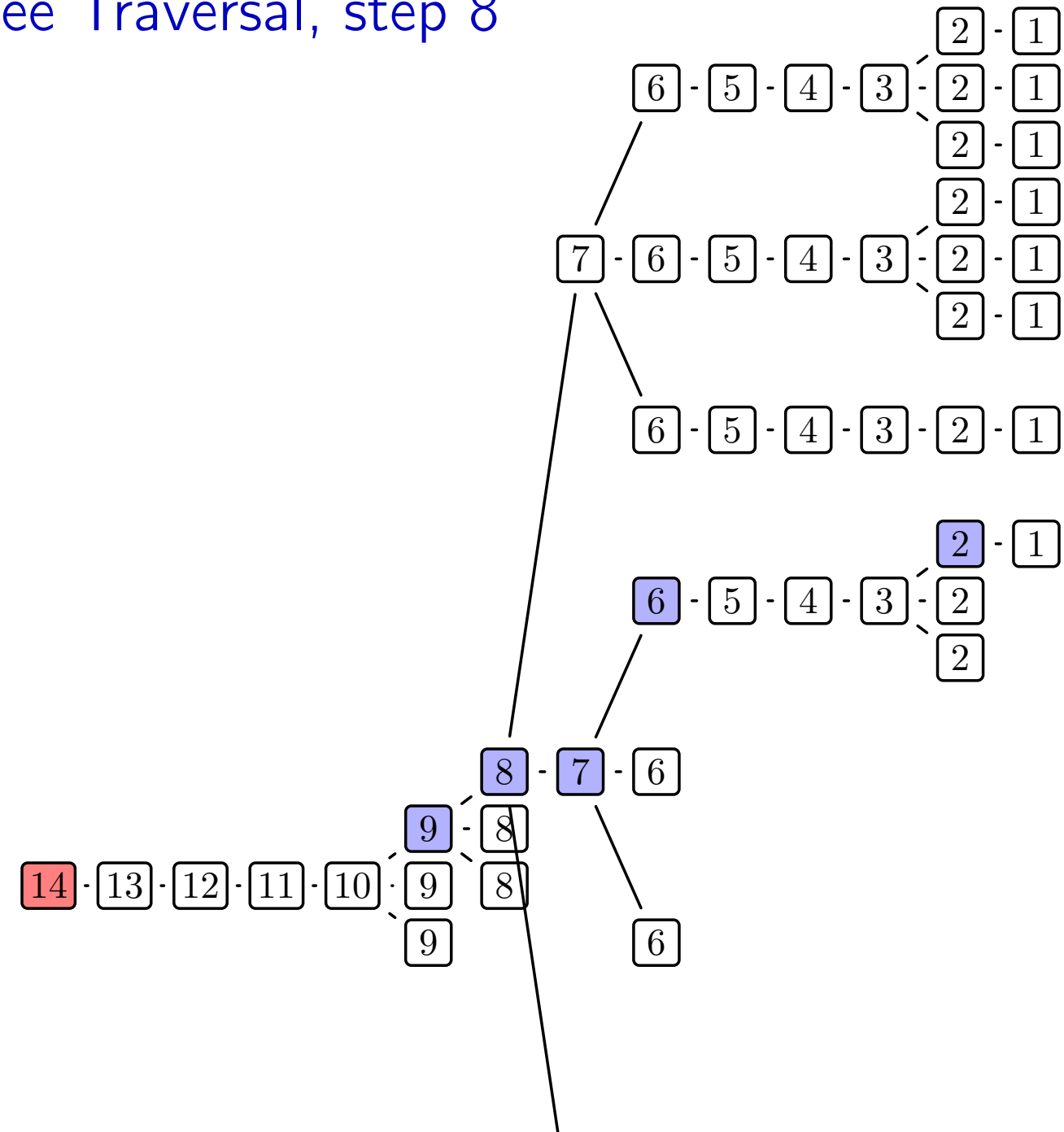
RLS Tree Traversal, step 6



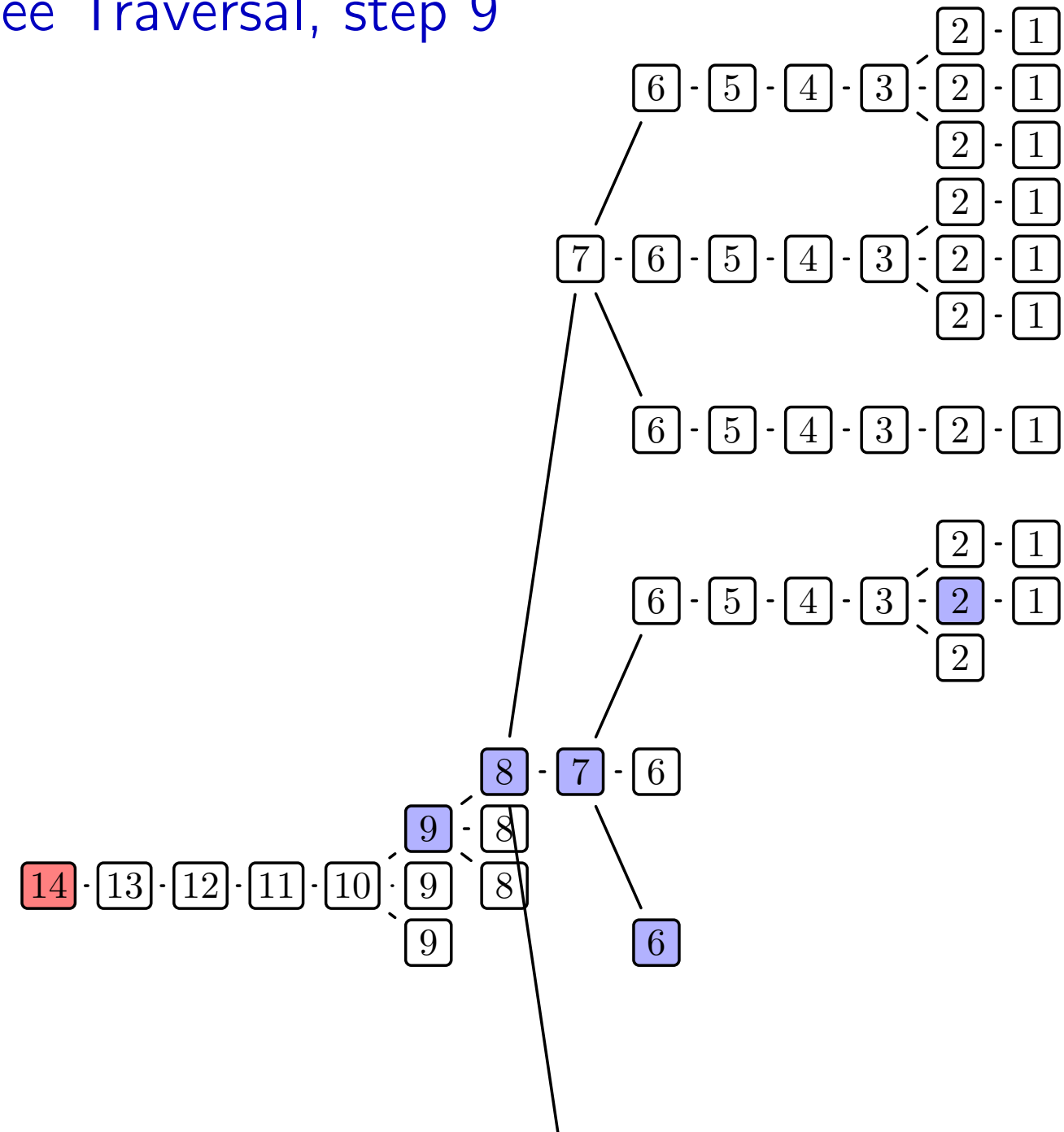
RLS Tree Traversal, step 7



RLS Tree Traversal, step 8



RLS Tree Traversal, step 9



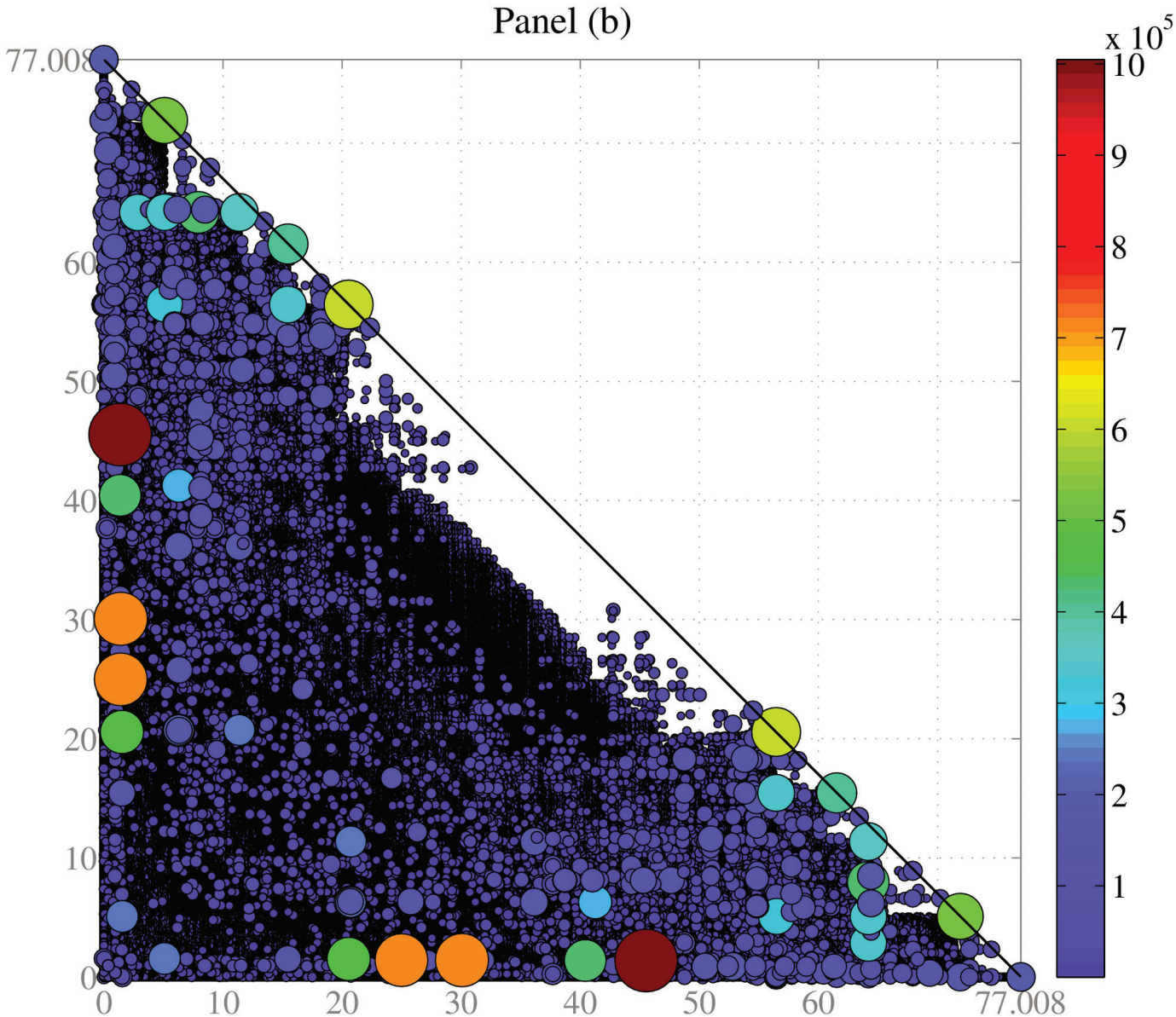
Recursive Lexicographic Search (RLS) algorithm

Theorem (RLS theorem)

Assume there exists an algorithm that can find all MPE of every stage game of the DDG, and that the number of these equilibria is finite in every stage game.
Then the RLS algorithm finds all MPE of the DDG in a finite number of steps, which equals the total number of MPE.

 Iskhakov, Rust and Schjerning, 2016, ReStud
"Recursive lexicographical search: Finding all markov perfect equilibria of finite state directional dynamic games."

Simultaneous move leapfrogging: 164,295,079 equilibria



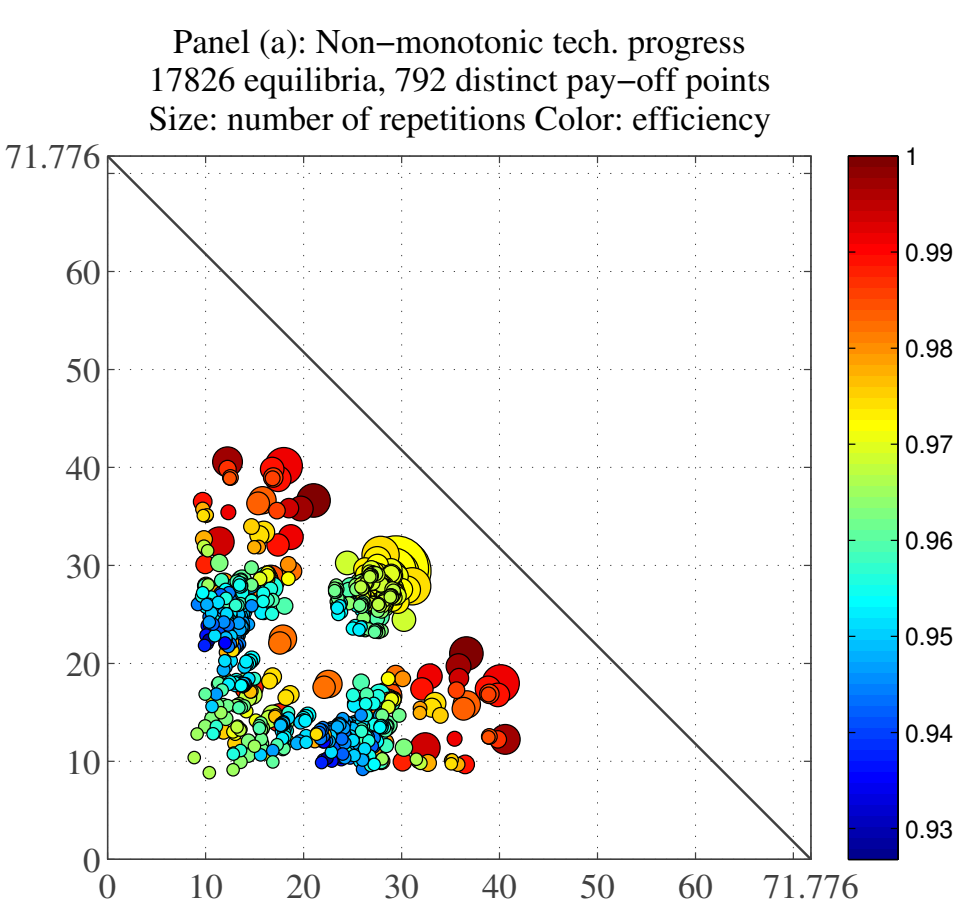
Theoretical results on Bertrand investment game

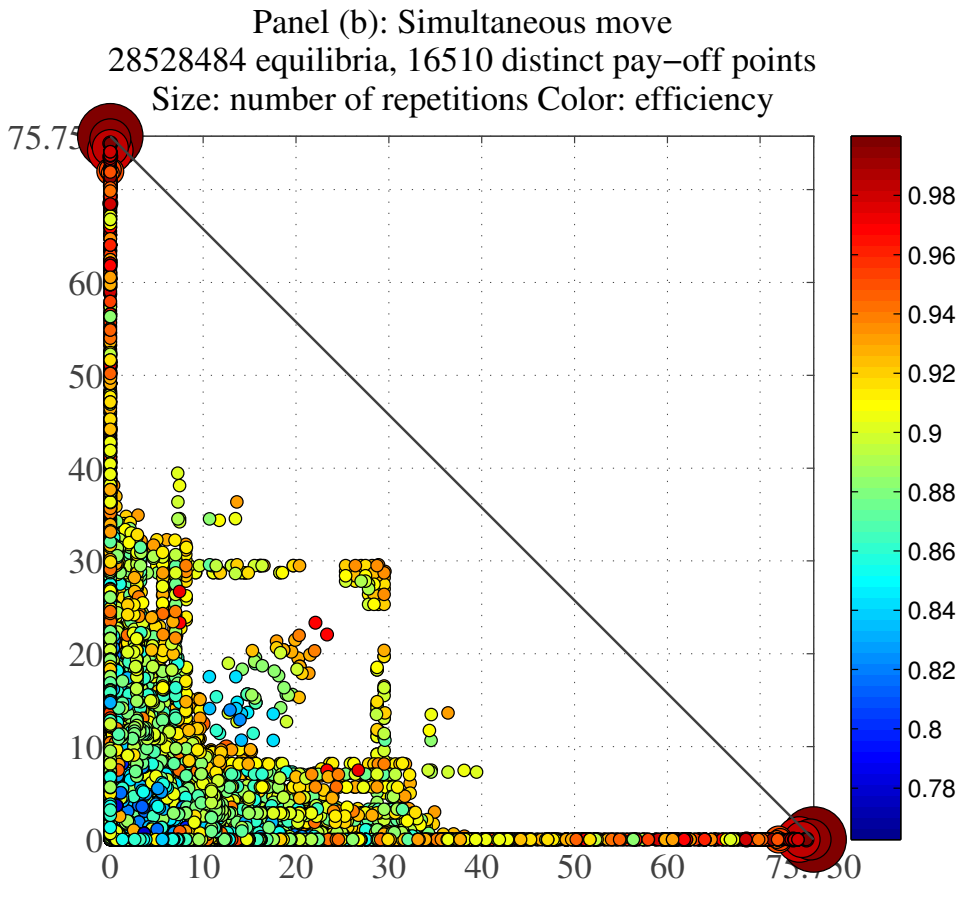
We show analytically or by example/counter-example:

