Dynamic Games with Continuous Actions

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Plan

Outline of lecture:

- Model
- Estimation
- Summary

Main references:

- Srisuma, S. (2013): Minimum Distance Estimators for Dynamic Games, *Quantitative Economics*
- Bajari, P., C. L. Benkard, and J. Levin (2007): Estimating Dynamic Models of Imperfect Competition, *Econometrica*

Model

Single Agent Problem

- Action: $a_t \in \mathcal{A}$ *e.g. entry or investment decisions*
- States: $s_t = (x_t, \varepsilon_t) \in \mathcal{X} \times \mathcal{E}$ *e.g. market size, past actions, and idiosyncratic error*
- **Payoff functions**: $u : A \times S \to \mathbb{R}$
- **Discount factor**: $\beta \in [0, 1)$
- **Markov transition law**: $P\left(s_{t+1} | s_t, a_t\right)$

Primitives of the model are (u, β, P)

Single Agent Problem

Decision problem:

$$
\begin{array}{ccccccccc}
s_t & \longrightarrow & s_{t+1} & \longrightarrow & s_{t+2} & \longrightarrow & s_{t+3} & \cdots \\
 & \searrow & \uparrow & \searrow & \uparrow & \searrow & \uparrow & \\
 & & a_t & & a_{t+1} & & a_{t+2} &
\end{array}
$$

$$u(a_t, s_t) + \beta u(a_{t+1}, s_{t+1}) + \beta^2 u(a_{t+2}, s_{t+2}) + \cdots$$

where at time t, $a_t = \alpha(s_t)$ s.t.

$$
\begin{aligned}
\alpha(s_t) &= \arg\max_{a \in \mathcal{A}} \{u(a, s_t) + \beta E[V(s_{t+1}) | s_t, a_t = a]\}, \\
V(s_t) &= \max_{a \in A} \{u(a, s_t) + \beta E[V(s_{t+1}) | s_t, a_t = a]\}.
\end{aligned}
$$

Suppose $(u_0, \beta_0, P_0) \mapsto \{(a_{mt}, x_{mt})\}_{m=1,t=1}^{M,T}$, the econometric goal is to learn about (u_0, β_0, P_0).

Ordered Choice Model

$\mathcal{A} \subseteq \mathbb{R}$ can be:

- finite or infinite (countably or otherwise)
- unordered or ordered (think multinomial vs ordered probit/logit models)

Examples of ordered outcomes:

- Investment units
- Monetary values

What change then?

Ordered Choice Model

We want $\alpha(s) \in \mathbb{R}$ to be *monotone* in $\varepsilon \in \mathbb{R}$. Not all ordered choice specify this (e.g., Euler equation).
Omit x_t and set $\beta = 0$ for now. Let
$\Delta u(a, a', \varepsilon) := u(a, \varepsilon) - u(a', \varepsilon)$.

Definition

u has increasing differences in (a, ε) if $\Delta u(a, a', \varepsilon) > \Delta u(a, a', \varepsilon')$ whenever $a > a'$ and $\varepsilon > \varepsilon'$.

If u is differentiable, ID is equivalent to $\frac{\partial^2}{\partial a \partial \varepsilon} u(a, \varepsilon) > 0$.

Theorem

Suppose u has increasing differences in (a, ε), then $\alpha(\varepsilon) > \alpha(\varepsilon')$ when $\varepsilon > \varepsilon'$.

Elements of the Game

- Players: $\mathcal{I} = \{1, \ldots, I\}$
- States: $s_{it} = (x_t, \varepsilon_{it}) \in X \times E_i$
- Actions: $a_{it} \in A_i, \mathbf{a}_{-it} \in A_{-i}, \mathbf{a}_t \in A_1 \times \cdots \times A_I$
- Payoff functions: $u_i : A \times S_i \to \mathbb{R}$
- Discounting factor: $\{\beta_i\}_{i=1}^{I}$
- Markov transition law: $P\left(ds_{it+1} | s_{it}, \mathbf{a}_t\right)$

Game progression:
players observe $\{s_{it}\} \to$ choose actions $\{a_{it}\} \to$ yield $u_i\left(\mathbf{a}_t, s_{it}\right) \to$ state evolves $\{s_{it+1}\} \to$ game repeats...

