Identification and Estimation of Dynamic Games with Unknown Information Structure

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Motivation

Econometric assumptions about what players do or don't observe are often too strong

Empirical models of dynamic competition have become standard

- Entry/exit, investment, innovation, etc.

It is standard to use dynamic games with a particular "incomplete information" assumption

- Assume players observe their shocks but not their rivals'
- Less flexibility than static analogs in terms of possible informational assumptions

Research Question: How strong are these informational assumptions?

- Robust estimation: How can we relax the standard informational assumption?
- Sensitivity analysis: How strong is the standard informational assumption?

Main Contributions

Estimating dynamic games when the researcher does not know what players (don't) know

We develop an empirical framework for informationally robust analysis of dynamic discrete games

- We assume the researcher only knows minimal information available to the players

1. Game Theory: We develop a new solution concept dubbed Markov correlated equilibrium

- A dynamic Markovian extension of Bayes correlated equilibrium (Bergemann & Morris, 2016)
- Contributes to the theory literature on information design and dynamic games (Doval & Ely, 2020; Makris & Renou, 2023)

2. Econometrics: We develop a computationally tractable routine for estimation/inference

- Mathematical programming with equilibrium constraint (Su & Judd, 2012)
- Contributes to the econometric literature on dynamic games and informationally robust analysis (Aguirregabiria & Mira, 2007; Magnolfi & Roncoroni, 2023)

3. Empirical Application to dynamic entry/exit game of Starbucks and Dunkin'

- We find that the standard informational assumption is very strong
- Contributes to the empirical literature on dynamic/spatial retail competition (Aguirregabiria & Suzuki, 2016; Arcidiacono et al., 2016; Jia, 2008)

Dynamic Entry Game of Starbucks vs. Dunkin'

