EULER (FINITE DEPENDENCE) OPERATORS FOR SOLVING DYNAMIC DISCRETE CHOICE MODELS

LECTURE 6

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SOLVING DYNAMIC DISCRETE CHOICE (DDC) MODELS

- The literature has developed computationally efficient methods for estimating the structural parameters of DDC that do not require solving the DP problem even once.

- The **solution of the DP problem** and its **curse of dimensionality** still play an important role in empirical applications because:

  1. **Counterfactual Experiments** using the estimated model.

  2. **Full Solution estimation methods** such as NFXP and NPL that can reduce substantial (finite sample) bias in two-step CCP methods.

- For this purpose, applications use one of these **solution methods**:

  a. Value Function iteration (VFI).

  b. Policy Function iteration (PFI) – also called Newton-Kantorovich.

  c. A Hybrid combination of VFI and PFI.

Solution Methods based on Euler Equations (Finite Dependence)

- For models with a Finite Dependence property, Euler equations (EE) representations of the optimal conditions of optimality.

  a. They are always necessary conditions of optimality, and often necessary & sufficient.

  b. **They do not involve computing Present Values** but only next period expected payoff.

- In this lecture, I review **recent methods for the solution of DDC models** that use Euler Equations (EE).

- This method is **computationally more efficient than VFI and PFI**:

  1. A single iteration has the same cost as VFI ($<<<$ than PFI).

  2. The contraction mapping has modulus $<<<$ modulus of VFI (β) and similar to the modulus of PFI.

OUTLINE

1. Dynamic Discrete Choice models.
   1.1. Model.
   1.2. VFI and PFI operators & algorithms.
2. Finite Dependence & Euler Mapping
   2.1. Derivation of the mapping.
   2.2. Contraction with modulus $< \beta$.
3. Analogy to Euler Equations in Continuous-Choice Models
   - Representation with CCPs as Continuous Decisions
4. Numerical Examples

1. MODEL

FRAMEWORK: Payoffs, Actions, States

- Markov Decision Process (MDP) with a discrete action: Rust (1994).
- Each discrete time period t, an agent selects an action a_t to maximize expected intertemporal payoff:

$$\mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j \, \Pi_{a_{t+j}}(\mathbf{s}_{t+j}) \right], \text{ with } \Pi_{a_t}(\mathbf{s}_t) = \text{per-period payoff function}$$

- $\mathbf{s}_t \in \mathcal{S}$ $=$ state vector, with transition probability:

$$\Pr(\mathbf{s}_{t+1} \mid a_t = a, \mathbf{s}_t) = f_a(\mathbf{s}_{t+1} \mid \mathbf{s}_t)$$

- The action a_t is drawn from a finite set $\mathcal{A} = \{0, 1, \ldots, J\}$.

Value Function, Bellman Equation, & Optimal Decision Rule

- The value function $V(\mathbf{s}_t)$ satisfies the Bellman equation:

$$V(\mathbf{s}_t) = \max_{a_t \in \mathcal{A}} \left\{ \Pi_{a_t}(\mathbf{s}_t) + \beta \int V(\mathbf{s}_{t+1})\, f_{a_t}(\mathbf{s}_{t+1} \mid \mathbf{s}_t)\, d\mathbf{s}_{t+1} \right\}.$$

- The optimal decision rule $\alpha(\mathbf{s}_t) : \mathcal{S} \to \mathcal{A}$ selects the action that maximizes the right-hand side of the Bellman equation.

Rust's Model: Assumptions & Integrated Bellman Equation

- We partition the state vector as $\mathbf{s}_t = (\mathbf{x}_t, \boldsymbol{\varepsilon}_t)$, where:
  - $\mathbf{x}_t \in \mathcal{X}$ is observed by the researcher
  - $\boldsymbol{\varepsilon}_t = \{\varepsilon_{a,t} : a \in \mathcal{A}\}$ are action-specific unobserved shocks.