Elements of the Game

Game progression:

$$
\begin{array}{ccccccc}
 & (s_{1t}) & a_{1t} & (s_{1t}) & a_{1t+1} & (s_{1t+1}) & \\
 & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \\
\{s_{it}\}_{i=1}^{I} & \longrightarrow & \{s_{it+1}\}_{i=1}^{I} & \longrightarrow & \{s_{it+2}\}_{i=1}^{I} & \longrightarrow & \cdots \\
 & \searrow & \uparrow & \searrow & \uparrow & \searrow & \\
 & (s_{It}) & a_{It} & (s_{It}) & a_{It+1} & (s_{It+1}) &
\end{array}
$$

Player i's problem is to choose a_i to maximize

$$
\begin{aligned}
\Lambda_i\left(a_i, s_{it}; \sigma_i\right) = \; & E_{\sigma_i}\left[\left. u_i\left(a_{it}, \mathbf{a}_{-it}, s_{it}\right)\right| s_{it}, a_{it} = a_i\right] \\
& + \beta_i E_{\sigma_i}\left[\left. V_i\left(s_{it+1}; \sigma_i\right)\right| s_{it}, a_{it} = a_i\right]
\end{aligned}
$$

based on beliefs σ_i

The *primitives* of the game are $\left(\{u_i\}_{i=1}^{I}, \{\beta_i\}_{i=1}^{I}, P\right)$

Elements of the Game

Game progression:

$$
\begin{array}{ccccccccc}
 & (s_{1t}) & & a_{1t} & & (s_{1t}) & & a_{1t+1} & & (s_{1t+1}) & \\
 & \nearrow & & \downarrow & & \nearrow & & \downarrow & & \nearrow & \\
\{s_{it}\}_{i=1}^{I} & \longrightarrow & & \{s_{it+1}\}_{i=1}^{I} & & \longrightarrow & & \{s_{it+2}\}_{i=1}^{I} & & \longrightarrow & \cdots \\
 & \searrow & & \uparrow & & \searrow & & \uparrow & & \searrow & \\
 & (s_{It}) & & a_{It} & & (s_{It}) & & a_{It+1} & & (s_{It+1}) &
\end{array}
$$

Player i's problem is to choose a_i to maximize

$$
\begin{aligned}
\Lambda_{i,\theta}(a_i, s_{it}; \sigma_i) &= E_{\sigma_i}\left[u_{i,\theta}(a_{it}, a_{-it}, s_{it}) \right| s_{it}, a_{it} = a_i] \\
&\quad + \beta_i E_{\sigma_i}\left[V_{i,\theta}(s_{it+1}; \sigma_i) \right| s_{it}, a_{it} = a_i]
\end{aligned}
$$

based on beliefs σ_i

The *primitives* of the game is $\left(\{u_{i,\theta}\}_{i=1}^{I}, \{\beta_i\}_{i=1}^{I}, P \right)$

Modeling Assumptions - Cts Monotone Choice

M1. Conditional Independence:

$$P\left(x_{t+1}, \boldsymbol{\varepsilon}_{t+1} | x_t, \boldsymbol{\varepsilon}_t, \mathbf{a}_t\right) = Q\left(\boldsymbol{\varepsilon}_{t+1} | x_{t+1}\right) G\left(x_{t+1} | x_t, \mathbf{a}_t\right)$$

M2. Independent Private Values with known distributions:

$$Q\left(\boldsymbol{\varepsilon}_t | x_t\right) = \prod_{i=1}^{I} Q_i\left(\boldsymbol{\varepsilon}_{it} | x_t\right)$$

M3. Discrete Public Values: $X = \left\{x^1, \ldots, x^J\right\}$ for some $J < \infty$
M4. Increasing Differences Utility Functions: for all $(i, \mathbf{a}_{-i}, x)$

$$u_i\left(a_i, \mathbf{a}_{-i}, x, \varepsilon_i\right) - u_i\left(a_i', \mathbf{a}_{-i}, x, \varepsilon_i\right) > u_i\left(a_i, \mathbf{a}_{-i}, x, \varepsilon_i'\right) - u_i\left(a_i', \mathbf{a}_{-i}, x, \varepsilon_i'\right)$$

for any $a_i > a_i', \varepsilon_i > \varepsilon_i'$.
M5. The discount factors are known.

For simplicity, we set Q_i (indp of x_t) and β_i to be the same for player.

Markovian Framework

- **Markov strategies**, whenever $s_{it} = s_{it'}$ for any t, t'

$$a_{it} = \alpha_i\left(s_{it}\right) = \alpha_i\left(s_{it'}\right) = a_{it'}.$$

- **Markov beliefs**, σ_i, is distribution of $\mathbf{a}_t$ condition on x_t.