1. Many types of endogenous coordination are supported as MPE:
   - Leapfrogging (alternating investments)
   - Preemption (investment by cost leader)
   - Duplicative (simultaneous investments)
2. The equilibria are generally inefficient due to over-investment
   - Duplicative or excessively frequent investments
3. The (convex hull of the) set of the expected discounted equilibrium payoffs at the initial (apex) state is:
   - triangle with the vertices at the origin and the monopoly payoffs in the simultaneous move game;
   - proper subset thereof in the alternating move game.
4. Sufficient conditions for uniqueness of equilibrium:
   - Stochastic or deterministic alternation of moves;
   - Strictly monotone technological progress, $\pi(c_{t+1}|c_t) = 0$, $c_{t+1} \geq c_t$

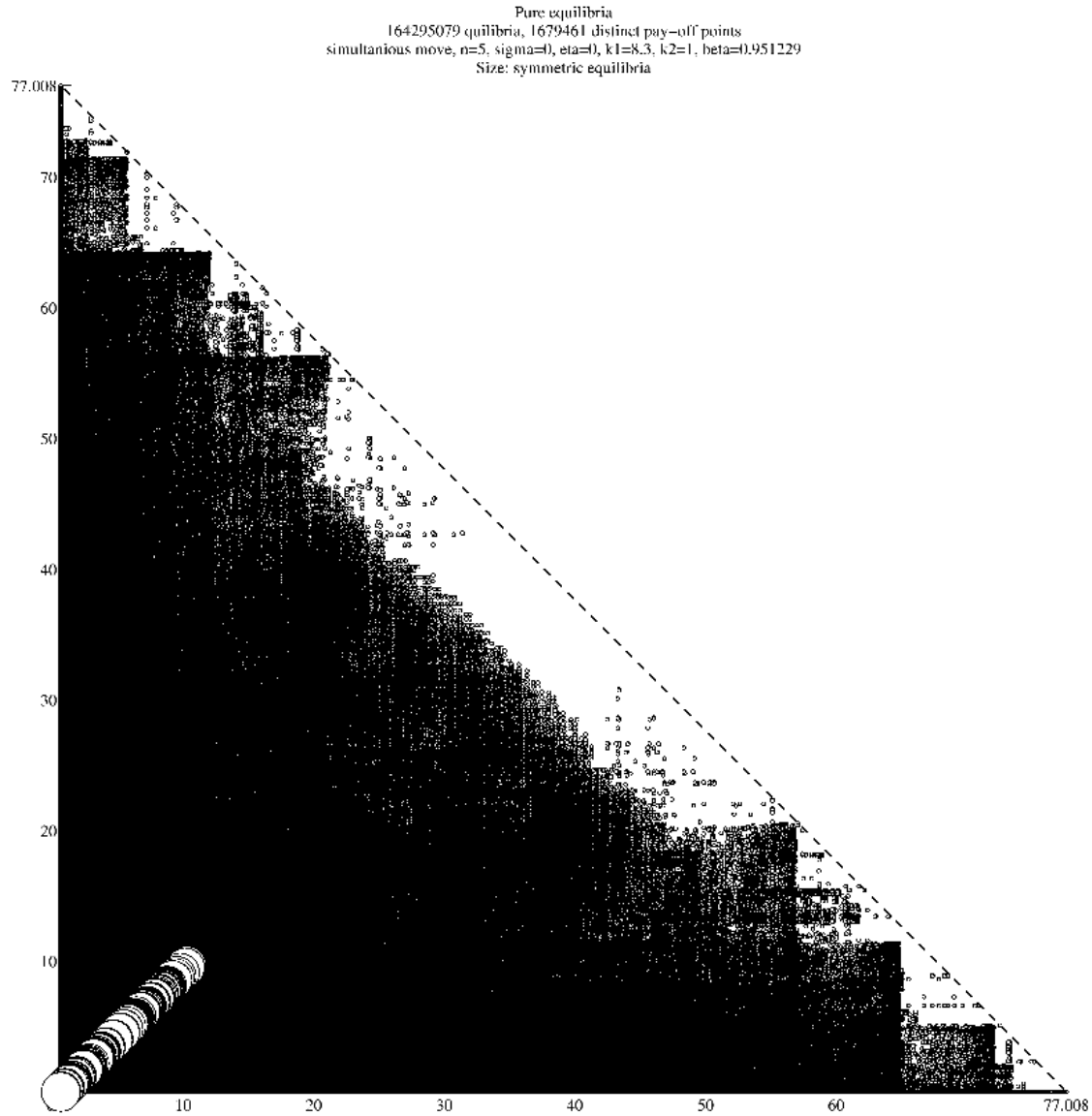
 Iskhakov, Rust and Schjerning (2018) *International Economic Review*
"The dynamics of Bertrand price competition with cost-reducing investments."

Pay-offs: alternating vs simultaneous move games





Symmetric equilibria: $V_1(c_1, c_2, c) = V_2(c_2, c_1, c)$



ROAD MAP

1. Solving directional dynamic games (DDGs):
   - Bertrand pricing and investment game
   - Solving once: State recursion algorithm
   - Solving for all MPE: Recursive lexicographical search (RLS) algorithm
2. Structural estimation of DDGs: nested MLE RLS
   - Is it even feasible computationally?
3. Monte Carlo to compare to exising estimators in dynamic games (two-step CCP, NPL, EPL, MPEC)
   - One equilibrium in the model and data
   - Multiplicity of equilibria at true parameter
   - Multiple equilibria in the data

Nested Recursive Lexicographical Search (NRLS)

- Data from M independent markets from T periods
  $Z = \{\bar{a}^{mt}, \bar{x}^{mt}\}_{m\in\{1,\dots,M\},t\in\{1,\dots,T\}}$
- Let the set of all MPE equilibria be $\mathcal{E} = \{1, \dots, K(\theta)\}$ such that

$$\mathrm{V} = \Psi^{\mathrm{V}}(\mathrm{V}, \mathrm{P}, \theta) \quad \text{and} \quad \mathrm{P} = \Psi^{\mathrm{P}}(\mathrm{V}, \mathrm{P}, \theta)$$

1. Outer loop
   Maximization of the likelihood function w.r.t. to structural parameters θ
$$\theta^{ML} = \arg\max_{\theta\in\Theta} \mathcal{L}(Z, \theta)$$
2. Inner loop
   Maximization of the likelihood function w.r.t. equilibrium selection
$$\mathcal{L}(Z, \theta) = \max_{k\in\{1,\dots,K(\theta)\}} \mathcal{L}(Z, \theta, P_\theta^k)$$

Max of a function on a discrete set organized into RLS tree

Likelihood over the state space

- Given equilibrium k choice probabilities $P_\theta^k(a|x) = \left(P_1^k(a|x,\theta), \ldots, P_N^k(a|x,\theta)\right)$, likelihood is

$$\mathcal{L}(Z, \theta, P_\theta^k) = \frac{1}{M} \sum_{m=1}^{M} \sum_{t=1}^{T} \sum_{i=1}^{N} \log P_i^k(\bar{a}_i^{mt} | \bar{x}^{mt}; \theta)$$

- Let ι index points in the state space, $\iota \in \{1, \ldots, S\}$
  $\iota = 1$ initial point, $\iota = S$ the terminal (absorbing) state
- Denote n_ι the number of observations in state x_ι and $n_\iota^{a_i}$ the number of observations of player i taking action a_i at x_ι

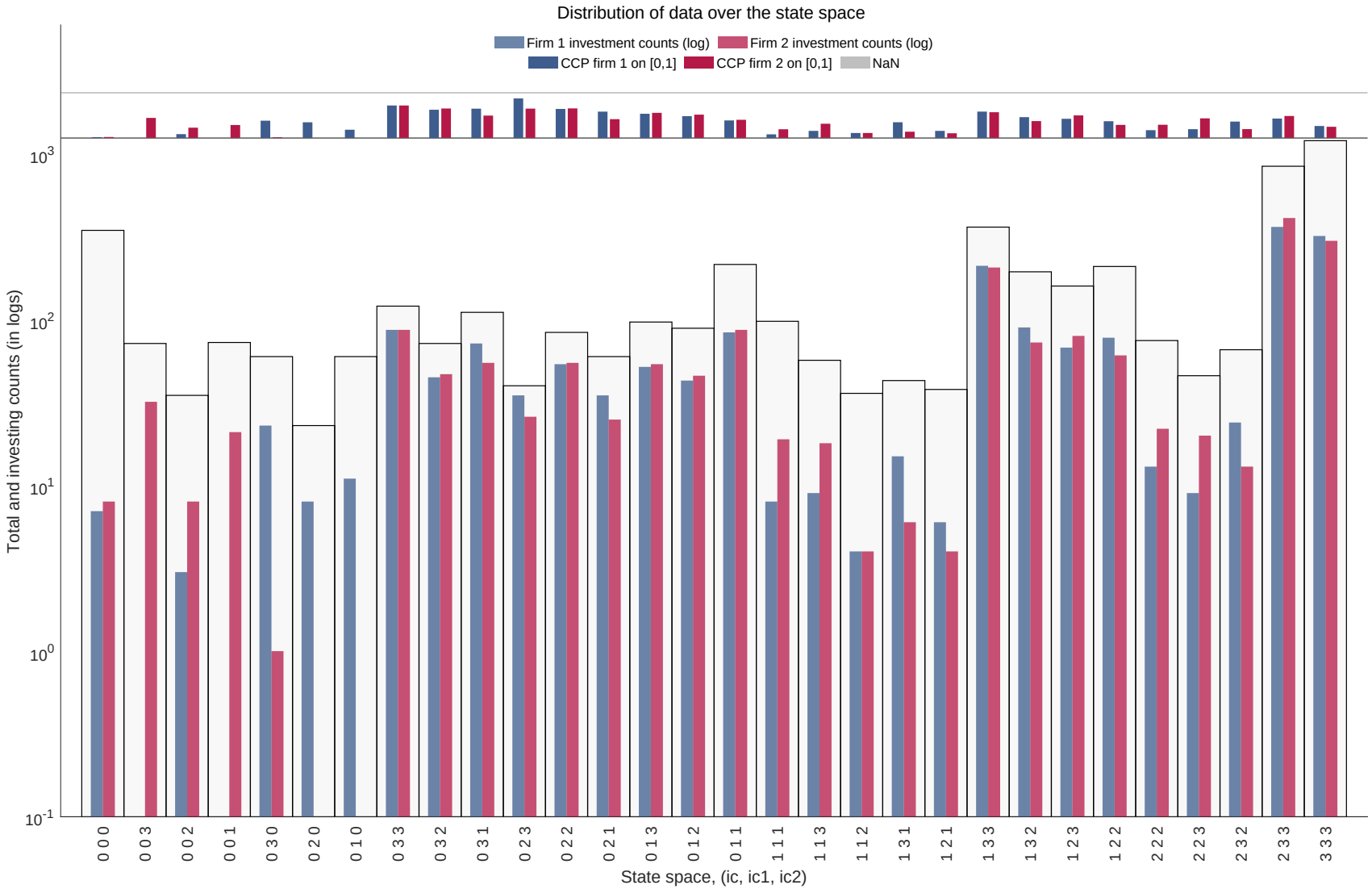
$$n_\iota = \sum_{m=1}^{M} \sum_{t=1}^{T} \mathbb{1}\{\bar{x}^{mt} = x_\iota\} \qquad n_\iota^{a_i} = \sum_{m=1}^{M} \sum_{t=1}^{T} \mathbb{1}\{\bar{a}_i^{mt} = a_i, \bar{x}^{mt} = x_\iota\}$$

- Then equilibrium-specific likelihood can be computed as

$$\mathcal{L}(Z, \theta, P_\theta^k) = \frac{1}{M} \sum_{\iota=1}^{S} \sum_{i=1}^{N} \sum_{a} n_\iota^{a_i} \log P_i^k(a|x_\iota; \theta)$$

Data distribution over the state space

1000 markets, 5 time periods, init at apex of the pyramid



Branch and bound (BnB) method

Old method for solving integer programming problems

1. Form a tree of subdivisions of the set of admissible plans
   $\Longrightarrow$ RLS tree
2. Specify a bounding function representing the best attainable objective on a given subset (branch)
   $\Longrightarrow$ Partial likelihood function computed on the subset of states

$$\mathcal{L}^{\text{part}}(Z^{\mathcal{S}}, \theta, V_\theta^k) = \sum_{\iota \in \mathcal{S}} \sum_{i=1}^{N} \sum_{a} n_\iota^{a_i} \log P_i^k(a|x_\iota; \theta)$$

where $Z^{\mathcal{S}} = \{(a, x) : x \in \mathcal{S} \subset \{1, \ldots, S\}\}$ denotes data on $\mathcal{S}$

- Monotonic decreasing in cardinality of $\mathcal{S}$
  (declines as more data is added)
- Equals to the full log-likelihood on the full state space when $Z^{\mathcal{S}} = Z$
  (at the leafs of RLS tree)

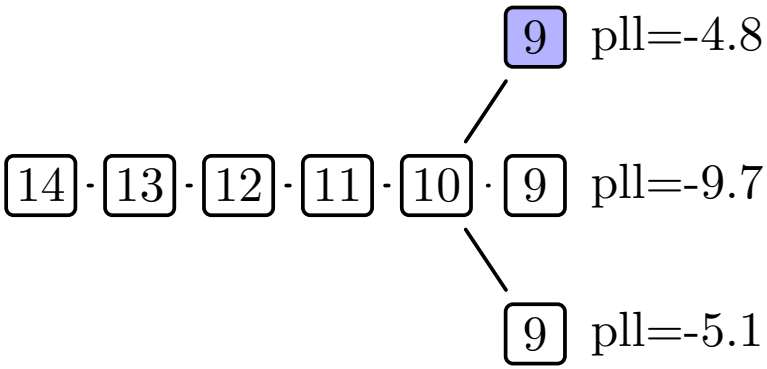
3. Dismiss the subsets of the plans where the bound is below the current best attained value of the objective