- *Additive Separability*: $\quad \Pi_{a_t}(\mathbf{s}_t) \; = \; \pi_{a_t}(\mathbf{x}_t) + \varepsilon_{a_t,t}$

- *Conditional Independence*: $f_{a_t}(\mathbf{s}_{t+1} \mid \mathbf{s}_t) \; = \; f_{a_t}(\mathbf{x}_{t+1} \mid \mathbf{x}_t) \cdot g(\boldsymbol{\varepsilon}_{t+1})$

- $V^\sigma(\mathbf{x}_t) \equiv \int V(\mathbf{x}_t, \boldsymbol{\varepsilon}_t)\, g(\boldsymbol{\varepsilon}_t)\, d\boldsymbol{\varepsilon}_t$ is the *integrated value function*.

- The *integrated Bellman equation* is then given by:

$$V^\sigma(\mathbf{x}_t) = \int \max_{a \in \mathcal{A}} \{ v_a(\mathbf{x}_t) + \varepsilon_{a,t} \} \; g(\boldsymbol{\varepsilon}_t)\, d\boldsymbol{\varepsilon}_t,$$

  where $v_a(\mathbf{x}_t)$ denotes the *choice-specific value function*, defined as:

$$v_a(\mathbf{x}_t) \equiv \pi_a(\mathbf{x}_t) + \beta \sum_{\mathbf{x}_{t+1} \in \mathcal{X}} V^\sigma(\mathbf{x}_{t+1})\, f_a(\mathbf{x}_{t+1} \mid \mathbf{x}_t).$$

McFadden's Social Surplus function

- Given a vector of choice-specific values $\mathbf{v} = \{v_a : a \in \mathcal{A}\} \in \mathbb{R}^{J+1}$, the social surplus is defined as:

$$S(\mathbf{v}) \equiv \int \max_{a \in \mathcal{A}} \{v_a + \varepsilon_{a,t}\} \ g(\boldsymbol{\varepsilon}_t) \, d\boldsymbol{\varepsilon}_t .$$

- Using Social Surplus, the integrated Bellman equation can be written:

$$V^{\sigma}(\mathbf{x}_t) \ = \ S(\mathbf{v}(\mathbf{x}_t))$$

where $\mathbf{v}(\mathbf{x}_t) = \{v_a(\mathbf{x}_t) : a \in \mathcal{A}\}$ = vector of choice-specific values.

VALUE FUNCTION OPERATOR

- The integrated Bellman equation defines a fixed-point mapping in the space of value functions: the *value function (VF) operator*:

$$\mathbf{V}^{\sigma} = \Gamma_{VF}\left(\mathbf{V}^{\sigma}\right) = S\left(\left\{\boldsymbol{\pi}_a + \beta\, \boldsymbol{F}_a \mathbf{V}^{\sigma} \text{ for } a \in \mathcal{A}\right\}\right)$$

where $\mathbf{V}^{\sigma} = \{V^{\sigma}(\mathbf{x}_t) : \mathbf{x}_t \in \mathcal{X}\}$.

- VFI iteration: $\mathbf{V}_n^{\sigma} = \Gamma_{VF}\left(\mathbf{V}_{n-1}^{\sigma}\right)$

- Properties:

   a. Contraction with modulus (Lipschitz constant) equal β.

   b. Linear convergence. Slow, especially if β is close to one.

   c. Cost per iteration: linear in $|\mathcal{X}|$.

POLICY FUNCTION (NEWTON-KANTOROVICH) OPERATOR

- It is a fixed-point mapping in the space of *policies* or CCPs:

$$\mathbf{P} = \Gamma_{PF}(\mathbf{P})$$

with $\mathbf{P} = \{\mathbf{P}_a(x) : (a, x) \in \mathcal{A} \setminus \{0\} \times \mathcal{X}\}$, and, in vector form:

$$\Gamma_{PF}(\mathbf{P})(a) = \int 1\left\{\boldsymbol{\pi}_a + \beta\, \mathbf{F}_a\, \mathbf{W}^{\mathbf{P}} + \varepsilon_a \geq \boldsymbol{\pi}_j + \beta\, \mathbf{F}_j\, \mathbf{W}^{\mathbf{P}} + \varepsilon_j,\ \forall j\right\} g(\boldsymbol{\varepsilon})$$

- $\mathbf{W}^{\mathbf{P}}$ is the vector of present discounted values of future payoffs if the agent's future behavior follows the vector of choice probabilities $\mathbf{P}$.

