

# Estimating **STATIC** discrete-choice games of incomplete information and Multiplicity of equilibria

Bertel Schjerning, University of Copenhagen

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# Estimating Games of Incomplete Information



Pakes, Ostrovsky and Berry, 2007

Various 2-Step CCP estimators: pseudo-ML, MoM,  $\min \chi^2$



Aguirregabiria and Mira, 2007

Recursive 2-Step CCP estimator, Nested Pseudo-Likelihood (NPL)



Pesendorfer and Schmidt-Dengler, 2008

2-Step least squares



Kasahara and Shimotsu, 2012

Modified (dampened) NPL



Su, 2013 and Egedal, Lai and Su, 2015

Constrained optimization (MPEC)



Blevins and Dearing, 2025

Efficient Pseudo-Likelihood (EPL)

## Example: Static Entry Game

- ▶ Two firms:  $a$  and  $b$
- ▶ Actions: each firm has two possible actions:

$$d_a = \begin{cases} 1, & \text{if firm } \underline{a} \text{ choose to enter the market} \\ 0, & \text{if firm } \underline{a} \text{ choose not to enter the market} \end{cases} \quad (1)$$

$$d_b = \begin{cases} 1, & \text{if firm } \underline{b} \text{ choose to enter the market} \\ 0, & \text{if firm } \underline{b} \text{ choose not to enter the market} \end{cases} \quad (2)$$

## Example: Static Entry Game of Incomplete Information

Utility: Ex-post payoff to firms

$$u_a(d_a, d_b, x_a, \epsilon_a) = \begin{cases} [\alpha + d_b * (\beta - \alpha)]x_a + \epsilon_{a1}, & \text{if } d_a = 1 \\ 0 + \epsilon_{a0}, & \text{if } d_a = 0 \end{cases}$$
$$u_b(d_a, d_b, x_a, \epsilon_b) = \begin{cases} [\alpha + d_a * (\beta - \alpha)]x_b + \epsilon_{b1}, & \text{if } d_b = 1 \\ 0 + \epsilon_{b0}, & \text{if } d_b = 0 \end{cases}$$

- ▶  $(\alpha, \beta)$ : structural parameters to be estimated
- ▶  $(x_a, x_b)$ : firms' observed types; **common knowledge**
- ▶  $\epsilon_a = (\epsilon_{a0}, \epsilon_{a1})$ ,  $\epsilon_b = (\epsilon_{b0}, \epsilon_{b1})$ : firms' unobserved types, **private information**
- ▶  $(\epsilon_a, \epsilon_b)$  are observed only by each firm, but not by their opponent firm nor by the econometrician

## Example: Static Entry Game of Incomplete Information

- ▶ Assume the error terms  $(\epsilon_a, \epsilon_b)$  have EV Type I distribution
- ▶ A Bayesian Nash equilibrium  $(p_a, p_b)$  satisfies

$$\begin{aligned} p_a &= \frac{\exp[p_b \beta x_a + (1 - p_b) \alpha x_a]}{1 + \exp[p_b \beta x_a + (1 - p_b) \alpha x_a]} \\ &= \frac{1}{1 + \exp[-\alpha x_a + p_b x_a (\alpha - \beta)]} \\ &\equiv \Psi_a(p_b, x_a; \alpha, \beta) \end{aligned}$$

$$\begin{aligned} p_b &= \frac{1}{1 + \exp[-\alpha x_b + p_a x_b (\alpha - \beta)]} \\ &\equiv \Psi_b(p_a, x_b; \alpha, \beta) \end{aligned}$$

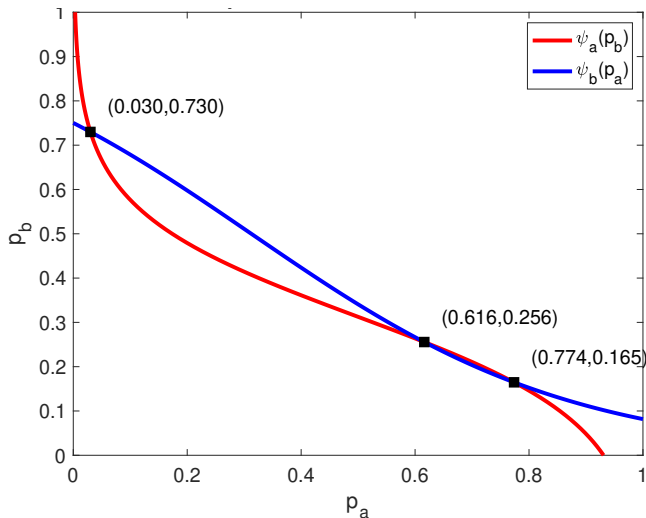
## Static Game Example: Parameters

We consider a very contestable game throughout

- ▶ Monopoly profits:  $\alpha * x_j = 5 * x_j$
- ▶ Duopoly profits:  $\beta * x_j = -11 * x_j$
- ▶ Firm types:  $(x_a, x_b) = (0.52, 0.22)$