Definition. *(Markov Perfect Equilibrium)*
A collection $(\boldsymbol{\alpha}, \boldsymbol{\sigma})$ *is a MPE if*

1. *for all* i, α_i *is a best response to* $\boldsymbol{\alpha}_{-i}$ *given the beliefs* σ_i *on* X;
2. *all players use Markov strategies;*
3. *for all* i, *the beliefs* σ_i *are consistent with the strategies* α_i.

Existence of PS-MPE

Under M1-M4 and some other regularity conditions, Srisuma (2013) showed the following:

- (Lemma 1) The discounted expected payoffs (Λ_i) satisfies increasing differences in (a_i, ε_i).
- (Lemma 2) Best responses are singleton sets almost surely.
- (Propositions 1&2) PS-MPE where players take monotone actions exists.

Estimation

Some Options

Focus on $u_i \mapsto u_{i,\theta}$ and the estimation of θ.

In the (unordered) discrete choice setting, we have seen:

- Full-solution approaches (cf. Rust (1987)).
- Two-or iterative-approaches (cf. Hotz and Miller (1993), Aguirregabiria and Mira (2002,2007)).

We are going to focus on the two-step approach that does not involve solving the dynamic model.

We will assume data to have been generated from a single equilibrium.

Data and Model CCDF

- Structural assumption: $(\boldsymbol{\alpha}_0, \boldsymbol{\sigma}_0)$ is consistent with the model for some $\theta_0 \in \Theta$.
- Beliefs can be represented by $\mathbf{F}$, vectorization of conditional distribution of $\mathbf{a}_t$ condition on x_t.
- Given $\{\mathbf{a}_n, x_n, x_n'\}_{n=1}^{N}$, we want to construct an objective function based on a distance:

$$\min_{\theta} \mu\left(\mathbf{F}, \Psi_{\theta}\left(\mathbf{F}\right)\right),$$

Ψ_{θ} maps beliefs into distributions of best responses where $F_i = F_{i,\theta_0}$.

Do you see any similarities between Ψ_{θ} here and the best response mapping in Aguirregabiria and Mira (2007)?

Two-Step Approach

Estimation proceeds in two-stages:

1. Estimation of the best response map and implied distribution, $\widehat{\Psi}_\theta\left(\widehat{\mathbf{F}}\right)$.
2. Minimization of distribution functions implied by the data vs model $\min_\theta \mu\left(\widehat{\mathbf{F}}, \widehat{\Psi}_\theta\left(\widehat{\mathbf{F}}\right)\right)$.

Compared to unordered discrete choice: the first step is analogous to CCP and formation of discounted payoffs off optimal paths, and the second step is the optimization stage. This is the spirit of the ALS estimator of Pesendorfer and Schmidt-Dengler (2008).

How to estimate are $\mathbf{F}$ and $\Psi_\theta\left(\mathbf{F}\right)$?

Pseudo-Decision Problem

Let's first define the expected payoffs a player gets from taking action a_i given state s_i with σ_{i0} for any θ:

$$\begin{aligned}\Lambda_{i,\theta}\left(a_i, x, \varepsilon_i\right) &= E\left[u_{i,\theta}\left(a_i, \mathbf{a}_{-in}, x_n, \varepsilon_i\right) | x_n = x\right] \\ &+ \beta E\left[V_{i,\theta}\left(s'_{in}\right) | s_{in} = s_i, a_{in} = a_i\right].\end{aligned}$$

Define the following policy function:

$$\alpha_{i,\theta}\left(x, \varepsilon_i\right) = \arg\max_{a_i} \Lambda_{i,\theta}\left(a_i, x, \varepsilon_i\right).$$

The model implied CDF is therefore

$$F_{i,\theta}\left(a_i | x\right) = \Pr\left[\alpha_{i,\theta}\left(x_n, \varepsilon_{in}\right) \leq a_i | x_n = x\right].$$

We need to estimate $\Lambda_{i,\theta}$ then $\alpha_{i,\theta}$.

Pseudo-Decision Problem

How to estimate $E\left[V_{i,\theta}\left(s'_{in}\right)|s_{in}=s_i,a_{in}=a_i\right]$?