- Vector of present values $\mathbf{W}^{\mathbf{P}}$ is obtained solving the system of $|\mathcal{X}|$ linear equations:

$$\mathbf{W}^{\mathbf{P}} = \sum_{a=0}^{J} \mathbf{P}_a * \left[\boldsymbol{\pi}_a + \mathbf{e}_a(\mathbf{P}) + \beta\, \mathbf{F}_a\, \mathbf{W}^{\mathbf{P}}\right],$$

where $*$ is the element-by-element product, and $\mathbf{e}_a(\mathbf{P}(y))$ is the vector of conditional expectations of $\varepsilon_{a,t}$ conditional on alternative a being the optimal choice.

POLICY FUNCTION ITERATION ALGORITHM

- Fixed Point iterations in PF operator Γ_{PF}.
- Given $\boldsymbol{P}_{n-1}$, at iteration n we apply 2 steps:

  1. Valuation Step: Solve system of linear equations to obtain $\boldsymbol{W}_n^P$:

$$\mathbf{W}_n^{\mathbf{P}} = \sum_{a=0}^{J} \mathbf{P}_{a,n-1} * \left[\boldsymbol{\pi}_a + \mathbf{e}_a(\mathbf{P}_{n-1}) + \beta \, \mathbf{F}_a \; \mathbf{W}_n^{\mathbf{P}} \right]$$

  2. Policy improvement Step: Given $\mathbf{W}_n^{\mathbf{P}}$, calculate "best response" CCPs.

- Properties:

  a. Contraction with modulus $<<< \beta$.

  b. Quadratic convergence. Fast, regardless β close to one.

  c. Cost per iteration: cubic in $|\mathcal{X}|$ – because the solution of the linear system in the valuation step.

2. FINITE DEPENDENCE & EULER MAPPING

CHOICE-SPECIFIC VALUE DIFFERENCES

- Define the value difference $\widetilde{v}_a(\mathbf{x}_t) \equiv v_a(\mathbf{x}_t) - v_0(\mathbf{x}_t)$, where alternative 0 is chosen as the baseline without loss of generality.

$$\widetilde{\mathbf{v}}(\mathbf{x}_t) = \{\widetilde{v}_a(\mathbf{x}_t) : a \in \mathcal{A} \setminus \{0\}\}$$

- Using value differences, we have:

$$V^{\sigma}(\mathbf{x}_t) \;=\; v_0(\mathbf{x}_t) + S([0, \widetilde{\mathbf{v}}(\mathbf{x}_t)])$$

where $[0, \widetilde{\mathbf{v}}]$ is vector of choice-specific values with $v_0(\mathbf{x}_t)$ set to zero.

- Using the definition of value difference $\widetilde{v}_a(\mathbf{x}_t)$, we have:

$$\widetilde{v}_a(\mathbf{x}_t) = \widetilde{\pi}_a(\mathbf{x}_t) + \beta \sum_{\mathbf{x}_{t+1}} \left[v_0(\mathbf{x}_{t+1}) + S([0, \widetilde{\mathbf{v}}(\mathbf{x}_{t+1})]) \right] \widetilde{f}_a \left(\mathbf{x}_{t+1} \mid \mathbf{x}_t\right)$$

with:

  - $\widetilde{\pi}_a(\mathbf{x}_t) \equiv \pi_a(\mathbf{x}_t) - \pi_0(\mathbf{x}_t)$.
  - $\widetilde{f}_a \left(\mathbf{x}_{t+1} \mid \mathbf{x}_t\right) \equiv f_a \left(\mathbf{x}_{t+1} \mid \mathbf{x}_t\right) - f_0 \left(\mathbf{x}_{t+1} \mid \mathbf{x}_t\right)$

CHOICE SPECIFIC VALUE DIFFERENCES [2/2]