Land and Doig, 1960 *Econometrica*

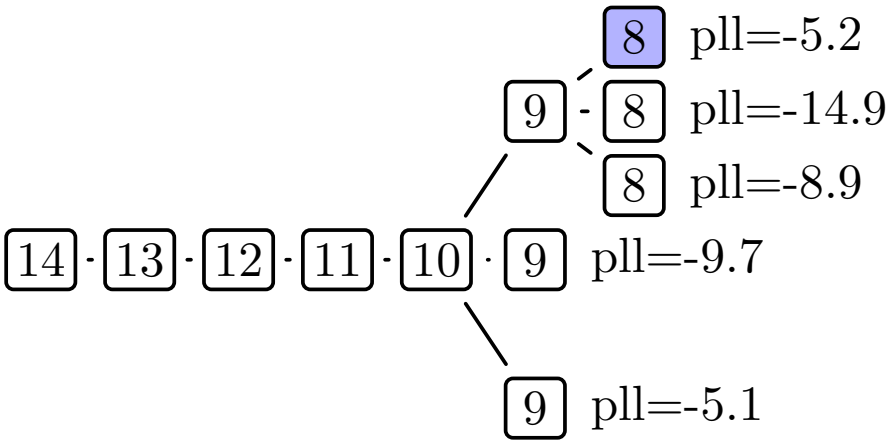
BnB on RLS tree, step 1

14 · 13 · 12 · 11 · 10 Partial loglikelihood = -3.2

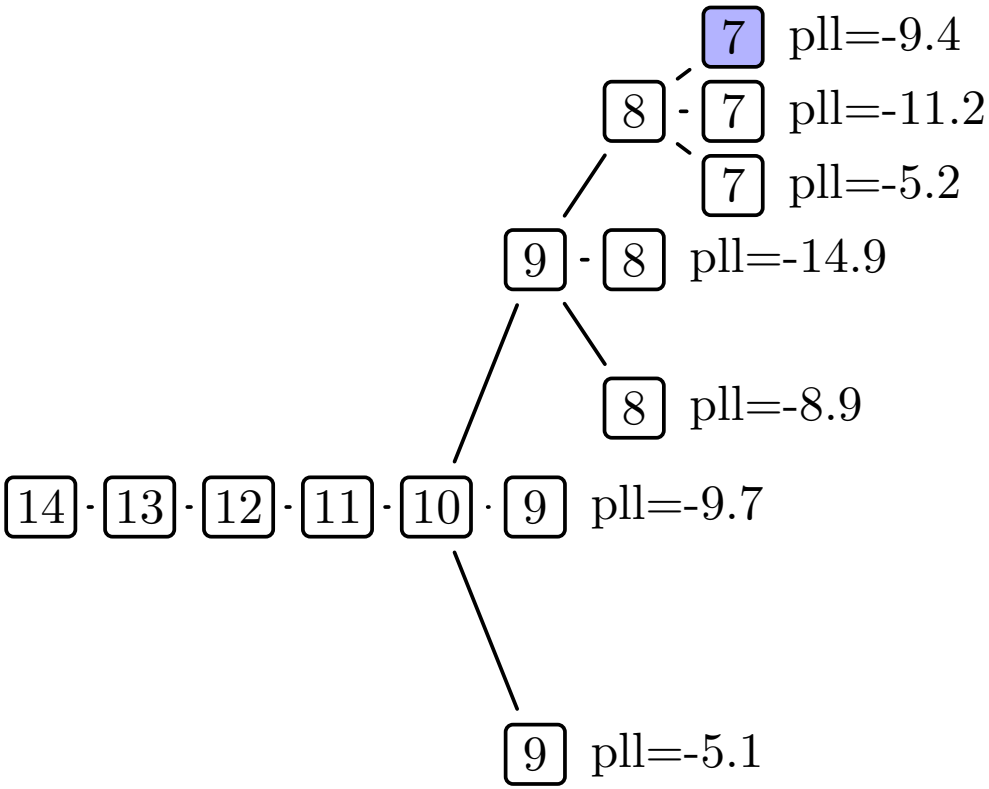
BnB on RLS tree, step 2



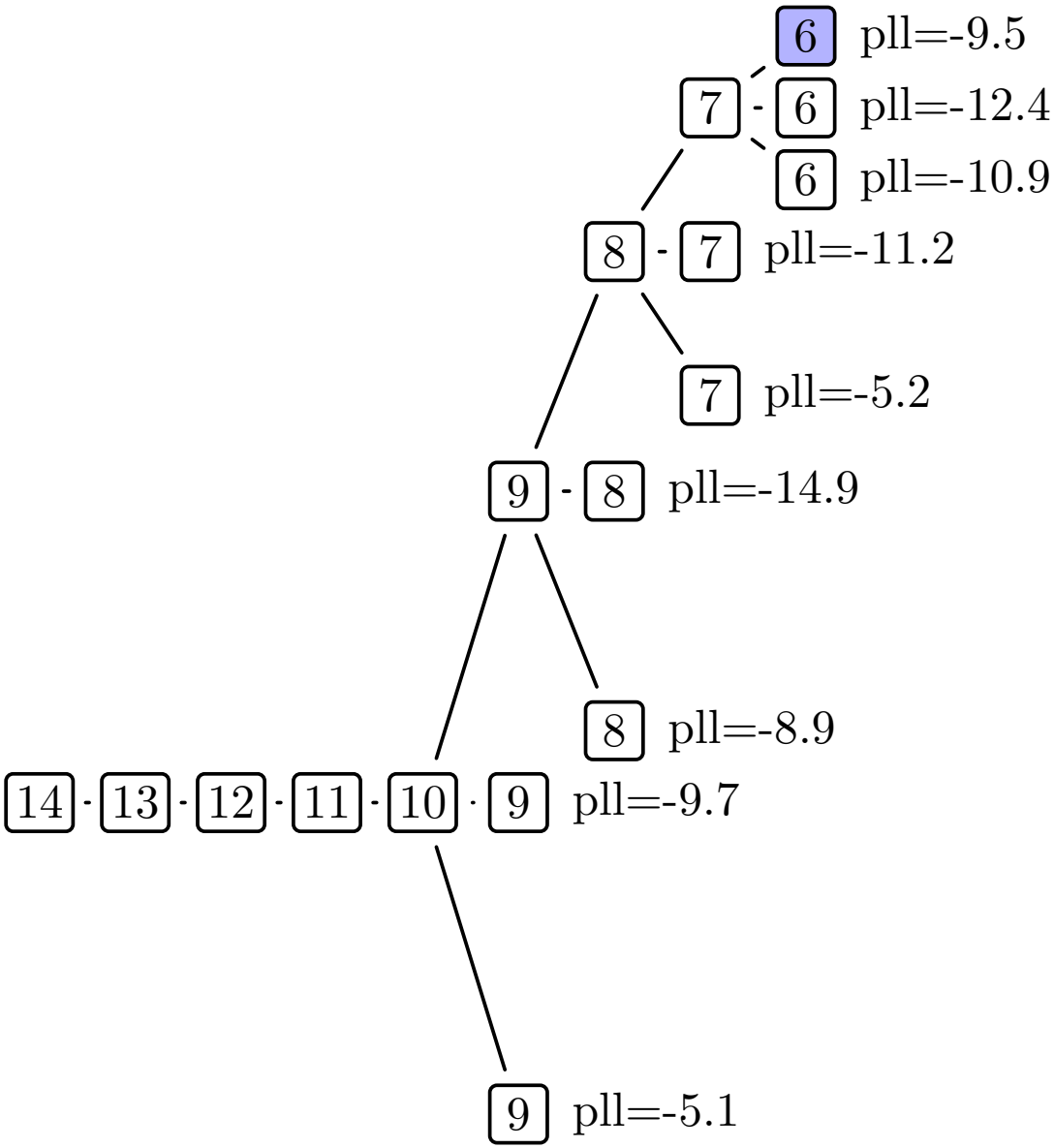
BnB on RLS tree, step 3



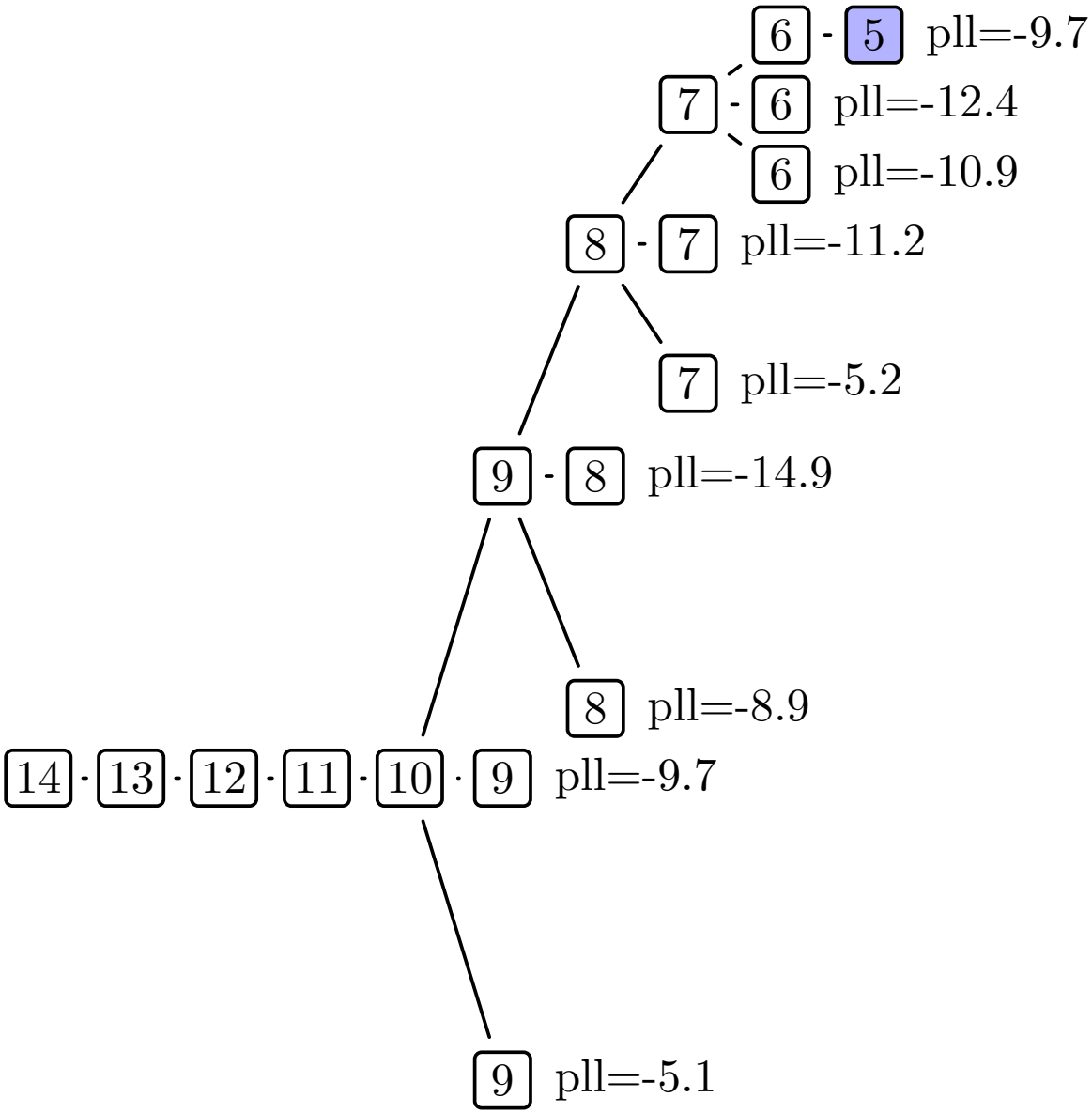
BnB on RLS tree, step 4



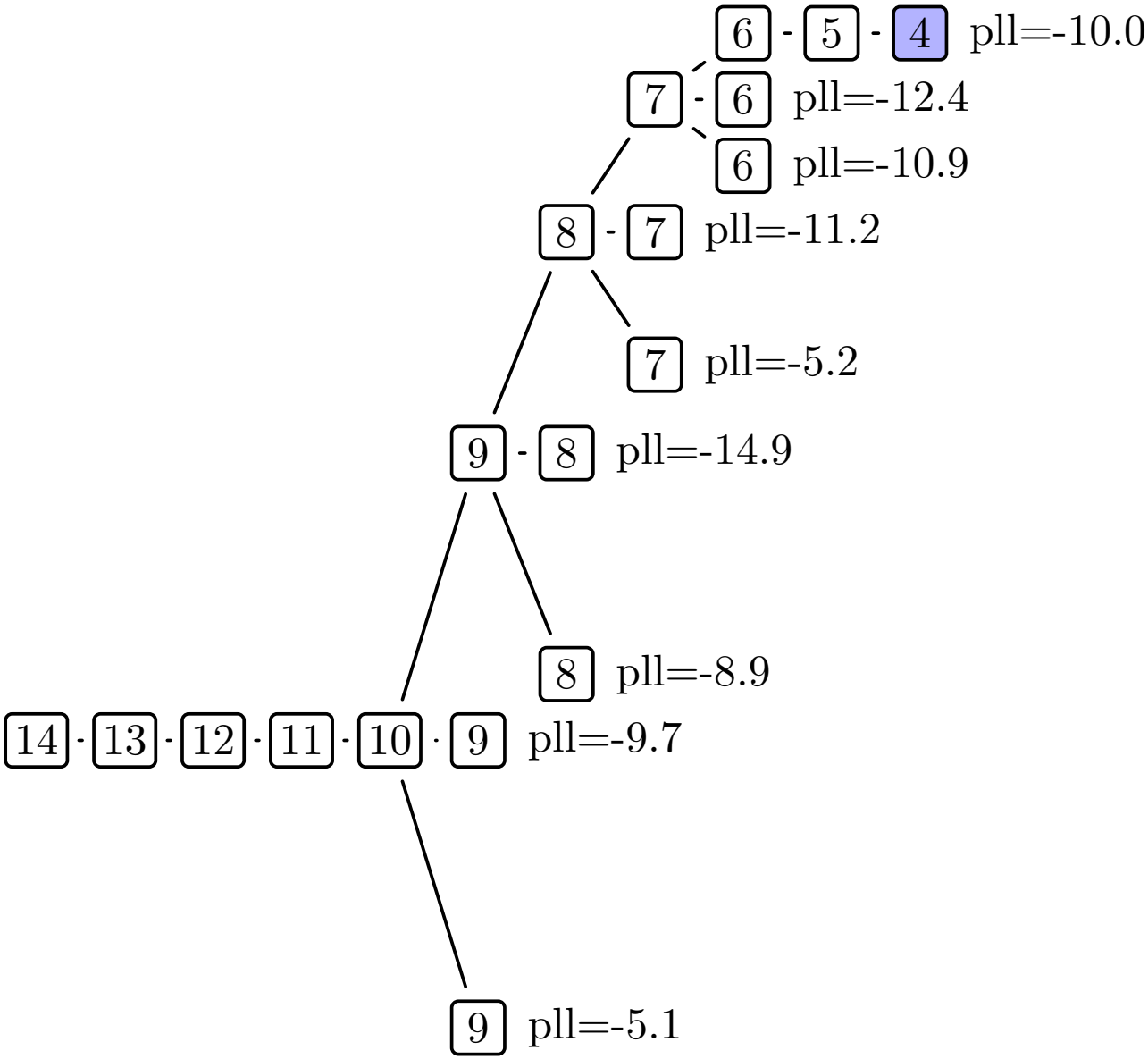
BnB on RLS tree, step 5



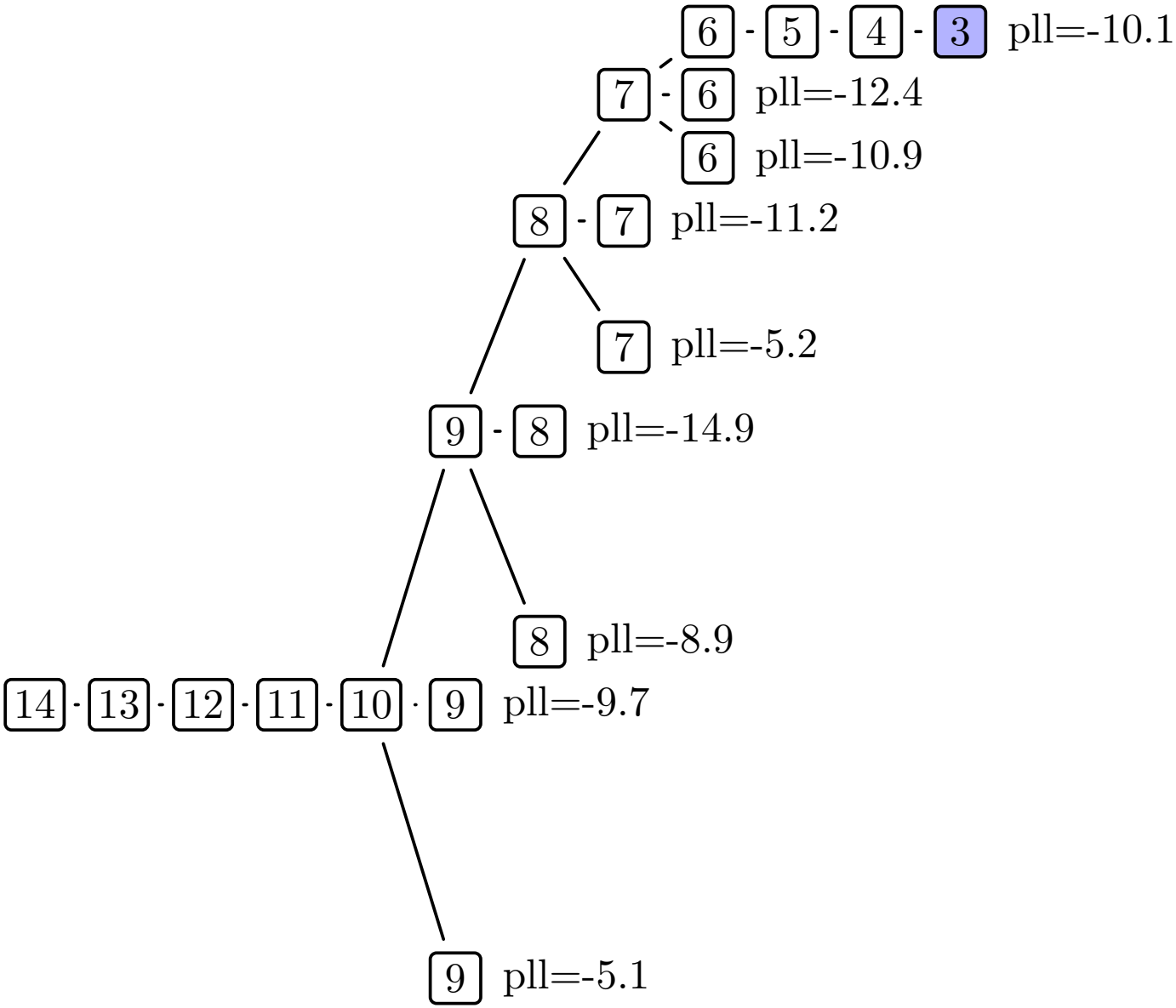
BnB on RLS tree, step 6



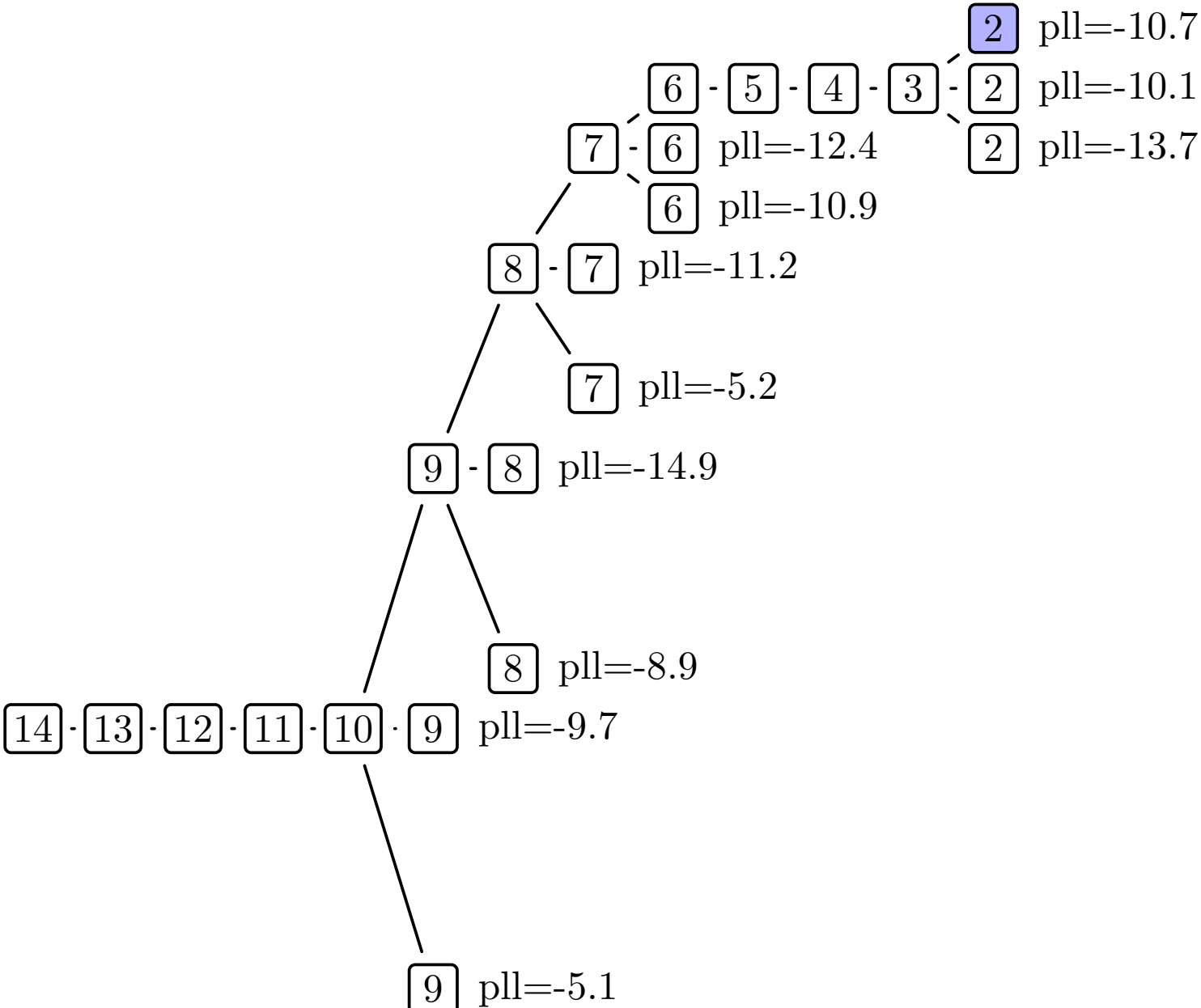
BnB on RLS tree, step 7



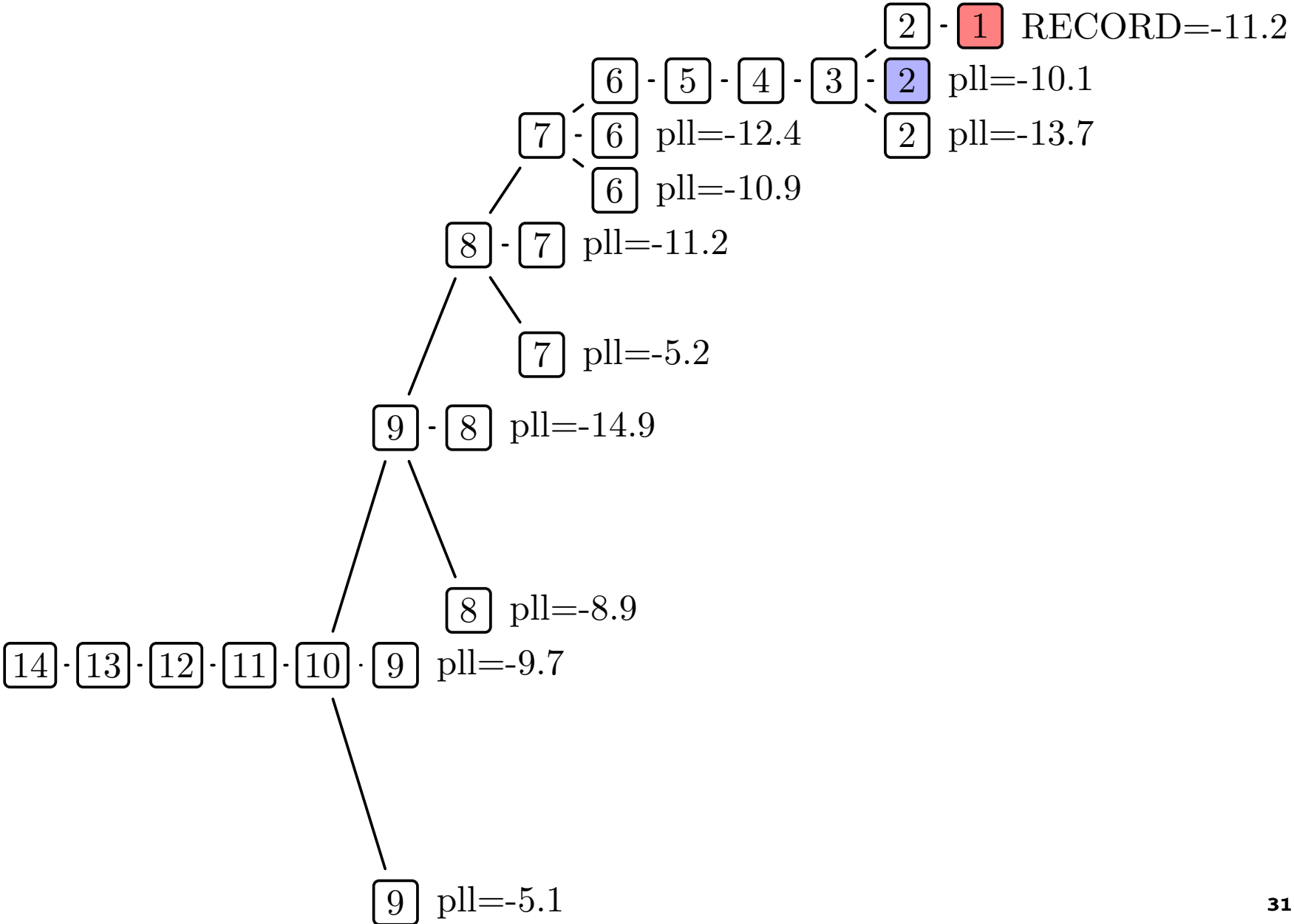
BnB on RLS tree, step 8



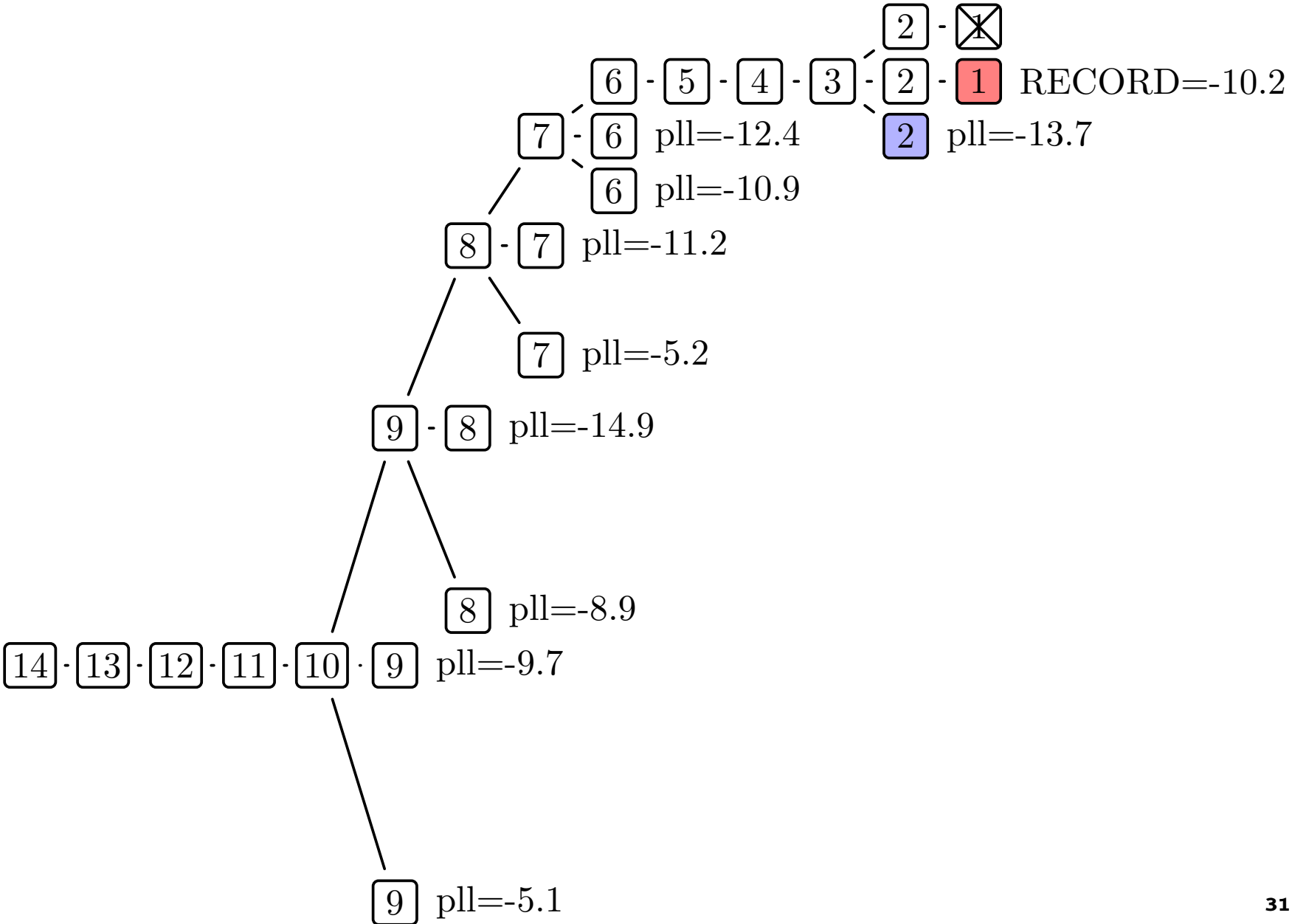
BnB on RLS tree, step 9



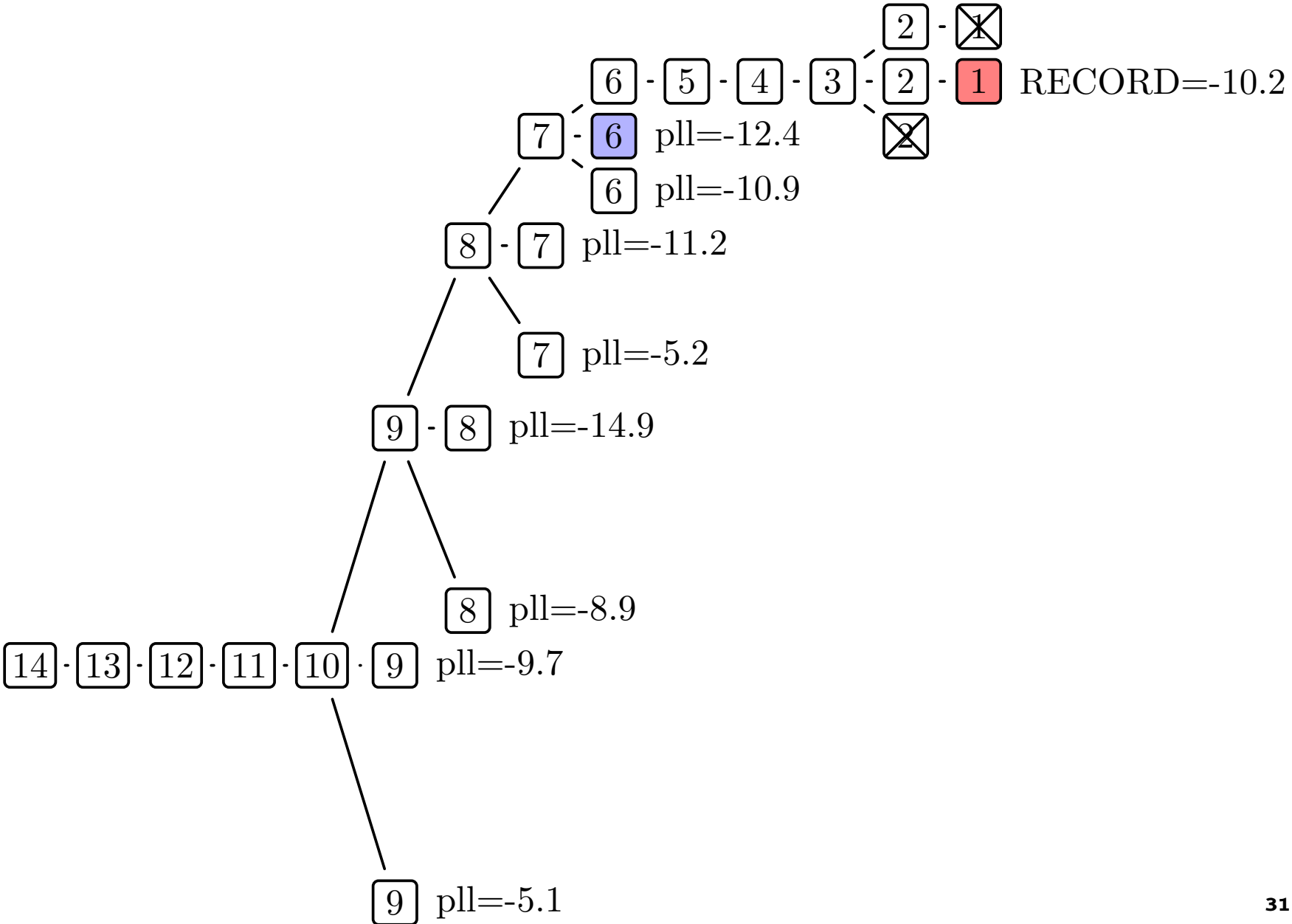