This is the *choice specific continuation value* that also appears in discrete games. It can be estimated in a similar way, by noting that

$$\begin{aligned} & E\left[V_{i,\theta}\left(s'_{in}\right)|s_{in}=s_i,a_{in}=a_i\right] \\ = \ & E\left[V_{i,\theta}\left(s'_{in}\right)|x_n=x,a_{in}=a_i\right] \\ = \ & E\left[E\left[V_{i,\theta}\left(s'_{in}\right)|x'_n\right]|x_n=x,a_{in}=a_i\right]. \end{aligned}$$

In what follows, it will be convenient to write

$$\begin{aligned} m_{i,\theta} \ &: \ = E\left[V_{i,\theta}\left(s_{in}\right)|x_n=\cdot\right], \\ g_{i,\theta} \ &: \ = E\left[V_{i,\theta}\left(s'_{in}\right)|x_n=\cdot,a_{in}=\cdot\right]. \end{aligned}$$

Pseudo-Decision Problem

Consider the value function, define for any θ,

$$V_{i,\theta}\left(s_i\right) = \sum_{\tau=0}^{\infty} \beta^{\tau} E\left[u_{i,\theta}\left(\mathbf{a}_{\tau}, s_{i\tau}\right) | s_{i0} = s_i\right].$$

Take conditional expectation and use a recursive form:

$$\begin{aligned} E\left[V_{i,\theta}\left(s_{in}\right) | x_n = x\right] &= E\left[u_{i,\theta}\left(\mathbf{a}_n, s_{in}\right) | x_n = x\right] \\ &\quad +\beta E\left[V_{i,\theta}\left(s'_{in}\right) | x_n = x\right]. \end{aligned}$$

Moreover, by M1 and LIE:

$$E\left[V_{i,\theta}\left(s'_{in}\right) | x_n = x\right] = E\left[E\left[V_{i,\theta}\left(s'_{in}\right) | x'_n\right] | x_n = x\right].$$

Pseudo-Decision Problem

We can then write,

$$\begin{aligned} E\left[V_{i,\theta}\left(s_{in}\right)|x_n = x\right] &= E\left[u_{i,\theta}\left(\mathbf{a}_n, s_{in}\right)|x_n = x\right] \\ &\quad +\beta E\left[V_{i,\theta}\left(s'_{in}\right)|x_n = x\right]. \end{aligned}$$

which can be vectorized over x and represented by a matrix equation:

$$\begin{aligned} m_{i,\theta} &= r_{i,\theta} + \beta\mathcal{L}m_{i,\theta} \\ &= \left(I - \beta\mathcal{L}\right)^{-1} r_{i,\theta}, \end{aligned}$$

where $r_{i,\theta} := E\left[u_{i,\theta}\left(a_n, s_{in}\right)|x_n = \cdot\right]$ and $\mathcal{L}$ is a matrix that serves as conditional expectation operator, i.e. $\mathcal{L}\phi$ gives $E\left[\phi\left(x'_n\right)|x_n = \cdot\right]$.

$r_{i,\theta}$ and $\mathcal{L}$ are np identified, and $m_{i,\theta}$ is identified for all i, θ.

Pseudo-Decision Problem

For the choice specific continuation value, recall that

$$\begin{aligned} & E\left[V_{i,\theta}\left(s'_{in}\right)|x_n = x, a_{in} = a_i\right] \\ = \quad & E\left[E\left[V_{i,\theta}\left(s'_{in}\right)|x'_n\right]|x_n = x, a_{in} = a_i\right], \end{aligned}$$

we have in a matrix form:

$$\begin{aligned} g_{i,\theta} &= \mathcal{H}_i m_{i,\theta} \\ &= \mathcal{H}_i \left(I - \beta\mathcal{L}\right)^{-1} m_{i,\theta}, \end{aligned}$$

where $\mathcal{H}_i$ is a matrix that serves as conditional expectation operator, i.e. $\mathcal{H}_i\phi$ gives $E\left[\phi\left(x'_n\right)|x_n = \cdot, a_{in} = \cdot\right]$.

Since $m_{i,\theta}$ and $\mathcal{H}_i$ are np identified, $g_{i,\theta}$ is identified for all i, θ.