Starbucks and Dunkin' have significantly expanded their presence across the US

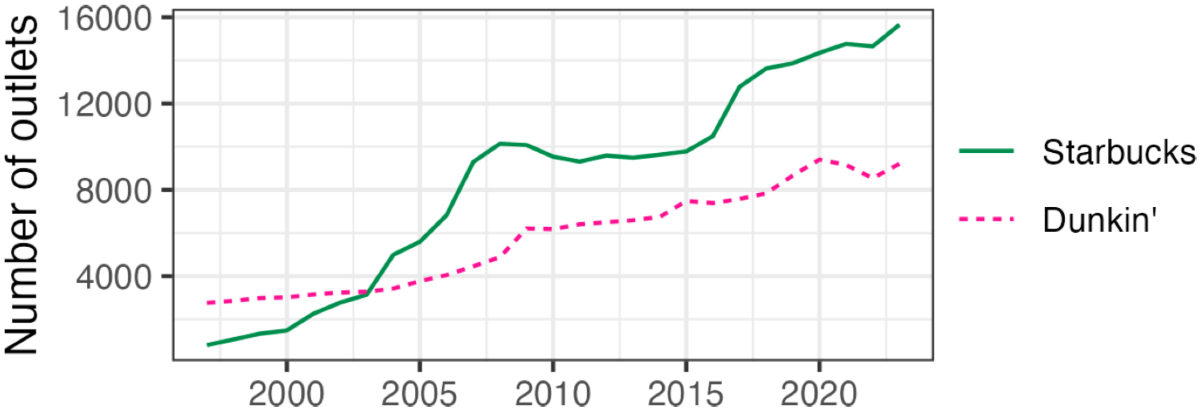

Figure. The number of Starbucks and Dunkin' stores in the US over time

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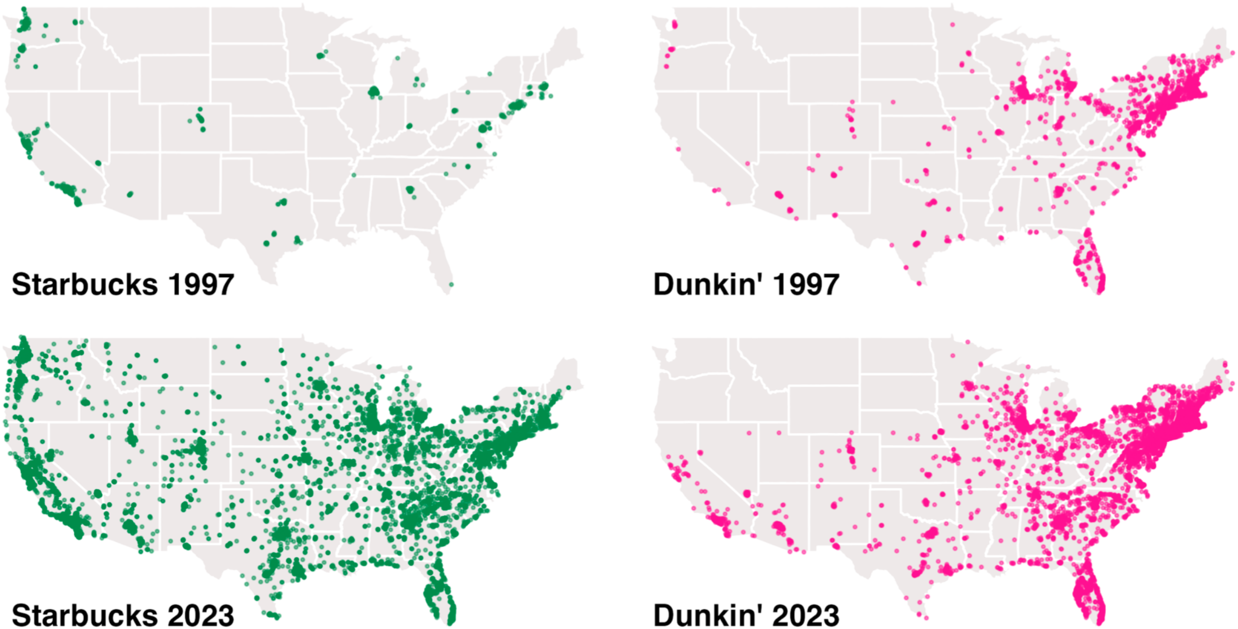


A Dynamic Game Model for Starbucks vs. Dunkin'

Two-player dynamic entry game

In each local geographic market m and period t, each firm i makes a binary decision

$$a_{imt} = \begin{cases} 1 & \text{if active} \\ 0 & \text{if not active} \end{cases}$$

Firm i's flow payoff at period t is

$$u_i(a_{imt}, a_{jmt}, z_{imt}, \varepsilon_{imt}) = \begin{cases} \theta_w^\top w_m + \theta_{i,v} v_{im} + \theta_{ec}(1 - z_{imt}) + \theta_{i,ce} a_{jmt} + \varepsilon_{imt} & \text{if } a_{imt} = 1 \\ 0 & \text{if } a_{imt} = 0 \end{cases}$$

- w_m: common market characteristics (observed)
- v_{im}: firm i-specific characteristic (observed)
- z_{imt}: firm i's incumbency status ($= a_{imt-1}$) (observed)
- ε_{imt}: latent payoff shock (unobserved)

Firms play a Markov perfect equilibrium

Econometrician wants to estimate θ from the panel data of actions and market characteristics

Econometrician's Identification Problem

Standard informational assumption is often arbitrary

The standard approach is to assume a particular "incomplete information" structure

- Each firm i observes its payoff shock ε_{imt} but not its opponents'

Thus, the standard identification approach is to obtain $\Theta_I^{MPE}(S^P)$ by inverting

$$f^{MPE} : (\Theta, S^P) \rightrightarrows \Phi$$

- Θ: Model parameters
- S^P: Standard "incomplete information" structure
- f^{MPE}: Markov perfect equilibria
- Φ: Distribution of actions and observable states (conditional choice probabilities)