BnB on RLS tree, step 10



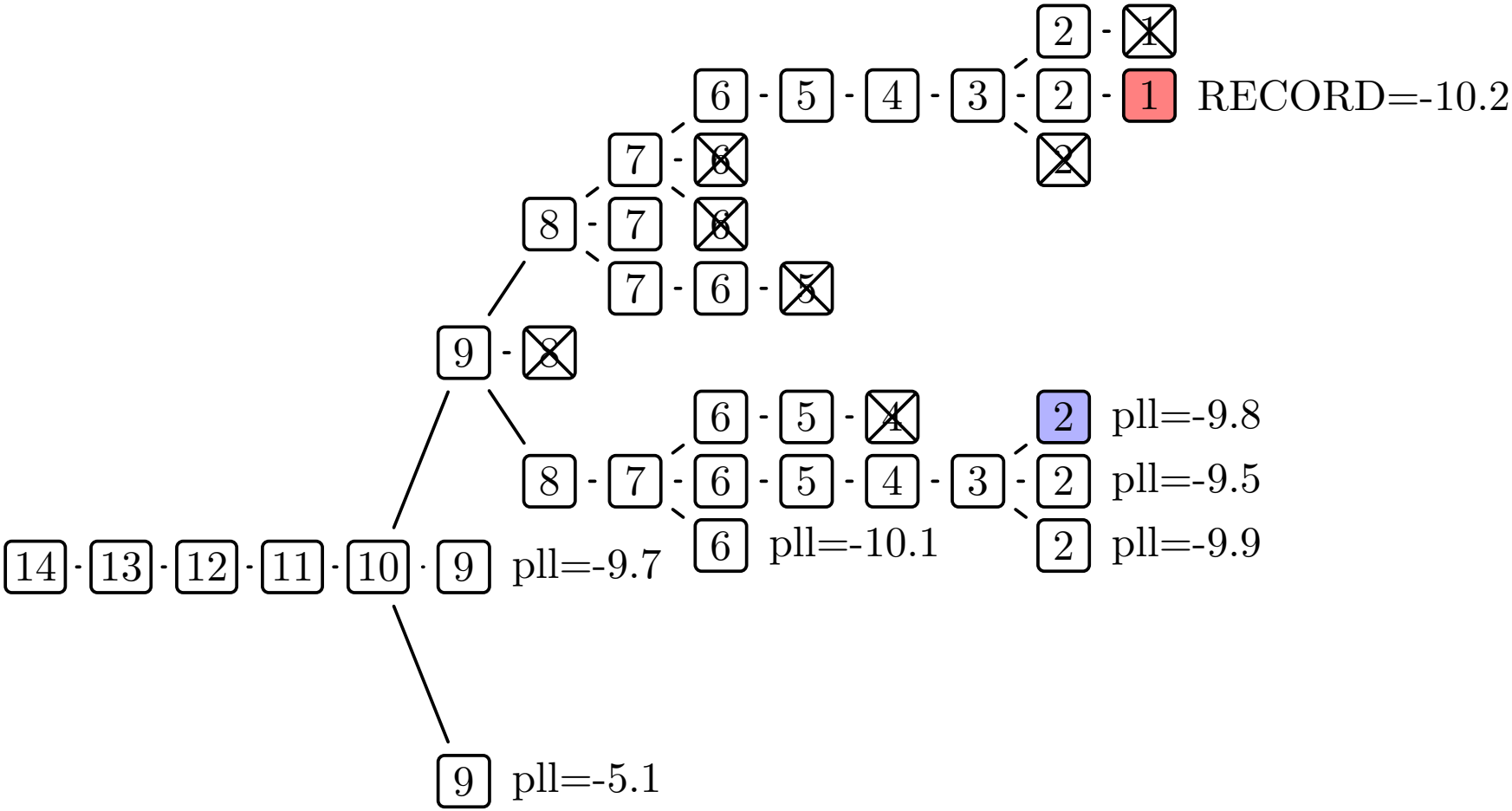
BnB on RLS tree, step 11



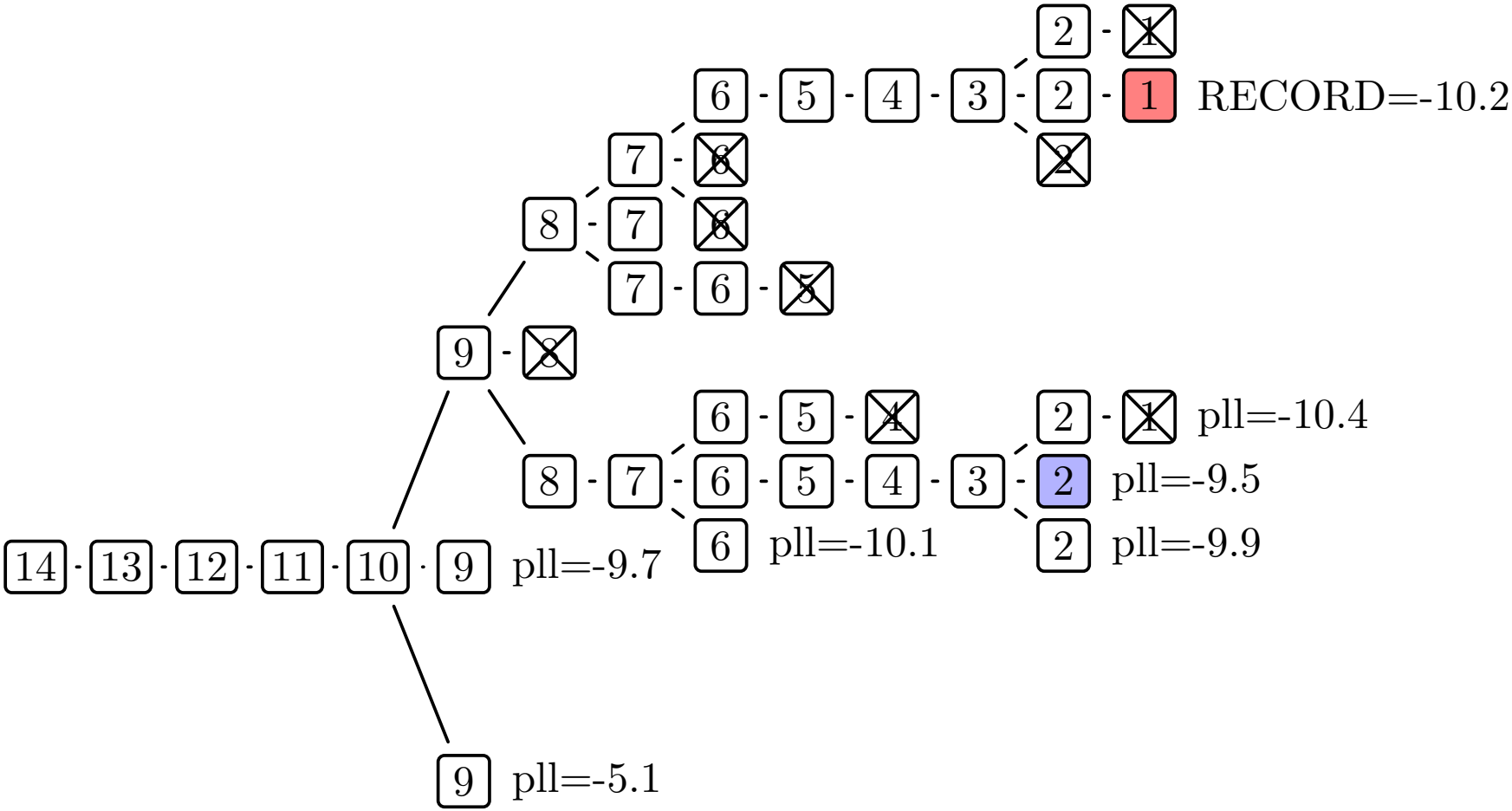
BnB on RLS tree, step 12



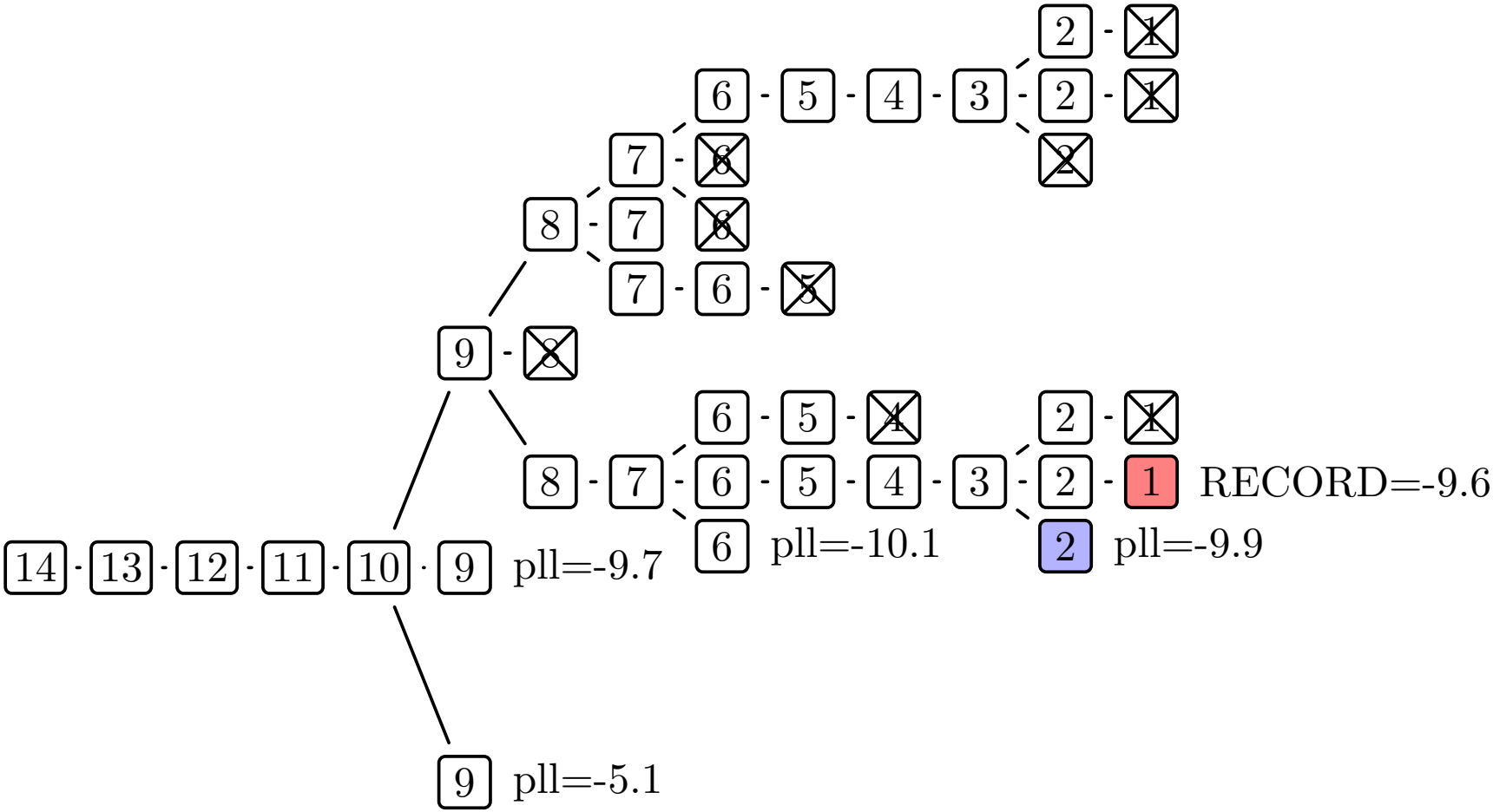
BnB on RLS tree, step 28



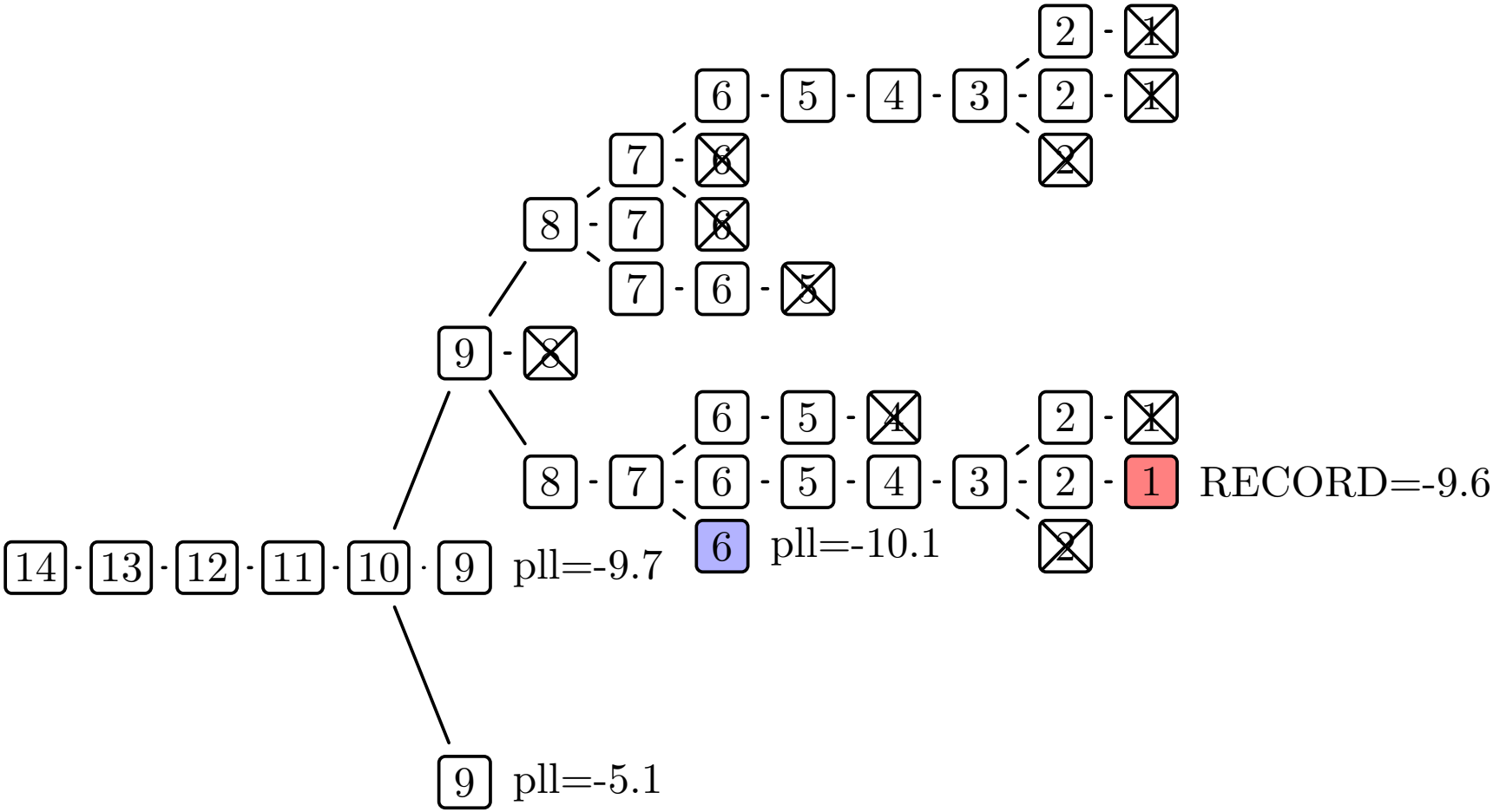
BnB on RLS tree, step 29



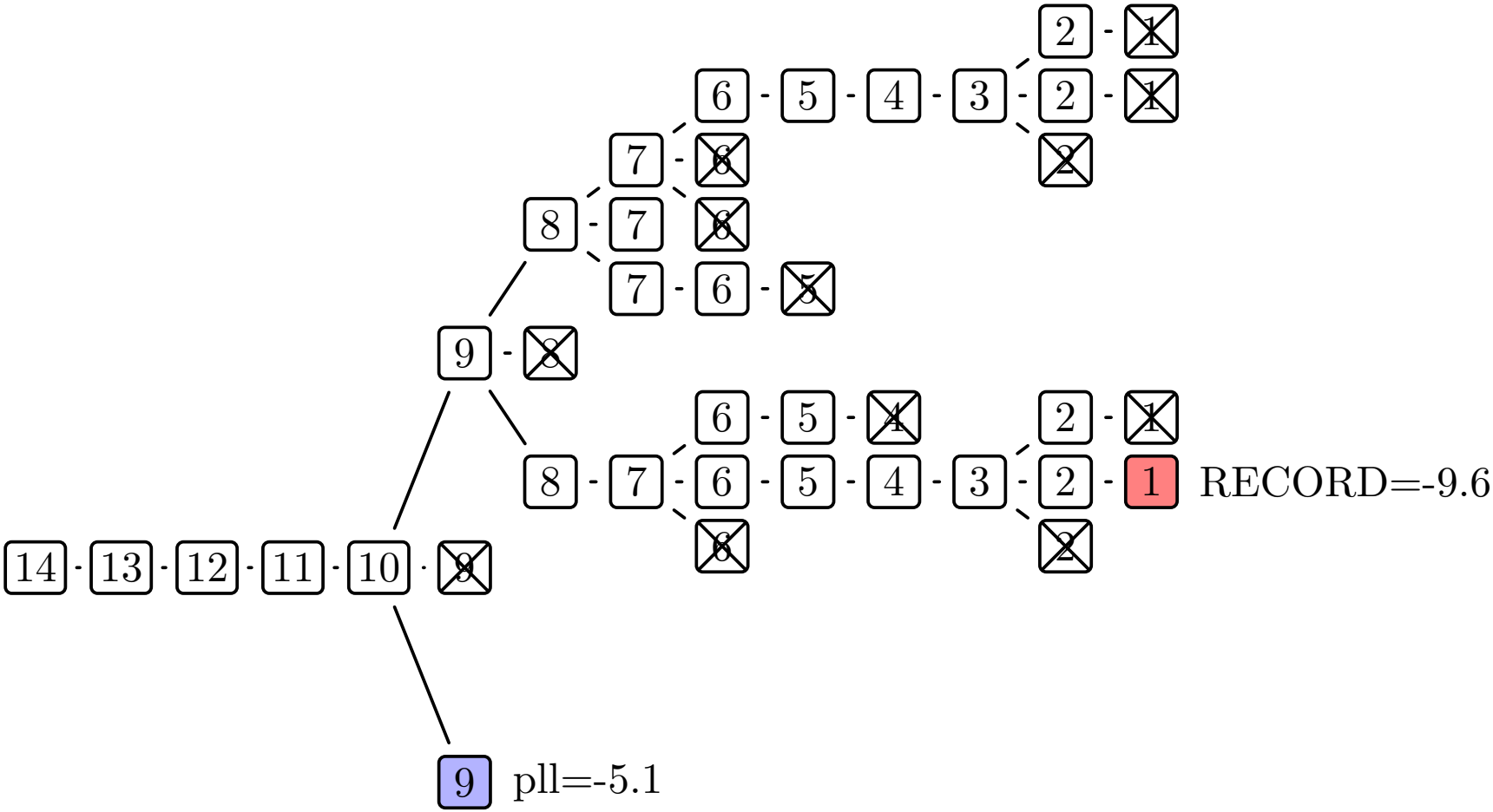
BnB on RLS tree, step 30



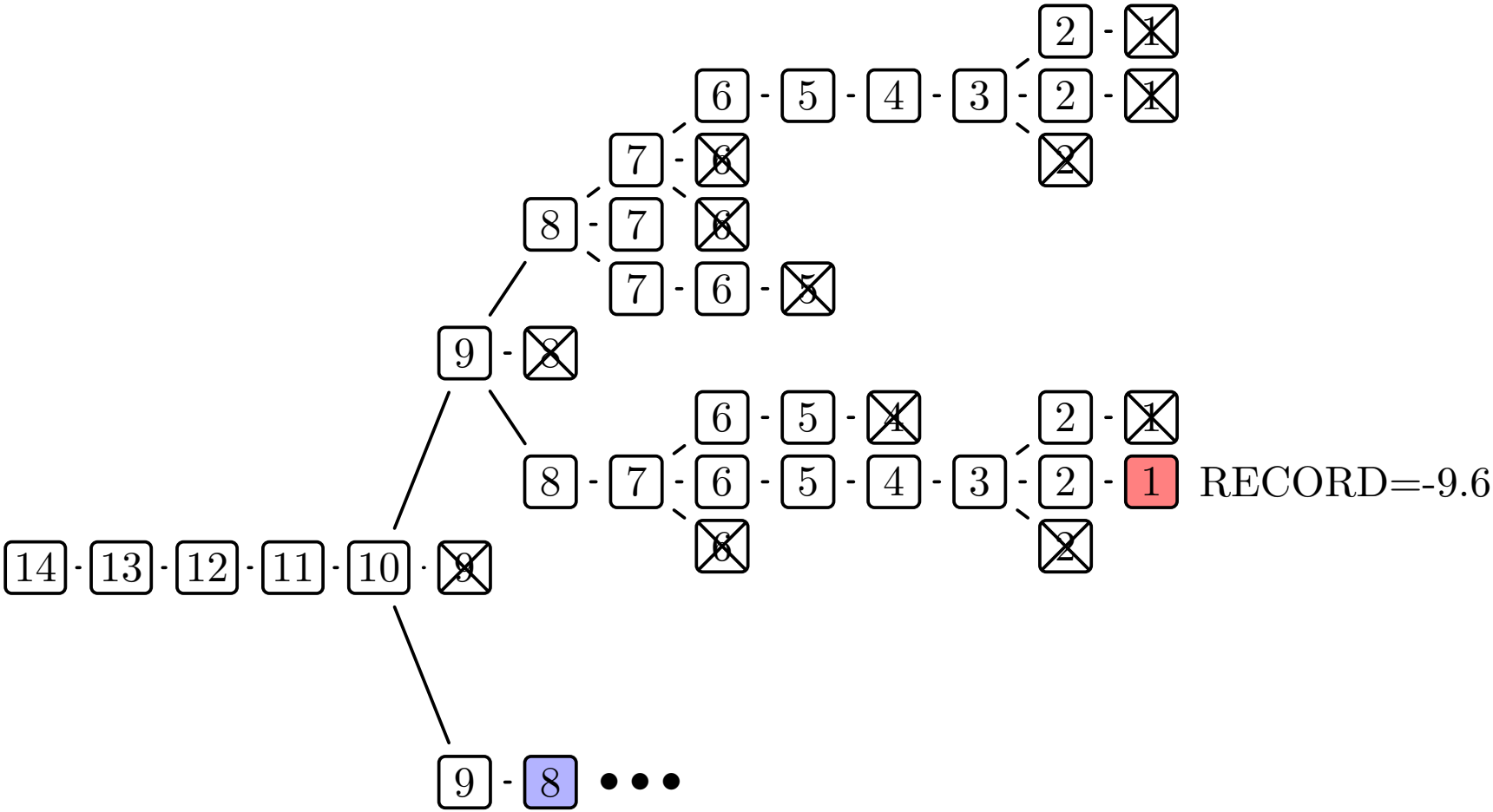
BnB on RLS tree, step 31



BnB on RLS tree, step 33



BnB on RLS tree, step 34



Non-parametric likelihood bounding

- Idea: augment the partial likelihood bounding function with non-parametric likelihood computed on the "rest of the data"
- Parametric choice probabilities $P_i^k(a|x_\iota;\theta) \to$ frequencies $n_\iota^{a_i}/n_\iota$