Pseudo-Best Response Map

Back to $\Psi_\theta(\mathbf{F})$:

$$\begin{aligned}
\alpha_{i,\theta}(x,\varepsilon_i) &= \arg\max_{a_i} \Lambda_{i,\theta}(a_i,x,\varepsilon_i), \\
\Lambda_{i,\theta}(a_i,x,\varepsilon_i) &= E\left[u_{i,\theta}(a_i,\mathbf{a}_{-in},x_n,\varepsilon_i)\,|x_n=x\right]+\beta g_{i,\theta}(a_i,x), \\
F_{i,\theta}(a_i|x) &= \Pr\left[\alpha_{i,\theta}(x_n,\varepsilon_{in})\le a_i|x_n=x\right].
\end{aligned}$$

- $\Lambda_{i,\theta}$ has strict increasing differences in (a_i,ε_i) and a unique solution.
- By construction: $F_{i,\theta}$ equals $F_i = F_{i,\theta_0}$
- By the *quantile invariance principle*: $F_i(a_{in}|x_n) = Q(\varepsilon_{in})$. I.e., $\varepsilon_{in} = Q^{-1}(F_i(a_{in}|x_n))$, which is how we identify $r_{i,\theta}$ (cf. Hotz-Miller inversion).

Empirical Analogs

True Functions	Nonparametric Estimators
$F_i\left(a_i \vert x\right)$	$\frac{\frac{1}{N}\sum_n \mathbf{1}[a_{in} \leq a_i, x_n = x]}{\frac{1}{N}\sum_n \mathbf{1}[x_n = x]}$
$E\left[u_{i,\theta}\left(a_n, s_{in}\right) \vert x_n = x\right]$	$\frac{\frac{1}{N}\sum_n u_{i,\theta}(\mathbf{a}_n, x_n, \widehat{\varepsilon}_{in})\mathbf{1}[x_n = x]}{\frac{1}{N}\sum_n \mathbf{1}[x_n = x]}$
$\Pr\left[x_n' = x' \vert x_n = x\right]$	$\frac{\frac{1}{N}\sum_n \mathbf{1}[x_n' = x', x_n = x]}{\frac{1}{N}\sum_n \mathbf{1}[x_n = x]}$
$E\left[V_{i,\theta}\left(s_n'\right) \vert x_n = x, a_{in} = a_i\right]$	$\sum_{x' \in X} \widehat{m}_{i,\theta}\left(x'\right) \frac{\frac{1}{N}\sum_n \mathbf{1}[x_n' = x', x_n = x] K_h(a_{in} - a_i)}{\frac{1}{N}\sum_n \mathbf{1}[x_n = x] K_h(a_{in} - a_i)}$

$\widehat{\varepsilon}_{in}$ is identified under monotone choice from

$$\widehat{\varepsilon}_{in} = Q_i^{-1}\left(\widehat{F}_i\left(a_{in} \vert x_n\right)\right)$$

Empirical Analogs

We need to provide estimates of $(r_{i,\theta}, \mathcal{L}_i, \mathcal{H}_i)$ since

$$
\begin{aligned}
\widehat{m}_{i,\theta} &= \widehat{r}_{i,\theta} + \beta \widehat{\mathcal{L}} \widehat{m}_{i,\theta} \\
&= \sum_{\tau=0}^{\infty} \beta^{\tau} \widehat{\mathcal{L}}^{\tau} \widehat{r}_{i,\theta} \\
&= \left(I - \beta \widehat{\mathcal{L}} \right)^{-1} \widehat{r}_{i,\theta} \\
\widehat{g}_{i,\theta} &= \widehat{\mathcal{H}}_i \widehat{m}_{i,\theta}
\end{aligned}
$$

$\widehat{g}_{i,\theta}$ can be estimatoed for any i, θ without any optimization.

First Stage: Nonparametric Estimation

Empirical pseudo-decision problem:

- Obtain nonparametric estimates $\widehat{g}_{i,\theta}$ (easy part).
- Use $\widehat{g}_{i,\theta}$ to approximate $\widehat{\alpha}_{i,\theta}$ (demanding part).
- Simulate $\{\varepsilon_i^r\}_{r=1}^R$ from Q_i to approximate $F_{i,\theta}(a_i|x)$, e.g.

$$\widetilde{F}_{i,\theta}(a_i|x) = \frac{1}{R}\sum_{r=1}^{R}\mathbf{1}\left[\widehat{\alpha}_{i,\theta}(x, \varepsilon_i^r) \le a_i\right]$$

$**$ This pseudo/2-step approach **does not solve a dynamic equilibrium** *and* **does not have an indeterminacy problem** $**$

Second Stage: Optimization

Minimization:

- Choose $\{\mu_{i,x}\}_{(i,x)\in\mathcal{I}\times X}$ to define a metric.
- Minimize distance w.r.t. θ,

$$M_N(\theta) = \sum_{i\in\mathcal{I}} \sum_{x\in X} \int_{\mathcal{A}_i} \left[\widetilde{F}_{i,\theta}(a_i|x) - \widehat{F}_i(a_i|x)\right]^2 \mu_{i,x}(da_i).$$

Choice of measures will affect consistency and efficiency - this paper only considers the former, where $\mu_{i,x}$ is dominated by the Lebesgue measure on $\mathcal{A}_i$. E.g., the empirical or uniform measures will do.

