

# Introduction to Dynamic Structural Econometrics

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# Bus Engines (Rust, 1987)

A renewal problem

- Mr Zurcher maximizes the expected discounted sum of payoffs:

$$E \left\{ \sum_{t=1}^{\infty} \beta^{t-1} [d_{t2}(\theta_1 x_t + \theta_2 s + \epsilon_{t2}) + d_{t1} \epsilon_{t1}] \right\}$$

where:

- $d_{t1} = 1$  and  $x_{t+1} = 1$  if Zurcher replaces the engine
  - $d_{t2} = 1$  and bus mileage advances to  $x_{t+1} = x_t + 1$  if he keeps the engine
  - buses are also differentiated by a fixed characteristic  $s \in \{0, 1\}$ .
  - the choice-specific shocks  $\epsilon_{tj}$  are *iid* Type 1 extreme value (T1EV).
- Define the conditional value function for each choice as:

$$v_j(x, s) = \begin{cases} \beta V(1, s) & \text{if } j = 1 \\ \theta_1 x + \theta_2 s + \beta V(x + 1, s) & \text{if } j = 2 \end{cases}$$

where  $V(x, s)$  denotes the social surplus function.

# Bus Engines

## The DGP and the CCPs

- We suppose the data comprises a cross section of  $N$  observations of buses  $n \in \{1, \dots, N\}$  reporting their:
  - fixed characteristics  $s_n$ ,
  - engine miles  $x_n$ ,
  - and maintenance decision  $(d_{n1}, d_{n2})$ .
- Let  $p_1(x, s)$  denote the conditional choice probability (CCP) of replacing the engine given  $x$  and  $s$ .
- Stationarity and T1EV imply that for all  $t$  :

$$\begin{aligned} p_1(x, s) &\equiv \int_{\epsilon_t} d_1^o(x, s, \epsilon_t) g(\epsilon_t) d\epsilon_t \\ &= \int_{\epsilon_t} \mathbf{1}\{\epsilon_{t2} - \epsilon_{t1} \leq v_1(x, s) - v_2(x, s)\} g(\epsilon_t | x_t) d\epsilon_t \\ &= \{1 + \exp[v_2(x, s) - v_1(x, s)]\}^{-1} \end{aligned}$$

- An ML estimator could be formed off this equation by nesting a fixed point algorithm inside the estimation algorithm.

# Bus Engines

## Exploiting the renewal property

- If  $\epsilon_{jt}$  is T1EV, then one can show that for all  $(x, s, j)$ :

$$V(x, s) = v_j(x, s) - \ln [p_j(x, s)] + 0.57 \dots$$

- Therefore the conditional value function of *not replacing* is:

$$\begin{aligned} v_2(x, s) &= \theta_1 x + \theta_2 s + \beta V(x, s + 1) \\ &= \theta_1 x + \theta_2 s + \beta \{v_1(x + 1, s) - \ln [p_1(x + 1, s)] + 0.57 \dots\} \end{aligned}$$

- Similarly:

$$v_1(x, s) = \beta V(1, s) = \beta \{v_1(1, s) - \ln [p_1(1, s)] + 0.57\} \dots$$

- Because bus engine miles is the only factor affecting bus value given  $s$ :

$$v_1(x + 1, s) = v_1(1, s)$$

# Bus Engines

Using CCPs to represent differences in continuation values

- Hence:

$$v_2(x, s) - v_1(x, s) = \theta_1 x + \theta_2 s + \beta \ln [p_1(1, s)] - \beta \ln [p_1(x + 1, s)]$$

- Therefore:

$$\begin{aligned} p_1(x, s) &= \frac{1}{1 + \exp [v_2(x, s) - v_1(x, s)]} \\ &= \frac{1}{1 + \exp \left\{ \theta_1 x + \theta_2 s + \beta \ln \left[ \frac{p_1(1, s)}{p_1(x + 1, s)} \right] \right\}} \end{aligned}$$

- Intuitively the CCP for current replacement is the CCP for a static model with an offset term.
- The offset term accounts for differences in continuation values using future CCPs that characterize optimal future replacements.

- Consider the following CCP estimator:

- Form a first stage estimator for  $p_1(x, s)$  from the relative frequencies:

$$\hat{p}_1(x, s) \equiv \frac{\sum_{n=1}^N d_{n1} I(x_n = x) I(s_n = s)}{\sum_{n=1}^N I(x_n = x) I(s_n = s)}$$

- Substitute  $\hat{p}_1(x, s)$  into the likelihood as incidental parameters to estimate  $(\theta_1, \theta_2, \beta)$  with a logit:

$$\frac{d_{n1} + d_{n2} \exp(\theta_1 x_n + \theta_2 s_n + \beta \ln \left[ \frac{\hat{p}_1(1, s_n)}{\hat{p}_1(x_n + 1, s_n)} \right])}{1 + \exp(\theta_1 x_n + \theta_2 s_n + \beta \ln \left[ \frac{\hat{p}_1(1, s_n)}{\hat{p}_1(x_n + 1, s_n)} \right])}$$

- Correct the standard errors for  $(\theta_1, \theta_2, \beta)$  induced by the first stage estimates of  $p_1(x, s)$ .
- Note that in the second stage  $\ln \left[ \frac{\hat{p}_1(1, s_n)}{\hat{p}_1(x_n + 1, s_n)} \right]$  enters the logit as an individual specific component of the data, the  $\beta$  coefficient entering in the same way as  $\theta_1$  and  $\theta_2$ .