However, often there is little justification for assuming S^P beyond analytic convenience

- Corresponding misspecification bias can be large in static settings (Gualdani & Sinha, 2024; Koh, 2023; Magnolfi & Roncoroni, 2023; Syrgkanis et al., 2017)

Identification with Weak Informational Assumption

Parameters become set-identified

To reflect econometrician's uncertainty, allow for a set of information structures **S**

But weakening the informational assumption leads to a hard partial identification problem

Q: For each candidate θ, is there an info structure S that can support the observed data ϕ?

For a set of candidate parameters $\theta^1, \theta^2, \theta^3, ...$, if

$$
\begin{aligned}
&\phi \in f(\theta^1, S^1) && \text{(i.e. } \exists S_1 \text{ that rationalizes data } \phi \text{ under candidate } \theta_1) \\
&\phi \in f(\theta^2, S^2) && \text{(i.e. } \exists S_2 \text{ that rationalizes data } \phi \text{ under candidate } \theta_2) \\
&\phi \in f(\theta^3, S^3) && \text{(i.e. } \exists S_3 \text{ that rationalizes data } \phi \text{ under candidate } \theta_3) \\
&\quad\vdots
\end{aligned}
$$

then $\{\theta^1, \theta^2, \theta^3, ...\}$ enter the identified set Θ_I

Identification with Weak Informational Assumption

Relaxing informational assumption is hard

Identified set inverts data-generating process $f^{MPE} : (\Theta, \mathbf{S}) \rightrightarrows \Phi$ (instead of $f^{MPE} : (\Theta, S^P) \rightrightarrows \Phi$)

Inversion yields the union of MPE-identified sets

$$\bigcup_{S \in \mathbf{S}} \Theta_I^{MPE}(S) \tag{1}$$

1. Pick a candidate $\theta^* \in \Theta$
2. Check whether there is an $S^* \in \mathbf{S}$ such that $f^{MPE}(\theta^*, S^*)$ is consistent with observed data
3. If yes, θ^* enters the identified set

Unfortunately, (1) is computationally intractable when $\mathbf{S}$ is large and ∞-dimensional

High-Level Description of Solution

Inverting MCE mapping is appropriate and feasible

We propose treating the data as if it was generated by a Markov correlated equilibrium

We show that MCE is *sharp* and *tractable*:

$$\Theta_I^{MCE} = \cup_{S \in \mathbf{S}} \Theta_I^{MPE}(S) \tag{2}$$

- *Sharp*: LHS is equal to RHS
- *Tractable*: LHS is computationally feasible

Thus, using MCE "solves" the information robustness problem when estimating dynamic games

A Numerical Example with Two-Player Entry/Exit Game of Pesendorfer and Schmidt-Dengler (2008)

Using MCE yields an informationally robust partially identified set

The per-period flow payoff is

$$u_i(a_{it}, a_{jt}, z_{it}, \varepsilon_{it}) = \begin{cases} (1-a_{jt})\pi^m + a_{jt}\pi^d + (1-z_{it})c + \varepsilon_{it} & \text{if } a_{it} = 1 \\ z_{it}\kappa & \text{if } a_{it} = 0 \end{cases}$$

Setting $(\pi^m, \pi^d, c, \kappa, \delta) = (1.2, -1.2, -0.2, 0.1, 0.9)$ and assuming perfectly private information yields

Table. Probability of entry in three MPE

	Eq'm (i)		Eq'm (ii)		Eq'm (iii)	
Action Profile	Firm 1	Firm 2	Firm 1	Firm 2	Firm 1	Firm 2
Out/Out	0.73	0.27	0.61	0.52	0.57	0.57
Out/In	0.61	0.42	0.31	0.84	0.30	0.84
In/Out	0.80	0.22	0.83	0.30	0.84	0.30
In/In	0.75	0.29	0.60	0.57	0.59	0.59