- Replacing $v_0(\mathbf{x}_{t+1})$ with its expression

$$v_0(\mathbf{x}_{t+1}) \;=\; \pi_0(\mathbf{x}_{t+1}) + \beta \sum_{\mathbf{x}_{t+2}} V^{\sigma}(\mathbf{x}_{t+2})\, f_0(\mathbf{x}_{t+2} \mid \mathbf{x}_{t+1})$$

we have:

$$\begin{aligned}
\widetilde{v}_a(\mathbf{x}_t) \;&=\; \widetilde{\pi}_a(\mathbf{x}_t) + \beta \sum_{\mathbf{x}_{t+1}} \left[\pi_0(\mathbf{x}_{t+1}) + S([0, \widetilde{\mathbf{v}}(\mathbf{x}_{t+1})])\right] \widetilde{f}_a\left(\mathbf{x}_{t+1} \mid \mathbf{x}_t\right) \\
&+\; \beta^2 \sum_{\mathbf{x}_{t+1}} \sum_{\mathbf{x}_{t+2}} V^{\sigma}(\mathbf{x}_{t+2})\, f_0\left(\mathbf{x}_{t+2} \mid \mathbf{x}_{t+1}\right)\; \widetilde{f}_a\left(\mathbf{x}_{t+1} \mid \mathbf{x}_t\right)
\end{aligned}$$

FINITE DEPENDENCE

- **DEFINITION. Two-Period Finite Dependence.**

- For arbitrary choices a_t and a_{t+1}, define the two-periods-forward transition probability of the state variables:

$$f_{a_t, a_{t+1}}^{(2)}\left(\mathbf{x}_{t+2} \mid \mathbf{x}_t\right) \equiv \sum_{\mathbf{x}_{t+1}} f_{a_{t+1}}\left(\mathbf{x}_{t+2} \mid \mathbf{x}_{t+1}\right) f_{a_t}\left(\mathbf{x}_{t+1} \mid \mathbf{x}_t\right)$$

- A controlled transition probability $f_{a_t}\left(\mathbf{x}_{t+1} \mid \mathbf{x}_t\right)$ has the (global) two-period finite dependence property if for every state $\mathbf{x}_t \in \mathcal{X}$ and choice $a \in \mathcal{A}$ we have:

$$f_{a_t=a,\ a_{t+1}}^{(2)}\left(\mathbf{x}_{t+2} \mid \mathbf{x}_t\right) = f_{a_t=0,\ a_{t+1}}^{(2)}\left(\mathbf{x}_{t+2} \mid \mathbf{x}_t\right), \text{ for any } \mathbf{x}_{t+2} \in \mathcal{X}$$

FINITE DEPENDENCE & EULER MAPPING

- Remember the general expression:

$$\begin{aligned}\widetilde{v}_a(\mathbf{x}_t) &= \widetilde{\pi}_a(\mathbf{x}_t) + \beta \sum_{\mathbf{x}_{t+1}} \Big[\pi_0(\mathbf{x}_{t+1}) + S([0, \widetilde{\mathbf{v}}(\mathbf{x}_{t+1})])\Big] \widetilde{f}_a\left(\mathbf{x}_{t+1} \mid \mathbf{x}_t\right) \\ &+ \beta^2 \sum_{\mathbf{x}_{t+1}} \sum_{\mathbf{x}_{t+2}} V^{\sigma}(\mathbf{x}_{t+2})\, f_0\left(\mathbf{x}_{t+2} \mid \mathbf{x}_{t+1}\right)\ \widetilde{f}_a\left(\mathbf{x}_{t+1} \mid \mathbf{x}_t\right)\end{aligned}$$

- Under the two-period FD property, the last term on this equation equals zero, and we have:

$$\widetilde{v}_a(\mathbf{x}_t) \;=\; c_a(\mathbf{x}_t) + \beta \sum_{\mathbf{x}_{t+1}} S([0, \widetilde{\mathbf{v}}(\mathbf{x}_{t+1})])\ \widetilde{f}_a\left(\mathbf{x}_{t+1} \mid \mathbf{x}_t\right)$$

where $c_a(\mathbf{x}_t)$ is closed-form of primitives:

$$c_a(\mathbf{x}_t) \;=\; \widetilde{\pi}_a(\mathbf{x}_t) + \beta \sum_{\mathbf{x}_{t+1}} \pi_0(\mathbf{x}_{t+1})\ \widetilde{f}_a\left(\mathbf{x}_{t+1} \mid \mathbf{x}_t\right)$$

EULER MAPPING

- Equation

$$\widetilde{v}_a(\mathbf{x}_t) \;=\; c_a(\mathbf{x}_t) + \beta \sum_{\mathbf{x}_{t+1}} S([0, \widetilde{\mathbf{v}}(\mathbf{x}_{t+1})]) \; \widetilde{f}_a \left(\mathbf{x}_{t+1} \mid \mathbf{x}_t\right)$$

defines a fixed-point mapping in the space of value differences.