# Monte Carlo Study (Arcidiacono and Miller, 2011)

## Modifying the bus engine problem

- Suppose bus type  $s \in \{0, 1\}$  is equally weighted.
- Two state variables affect wear and tear on the engine:

① total accumulated mileage:

$$x_{1,t+1} = \begin{cases} \Delta_t & \text{if } d_{1t} = 1 \\ x_{1t} + \Delta_t & \text{if } d_{2t} = 1 \end{cases}$$

② a permanent route characteristic for the bus,  $x_2$ , that systematically affects miles added each period.

- More specifically we assume:
  - $\Delta_t \in \{0, 0.125, \dots, 24.875, 25\}$  is drawn from a discretized truncated exponential distribution, with:

$$f(\Delta_t | x_2) = \exp[-x_2(\Delta_t - 25)] - \exp[-x_2(\Delta_t - 24.875)]$$

- $x_2$  is a multiple 0.01 drawn from a discrete equi-probability distribution between 0.25 and 1.25.

# Monte Carlo Study

Including the age of the bus in panel estimation

- Let  $\theta_{0t}$  denote other bus maintenance costs tied to its vintage.
- This modification renders the optimization problem nonstationary.
- The payoff difference from retaining versus replacing the engine is:

$$u_{t2}(x_{t1}, s) - u_{t1}(x_{t1}, s) \equiv \theta_{0t} + \theta_1 \min \{x_{t1}, 25\} + \theta_2 s$$

- Denoting  $x_t \equiv (x_{1t}, x_2)$ , this implies:

$$\begin{aligned} v_{t2}(x_t, s) - v_{t1}(x_t, s) &= \theta_{0t} + \theta_1 \min \{x_{t1}, 25\} + \theta_2 s \\ &\quad + \beta \sum_{\Delta_t \in \Lambda} \left\{ \ln \left[ \frac{p_{1t}(\Delta_t, s)}{p_{1t}(x_{1t} + \Delta_t, s)} \right] \right\} f(\Delta_t | x_2) \end{aligned}$$

# Monte Carlo Study

Extract from Table 1 of Arcidiacono and Miller (2011)

	DGP (1)	FIML (2)	CCP (3)
$\theta_0$ (intercept)	2	2.0100 (0.0405)	1.9911 (0.0399)
$\theta_1$ (mileage)	-0.15	-0.1488 (0.0074)	-0.1441 (0.0098)
$\theta_2$ (unobs. state)	1	0.9945 (0.0611)	0.9726 (0.0668)
$\beta$ (discount factor)	0.9	0.9102 (0.0411)	0.9099 (0.0554)
Time (minutes)		130.29 (19.73)	0.078 (0.0041)

# Dynamic Optimization

## Discrete choice in discrete time

- $t \in \{1, 2, \dots, T\}$  for  $T \leq \infty$  denotes the time period.
- $z_t \in Z$  denotes a state revealed to the individual at the beginning of  $t$ .
- $d_t \equiv (d_{1t}, \dots, d_{Jt})$  denotes an individual's choice
  - where  $d_{jt} \in \{0, 1\}$  and  $\sum_{j=1}^J d_{jt} = 1$ .
- $f_{jt}(z_{t+1}|z_t)$  denotes the probability (density function) of  $z_{t+1}$  occurring in period  $t+1$  when action  $j$  is taken at time  $t$ .
- For choices  $d_t$  in each period  $t$  (depending on  $z_t$ ), the individual's expected lifetime utility is:

$$E \left\{ \sum_{\tau=t}^T \sum_{j=1}^J \beta^{\tau-t} d_{j\tau} u_{j\tau}^*(z_\tau) | z_t \right\} \quad (1)$$

where:

- $\beta \in (0, 1)$  is a subjective discount factor.
- the expectation is taken over  $z_{t+1}, \dots, z_T$  in each period  $t$ .

# Dynamic Optimization

## Conditional Independence Assumption

- **John Rust** (Econometrica 1987) introduced the conditional independence assumption in his Harold Zurcher paper.
- Decompose  $z_t = (x_t, \epsilon_t)$  and write:

$$u_{j\tau}^*(z_\tau) = u_{j\tau}(x_\tau) + \epsilon_{j\tau}$$

where:

- $x_t \in \{1, \dots, X\}$  is an *observed variable*.
- $\epsilon_t \equiv (\epsilon_{1t}, \dots, \epsilon_{Jt})$  where  $\epsilon_{jt} \in \mathbb{R}$  is an *unobserved variable*.
- $g_t(\epsilon_t | x_t)$  is a probability density for  $\epsilon_t$  conditional on  $x_t$ .
- $f_{jt}(x_{t+1} | x_t)$  is a transition probability for  $x_{t+1}$  conditional on  $x_t$ .
- $g_{t,x,\epsilon}(x_{t+1}, \epsilon_{t+1} | x_t, \epsilon_t)$ , the joint mixed density function for the state in period  $t+1$  conditional on  $(x_t, \epsilon_t)$ .
- The *conditional independence assumption* is:

$$g_{t,j,x,\epsilon}(x_{t+1}, \epsilon_{t+1} | x_t, \epsilon_t) = g_{t+1}(\epsilon_{t+1} | x_{t+1}) f_{jt}(x_{t+1} | x_t)$$

- It is widely used when estimating dynamic discrete choice models.