A Numerical Example with Two-Player Entry/Exit Game of Pesendorfer and Schmidt-Dengler (2008)

Using MCE yields an informationally robust partially identified set

Table. MCE identified sets

	True Value	Eq'm (i)	Eq'm (ii)	Eq'm (iii)
π^m	1.2	[0.93, 1.83]	[0.88, 2.52]	[0.88, 2.52]
π^d	-1.2	[-1.87, -1.01]	[-3.16, -0.65]	[-3.16, -0.65]
c	-0.2	[-0.51, 0.37]	[-0.67, 1.02]	[-0.67, 1.02]

Note: $(\kappa, \delta) = (0.1, 0.9)$ assumed to be known.

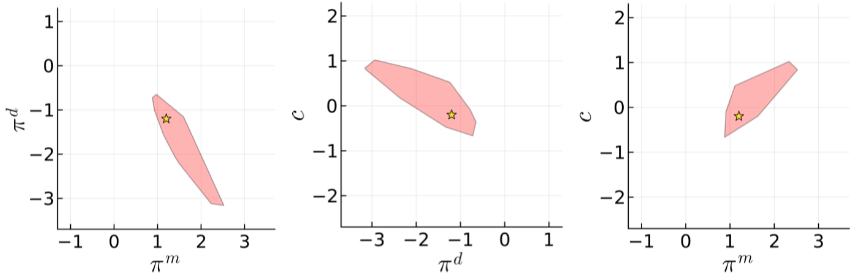


Figure. Convex hull of projected identified set under data generated from eq'm (ii)

So what is Markov Correlated Equilibrium and how does it work?

Let's review Bayes correlated equilibrium first

Bayes correlated equilibrium generalizes both Bayesian Nash equilibrium (BNE) and correlated equilibrium (CE) in static games

1. A mediator observes the state (types)
2. The mediator privately recommends actions to players according to some joint distribution
3. Players follow the recommendation because doing so is incentive compatible given their information

Bergemann and Morris (2016) Theorem 1 implies

$$\underbrace{\mathcal{P}^{BCE}(S^P)}_{\text{BCE-predictions when info structure is } S^P} = \underbrace{\cup_{S \succsim_E S^P} \mathcal{P}^{BNE}(S)}_{\text{BNE-predictions when players can observe more than prescribed in } S^P} \tag{3}$$

(3) justified using BCE as the solution concept for informationally robust identification

$$\underbrace{\Theta_I^{BCE}(S^P)}_{\text{BCE-identified set when info structure is } S^P} = \underbrace{\cup_{S \succsim_E S^P} \Theta_I^{BNE}(S)}_{\text{BNE-identified set when players can observe more than prescribed in } S^P}$$

Estimation with Bayes Correlated Equilibrium

BCE is computationally tractable due to its linear programming (LP) structure

$\theta \in \Theta_I^{BCE}(S^P)$ if and only if there exists a decision rule σ that satisfies

$$\sigma_{a|x,\varepsilon} \geq 0 \text{ for each } a \text{ and } \sum_a \sigma_{a|x,\varepsilon} = 1, \qquad \forall x, \varepsilon$$

$$\phi_{a|x} = \sum_\varepsilon \psi_{\varepsilon|x}\sigma_{a|x,\varepsilon}, \qquad \forall a, x$$

$$\sum_{\varepsilon_{-i}, a_{-i}} \psi_{\varepsilon|x}\sigma_{a|x,\varepsilon}(u_i^\theta(a_i', a_{-i}, x, \varepsilon_i) - u_i^\theta(a, x, \varepsilon_i)) \leq 0, \qquad \forall i, x, \varepsilon_i, a_i, a_i'$$

- ϕ represent the observed data (conditional choice probabilities)
- ψ represents the assumed distribution of latent states

Thus, we can compute the BCE-identified set by solving LPs at candidate parameters