Theorems 1&2 in Srisuma (2013) show that, under some regularity conditions, $\widehat{\theta} = \arg\min_\theta M_N(\theta)$ is consistent and has an asymptotic normal distribution.

Computationally Simpler Approach

Estimating games with continuous actions with a pseudo-model is more involved than for unordered discrete choice games (also see Jenkins, Liu, Matzkin and McFadden (2021, JE)).

Bajari, Benkard, and Levin (2007, ECMA) propose an innovative estimator that avoids solving pseudo-problems based moment inequality restrictions.

Inequality Restrictions

Let $\boldsymbol{\sigma}_0 = (\sigma_{0i}, \ldots, \sigma_{0I})$ equilibrium beliefs consistent with θ_0. Equilibrium condition implies for all i:

$$E\left[\left. V_{i,\theta_0}\left(s_{it}; \sigma_{0i}\right)\right| x_t\right] \geq E\left[\left. V_{i,\theta_0}\left(s_{it}; \sigma'_i\right)\right| x_t\right],\ \sigma'_i \in \Sigma_i.$$

Suppose the following identification condition holds

$$E\left[\left. V_{i,\theta}\left(s_{it}; \sigma_{0i}\right)\right| x_t\right] \geq E\left[\left. V_{i,\theta}\left(s_{it}; \sigma'_i\right)\right| x_t\right],\ \sigma'_i \in \Sigma_i, \forall i \ \text{ iff } \theta = \theta_0.$$

- $\boldsymbol{\sigma}_0$ is nonparametrically identified.
- Construct objective functions by comparing returns from $\boldsymbol{\sigma}_0$ with other σ'_i in Σ_i, and collect θ that respects optimality.
- How to construct $E\left[\left. V_{i,\theta}\left(s_{it}; \sigma_i\right)\right| x_t\right]$, and what are *alternative strategies?*

Inequality Restrictions

$$E\left[\left. V_{i,\theta}\left(s_{it};\sigma_i\right)\right| x_t = x\right] = \sum_{\tau=0}^{\infty} \beta^{\tau} E\left[u_{i,\theta}\left(\sigma_i\left(s_{i\tau}\right), s_{i\tau}\right)|x_0 = x\right],$$

given σ_i, starting with $s_{i0}^r = \left(x_0, \varepsilon_{i0}^r\right)$, one can forward simulate (cf. Hotz et al. (1994)) by:

$$\frac{1}{R}\sum_{r=1}^{R}\left(\sum_{\tau=0}^{T}\beta^{\tau} u_{i,\theta}\left(\sigma_i\left(s_{i\tau}^r\right), s_{i\tau}^r\right)\right).$$

Simulate $\left\{\varepsilon_{i\tau}^r\right\}_{\tau=0}^{T}$ for all i:

- For all i, $\sigma_i\left(s_{i0}^r\right) \mapsto s_{i1}^r \mapsto \sigma_i\left(s_{i1}^r\right) \mapsto s_{i2}^r \cdots$
- Collect the payoffs then average over r.
- Data implied strategy uses: $a_{i\tau} = F_i^{-1}\left(Q\left(\varepsilon_{i\tau}^r\right)|x_\tau\right)$.
- A popular alternative is $\left\{F_i^{-1}\left(Q\left(\varepsilon_{it}\right)|x_t\right) + \eta \text{ for } \eta \in \mathbb{R}\right\}$.

Inequality Restrictions

Estimate θ_0 by finding the set of $\theta's$ where estimated versions of

$$E\left[V_{i,\theta}\left(s_{it};\sigma_{0i}\right)\middle|\, x_t = x\right] \geq E\left[V_{i,\theta}\left(s_{it};\sigma_i'\right)\middle|\, x_t = x\right], \ \sigma_i' \in \Sigma_i, \tag{1}$$

hold for all i and x.

While this is very intuitive conceptually, efficiency consideration aside, an important question to ask is whether $\mathbb{S}_A = \left\{F_i^{-1}\left(Q\left(\varepsilon_{it}\right)|x_t\right) + \eta \text{ for } \eta \in \mathbb{R}\right\}$ is adequate for identifying θ_0.

The answer is: *not always*. I.e., it is possible there is a unique θ_0 that (1) holds. However, when only strategies like $\mathbb{S}_A$ are used, other $\theta \neq \theta_0$ also satisfies (1).