# Conditional Independence

## Optimization

- Denote the optimal decision rule at  $t$  as  $d_t^o(x_t, \epsilon_t)$ , with  $j^{th}$  element  $d_{jt}^o(x_t, \epsilon_t)$ , and define the *social surplus function* as:

$$V_t(x_t) \equiv E \left\{ \sum_{\tau=t}^T \sum_{j=1}^J \beta^{\tau-t-1} d_{j\tau}^o(x_\tau, \epsilon_\tau) (u_{j\tau}(x_\tau) + \epsilon_{j\tau}) \right\}$$

- The *conditional value function*,  $v_{jt}(x_t)$ , is defined as:

$$v_{jt}(x_t) \equiv u_{jt}(x_t) + \beta \sum_{x=1}^X V_{t+1}(x) f_{jt}(x|x_t)$$

- Integrating  $d_{jt}^o(x_t, \epsilon)$  over  $\epsilon \equiv (\epsilon_1, \dots, \epsilon_J)$  define the *conditional choice probabilities* CCPs by:

$$p_{jt}(x_t) \equiv E [d_{jt}^o(x_t, \epsilon) | x_t] = \int d_{jt}^o(x_t, \epsilon) g_t(\epsilon | x_t) d\epsilon$$

# Extension to Dynamic Markov Games

Players, choices and state variables

- Consider a dynamic game for  $I$  countable players:

- ①  $d_t^{(i)} \equiv (d_{t1}^{(i)}, \dots, d_{tJ}^{(i)})$  choice of player  $i$  in period  $t$ .
- ②  $d_t \equiv (d_t^{(1)}, \dots, d_t^{(I)})$  choices of all the players in period  $t$ .
- ③  $d_t^{(-i)} \equiv (d_t^{(1)}, \dots, d_t^{(i-1)}, d_t^{(i+1)}, \dots, d_t^{(I)})$  choices of all but  $i$  in  $t$ .
- ④  $x_t$  value of state variables of the game in period  $t$ .
- ⑤  $F(x_{t+1} | x_t, d_t)$  transition probability for  $x_{t+1}$  given  $(x_t, d_t)$ .
- ⑥  $F_j(x_{t+1} | x_t, d_t^{(-i)}) \equiv F(x_{t+1} | x_t, d_t^{(-i)}, d_{jt}^{(i)} = 1)$  transition probability for  $x_{t+1}$  given  $x_t$ ,  $i$  choosing  $j$ , and everyone else  $d_t^{(-i)}$ .

# Extension to Dynamic Markov Games

## Payoffs, information and CCPs

- The summed discounted payoff to  $i$  from playing the game is:

$$\sum_{t=1}^T \sum_{j=1}^J \beta^{t-1} d_{jt}^{(i)} \left[ U_j^{(i)} \left( x_t, d_t^{(-i)} \right) + \epsilon_{jt}^{(i)} \right]$$

where:

- 1  $U_j^{(i)} \left( x_t, d_t^{(-i)} \right)$  depends on the choices of all the players.
  - 2  $\epsilon_t^{(i)} \equiv \left( \epsilon_{1t}^{(i)}, \dots, \epsilon_{Jt}^{(i)} \right)$  is iid across  $i$  with density  $g \left( \epsilon_t^{(i)} | x_t \right)$ .
  - 3 neither  $d_t^{(-i)}$  nor  $\epsilon_t^{(-i)}$  are observed by  $i$ .
- Analogous to the single agent setup define:
    - 1  $p_j^{(i)}(x_t) = \int d_{jt}^{(i)} \left( x_t, \epsilon_t^{(i)} \right) g \left( \epsilon_t^{(i)} \right) d\epsilon_t^{(i)}$  as the CCP for the  $i$  choosing  $j$  in period  $t$ .
    - 2  $P \left( d_t^{(-i)} | x_t \right) = \prod_{i'=1, i' \neq i}^I \left( \sum_{j=1}^J d_{jt}^{(i')} p_j^{(i')}(x_t) \right)$  as the CCP for all the other players choosing  $d_t^{(-i)}$  in period  $t$ .

# Extension to Dynamic Markov Games

## Equilibrium defined

- Then  $\left(p_1^{(i)}(x_t), \dots, p_J^{(i)}(x_t)\right)$  is an equilibrium if  $d_j^{(i)}\left(x_t, \epsilon_t^{(i)}\right)$  solves the individual optimization problem (1) for each  $\left(i, x_t, \epsilon_t^{(i)}\right)$  when:

$$u_j^{(i)}(x_t) = \sum_{d_t^{(-i)}} P\left(d_t^{(-i)} | x_t\right) U_j^{(i)}\left(x_t, d_t^{(-i)}\right) \quad (2)$$

and:

$$f_j^{(i)}\left(x_{t+1} | x_t^{(i)}\right) = \sum_{d_t^{(-i)}} P\left(d_t^{(-i)} | x_t^{(i)}\right) F_j\left(x_{t+1} | x_t, d_t^{(-i)}\right) \quad (3)$$

- To analyze dynamic games taking this form:
  - 1 interpret  $u_j^{(i)}(x_t)$  with (2) and  $f_j^{(i)}\left(x_{t+1} | x_t^{(i)}\right)$  with (3)
  - 2 in estimation treat the *best reply function* as the solution to a dynamic discrete choice optimization problem within the equilibrium played out by the *data generating process* DGP.