$$\mathcal{L}^{\text{non-par}}(Z^{\mathcal{S}}) = \sum_{\iota\in\mathcal{S}}\sum_{i=1}^{N}\sum_{a} n_\iota^{a_i}\log(n_\iota^{a_i}/n_\iota)$$

- $\mathcal{L}^{\text{non-par}}(Z^{\mathcal{S}})$ depends only on the counts from the data!
- Not hard to show algebraically that for any $Z^{\mathcal{S}}$ ($\approx$ Gibbs inequality)

$$\mathcal{L}^{\text{non-par}}(Z^{\mathcal{S}}) > L^{\text{part}}(Z^{\mathcal{S}},\theta,P_\theta^k) \quad \forall S$$

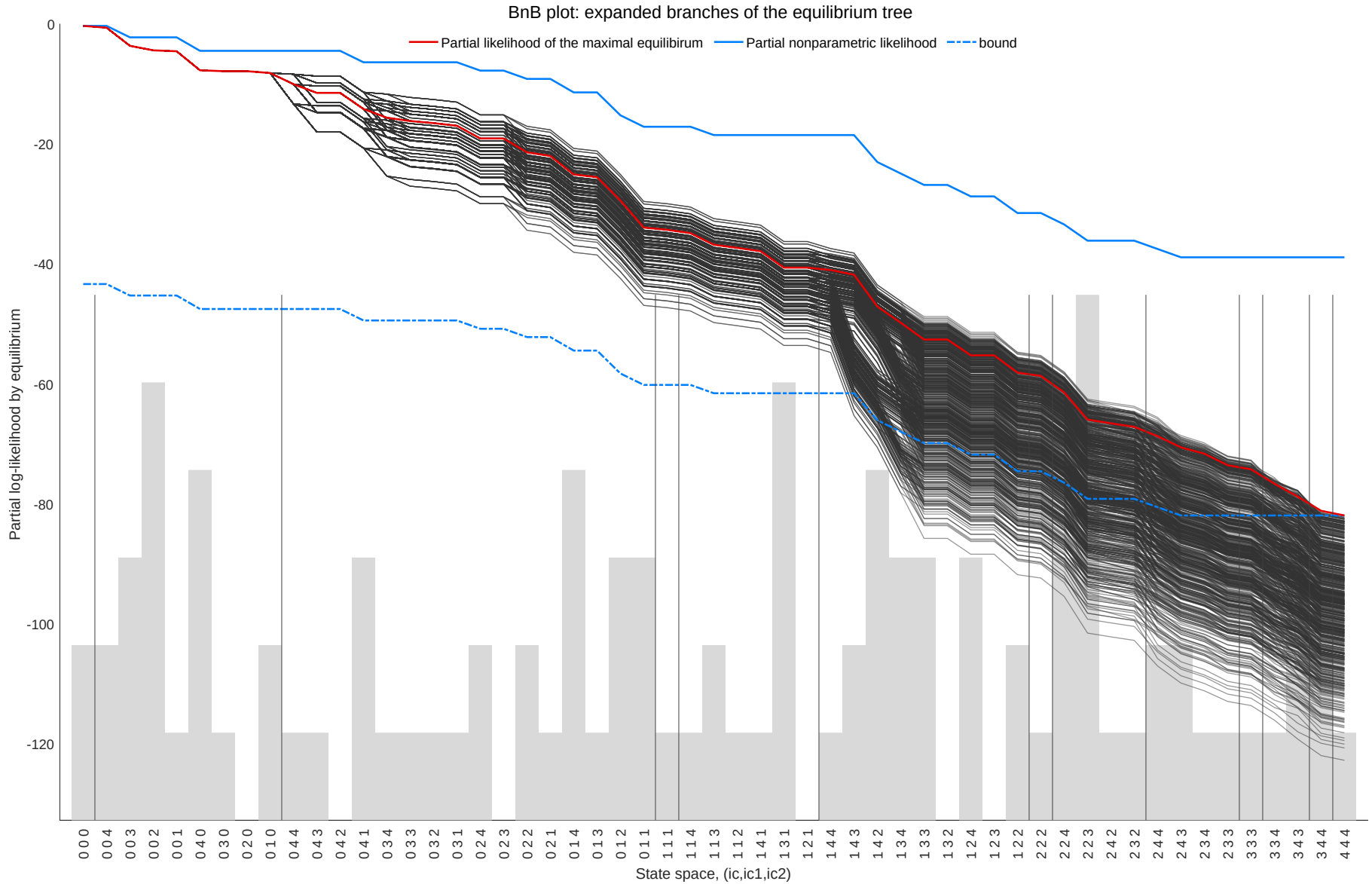
- Therefore partial likelihood can be optimistically extrapolated by empirical likelihood at any step ι of the RLS tree traversal

$$\mathcal{L}^{\text{part}}(Z^{\{S,S-1,\ldots,\iota\}},\theta,P_\theta^k) + \mathcal{L}^{\text{non-par}}(Z^{\{\iota-1,\ldots,1\}})$$