- This mapping is the *Euler mapping (EE)*. In vector form:

$$\widetilde{\mathbf{v}} \;=\; \Gamma_{EE}\left(\widetilde{\mathbf{v}}\right)$$

where $\widetilde{\mathbf{v}} = \{v_a(\mathbf{x}_t) : a \in \mathcal{A} \setminus \{0\}, \mathbf{x}_t \in \mathcal{X}\}$.

- Given the vector $\widetilde{\mathbf{v}}$ that solves this mapping, we have a complete solution to the dynamic programming problem.

EULER MAPPING IS A STRONGER CONTRACTION

- **PROPOSITION**. Consider a discrete choice MDP with the two-period FD property. The Euler mapping Γ_{EE} is a contraction in the complete metric space $(\mathcal{V}^R, \|.\|_\infty)$, and its Lipschitz constant δ is strictly smaller than β.

- That is, there is a constant $\delta \in (0,1)$ with $\delta < \beta$ such that for any pair $\widetilde{\mathbf{v}}$ and $\widetilde{\mathbf{w}}$ in $\mathcal{V}^R$,

$$\|\Gamma_{EE}(\widetilde{\mathbf{v}}) - \Gamma_{EE}(\widetilde{\mathbf{w}})\|_\infty \ \leq \ \delta \ \|\widetilde{\mathbf{v}} - \widetilde{\mathbf{w}}\|_\infty \qquad \blacksquare$$

- Comments:

  1. Proof applies **Willians-Daly-Zachary Theorem**: $\dfrac{\partial S}{\partial \boldsymbol{v}_a} = \boldsymbol{P}_a$.

  2. Stronger contraction: $\dfrac{\delta}{\beta} = ||\{\boldsymbol{P}_a : a > 0\}||$

EULER ITERATION ALGORITHM

- Fixed Point iterations in EE operator Γ_{EE}.

$$\widetilde{\boldsymbol{v}}_n = \Gamma_{EE}\left(\widetilde{\boldsymbol{v}}_{n-1}\right)$$

- Properties:

  a. Contraction with modulus $<<< \beta$.

  b. Close to quadratic convergence.

  c. Cost per iteration: linear in $|\mathcal{X}|$.

3. ANALOGY TO EULER EQUATIONS IN CONTINUOUS-CHOICE MODELS

CONTINUOUS CHOICE MDP

- Agent chooses $a_t \in \mathbb{R}$ to maximize $\mathbb{E}_t \left[\sum_{j=0}^{T} \beta^s \ \pi_t(a_{t+j}, \mathbf{s}_{t+j}) \right]$.

- $\mathbf{s}_t$ has two components: $\mathbf{s}_t = (k_t, \mathbf{z}_t)$.

- $\mathbf{z}_t$ = vector of exogenous state variables following a Markov process that can be non-homogeneous over time.

- k_t is an endogenous state variable, with transition rule:

$$k_{t+1} = f(a_t, \mathbf{s}_t) + \xi_{t+1}$$

  ξ_{t+1} is an innovation shock that does not depend on a_t or k_t.

- The value function $V_t(\mathbf{s}_t)$ solves the Bellman equation:

$$V_t(\mathbf{s}_t) = \max_{a_t \in \mathbb{R}} \left\{ \ \pi_t(a_t, \mathbf{s}_t) + \beta \ \mathbb{E}\left[V_{t+1}\left(\mathbf{s}_{t+1}\right) \ \middle| \ a_t, \mathbf{s}_t \right] \ \right\}$$

EULER EQUATION (EE)

- In this model, an EE is the combination of the **F.O.C. of optimality** at time t and an **Envelope Condition** at time t.