# Inversion

## Differences in conditional valuation functions

- The starting point for our analysis is to define differences in the conditional valuation functions as:

$$\Delta v_{jkt}(x) \equiv v_{jt}(x) - v_{kt}(x)$$

- Although there are  $J(J-1)$  differences all but  $(J-1)$  are linear combinations of the  $(J-1)$  basis functions.
- For example setting the basis functions as:

$$\Delta v_{jt}(x) \equiv v_{jt}(x) - v_{Jt}(x)$$

then clearly:

$$\Delta v_{jkt}(x) = \Delta v_{jt}(x) - \Delta v_{kt}(x)$$

- Without loss of generality we focus on this particular basis function.

# Inversion

Each CCP is a mapping of differences in the conditional valuation functions

- Using the definition of  $\Delta v_{jt}(x)$ :

$$\begin{aligned} p_{jt}(x) &\equiv \int d_{jt}^o(x, \epsilon) g_t(\epsilon | x) d\epsilon \\ &= \int I\{\epsilon_k \leq \epsilon_j + \Delta v_{jt}(x) - \Delta v_{kt}(x) \forall k \neq j\} g_t(\epsilon | x) d\epsilon \\ &= \int_{\epsilon_j = -\infty}^{\infty} \int_{-\infty}^{\epsilon_j + \Delta v_{jt}(x)} g_t(\epsilon | x) d\epsilon \end{aligned}$$

- Noting  $g_t(\epsilon | x) \equiv \partial^J G_t(\epsilon | x) / \partial \epsilon_1, \dots, \partial \epsilon_J$ , integrate over  $\epsilon$ .
- Denoting  $G_{jt}(\epsilon | x) \equiv \partial G_t(\epsilon | x) / \partial \epsilon_j$ , yields:

$$p_{jt}(x) = \int_{-\infty}^{\infty} G_{jt} \left( \begin{array}{c} \epsilon_j + \Delta v_{jt}(x) - \Delta v_{1t}(x), \dots \\ \dots, \epsilon_j, \dots, \epsilon_j + \Delta v_{jt}(x) \end{array} | x \right) d\epsilon_j$$

# Inversion

There are as many CCPs as there are conditional valuation functions

- For any vector  $J - 1$  dimensional vector  $\delta \equiv (\delta_1, \dots, \delta_{J-1})$  define:

$$Q_{jt}(\delta, x) \equiv \int_{-\infty}^{\infty} G_{jt}(\epsilon_j + \delta_j - \delta_1, \dots, \epsilon_j, \dots, \epsilon_j + \delta_j | x) d\epsilon_j$$

- We interpret  $Q_{jt}(\delta, x)$  as the probability taking action  $j$  in a static random utility model (RUM) where the payoffs are  $\delta_j + \epsilon_j$  and the probability distribution of disturbances is given by  $G_t(\epsilon | x)$ .
- It follows from the definition of  $Q_{jt}(\delta, x)$  that:

$$0 \leq Q_{jt}(\delta, x) \leq 1 \text{ for all } (j, t, \delta, x) \text{ and } \sum_{j=1}^{J-1} Q_{jt}(\delta, x) \leq 1$$

- In particular the previous slide implies that for any given  $(j, t, x)$ :

$$p_{jt}(x) = \int_{-\infty}^{\infty} G_{jt} \left( \begin{matrix} \epsilon_j + \Delta v_{jt}(x) - \Delta v_{1t}(x), \\ \dots, \epsilon_j, \dots, \epsilon_j + \Delta v_{jt}(x) \end{matrix} | x \right) d\epsilon_j \equiv Q_{jt}(\Delta v_t(x), x)$$

# Inversion (Hotz and Miller, 1993)

The CCPs invert

## Theorem (Inversion)

For each  $(t, \delta, x)$  define:

$$Q_t(\delta, x) \equiv (Q_{1t}(\delta, x), \dots, Q_{J-1,t}(\delta, x))'$$

Then the vector function  $Q_t(\delta, x)$  is invertible in  $\delta$  for each  $(t, x)$ .

- Note that  $p_{Jt}(x) = Q_{Jt}(\Delta v_t, x)$  is a linear combination of the other equations in the system because  $\sum_{k=1}^J p_k = 1$ .
- Let  $p \equiv (p_1, \dots, p_{J-1})$  where  $0 \leq p_j \leq 1$  for all  $j \in \{1, \dots, J-1\}$  and  $\sum_{j=1}^{J-1} p_j \leq 1$ . Denote the inverse of  $Q_{jt}(\Delta v_t, x)$  by  $Q_{jt}^{-1}(p, x)$ .
- The inversion theorem implies:

$$\begin{bmatrix} \Delta v_{1t}(x) \\ \vdots \\ \Delta v_{J-1,t}(x) \end{bmatrix} = \begin{bmatrix} Q_{1t}^{-1}[p_t(x), x] \\ \vdots \\ Q_{J-1,t}^{-1}[p_t(x), x] \end{bmatrix}$$