Cournot Game

Design: Each firm faces the following inverse demand function

$$
\begin{aligned}
D_{\theta}(\mathbf{a}, x) &= x\left(1-\theta_1\left(a_1+a_2\right)\right) \\
u_{i,\theta}\left(a_i, a_j, x, \varepsilon_i\right) &= a_i\left(D_{\theta}(\mathbf{a}, x)-\theta_2 \varepsilon_i\right)
\end{aligned}
$$

where

- $a_{in} \subset \mathbb{R}$
- $x_n \in \{2, 4\}$ with equal probability
- $\varepsilon_{in} \sim N(0, 1)$
- for $(\theta_1, \theta_2) \in \mathbb{R}^+ \times \mathbb{R}^+$

Cournot Game

By solving the model and find the optimal policy wrt to θ_0, $\mathbb{S}_A$ **cannot identify** θ_{02}, as

$$E\left[u_{i,\theta}\left(\sigma_0\left(\mathbf{a}_n\right), x_n, \varepsilon_{in}\right)|x_n\right] - E\left[u_{i,\theta}\left(\sigma'\left(\mathbf{a}_n\right), x_n, \varepsilon_{in}\right)|x_n\right]$$

does not depend on θ_2!

To see what happens numerically, let $\theta_0 = (0.2, 0.2)$ and generate $\{\mathbf{a}_n, x_n\}_{n=1}^N$ from the symmetric equilibrium with $N = 100, 500, 1000$ (500 simulations each).

Computed the following estimators:

1. $\widehat{\theta} = \arg\min_\theta M_N(\theta)$ with uniform & empirical measures with misspecification (random dynamics of order $N^{-1/2}$).
2. BBL with additive & multiplicative perturbations with correct specification.

Table 3: Cournot Game

N	$\widehat{\theta}_1$	Bias	Mbias	Std	Iqr	95%	Mse
100	UM	0.0000	-0.0001	0.0014	0.0013	0.9460	0.0000
	UM-M	0.0001	-0.0001	0.0026	0.0027	0.9460	0.0000
	EM	-0.0004	-0.0004	0.0014	0.0013	0.9400	0.0000
	EM-M	-0.0003	-0.0005	0.0026	0.0028	0.9580	0.0000
	AP-L	-0.0000	-0.0000	0.0016	0.0017	0.9560	0.0000
	AP-H	-0.0001	-0.0001	0.0017	0.0015	0.9540	0.0000
	MP-L	0.0002	0.0000	0.0020	0.0019	0.9440	0.0000
	MP-H	0.0001	-0.0000	0.0021	0.0020	0.9400	0.0000
500	UM	0.0000	0.0000	0.0007	0.0007	0.9580	0.0000
	UM-M	0.0000	-0.0000	0.0012	0.0012	0.9580	0.0000
	EM	-0.0001	-0.0000	0.0007	0.0007	0.9540	0.0000
	EM-M	-0.0000	-0.0001	0.0012	0.0012	0.9560	0.0000
	AP-L	0.0000	0.0000	0.0008	0.0008	0.9380	0.0000
	AP-H	-0.0000	-0.0000	0.0007	0.0008	0.9580	0.0000
	MP-L	0.0000	0.0000	0.0010	0.0009	0.9580	0.0000
	MP-H	-0.0000	0.0000	0.0010	0.0009	0.9460	0.0000
1000	UM	-0.0000	0.0000	0.0004	0.0004	0.9460	0.0000
	UM-M	-0.0000	0.0000	0.0008	0.0008	0.9420	0.0000
	EM	-0.0001	-0.0000	0.0004	0.0004	0.9480	0.0000
	EM-M	-0.0001	-0.0000	0.0008	0.0008	0.9480	0.0000
	AP-L	-0.0000	0.0000	0.0006	0.0005	0.9380	0.0000
	AP-H	0.0001	0.0000	0.0005	0.0005	0.9440	0.0000
	MP-L	0.0000	0.0000	0.0007	0.0005	0.9720	0.0000
	MP-H	0.0000	0.0000	0.0007	0.0007	0.9460	0.0000