- **F.O.C. of optimality**:

$$\frac{\partial \pi_t}{\partial a_t} + \beta \, \mathbb{E}_t \left[\frac{\partial V_{t+1}}{\partial k_{t+1}} \right] \frac{\partial f_t}{\partial a_t} = 0$$

- **Envelope Condition**:

$$\frac{\partial V_t}{\partial k_t} = \frac{\partial \pi_t}{\partial k_t} + \beta \, \mathbb{E}_t \left[\frac{\partial V_{t+1}}{\partial k_{t+1}} \right] \frac{\partial f_t}{\partial k_t}$$

- Combining these two conditions, we obtain the **Euler Equation**:

$$\frac{\partial \pi_t}{\partial a_t} + \beta \, \mathbb{E}_t \left[\frac{\partial \pi_{t+1}}{\partial k_{t+1}} - \frac{\partial \pi_{t+1}}{\partial a_{t+1}} \frac{\frac{\partial f_{t+1}}{\partial k_{t+1}}}{\frac{\partial f_{t+1}}{\partial a_{t+1}}} \right] \frac{\partial f_t}{\partial a_t} = 0$$

EXAMPLE: CONSUMPTION MODEL

- a_t = consumption, k_t = wealth. Payoff function (CRRA):

$$\pi_t(a_t, \mathbf{s}_t) = \frac{1}{\gamma+1} a_t^{\gamma+1}$$

- Transition rule for wealth:

$$k_{t+1} = f(a_t, \mathbf{s}_t) = R_t \ k_t + y_t - a_t$$

- The well-known EE is:

$$a_t^{\gamma} - \beta \ \mathbb{E}_t \left[a_{t+1}^{\gamma} \ R_{t+1} \right] = 0$$

PROBABILITY-CHOICE REPRESENTATION OF DDC MODELS

- Let $\mathbf{P}_t \in [0,1]^J$ be an arbitrary vector of CCPs (not necessarily optimal) for every action a.

- Given primitive f_x and CCPs $\mathbf{P}_t$, define transition probabilities:

$$f^P(\mathbf{x}_{t+1}|\mathbf{P}_t, \mathbf{x}_t) \equiv \sum_{a=0}^{J} P_t(a|\mathbf{x}_t)\ f_x(\mathbf{x}_{t+1}|a, \mathbf{x}_t)$$

- Given primitives π and f_ε and CCPs $\mathbf{P}_t$, define payoff function:

$$\Pi^P(\mathbf{P}_t, \mathbf{x}_t) \equiv \textstyle\sum_{a=0}^{J} P_t(d)\ [\pi(a, \mathbf{x}_t) + e(a, \mathbf{P}_t)],$$

with

$$e(a, \mathbf{P}_t) = \mathbb{E}\left[\varepsilon_{a,t} \mid \Lambda^{-1}(a, \mathbf{P}_t) + \varepsilon_{a,t} \geq \Lambda^{-1}(j, \mathbf{P}_t) + \varepsilon_{j,t} \ \ \forall j\right]$$

PROBABILITY-CHOICE REPRESENTATION OF DDC [2]

- Given $\{\beta, \Pi^P, f^P\}$, we can define a DP problem where the decision variable is the vector of CCPs, $\mathbf{P}_t$, the state variables are $\mathbf{x}_t$ and Bellman equation is:

$$V_t^P(\mathbf{x}_t) = \max_{\mathbf{P}_t \in [0,1]^J} \left\{ \Pi^P(\mathbf{P}_t, \mathbf{x}_t) + \beta \sum_{\mathbf{x}_{t+1}} V^P(\mathbf{x}_{t+1}) \; f^P(\mathbf{x}_{t+1} | \mathbf{P}_t, \mathbf{x}_t) \right\}$$

EQUIVALENCE OF PROB-CHOICE & DISCRETE-CHOICE

1. The optimal decision in probability-choice problem is equal to the optimal choice probability in the corresponding discrete choice problem:

2. Let $v^P(\mathbf{P}_t, \mathbf{x}_t)$ be the analog of the conditional choice value function but in the probability-choice DP. That is:

$$v^P(\mathbf{P}_t, \mathbf{x}_t) \equiv \Pi^P(\mathbf{P}_t, \mathbf{x}_t) + \beta \sum_{\mathbf{x}_{t+1}} V^P(\mathbf{x}_{t+1}) \ f^P(\mathbf{x}_{t+1}|\mathbf{P}_t, \mathbf{x}_t)$$