# Inversion

## Interpreting the inversion expression

- $Q_{jt}^{-1}(p, x)$  has an intuitive interpretation:
  - Given  $x$  and  $p(x)$  the agent is indifferent between the  $j^{th}$  and  $J^{th}$  choices for values of  $\epsilon'_{jt}$  and  $\epsilon'_{Jt}$  satisfying:

$$\begin{aligned}v_{jt}(x) + \epsilon'_{jt} &= v_{Jt}(x) + \epsilon'_{Jt} \\ \Rightarrow \Delta v_{jt}(x) &= \epsilon'_j - \epsilon'_J \\ &= Q_{jt}^{-1}[p_t(x), x]\end{aligned}$$

- Thus the value of  $Q_{jt}^{-1}[p_t(x), x]$  is the difference between the  $j^{th}$  and  $J^{th}$  taste shocks that would make the agent indifferent between those two choices.
- More generally the value of the vector mapping:

$$Q_t^{-1}[p_t(x), x] = \left( Q_{1t}^{-1}[p_t(x), x], \dots, Q_{J-1t}^{-1}[p_t(x), x] \right)$$

corresponds to the value of a vector  $\epsilon'_t \equiv (\epsilon'_{1t}, \dots, \epsilon'_{Jt})$  that renders the agent indifferent to all the choices.

# Inversion

## Using the inversion theorem

- The inversion theorem exploits conditional independence to finesse optimization and integration.
- More specifically we can use the inversion theorem to:
  - ① provide empirically tractable *representations of the conditional value functions*.
  - ② analyze *identification* in dynamic discrete choice models.
  - ③ provide convenient parametric forms for the density of  $\epsilon_t$  *generalizing T1EV*.
  - ④ generalize the renewal and terminal state properties often used in empirical work to *finite dependence*, by obtaining restrictions on the state variable transitions used to implement CCP estimators.
  - ⑤ introduce new methods for *incorporating unobserved state variables*.

# Conditional Value Function Correction

Definition of the conditional value function correction

- Define the conditional value function correction as:

$$\psi_{jt}(x) \equiv V_t(x) - v_{jt}(x)$$

- In stationary settings, we drop the  $t$  subscript and write:

$$\psi_j(x) \equiv V(x) - v_j(x)$$

- Suppose that instead of taking the optimal action she committed to taking action  $j$  instead. Then the expected lifetime utility would be:

$$v_{jt}(x_t) + E_t[\epsilon_{jt} | x_t]$$

so committing to  $j$  before  $\epsilon_t$  is revealed entails a loss of:

$$V_t(x_t) - v_{jt}(x_t) - E_t[\epsilon_{jt} | x_t] = \psi_{jt}(x) - E_t[\epsilon_{jt} | x_t]$$

- For example if  $E_t[\epsilon_t | x_t] = 0$ , the loss simplifies to  $\psi_{jt}(x)$ .

# Conditional Value Function Correction

## Identifying the conditional value function correction

- From their respective definitions:

$$\begin{aligned} & V_t(x) - v_{it}(x) \\ &= \sum_{j=1}^J \left\{ p_{jt}(x) [v_{jt}(x) - v_{it}(x)] + \int \epsilon_{jt} d_{jt}^o(x_t, \epsilon_t) g_t(\epsilon_t | x) d\epsilon_t \right\} \end{aligned}$$

- But:

$$v_{jt}(x) - v_{it}(x) = Q_{jt}^{-1}[p_t(x), x] - Q_{it}^{-1}[p_t(x), x]$$

and

$$\begin{aligned} & \int \epsilon_{jt} d_{jt}^o(x, \epsilon_t) g(\epsilon_t | x) d\epsilon_t \\ &= \int \prod_{k=1}^J 1 \left\{ \begin{array}{l} \epsilon_{kt} - \epsilon_{jt} \\ \leq Q_{jt}^{-1}[p_t(x), x] - Q_{kt}^{-1}[p_t(x), x] \end{array} \right\} \epsilon_{jt} g_t(\epsilon_t | x) d\epsilon_t \end{aligned}$$

- Therefore  $\psi_{it}(x) \equiv V_t(x) - v_{it}(x)$  is identified if  $G_t(\epsilon | x)$  is known and  $(x_t, d_t)$  is the DGP.

# Conditional Value Function Correction

Correction factor for extended nested logit

## Lemma

*For the nested logit  $G(\epsilon_t)$  defined above:*

$$\psi_j(p) = \gamma - \sigma \ln(p_j) - (1 - \sigma) \ln \left( \sum_{k \in \mathcal{J}} p_k \right)$$

- Note that  $\psi_j(p)$  only depends on the conditional choice probabilities for choices that are in the nest: the expression is the same no matter how many choices are outside the nest or how those choices are correlated.
- Hence,  $\psi_j(p)$  will only depend on  $p_{j'}$  if  $\epsilon_{jt}$  and  $\epsilon_{j't}$  are correlated. When  $\sigma = 1$ ,  $\epsilon_{jt}$  is independent of all other errors and  $\psi_j(p)$  only depends on  $p_j$ .