A Connection Between MCE and BCE

One-shot deviation principle (OSDP) allows us to leverage results from static settings

Want to show: For any Markov game with basic game G and information structure S,

$$\underbrace{\mathcal{P}^{MCE}(G,S)}_{\text{MCE predictions when info structure is } S} = \underbrace{\cup_{\tilde{S} \succsim_E S} \mathcal{P}^{MPE}(G,\tilde{S})}_{\text{MPE predictions when players can observe more than prescribed in } S}$$

Step 1 of Proof: Use OSDP to transform statements into "static" ones

1. A strategy profile β is a MPE of (G, S) iff β is a BNE of (G^β, S)
2. A decision rule σ is a MCE of (G, S) iff it is a BCE of (G^σ, S)
3. If β induces σ, then $G^\beta = G^\sigma$

Step 2 of Proof: Invoke Theorem 1 of Bergemann and Morris (2016) $\square$

Estimation with Markov Correlated Equilibrium

We estimate MCE-identified set with nonlinear programming (NLP) formulation

$\theta \in \Theta_I^{MCE}(S^P)$ if and only if there exists a decision rule σ and ex-ante value function V that satisfies

$$\sigma_{a|x,\varepsilon} \geq 0 \text{ for each } a \text{ and } \sum_a \sigma_{a|x,\varepsilon} = 1, \qquad \forall x, \varepsilon$$

$$\phi_{a|x} = \sum_\varepsilon \psi_{\varepsilon|x} \sigma_{a|x,\varepsilon}, \qquad \forall a, x$$

$$\sum_{\varepsilon_{-i}, a_{-i}} \psi_{\varepsilon|x} \sigma_{a|x,\varepsilon} (v_i^\theta(a_i', a_{-i}, x, \varepsilon_i) - v_i^\theta(a, x, \varepsilon_i)) \leq 0, \qquad \forall i, x, \varepsilon_i, a_i, a_i'$$

$$V_{i,x} = \sum_{\varepsilon, a} \psi_{\varepsilon|x} \sigma_{a|x,\varepsilon} u_i^\theta(a, x, \varepsilon_i) + \delta \sum_{a,x'} \phi_{a|x} V_{i,x'} f_{x'|a,x}, \qquad \forall i, x$$

- ϕ is the observed data (conditional choice probabilities)
- ψ is the (assumed) distribution of ε
- δ is the discount factor
- $f_{x'|a,x}$ is the state transition rule
- $v_i^\theta(a, x, \varepsilon_i) \equiv u_i^\theta(a, x, \varepsilon_i) + \delta \sum_{x'} V_{i,x'} f_{x'|a,x}$ is the outcome-specific value function

Thus, we can compute the MCE-identified set by solving NLPs at candidate parameters

ESTIMATION WITH MARKOV CORRELATED EQUILIBRIUM

ESTIMATION OF FULLY ROBUST IDENTIFIED SET IS EASIER

$\theta \in \Theta_I^{MCE}(S^{null})$ if and only if there exists a decision rule σ and ex-ante value function V that satisfies

$$\sigma_{a|x,\varepsilon} \geq 0 \text{ for each } a \text{ and } \sum_a \sigma_{a|x,\varepsilon} = 1, \qquad \forall x, \varepsilon$$

$$\phi_{a|x} = \sum_\varepsilon \psi_{\varepsilon|x}\sigma_{a|x,\varepsilon}, \qquad \forall a, x$$

$$\sum_{\varepsilon,a_{-i}} \psi_{\varepsilon|x}\sigma_{a|x,\varepsilon}\partial u_i^\theta(a_i', a, x, \varepsilon_i) + \delta \sum_{a_{-i}} \phi_{a|x} V_{i,x'} \partial f_{x'|a_i',a,x} \leq 0, \qquad \forall i, x, a_i, a_i'$$

$$V_{i,x} = \sum_{\varepsilon,a} \psi_{\varepsilon|x}\sigma_{a|x,\varepsilon} u_i^\theta(a, x, \varepsilon_i) + \delta \sum_{a,x'} \phi_{a|x} V_{i,x'} f_{x'|a,x}, \qquad \forall i, x$$