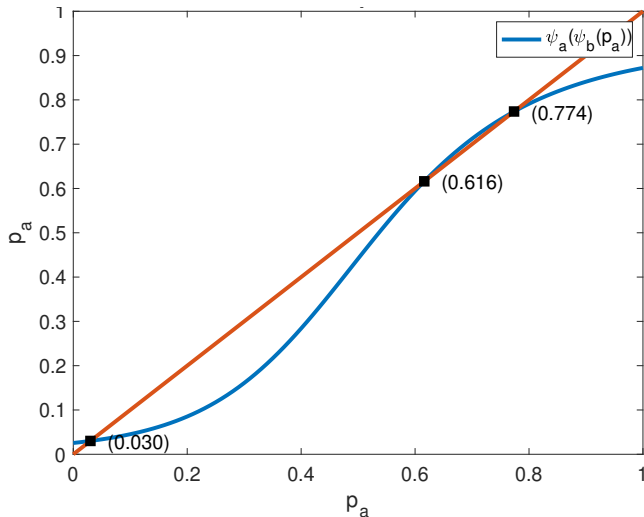
# Static Game Example: Three Bayesian Nash Equilibria

Figure: Equilibria at intersections of best response functions



# Static Game Example: Solving for Equilibria

Figure: Fixed points on second order best response function





# Static Game Example: Solving for Equilibria

**Solution method:** Combination of successive approximations and bisection algorithm

## Successive approximations (SA)

- ▶ Converge to nearest stable equilibrium.
- ▶ Start SA at  $p_a = 0$  and  $p_a = 1$ .
- ▶ Unique equilibrium ( $K=1$ ): SA will converge to it from anywhere.
- ▶ Three equilibria ( $K=3$ ): Two will be stable, and one will be unstable.
- ▶ More equilibria ( $K>3$ ): Not in this model.

## Bisection method

- ▶ Use this to find the unstable equilibrium (if  $K=3$ ).
- ▶ The bisection method that repeatedly bisects an interval and then selects a subinterval in which the fixed point (or root) must lie.
- ▶ The two stable equilibria, defines the initial interval to search over.
- ▶ The bisection method is a very simple and robust method, but it is also relatively slow.

# Static Game Example: Data Generation and Identification

- ▶ Data Generating Process (DGP): the data are generated by a single equilibrium
- ▶ The two players use the **same** equilibrium to play 1000 times
- ▶ Data:  $X = \{(d_a^i, d_b^i)_{i=1}^{1000}, (x_a, x_b) = (0.52, 0.22)\}$
- ▶ Given data  $X$ , we want to recover structural parameters  $\alpha$  and  $\beta$

# Static Game Example: Maximum Likelihood Estimation

- Maximize the likelihood function

$$\begin{aligned} \max_{\alpha, \beta} \quad & \log \mathcal{L}(p_a(\alpha, \beta); X) \\ = \quad & \sum_{i=1}^N (d_a^i * \log(p_a(\alpha, \beta)) + (1 - d_a^i) * \log(1 - p_a(\alpha, \beta))) \\ & + \sum_{i=1}^N (d_b^i * \log(p_b(\alpha, \beta)) + (1 - d_b^i) * \log(1 - p_b(\alpha, \beta))) \end{aligned}$$

- $p_a(\alpha, \beta)$  and  $p_b(\alpha, \beta)$  are the solutions of the Bayesian-Nash Equilibrium equations

$$\begin{aligned} p_a &= \frac{1}{1 + \exp[-\alpha x_a + p_b x_a (\alpha - \beta)]} \equiv \Psi_a(p_b, x_a; \alpha, \beta) \\ p_b &= \frac{1}{1 + \exp[-\alpha x_b + p_a x_b (\alpha - \beta)]} \equiv \Psi_b(p_a, x_b; \alpha, \beta) \end{aligned}$$

# Static Game Example: MLE via NFXP

- ▶ Outer Loop

- ▶ Choose  $(\alpha, \beta)$  to maximize the likelihood function

$$\log \mathcal{L}(p_a(\alpha, \beta), p_b(\alpha, \beta); X)$$

- ▶ Inner loop:

- ▶ For a given  $(\alpha, \beta)$ , solve the BNE equations for **ALL** equilibria:  
 $(p_a^k(\alpha, \beta), p_b^k(\alpha, \beta)), k = 1, \dots, K$
  - ▶ Choose the equilibrium that gives the highest likelihood value:

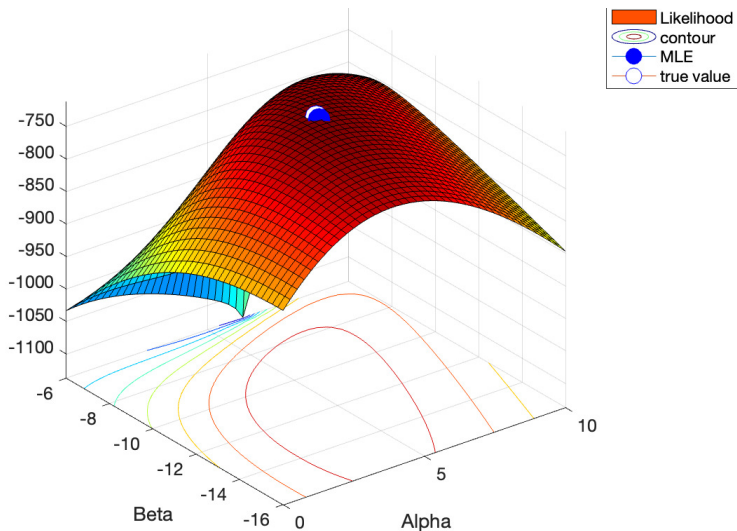
$$k^* = \arg \max_{k=1, \dots, K} \log \mathcal{L}(p_a^k(\alpha, \beta), p_b^k(\alpha, \beta); X)$$

such that

$$(p_a(\alpha, \beta), p_b(\alpha, \beta)) = (p_a^{k^*}(\alpha, \beta), p_b^{k^*}(\alpha, \beta))$$

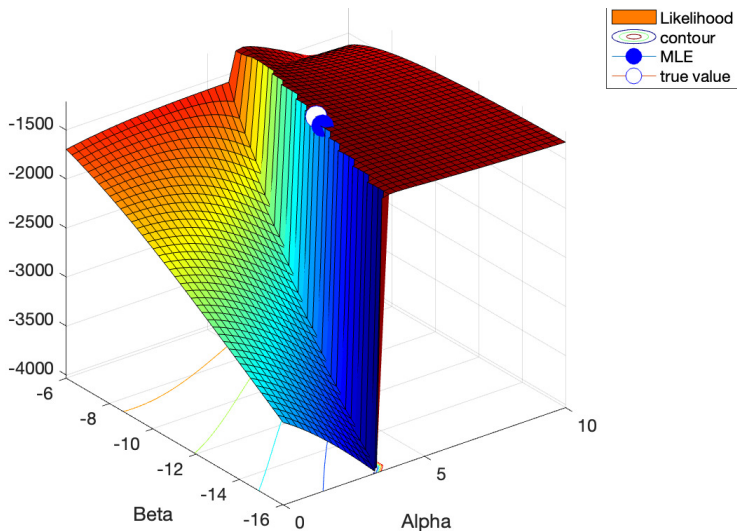
# NFXP's Likelihood as a Function of $(\alpha, \beta)$ - Eq 1

Figure: Data generated from equilibrium 1



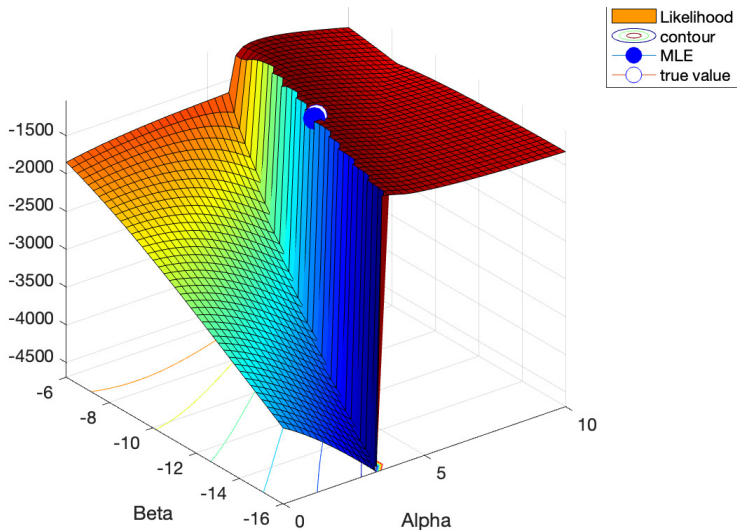
## NFXP's Likelihood as a Function of $(\alpha, \beta)$ - Eq 2