- Augmented partial likelihood is much more powerful bound for BnB

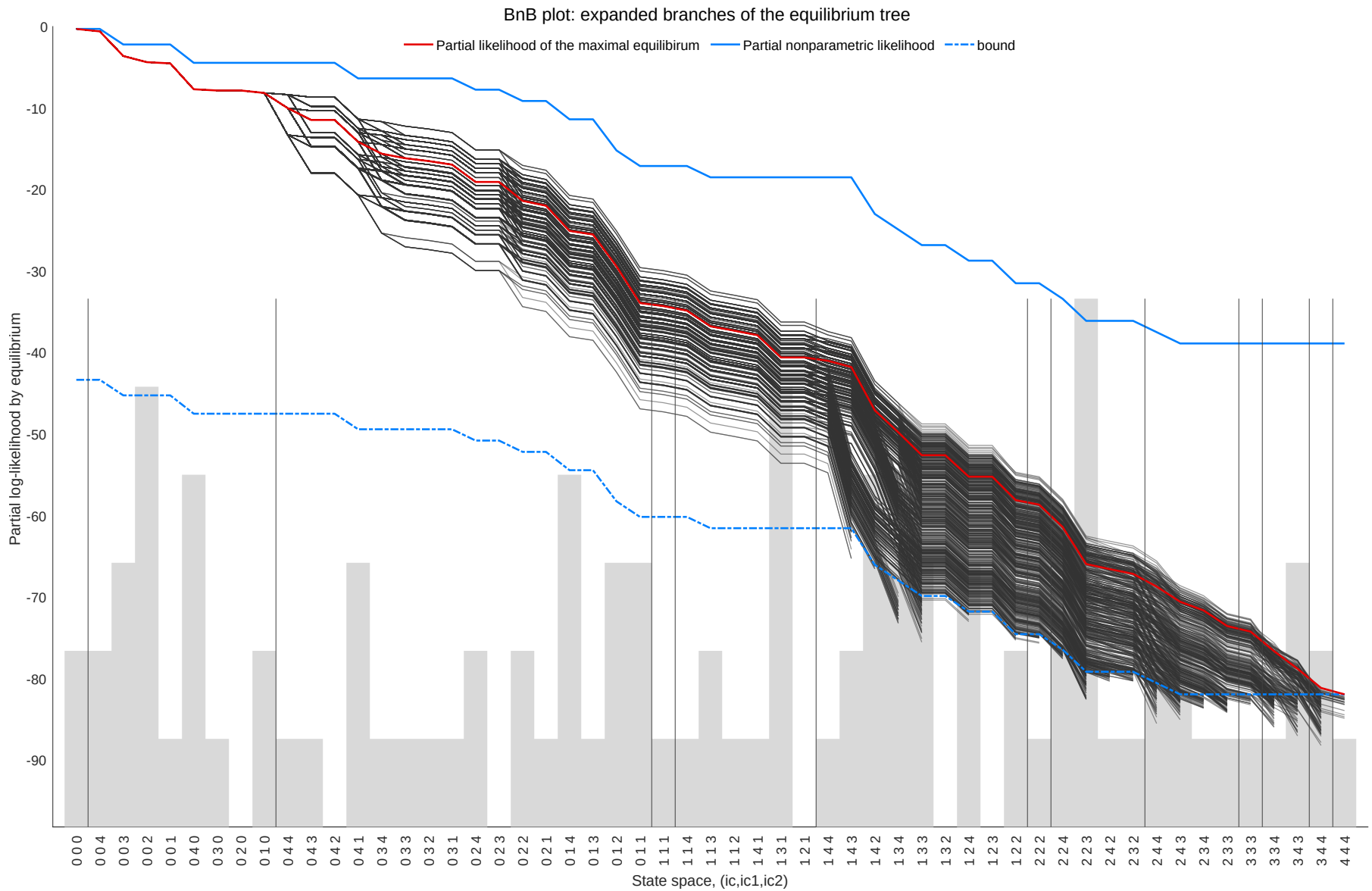
Non-parameteric likelihood bounding

$\iota = S = 14$ (terminal state) on the left, $\iota = 1$ (initial state) on the right



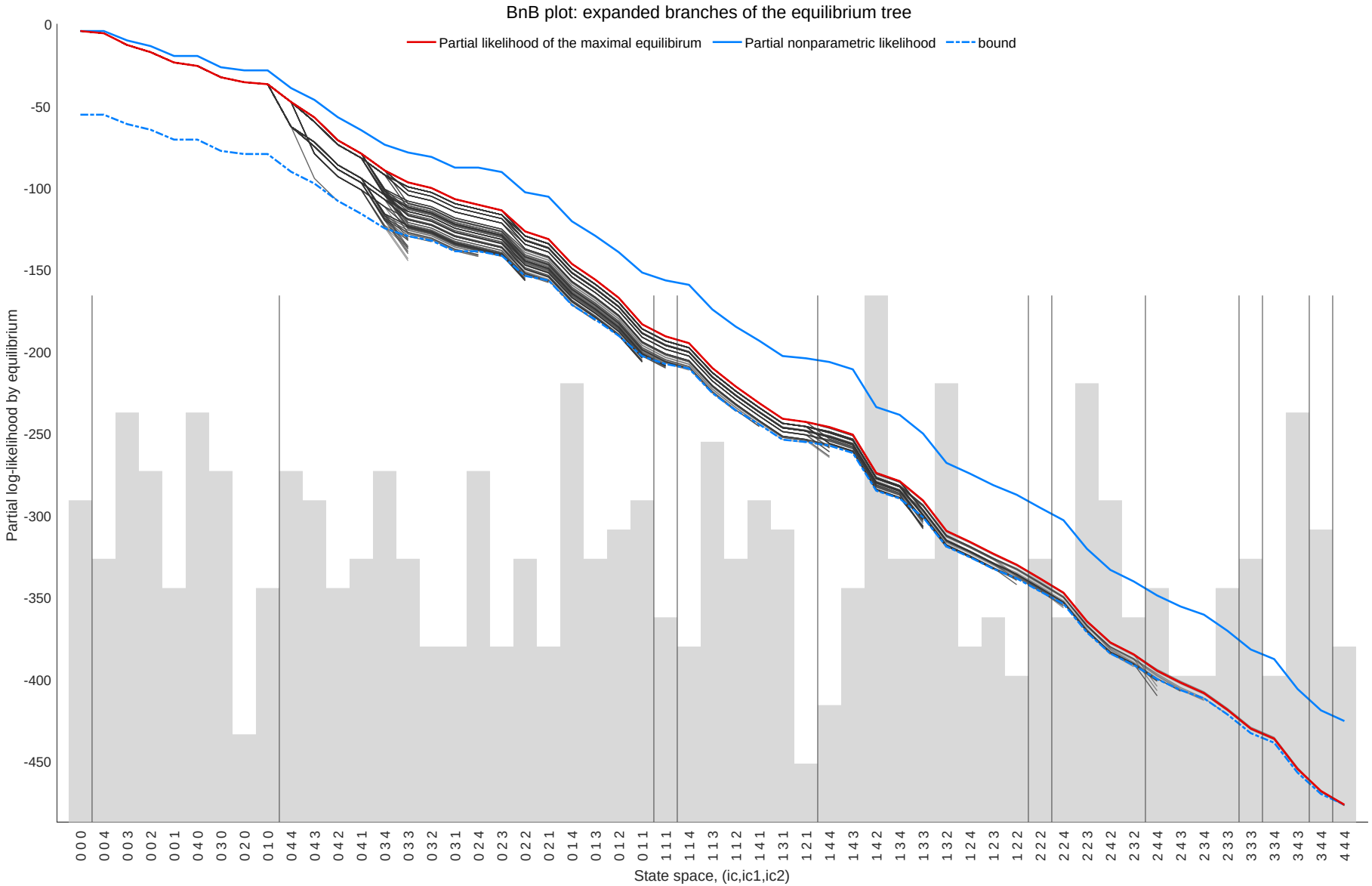
BnB with non-parameteric likelihood bound

Greedy traversal + non-parameteric likelihood bound



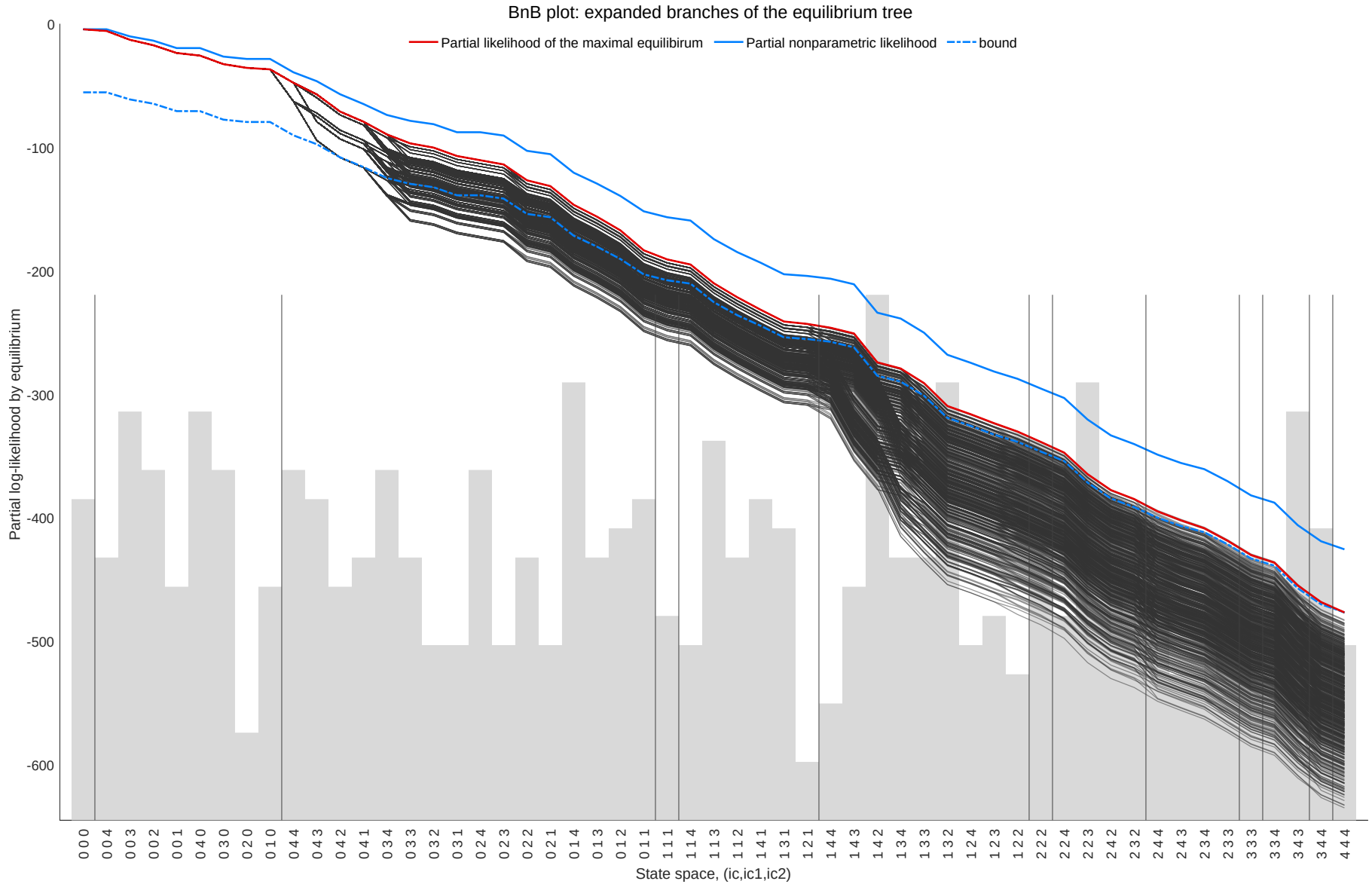
BnB with non-parameteric likelihood bound, larger sample

Non-parametric $\rightarrow$ parametric likelihood as $N \rightarrow \infty$ at true $\theta \Rightarrow$ even less computation



Full enumeration RLS in larger sample

Comparing with the previous slide most of the computation is avoided!