# Conditional Valuation Function Representation

Telescoping one period forward

- From its definition:

$$v_{jt}(x_t) = u_{jt}(x_t) + \beta \sum_{x=1}^X V_{t+1}(x) f_{jt}(x_{t+1}|x_t)$$

- Substituting for  $V_{t+1}(x_{t+1})$  using conditional value function correction we obtain for any  $k$ :

$$v_{jt}(x_t) = u_{jt}(x_t) + \beta \sum_{x=1}^X [v_{k,t+1}(x) + \psi_{k,t+1}(x)] f_{jt}(x|x_t)$$

- We could repeat this procedure ad infinitum, substituting in for  $v_{k,t+1}(x)$  by using the definition for  $\psi_{kt}(x)$ .

# Conditional Valuation Function Representation

Recursively defining the distribution of future state variables

- To formalize this idea, consider a random sequence of weights from  $t$  to  $T$  which begins with  $\omega_{jt}(x_t, j) = 1$ .
- For periods  $\tau \in \{t + 1, \dots, T\}$ , the choice sequence maps  $x_\tau$  and the initial choice  $j$  into

$$\omega_\tau(x_\tau, j) \equiv \{\omega_{1\tau}(x_\tau, j), \dots, \omega_{J\tau}(x_\tau, j)\}$$

where  $\omega_{k\tau}(x_\tau, j)$  may be negative or exceed one but:

$$\sum_{k=1}^J \omega_{k\tau}(x_\tau, j) = 1$$

- The weight of state  $x_{\tau+1}$  conditional on following the choices in the sequence is recursively defined by  $\kappa_\tau(x_{\tau+1}|x_t, j) \equiv f_{jt}(x_{\tau+1}|x_t)$  and for  $\tau = t + 1, \dots, T$ :

$$\kappa_\tau(x_{\tau+1}|x_t, j) \equiv \sum_{x_\tau=1}^X \sum_{k=1}^J \omega_{k\tau}(x_\tau, j) f_{k\tau}(x_{\tau+1}|x_\tau) \kappa_{\tau-1}(x_\tau|x_t, j)$$

# Conditional Valuation Function Representation

Theorem 1 of Arcidiacono and Miller (2011)

## Theorem (Representation)

*For any state  $x_t \in \{1, \dots, X\}$ , choice  $j \in \{1, \dots, J\}$  and weights  $\omega_\tau(x_\tau, j)$  defined for periods  $\tau \in \{t, \dots, T\}$ :*

$$v_{jt}(x_t) = u_{jt}(x_t) + \sum_{\tau=t+1}^T \sum_{k=1}^J \sum_{x=1}^X \beta^{\tau-t} [u_{k\tau}(x) + \psi_k[p_\tau(x)]] \omega_{k\tau}(x, j) \kappa_{\tau-1}(x | x_t, j)$$

- The theorem yields an alternative expression for  $v_{jt}(x_t)$  that dispenses with recursive maximization.
- Intuitively, the individuals have already solved their optimization problem, so their decisions, as reflected in their CCPs, are informative of their value functions.

# Identifying the Primitives

Identifying assumptions and data generating process

- The optimization model is fully characterized by the time horizon, the utility flows, the discount factor, the transition matrix of the observed state variables, and the distribution of the unobserved variables, summarized with the notation  $(T, \beta, f, g, u)$ .
- The data comprise observations for a real or synthetic panel on the observed part of the state variable,  $x_t$ , and decision outcomes,  $d_t$ .
- Following most of the empirical work in this area we consider identification when  $(T, \beta, f, g)$  are assumed to be known.
- Thus the goal is to identify  $u$  from  $(x_t, d_t)$  when  $(T, \beta, f, g)$  is known.

# Identifying the Primitives

## Observational Equivalence

- It is widely believed that  $u$  is only identified relative to one choice per period for each state.
- Can we say more than that?
- For each  $(x, t)$  let  $l(x, t) \in \{1, \dots, J\}$  denote any arbitrarily defined normalizing action and  $c_t(x) \in \mathbb{R}$  its associated benchmark flow utility, meaning  $u_{l(x,t),t}^*(x) \equiv c_t(x)$ .
- Assume  $\{c_t(x)\}_{t=1}^T$  is bounded for each  $x \in \{1, \dots, X\}$ .
- Let  $\kappa_\tau^*(x_{\tau+1}|x_t, j)$  denote the probability distribution of  $x_{\tau+1}$ , given a state of  $x_t$  taking action  $j$  at  $t$ , and then repeatedly taking the normalized action from period  $t+1$  through to period  $\tau$ .
- Thus  $\kappa_t^*(x_{t+1}|x_t, j) \equiv f_{jt}(x_{t+1}|x_t)$  and for  $\tau \in \{t+1, \dots, T\}$ :

$$\kappa_\tau^*(x_{\tau+1}|x_t, j) \equiv \sum_{x=1}^X f_{l(x,\tau),\tau}(x_{\tau+1}|x) \kappa_{\tau-1}^*(x|x_t, j) \quad (4)$$