Table 4: Cournot Game

N	$\widehat{\theta}_2$	Bias	Mbias	Std	Iqr	95%	Mse
100	UM	-0.0009	-0.0011	0.0119	0.0131	0.9580	0.0001
	UM-M	0.0054	0.0053	0.0142	0.0132	0.9500	0.0002
	EM	0.0033	0.0029	0.0140	0.0139	0.9360	0.0002
	EM-M	0.0205	0.0164	0.0255	0.0206	0.8960	0.0011
	AP-L	-0.0174	-0.0421	0.2613	0.2153	0.9300	0.0686
	AP-H	0.0008	-0.0062	0.1520	0.1390	0.9480	0.0231
	MP-L	0.0268	0.0047	0.2623	0.2009	0.9400	0.0695
	MP-H	0.0217	0.0064	0.2753	0.2623	0.9500	0.0762
500	UM	-0.0002	-0.0003	0.0052	0.0050	0.9580	0.0000
	UM-M	0.0005	0.0006	0.0055	0.0053	0.9500	0.0000
	EM	0.0006	0.0002	0.0059	0.0055	0.9520	0.0000
	EM-M	0.0036	0.0036	0.0070	0.0069	0.9320	0.0001
	AP-L	-0.0241	-0.0828	0.2012	0.1380	0.9380	0.0411
	AP-H	-0.0150	-0.0248	0.1388	0.1097	0.9340	0.0195
	MP-L	-0.0010	0.0039	0.0945	0.0117	0.9260	0.0089
	MP-H	0.0046	0.0043	0.1191	0.0841	0.9400	0.0142
1000	UM	0.0001	-0.0000	0.0037	0.0038	0.9460	0.0000
	UM-M	0.0004	0.0006	0.0039	0.0039	0.9600	0.0000
	EM	0.0006	0.0004	0.0042	0.0046	0.9560	0.0000
	EM-M	0.0019	0.0022	0.0047	0.0045	0.9380	0.0000
	AP-L	-0.0288	-0.0943	0.1833	0.1024	0.9400	0.0344
	AP-H	-0.0168	-0.0295	0.1284	0.1141	0.9360	0.0168
	MP-L	0.0021	0.0000	0.0643	0.0046	0.9280	0.0041
	MP-H	-0.0054	0.0005	0.0820	0.0455	0.9080	0.0068

Loss of Identifying Information

Domínguez and Lobato (2004, ECMA) give explicit examples where

$$E\left[\rho_\theta\left(w_t\right)\left|y_t\right.\right]=0 \text{ iff } \theta=\theta_0 \not\Rightarrow E\left[\phi\left(y_t\right)\otimes\rho_\theta\left(w_t\right)\right]=0 \text{ iff } \theta=\theta_0$$

- Domínguez and Lobato illustrate how to choose ϕ to preserve the identifying information (at least asymptotically) - e.g., $\phi\left(y_t\right)=1\left[y_t\leq y\right]$ for all $y\in\mathcal{Y}$ works.
- Khan and Tamer (2009, JE) use same idea for moment inequality models

These ideas cannot apply here, as Σ_i as is a much large space than $\mathbb{R}^k$.

Suggestions in Srisuma (2013)

- Consider a variety of *distinct* classes of strategies for cts actions. E.g., $\mathbb{S}_M = \left\{ F_i^{-1} \left(Q \left(\varepsilon_{it} \right) | x_t \right) \times \eta \text{ for } \eta \in \mathbb{R} \right\}$ or $\mathbb{S}_{Sq} = \left\{ \left(F_i^{-1} \left(Q \left(\varepsilon_{it} \right) | x_t \right) \right)^2 \right\}$ etc – get creative!
- For discrete ordered choice, $\{1, ..., K\}$, given x_t it is optimal to choose k if $\varepsilon_{it} \in \left[Q^{-1} \left(F_i \left(k - 1 | x_t \right) \right), Q^{-1} \left(F_i \left(k | x_t \right) \right) \right)$ and $\mathbb{S}_O = \left\{ Q^{-1} \left(F_i \left(k | x_t \right) \right) + \eta \text{ for } \eta \in \left(-\omega, \omega \right) \right\}$ works.
- For unordered discrete choice, additive perturbation applied to the vector of differences in expected payoffs (used in HM inversion) where perturbations are taken from a ball that contains zero works.
- See Appendix A in Srisuma (2013) for more details.

Summary

Summary

- How to model and estimate continuous action games with a two-step approach.
- There are often many estimators available for a structural model, one may have to balance between computational and other considerations.
- It is possible to mix-match different action choices. E.g., one can build an entry model followed by investment decision.
- For some recents work with dynamic ordered choice problems see Caoui (2023, ReStud), Forte and Guanja (2024, WP), Gowrisankaran and Schmidt-Dengler (2025, WP).