- Then, $v^P(\mathbf{P}_t, \mathbf{x}_t)$ is twice continuously differentiable, globally concave in $\mathbf{P}_t$, and the optimal CCP $\mathbf{P}_t$ is uniquely characterized by the marginal condition:

$$\frac{\partial v^P(\mathbf{P}_t, \mathbf{x}_t)}{\partial \mathbf{P}_t} = 0$$

EULER EQUATIONS IN DDC MODELS

- We derive EE of the probability choice DP problem by using the Lagrange conditions of a **Constrained optimization problem.**

$$\max_{\{\mathbf{P}_t, \mathbf{P}_{t+1}\}} \quad \Pi^P(\mathbf{P}_t, \mathbf{x}_t) + \beta \sum_{x_{t+1}} \Pi^P(\mathbf{P}_{t+1}, \mathbf{x}_{t+1}) \; f^P(\mathbf{x}_{t+1} \mid \mathbf{P}_t, \mathbf{x}_t)$$

$$\text{subject to:} \quad f^P_{(2)}(\mathbf{x}_{t+2} \mid \mathbf{P}_t, \mathbf{P}_{t+1}, \mathbf{x}_t) = f^P(\mathbf{x}_{t+2} \mid \mathbf{P}^*_t, \mathbf{P}^*_{t+1}, \mathbf{x}_t)$$

- By one-period deviation principle, the optimal $\mathbf{P}(.)$ of the DP problem is the unique solution to this constrained optimization problem.

- We obtain EEs by combining Lagrange conditions in such a way that we get rid of Lagrange multipliers.

4. NUMERICAL EXAMPLES

NUMERICAL EXAMPLES

- We present numerical examples that illustrate large computational advantages of the EE operator for:

- **[1] The exact solution of the DP problem** (relative to Value function iteration (VFI), Relative Value function iterations (RVFI), and Policy function iterations (PFI).

- **[2] The estimation of structural parameters** (relative to Hotz-Miller, and to Nested Pseudo Likelihood methods).

- **[3] The approximation of counterfactuals in applications where the model cannot be solved exactly (**relative to Approximate VFI, and Approximate PFI).

DATA GENERATING PROCESS

- Binary choice model of market entry-exit.

- Inactive firms get profit $\pi(0, \mathbf{x}_t) + \varepsilon_t(0)$, with $\pi(0, \mathbf{x}_t) = 0$, and active firms earn profit $\pi(1, \mathbf{x}_t) + \varepsilon_t(1)$, with:

$$\pi(1, \mathbf{x}_t) = VP_t - FC_t - EC_t * (1 - d_{t-1})$$

and

$$\begin{aligned} VP_t &= \left[\theta_0^{VP} + \theta_1^{VP} z_{1t} + \theta_2^{VP} z_{2t}\right] \exp\left(\omega_t\right) \\ FC_t &= \theta_0^{FC} + \theta_1^{FC} z_{3t} \\ EC_t &= \theta_0^{EC} + \theta_1^{EC} z_{4t} \end{aligned}$$

DATA GENERATING PROCESS

Table 1
Parameters in the DGP

Payoff Parameters:	$\theta_0^{VP} = 0.5$; $\theta_1^{VP} = 1.0$; $\theta_2^{VP} = -1.0$
	$\theta_0^{FC} = 0.5$; $\theta_1^{FC} = 1.0$
	$\theta_0^{EC} = 1.0$; $\theta_1^{EC} = 1.0$
Each z_j state variable:	z_{jt} is AR(1), $\gamma_0^j = 0.0$; $\gamma_1^j = 0.6$
Productivity :	ω_t is AR(1), $\gamma_0^\omega = 0.2$; $\gamma_1^\omega = 0.9$
Low persistence model:	$\sigma_e^\omega = \sigma_e = 1$
High persistence model:	$\sigma_e^\omega = \sigma_e = 0.01$
Discount factor	$\beta = 0.95$