## Theorem (Observational Equivalence, Arcidiacono and Miller, 2020)

For each  $R \in \{1, 2, \dots\}$ , define for all  $x \in \{1, \dots, X\}$ ,  $j \in \{1, \dots, J\}$  and  $t \in \{1, \dots, R\}$ :

$$u_{jR}^*(x) \equiv u_{jR}(x) + c_R(x) - u_{l(x,R),R}(x) \quad (5)$$

$$u_{jt}^*(x) \equiv u_{jt}(x) + c_t(x) - u_{l(x,t),t}(x) \quad (6)$$

$$+ \sum_{\tau=t+1}^R \sum_{x'=1}^X \beta^{\tau-t} \left\{ \begin{aligned} & \left[ c_{\tau}(x') - u_{l(x,\tau),\tau}(x') \right] \times \\ & \left[ \kappa_{\tau-1}^*(x'|x_t, l(x, t)) - \kappa_{\tau-1}^*(x'|x_t, j) \right] \end{aligned} \right\}$$

$(T, \beta, f, g, u^*)$ , is observationally equivalent to  $(T, \beta, f, g, u)$  in the limit of  $R \rightarrow T$ . Conversely suppose  $(T, \beta, f, g, u^*)$  is observationally equivalent to  $(T, \beta, f, g, u)$ . For each date and state select any action  $l(x, t) \in \{1, \dots, J\}$  with payoff  $u_{l(x,t),t}^*(x) \equiv c_t(x) \in \mathbb{R}$ . Then (5) and (6) hold for all  $(t, x, j)$ .

# Identifying the Primitives

Identification off long panels (Arcidiacono and Miller, 2020)

## Theorem (Identification)

For all  $j$ ,  $t$ , and  $x$ :

$$u_{jt}(x) = u_{1t}(x) + \psi_{1t}(x) - \psi_{jt}(x) + \sum_{\tau=t+1}^T \sum_{x_{\tau}=1}^X \beta^{\tau-t} \left\{ \begin{array}{l} [u_{1\tau}(x_{\tau}) + \psi_{1\tau}(x_{\tau})] \times \\ [\kappa_{\tau-1}(x_{\tau}|x, 1) - \kappa_{\tau-1}(x_{\tau}|x, j)] \end{array} \right\} \quad (7)$$

- If  $(T, \beta, f, g)$  is known, and if a payoff, say the first, is also known for every state and time, then  $u$  is identified.

# Identifying the Primitives

## Asymptotic efficiency of the unrestricted estimator

- By the Law of Large Numbers the cell estimators  $\hat{f}_{jt}(x' | x)$  and  $\hat{p}_{jt}(x)$  converge to their population analogues
- By the Central Limit Theorem both estimators converge at  $\sqrt{N}$  and have asymptotic normal distributions.
- Both  $\hat{f}_{jt}(x' | x)$  and  $\hat{p}_{jt}(x)$  are ML estimators for  $f_{jt}(x' | x)$  and  $p_{jt}(x)$  and obtain the Cramer-Rao lower bound asymptotically.
- Since and  $u_{jt}(x)$  is exactly identified, it follows by the *invariance principle* that  $\hat{u}_{jt}(x)$  is consistent and asymptotically efficient for  $u_{jt}(x_t)$ , also attaining its Cramer-Rao lower bound.
- Greater efficiency can only be obtained by making functional form assumptions about  $u_{jt}(x_t)$  and  $f_{jt}(x' | x)$ .

# Restricting the Parameter Space

## Parameterizing the primitives

- In practice applications further restrict the parameter space.
- For example assume  $\theta \equiv (\theta^{(1)}, \theta^{(2)}) \in \Theta$  is a closed convex subspace of Euclidean space, and:
  - $u_{jt}(x) \equiv u_j(x, \theta^{(1)})$
  - $f_{jt}(x|x_{nt}) \equiv f_{jt}(x|x_{nt}, \theta^{(2)})$
- We now define the model by  $(T, \beta, \theta, g)$ .
- Assume the DGP comes from  $(T, \beta, \theta_0, g)$  where:

$$\theta_0 \equiv (\theta_0^{(1)}, \theta_0^{(2)}) \in \Theta^{(interior)}$$

- For example many applications assume:
  - $u_{jt}(x) \equiv x' \theta_j^{(1)}$  is linear in  $x$  and does not depend on  $t$
  - $f_{jt}(x|x_{nt})$  is degenerate,  $x$  following a deterministic law of motion that does not depend on  $t$ .

# Quasi Maximum Likelihood (Hotz and Miller, 1993)

## Overview of the steps

- A Quasi Maximum Likelihood (QML) estimator can be obtained by estimating:
  - 1  $\theta_0^{(2)}$  with  $\theta_{LIML}^{(2)}$  from the data on  $f_{jt}(x|x_t, \theta^{(2)})$ .
  - 2  $\kappa_\tau(x|t, x_t, k, \theta_0^{(2)})$  with  $\hat{\kappa}_\tau(x|t, x_t, k, \theta_{LIML}^{(2)})$  using  $f_{jt}(x|x_t, \theta_{LIML}^{(2)})$ .
  - 3  $\psi_{1t}(x)$  with  $\hat{\psi}_{1t}(x)$  by substituting cell estimators  $\hat{p}_{jt}(x)$  for  $p_{jt}(x)$ .
  - 4  $v_{jt}(x, \theta^{(1)}, \theta_0^{(2)})$  with  $\hat{v}_{jt}(x, \theta^{(1)}, \theta_{LIML}^{(2)})$  for any given  $\theta^{(1)}$ , given below.
  - 5  $p_{jt}(x, \theta^{(1)}, \theta_0^{(2)})$  with  $\hat{p}_{jt}(x, \theta^{(1)}, \theta_{LIML}^{(2)})$  by substituting  $\hat{v}_{jt}(x, \theta^{(1)}, \theta_{LIML}^{(2)})$  for  $v_{jt}(x, \theta^{(1)}, \theta_0^{(2)})$  in ML estimator.