- $\partial u_i^\theta(a_i', a, x, \varepsilon_i) \equiv u_i^\theta(a_i', a_{-i}, x, \varepsilon_i) - u_i^\theta(a, x, \varepsilon_i)$
- $\partial f_{x'|a_i',a,x} \equiv f_{x'|a_i',a_{-i},x} - f_{x'|a,x}$

Thus, we can compute the fully information-robust identified set by solving LPs at candidate parameters

Empirical Application to Starbucks vs. Dunkin'

What is the implication of weakening the informational assumption?

Starbucks and Dunkin' are the two largest coffee chains in the US by a wide margin[1]

Using a dynamic discrete game model to analyze their entry/exit seems appropriate

But whether the standard "incomplete information" assumption S^P is appropriate is unclear

We use MCE to test the sensitivity of empirical conclusions w.r.t. informational assumptions

[1] SB and DK control 65% and 28% (resp. 55% and 35%) of total sales (resp. outlets) from the top 8 coffee chains in the US in 2019

Data and Descriptive Statistics

We construct an annual balanced panel of $M = 38{,}687$ markets over 2003-2017 ($T = 15$)

- Establishment location data from Data Axle's *U.S. Historic Business Database*
- Market characteristics from *NaNDA*, assumed to be time-invariant
- Local markets are defined as the *four-digit 2010 census tracts* ($\approx$4,000 people)
- Define a firm as a potential entrant if operating in the county

Table. Summary statistics of 38,687 coffee chain markets from 2003 to 2017

Panel A. Potential entrants	Both	SB only	DK only
Share of markets	0.786	0.172	0.042
Panel B. Player decisions	SB	DK	
Probability of active	0.138	0.113	
Probability of entry	0.011	0.007	
Probability of exit	0.002	0.002	
Panel C. Market characteristics	Mean	SD	Range
Number of eating places	15.6	25.2	[0.1, 649.8]
SB outlets in county	592.0	1243.1	[0.0, 6423.0]
DK outlets in county	354.9	750.9	[0.0, 3948.0]

Econometric Assumptions

An active firm's flow profit is

$$u_i(a_{imt}=1, a_{jmt}, x_{mt}, \varepsilon_{imt}) = \theta_0 + \underbrace{\theta_w w_m}_{\text{common}} + \underbrace{\theta_{i,v} v_{im}}_{\text{excluded}} + \underbrace{\theta_{ec}(1-a_{imt-1})}_{\text{entry cost}} + \underbrace{\theta_{i,ce} a_{jmt}}_{\text{competitive effect}} + \underbrace{\varepsilon_{imt}}_{\text{idio. payoff. shock}}$$

- w_m is the log number of eating places
- $v_{i,m}$ is the log number of own outlets in county

We assume $\delta = 0.9$ and discretize continuous variables

We estimate the competitive effects parameters with MCE

- We use markets with single potential entrants to estimate θ except competitive effects $\theta_{i,ce}$
- We then estimate the competitive effects using MCE

Estimation Results

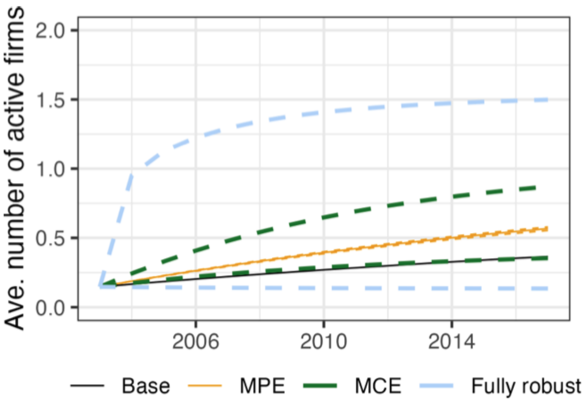
The size and signs of competitive effects become ambiguous