COMPARING CONTRACTION PROPERTIES (MODULUS)

Table 2
Degree of Contraction (Lipschitz)

	Low Persistence			High Persistence		
$\lvert\mathcal{X}\rvert$	**EE**	**VF**	**RVF**	**EE**	**VF**	**RVF**
64	0.20	0.95	0.59	0.34	0.95	0.95
486	0.18	0.95	0.54	0.34	0.95	0.95
2,032	0.18	0.95	0.53	0.31	0.95	0.95
6,250	0.18	0.95	0.53	0.32	0.95	0.95
15,552	0.18	0.95	0.53	0.28	0.95	0.95
200,000	0.18	0.95	0.53	0.28	0.95	0.95

COMPARING COMPUTING TIMES

Table 3(a)
Comparison of Solution Methods
(Model with low persistence)

# states	Number iters.				Time per iter. (secs)			
$\lvert\mathcal{X}\rvert$	EE	PF	VF	RVF	EE	PF	VF	RVF
64	13	5	351	36	< 0.001	0.01	< 0.001	< 0.001
486	13	5	246	31	< 0.001	0.60	< 0.001	< 0.001
2048	13	5	345	30	0.005	16.29	0.01	0.01
6250	13	5	344	30	0.04	150.0	0.08	0.08
15552	13	5	344	30	0.17	916.2	0.46	0.46
200,000	13	5	344	30	21.05	198,978	67.64	64.47

COMPARING COMPUTING TIMES

Table 3(b)
Comparison of Solution Methods
(Model with low persistence)

# states	**Total Time (in seconds)**				**Time Ratios**			
$\lvert\mathcal{X}\rvert$	EE	PF	VF	RVF	$\frac{EE}{EE}$	$\frac{PF}{EE}$	$\frac{VF}{EE}$	$\frac{RVF}{EE}$
64	<0.001	0.05	0.03	<0.001	1	60	37.0	4.0
486	0.01	3.02	0.35	0.03	1	300	26.6	3.4
2048	0.05	81.44	2.76	0.27	1	1320	53.1	5.2
6250	0.47	750.0	26.14	2.37	1	1,595	55.9	5.1
15552	2.25	4,581	158.9	13.89	1	2,040	70.7	6.2
200,000	273.6	$\simeq$ 1M*	23,270	1,934	1	3,630	85.0	7.1

COMPARING COMPUTING TIMES

Table 4
Comparison of EE and RVF Solution Methods
(Model with high persistence)

Number of states $\lvert\mathcal{X}\rvert$	**Euler** # iter.	**Euler** Time-per-iter.	**Relative value** # iter.	**Relative value** Time-per-iter.	**Ratio total time RVF / EE-v**
64	24	< 0.001	378	< 0.001	24.5
486	24	< 0.001	350	< 0.001	30.4
2048	17	0.01	341	0.02	38.8
6250	18	0.06	322	0.12	38.0
15552	17	0.43	333	0.87	40.5
200,000	16	46.2	319	98.5	42.5

PARAMETER ESTIMATION

Table 5(c)
Monte Carlos: Estimation of Parameters
$N = 1,000$ & $T = 2$; Monte Carlo rep. $= 1,000$

	Root Mean Squared Error			
Parameter (True value)	2-step Eff. HM	2-step EE	MLE (NPL)	K-step EE
Total RMSE	0.703	0.935	0.680	0.684
Ratio $\frac{RMSE\text{ HM}}{RMSE\text{ EE}}$	0.75		0.99	
Time (in secs)	1067.80	0.218	7345.79	3.261
Ratio $\frac{Time\text{ PF}}{Time\text{ EE}}$	4898		2252	

PARAMETER ESTIMATION

Table 8
Monte Carlo: Counterfactual Estimates

	Prob. Being Active	Entry Prob.	Exit Prob.	State Persist.	Total Output
True Policy Effect	-0.065	-0.117	-0.068	+0.116	-0.106
Root Mean Square Error					
VF iterations	67.5%	17.1%	30.1%	11.8%	82.8%
PF iterations	67.5%	17.1%	30.1%	11.8%	82.8%
EE-value iterations	37.1%	6.3%	14.8%	7.2%	38.6%