# Quasi Maximum Likelihood Estimation

Elaborating the last steps in QML estimation

- With respect to the last two steps:
  4. Appealing to the Representation theorem:

$$\hat{v}_{jt} \left( x, \theta^{(1)}, \theta_{LIML}^{(2)} \right) = u_{jt}(x, \theta^{(1)}) + \hat{h}_{jt}(x)$$

where the numeric *dynamic correction factor*  $\hat{h}_{kt}(x)$  is defined:

$$\hat{h}_{jt}(x) \equiv \sum_{\tau=t+1}^T \sum_{x_{\tau}=1}^X \beta^{\tau-t} \hat{\psi}_{1\tau}(x_{\tau}) \hat{\kappa}_{\tau-1}(x_{\tau} | t, x, j, \theta_{LIML}^{(2)})$$

5. In T1EV applications:

$$\hat{p}_{jt}(x, \theta^{(1)}, \theta_{LIML}^{(2)}) = \frac{\exp \left[ u_{jt}(x, \theta^{(1)}) + \hat{h}_{jt}(x) \right]}{\sum_{k=1}^J \exp \left[ u_{kt}(x, \theta^{(1)}) + \hat{h}_{kt}(x) \right]}$$

# Minimum Distance Estimators (Altug and Miller, 1998)

Minimizing the difference between unrestricted and restricted current payoffs

- Another approach is to match up the parametrization of  $u_{jt}(x_t)$ , denoted by  $u_{jt}(x_t, \theta^{(1)})$ , to its representation as closely as possible:

- 1 Form the vector function where  $\Psi(p, f)$  by stacking:

$$\begin{aligned}\Psi_{jt}(x_t, p, f) \equiv & \psi_{1t}(x_t) - \psi_{jt}(x_t) \\ & + \sum_{\tau=1}^{T-t} \sum_{x=1}^X \beta^\tau \psi_{1,t+\tau}(x) \begin{bmatrix} \kappa_{kt,\tau-1}(x|x_t) \\ -\kappa_{jt,\tau-1}(x|x_t) \end{bmatrix}\end{aligned}$$

- 2 Estimate the reduced form  $\hat{p}$  and  $\hat{f}$ .
- 3 Minimize the quadratic form to obtain:

$$\theta_{MD}^{(1)} = \arg \min_{\theta^{(1)} \in \Theta^{(1)}} \left[ u(x, \theta^{(1)}) - \Psi(\hat{p}, \hat{f}) \right]' \widetilde{W} \left[ u(x, \theta^{(1)}) - \Psi(\hat{p}, \hat{f}) \right]$$

where  $\widetilde{W}$  is a square  $(J-1)TX$  weighting matrix.

- Note  $\theta_{MD}^{(1)}$  has a closed form if  $u(x; \theta_0^{(1)})$  is linear in  $\theta_0^{(1)}$ .

# Simulated Moments Estimators

A simulated moments estimator (Hotz, Miller, Sanders and Smith, 1994)

- We could form a Methods of Simulated Moments (MSM) estimator from:
  - 1 Simulate a lifetime path from  $x_{nt_n}$  onwards for each  $j$ , using  $\hat{f}$  and  $\hat{p}$ .
  - 2 Obtain estimates of  $\hat{E} \left[ \epsilon_{jt} \mid d_{jt}^o = 1, x_t \right]$  from  $\hat{p}$ .
  - 3 Stitch together a simulated lifetime utility outcome from the  $j^{th}$  choice at  $t_n$  onwards for  $n$ , to form  $\hat{v}_{jt} \left( x_{nt_n}, \theta^{(1)}, \hat{f}, \hat{p} \right)$ .
  - 4 Form the  $J - 1$  dimensional vector  $l_n \left( x_{nt_n}; \theta^{(1)}, \hat{f}, \hat{p} \right)$  from:

$$l_{nj} \left( x_{nt_n}; \theta^{(1)}, \hat{f}, \hat{p} \right) \equiv \hat{v}_{jt_n} \left( x_{nt_n}, \theta^{(1)}, \hat{f}, \hat{p} \right) - \hat{v}_{Jt_n} \left( x_{nt_n}, \theta^{(1)}, \hat{f}, \hat{p} \right) + \hat{\psi}_{jt} (x_{nt_n}) - \hat{\psi}_{Jt} (x_{nt_n})$$

- 5 Given a weighting matrix  $W_S$  and an instrument vector  $z_n$  minimize:

$$N^{-1} \left[ \sum_{n=1}^N z_n l_n \left( x_{nt_n}; \theta^{(1)}, \hat{f}, \hat{p} \right) \right]' W_S \left[ \sum_{n=1}^N z_n l_n \left( x_{nt_n}; \theta^{(1)}, \hat{f}, \hat{p} \right) \right]$$