BnB refinement with non-parametric likelihood

- For any amount of data the non-parametric likelihood is greater or equal to the parametric likelihood *algebraically*
- BnB augmented with non-parameteric likelihood bound gives sharper Bounding Rules $\rightarrow$ less computation

- Wih more data as $M \rightarrow \infty$
- Non-parametric log-likelihood converges to the true partial likelihood

  (assuming the model is well-specified and in the neighborhood of the true parameter)
- The width of the band between the blue lines in the plots decreases

  $\rightarrow$ Even sharper Bounding Rules
  $\rightarrow$ Even less computation
  $\rightarrow$ Still delivering exact maximum

MLE for any sample size, but much easier to compute with more data!

ROAD MAP

1. Solving directional dynamic games (DDGs):
   - Simple example: Bertrand pricing and investment game
   - State recursion algorithm
   - Recursive lexicographical search (RLS) algorithm
2. Structural estimation of DDGs using Nested RLS
   - Branch-and-bound on RLS tree
   - Non-parametric likelihood bounding
3. Monte Carlo: (Compare NRLS, two-step CCP, NPL, EPL, MPEC)
   - One equilibrium in the model and data
   - Multiplicity of equilibria at true parameter
   - (Multiple equilibria in the data)

Monte Carlo simulations

A

Single equilibrium in the model
One equilibrium in the data

B

Multiple equilibria in the model
Same equilibrium played the data

C

Multiple equilibria in the model
Multiple equilibria in the data:

- Long panels, each market plays their own equilibrium
- Groups of markets play the same equilibrium

Implementation details:

- Leapfrogging model with $N = 2$ Bertrand competitors deciding whether to invest in cost-reducing technology (IRS, 2016)
- k_1 parameter in investment cost function
- $M = 1000$, $T = 5$
- All methods are initialized with 2-step CCP estimator

Monte Carlo A: no multiplicity

Number of equilibria at true parameter: 1
Number of equilibria in the data: 1

	2step	NPL	EPL	MPEC-VP	MPEC-P	NRLS
True $k_1 = 3.5$	3.52786	3.49714	3.49488	3.49488	3.49486	3.49488
Bias	0.02786	-0.00286	-0.00512	-0.00512	-0.00514	-0.00512
MCSD	0.10037	0.06522	0.07042	0.07042	0.07078	0.07042
ave log-like	-1.16661	-1.16144	-1.16143	-1.16143	-1.16139	-1.16143
log-likelihood	-5833.07	-5807.21	-5807.16	-5807.16	-5806.95	-5807.16
log-like short	-	-0.050	-0.000	-0.000	-0.000	-0.000
KL divergence	0.03254	0.00021	0.00024	0.00024	0.00024	0.00024
$\|\|P - P0\|\|$	0.11270	0.00469	0.00495	0.00495	0.00500	0.00495
$\|\|\Psi(P) - P\|\|$	0.16185	0.0000	0.0000	0.0000	0.0000	0.0000
$\|\|\Gamma(v) - v\|\|$	0.87095	0.00000	0.00000	0.00000	0.00000	0.00000
Converged of 100	-	100	100	100	99	100

- Equilibrium conditions satisfied (except 2step)
- Nearly all MLE estimators identical to the last digit
- NPL and EPL estimators approach MLE

Monte Carlo B, run 1: little multiplicity

Number of equilibria at true parameter: 3
Number of equilibria in the data: 1
Data generating equilibrium: stable

	2step	NPL	EPL	MPEC-VP	MPEC-P	NRLS
True k1=7.5	7.55163	7.49844	7.49918	7.65318	7.35124	7.49919
Bias	0.05163	-0.00156	-0.00082	0.15318	-0.14876	-0.00081
MCSD	0.17875	0.06062	0.03413	0.99742	0.47136	0.03413
ave log-like	-0.84779	-0.84425	-0.84421	-0.88682	-0.87541	-0.84421
log-likelihood	-21194.86	-21106.33	-21105.13	-22170.40	-21885.37	-21105.13
log-like short	-	-1.206	-0.000	-1062.740	-776.809	-0.000
KL divergence	0.02557	0.00040	0.00013	0.23536	0.16051	0.00013
$\|\|P - P_0\|\|$	0.11085	0.00490	0.00280	0.17466	0.20957	0.00280
$\|\|\Psi(P) - P\|\|$	0.170940	0.000000	0.000000	0.000000	0.000000	0.000000
$\|\|\Gamma(v) - v\|\|$	1.189853	0.000000	0.000000	0.000000	0.000000	0.000001
N runs of 100	100	100	100	98	97	100

- MPEC convergence deteriorates
- Equilibrium conditions are satisfied, but estimators start to converge to *wrong* equilibria
  (as seen from KL divergence from the data generating equilibrium)

Monte Carlo B, run 2: little multiplicity, unstable

Number of equilibria at true parameter: 3
Number of equilibria in the data: 1
Data generating equilibrium: unstable

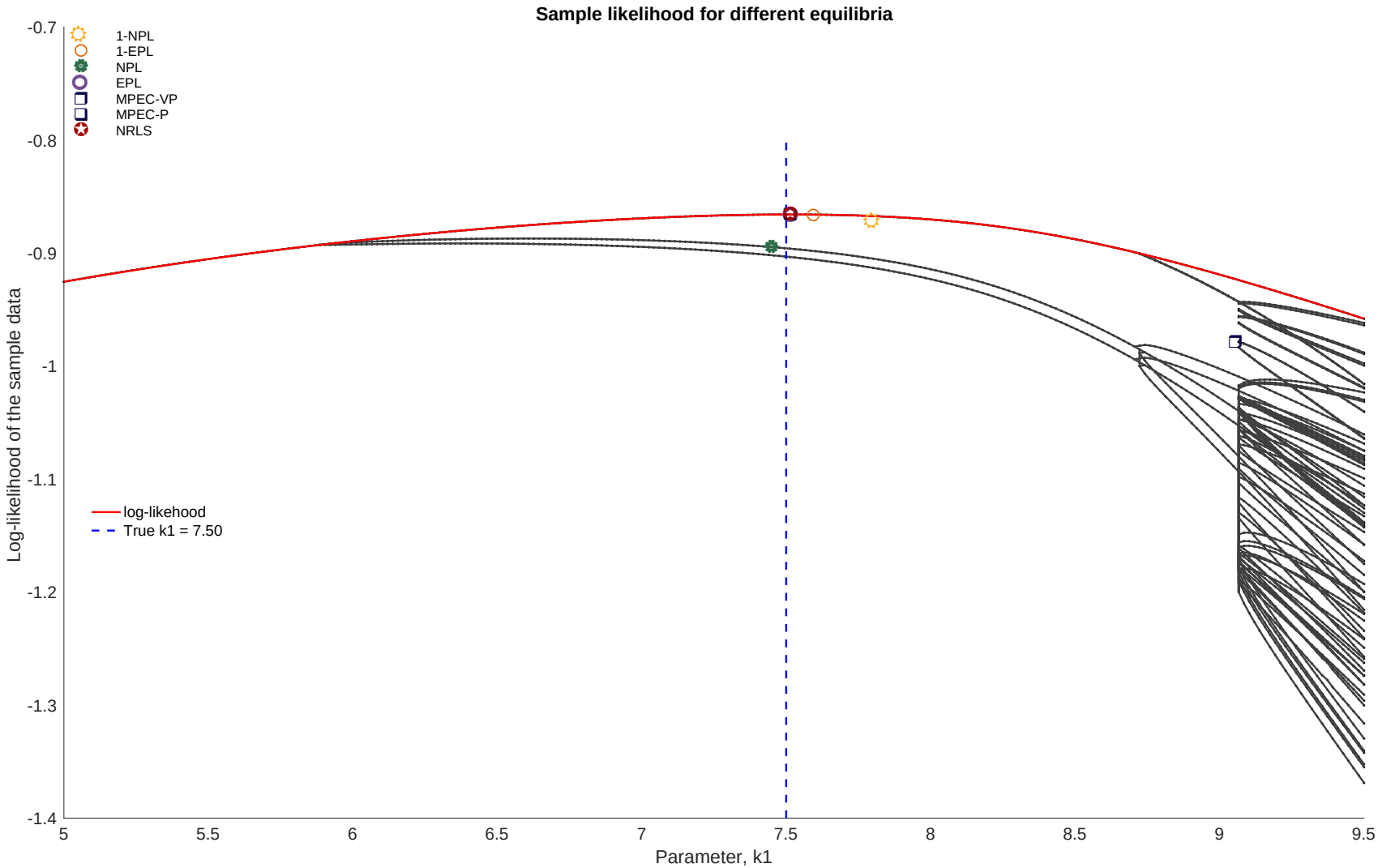
	2step	NPL	EPL	MPEC-VP	MPEC-P	NRLS
True k1=7.5	7.54238	7.39276	7.48044	7.73133	7.63100	7.50176
Bias	0.04238	-0.10724	-0.01956	0.23133	0.13100	0.00176
MCSD	0.17145	0.05608	0.15801	0.72988	0.89874	0.03820
ave log-like	-0.86834	-0.89374	-0.86550	-0.88512	-0.90196	-0.86504
log-likelihood	-21708.60	-22343.58	-21637.54	-22127.91	-22549.06	-21626.12
log-like short	-	-765.242	-11.413	-502.121	-920.643	-0.000
KL divergence	0.02271	0.15996	0.00257	0.11452	0.20182	0.00012
$\|\|P - P_0\|\|$	0.09757	0.20709	0.00619	0.03860	0.02504	0.00307
$\|\|\Psi(P) - P\|\|$	0.160102	0.000000	0.000000	0.000000	0.000000	0.000000
$\|\|\Gamma(v) - v\|\|$	1.126738	0.000000	0.000000	0.000000	0.000000	0.000001
N runs of 100	100	18	100	99	98	100

- NPL estimator fails to converge
- Similar convergence issues for MPEC
- EPL estimator performs well

Aguirregabiria, Marcoux (2021)

Likelihood correspondence

Lines are costructed using symmetric KL-divergence



Monte Carlo B: discontinuous likelihood

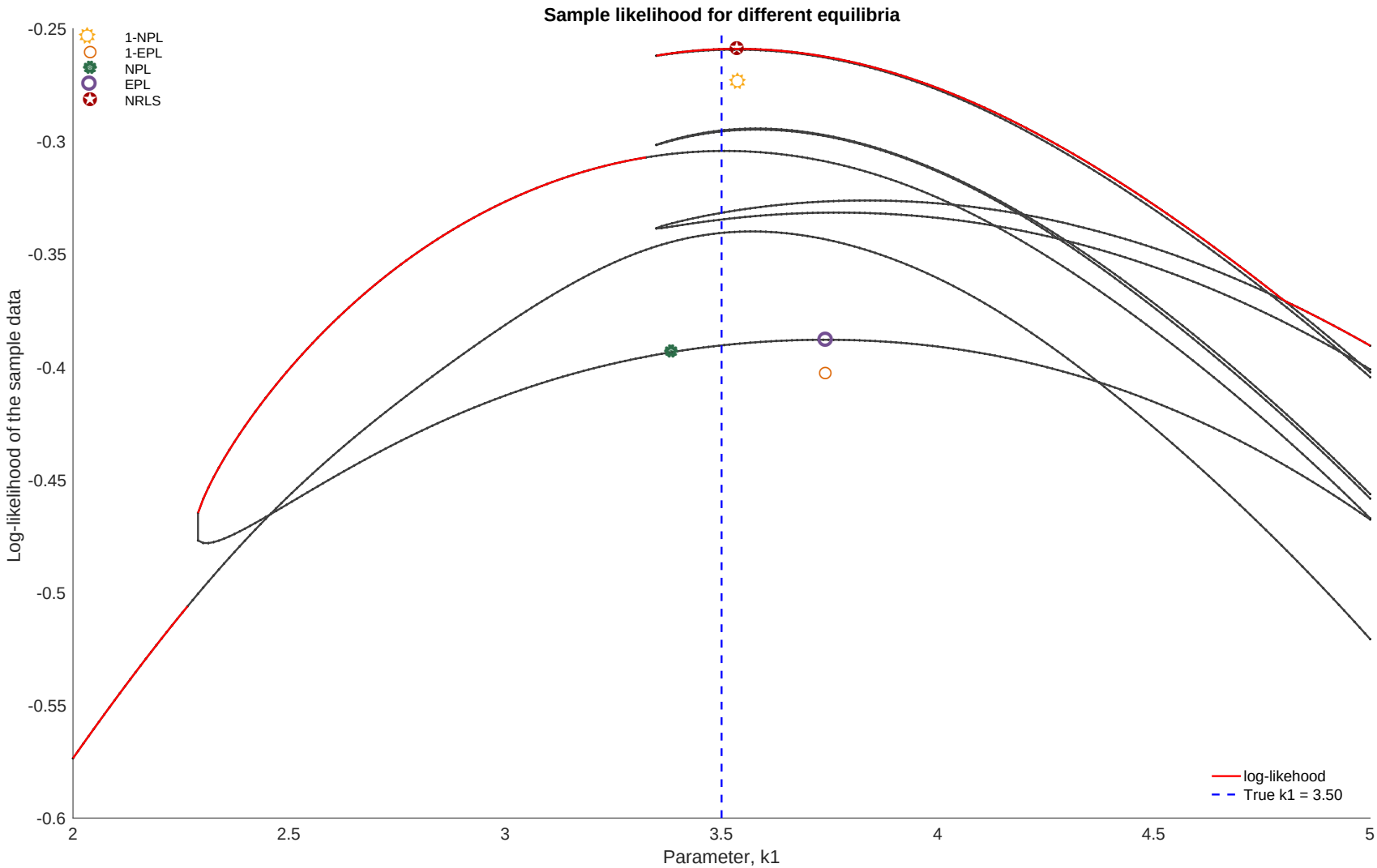
Number of equilibria at true parameter: 9
Number of equilibria in the data: 1
Data generating equilibrium: unstable, near "cliffs"

	2step	NPL	EPL	MPEC-VP	MPEC-P	NRLS
True k1=3.5	3.49739	3.55144	3.64772	3.65943	3.67027	3.50212
Bias	-0.00261	0.05144	0.14772	0.15943	0.17027	0.00212
MCSD	0.13999	0.07133	0.12900	0.12693	0.11583	0.03255
ave log-like	-0.27494	-0.29474	-0.29528	-0.30330	-0.30257	-0.25086
log-likelihood	-1374.721	-1473.695	-1476.425	-1516.503	-1512.847	-1254.320
log-like short	-	-219.375	-222.104	-270.999	-267.523	-0.000
KL divergence	0.01512	0.04889	0.04495	0.04102	0.04078	0.00016
$\|\|P - P_0\|\|$	0.62850	0.86124	0.83062	0.66562	0.65879	0.01610
$\|\|\Psi(P) - P\|\|$	0.763764	0.000000	0.000000	0.000000	0.000000	0.000002
$\|\|\Gamma(v) - v\|\|$	0.852850	0.000000	0.000000	0.000000	0.000000	0.000005
N runs of 100	100	100	100	28	27	100