	MPE Private info (1)	MCE Private info (2)	MCE Fully robust (3)
Non-competitive effects parameters			
Intercept	-0.016	0.000	
	$[-0.032, -0.002]$	$[-0.040, 0.040]$	
Eating places (log)	0.106	0.127	
	$[0.103, 0.110]$	$[0.117, 0.136]$	
Starbucks outlets in county (log)	0.029	0.024	
	$[0.027, 0.031]$	$[0.019, 0.029]$	
Dunkin' outlets in county (log)	0.029	0.038	
	$[0.027, 0.031]$	$[0.029, 0.048]$	
Entry cost	-8.408	-8.165	
	$[-8.473, -8.365]$	$[-8.323, -8.008]$	
Competitive effects parameters			
Starbucks competitive effect	0.040	-	-
	$[0.027, 0.049]$	$[-0.123, 0.143]$	$[-0.257, 0.449]$
Dunkin' competitive effect	0.024	-	-
	$[0.011, 0.035]$	$[-0.299, 0.038]$	$[-0.300, 0.074]$

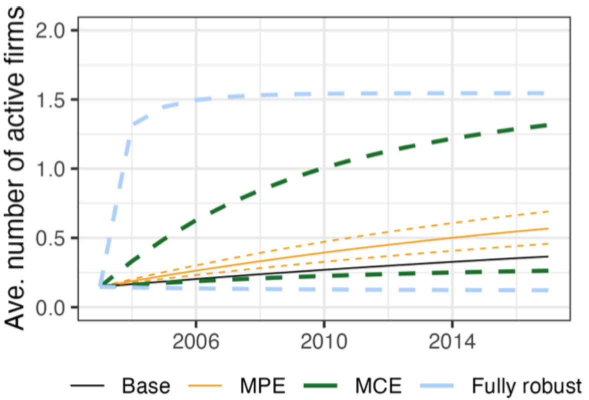
Note: 95% CI in brackets. Non-competitive parameters under MCE are estimated using the single-agent dynamic discrete choice framework.

Counterfactual Experiment: What is the effect of a 10% reduction in entry cost on the expected number of active firms?

Weakening informational assumption translates to a large uncertainty in counterfactual predictions



(a) Use CIs of competitive effects



(b) Use CIs of all parameters

Figure. Bounds on counterfactual average number of firms over time

Counterfactual Experiment

The large width of counterfactual bounds is mostly due to set identification

Counterfactual bounds can be large due to

- large identified set
- large set of predictions at each parameter

Table. Decomposition of counterfactual bounds: set identification vs. set predictions

	MCE standard info		MCE fully robust	
Year	w/ CI lower	w/ CI upper	w/ CI lower	w/ CI upper
2003	0.15	0.15	0.15	0.15
2010	[0.29, 0.32]	[0.64, 0.65]	[0.14, 0.52]	[0.33, 1.41]
2017	[0.35, 0.38]	[0.87, 0.87]	[0.13, 0.54]	[0.46, 1.50]

Note: Table reports (bounds on) the expected number of active firms after a counterfactual 10% reduction in the entry cost.

We find that the former (parameters are set identified) is more important than the latter (multiple equilibria can exist)

Conclusion

We propose an empirical framework for informationally robust estimation of dynamic games

- Develop a theory of Markov correlated equilibrium
- Propose a computationally tractable estimation/inference strategy

Key lesson: *Standard informational assumptions are far from innocuous!*

Future directions:

1. Develop a refinement of BCE (b/c BCE/MCE allows for too much)
2. Increase computational tractability (b/c BCE/MCE can still be computationally intensive)

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