- Equilibrium conditions are satisfied, but estimators converge to *wrong* equilibria as seen from KL divergence from DGP equilibria
- Biased estimates by EPL, NPL and MPEC
  (constraints are satisfied, yet low likelihood and high KL divergence)

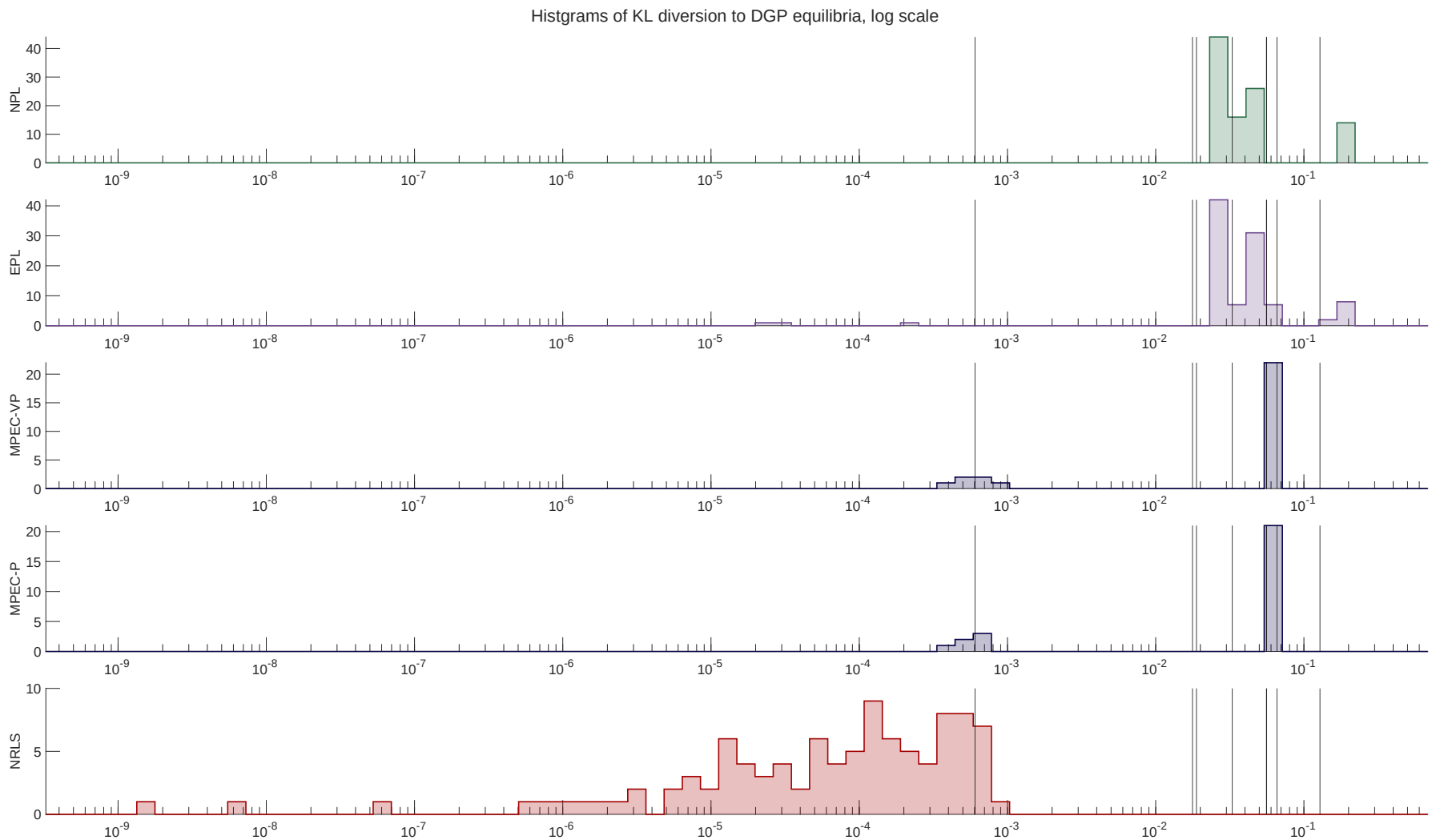
Likelihood correspondence

Lines are costructed using symmetric KL-divergence



Equilibrium selection estimates

Distribution of KL-divergence to the DGP equilibrium, vertical lines represent other equilibria



Monte Carlo B: massive multiplicity

Number of equilibria at true parameter: 2455
Number of equilibria in the data: 1
Time to enumerate all equilibria (RLS) once: 10m 39s

	1-NPL	NPL	EPL	NRLS
True k1=3.75	3.70959	3.71272	3.78905	3.74241
Bias	-0.04041	-0.03728	0.03905	-0.00759
MCSD	0.11089	0.06814	0.40716	0.03032
ave log-likelihood	-0.38681557	-0.37348793	-0.45256293	-0.35998461
log-likelihood	-1934.078	-1867.440	-2262.815	-1799.923
log-like shortfall	-	-66.529	-467.607	-0.000
KL divergence	Inf	14.07523	12231.59186	0.32429
$\|\|P - P_0\|\|$	0.82204	0.65580	0.79241	0.07454
$\|\|\Psi(P) - P\|\|$	0.963574	0.000000	0.000000	0.000006
$\|\|\Gamma(v) - v\|\|$	7.020899	0.000000	0.000000	0.000008
N runs of 100	100	18	68	100
CPU time	0.159s	11.262s	4.013s	4.731s

- Severe convergence problems for NPL and EPL
- Poor eqb identification (low likelihood and high KL divergence)
- NRLS has comparable CPU time (much faster than full enumeration)

Monte Carlo C, multiple equilibria in the data

- Assume that the same equilibrium is played in each market over time
- Grouped fixed-effects, groups defined by the equilibria played

1. Joint grouped fixed-effects estimation
   - Estimate the partition of the markets into groups playing different equilibria together with θ
   - For each market compute maximum likelihood over all equilibria and "assign" it to the relevant group (estimation+classification)
   - Computationally very demanding: BnB market-by-market, non-parametric refinement has no bite
2. Two-step grouped fixed-effects estimation
   - Step 1: partition the markets based on some observable characteristics (K-means clustering) (Outside of Monte Carlo)
   - Step 2: estimate θ allowing different equilibria in different groups
   - Small additional computational cost for NRLS!

 Bonhomme, Manresa (2015); Bonhomme, Lamadon, Manresa (2022)

Monte Carlo C: multiple equilibria in the data

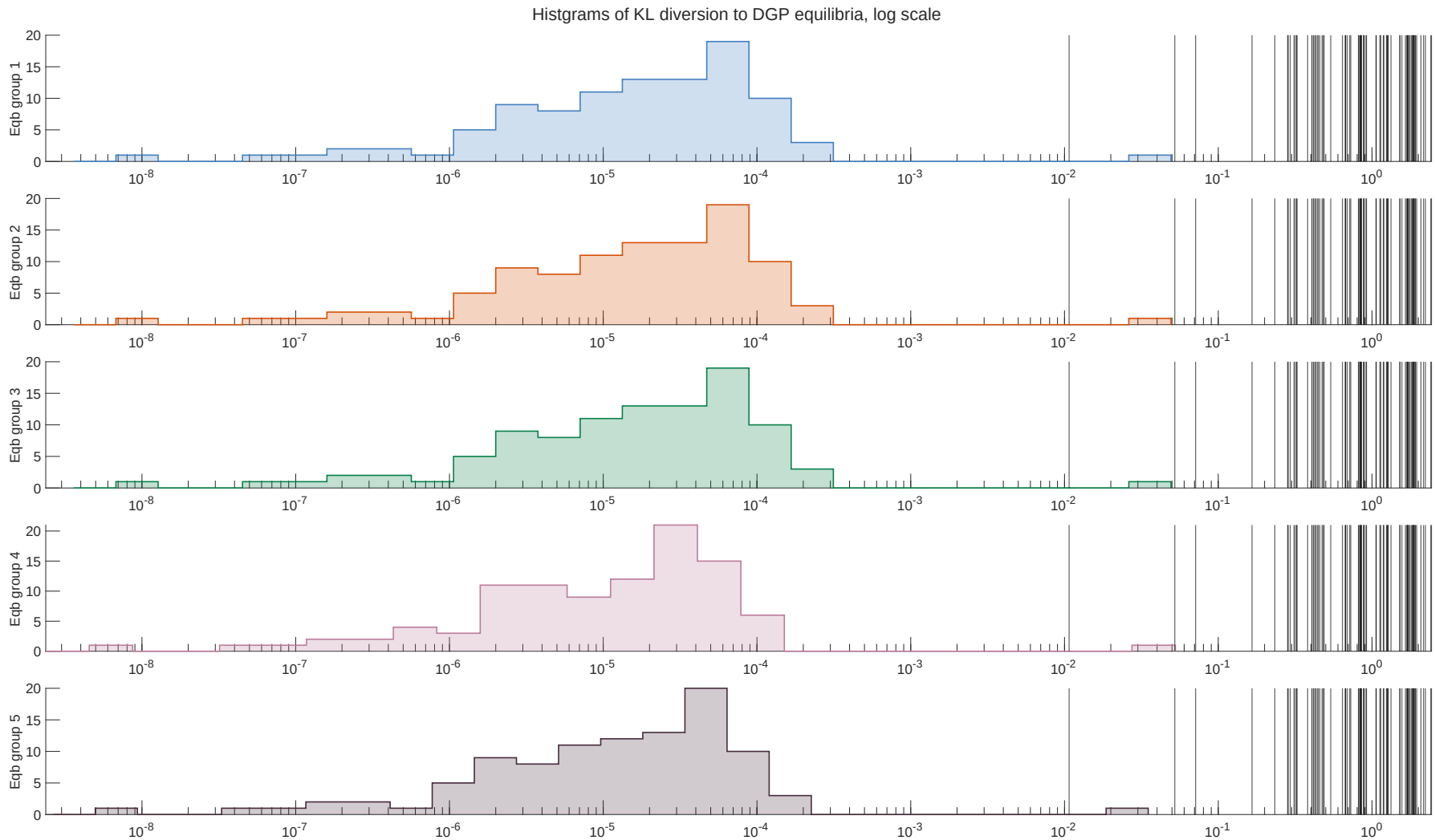
Number of equilibria at true parameter: 81
Number of equilibria in the data: 5
Number of unique equilibria in the data: 3

	1-NPL	NRLS				
True k1=9.25	9.20991	9.25449				
Bias	-0.04009	0.00449				
MCSD	0.15021	0.04109				
ave log-likelihood	-0.798223	-0.707174				
log-likelihood	-19955.57	-17679.36				
log-like shortfall	-	0.000				
KL divergence	0.32943	0.00039	0.00039	0.00039	0.00040	0.00028
$\|P - P_0\|$	0.32787	0.00287	0.00287	0.00287	0.00252	0.00240
$\|\Psi(P) - P\|$	0.460870	0.000000	0.000000	0.000000	0.000000	0.000000
$\|\text{Bellman}(V) - V\|$	5.438776	0.000000	0.000000	0.000000	0.000000	0.000000
# converged of 100	100	100				
CPU time, sec	0.023	20.695				

- All 5 equilibria were identified correctly as seen from KL divergence
- The first three equilibria are the same in DGP, and have the same KL and L1 divergence
- Similar results in runs with many more equilibria in the data

Equilibrium selection estimates

Distribution of KL-divergence to the DGP equilibrium, vertical lines represent other equilibria



Monte Carlo C, run 2: many more equilibria in the data

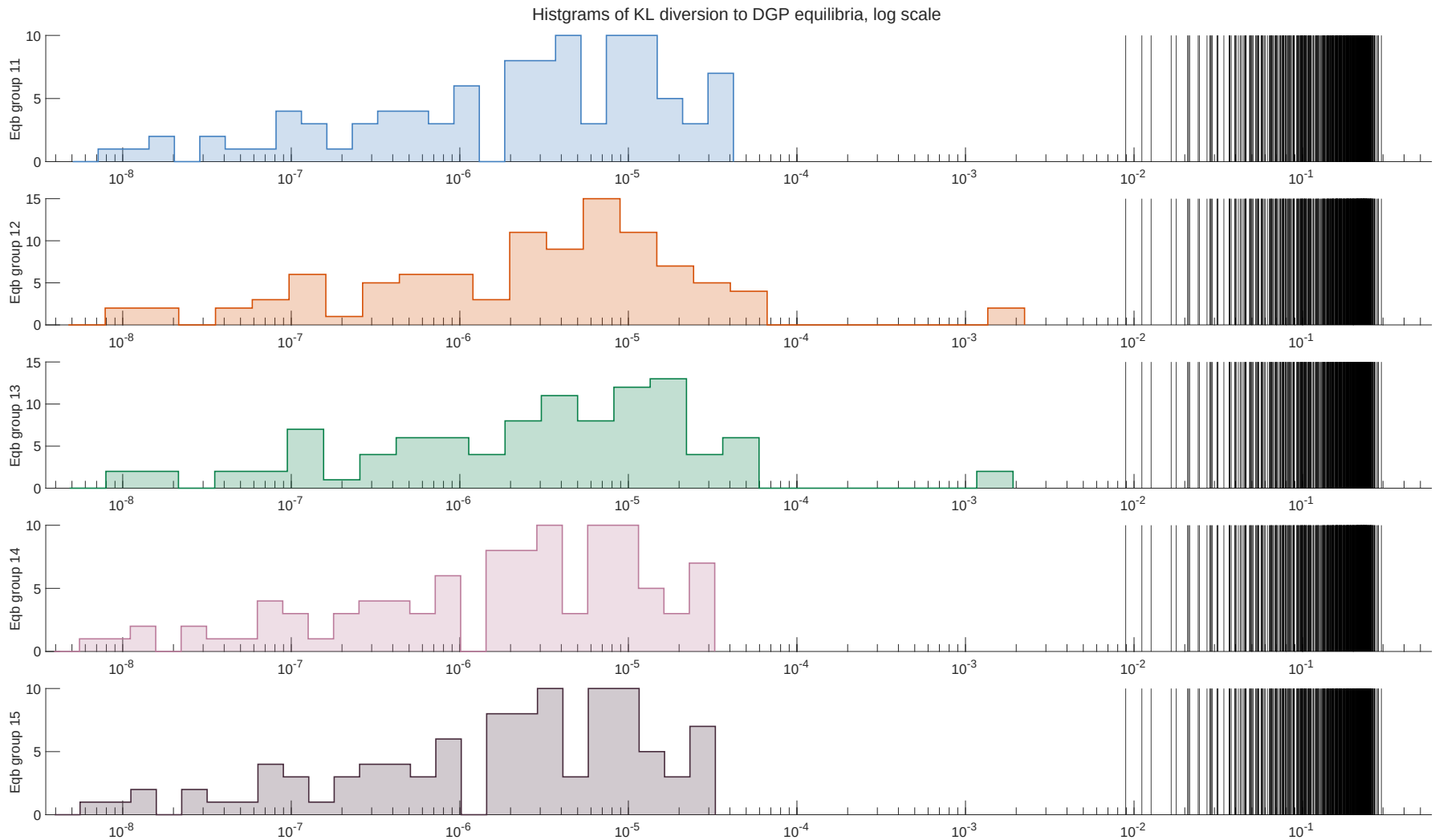
Number of equilibria at true parameter: 19,683
Number of equilibria in the data: 25

	1-NPL	EPL	NRLS
True k1=9.5	9.12069	9.40900	9.49992
Bias	0.03918	-0.09100	-0.00008
MCSD	0.02864	0.05586	0.00414
ave log-likelihood	-0.511188	-0.515356	-0.504482
log-likelihood	-25559.42	-25767.79	-25224.10
log-like shortfall	-	-543.690	0.000
KL divergence	0.05753	0.05045	0.0 to 0.0001
$\|P - P_0\|$	0.18517	0.26430	0.00059 to 0.00572
$\|\Psi(P) - P\|$	0.186981	0.000000	0.0 for all
$\|\text{Bellman}(V) - V\|$	2.577006	0.000000	0.0 for all
# converged of 100	100	100	100
CPU time, sec	0.047	1.041	3m 9.4s

- All 25 equilibria were identified correctly
- Largest average KL divergence 10^{-4} whereas the closest to DGP equilibrium at true θ has KL=10^{-2}
- Largest error in choice probabilities across the state space 0.00572

Equilibrium selection estimates

Distribution of KL-divergence to the DGP equilibrium, vertical lines represent other equilibria



Shown are the first five identified equilibria, other twenty are similar

NRLS estimator for directional dynamic games

Complicated computational task involving maximization over the large finite set of all MPE equilibria $\rightarrow$ branch-and-bound algorithm with refined bounding rule

NRLS nested structure:

1. Each stage game $\rightarrow$ non-linear solver, specific to the model
2. Combining stage game solutions to full game MPEs $\rightarrow$ State Recursion algorithm
3. Solving for all MPE equilibria $\rightarrow$ Recursive Lexicographic Search
4. Structural estimation $\rightarrow$ Nested loop MLE with BnB on RLS tree

Performance of NRLS

- Implementation of statistically efficient estimator (MLE)
- BnB with NRLS avoids full enumeration at no additional cost
- BnB augmented with non-parameteric likelihood bound gives sharper Bounding Rules $\rightarrow$ less computation larger samples
- Computationally tractable
- Fully robust to multiplicity of equilibria