Figure: Data generated from equilibrium 2



## NFXP's Likelihood as a Function of $(\alpha, \beta)$ - Eq 3

Figure: Data generated from equilibrium 3



# Static Game Example: MLE via MPEC

- Maximize the likelihood function

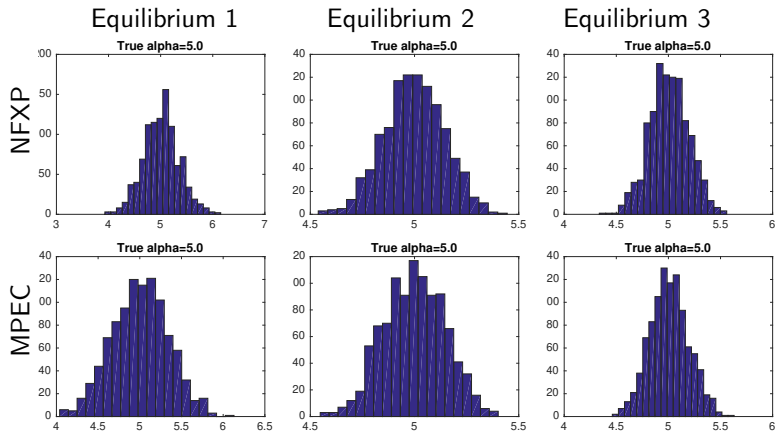
$$\begin{aligned} \max_{\alpha, \beta, p_a, p_b} \quad & \log \mathcal{L}(p_a; X) \\ = \quad & \sum_{i=1}^N (d_a^i * \log(p_a) + (1 - d_a^i) * \log(1 - p_a)) \\ & + \sum_{i=1}^N (d_b^i * \log(p_b) + (1 - d_b^i) * \log(1 - p_b)) \end{aligned}$$

- Subject to  $p_a$  and  $p_b$  are the solutions of the Bayesian-Nash Equilibrium equations

$$\begin{aligned} p_a &= \frac{1}{1 + \exp[-\alpha x_a + p_b x_a (\alpha - \beta)]} \\ p_b &= \frac{1}{1 + \exp[-\alpha x_b + p_a x_b (\alpha - \beta)]} \\ 0 &\leq p_a, p_b \leq 1 \end{aligned}$$



# Monte Carlo Results



Estimates of parameter  $\alpha$

Data generated from each of the three equilibria

# Least Square Estimators

Pesendofer and Schmidt-Dengler (2008)

- ▶ Step 1: Estimate  $\hat{p} = (\hat{p}_a, \hat{p}_b)$  from the data
- ▶ Step 2: Solve

$$\min_{\alpha, \beta} \{ (\hat{p}_a - \Psi_a(\hat{p}_b, x_a; \alpha, \beta))^2 + (\hat{p}_b - \Psi_b(\hat{p}_a, x_b; \alpha, \beta); X))^2 \}$$

For dynamic games, Markov perfect equilibrium conditions are characterized by

$$p = \Psi(p, \theta)$$

- ▶ Step 1: Estimate  $\hat{p}$  from the data
- ▶ Step 2: Solve

$$\min_{\alpha, \beta} [\hat{p} - \Psi(\hat{p}; \theta)]' W [\hat{p} - \Psi(\hat{p}; \theta)]'$$

## 2-Step Methods: Pseudo Maximum Likelihood

In 2-step methods

- ▶ Step 1: Estimate  $\hat{p} = (\hat{p}_a, \hat{p}_b)$  from the data
- ▶ Step 2: Solve

$$\max_{\alpha, \beta, p_a, p_b} \log \mathcal{L}(p_a, p_b; X)$$

subject to

$$\begin{aligned} p_a &= \Psi_a(\hat{p}_a, x_a; \alpha, \beta) \\ p_b &= \Psi_b(\hat{p}_b, x_b; \alpha, \beta) \\ 0 &\leq p_a^m, p_b^m \leq 1, m = 1, \dots, M \end{aligned}$$

Or equivalently

- ▶ Step 1: Estimate  $\hat{p} = (\hat{p}_a, \hat{p}_b)$  from the data
- ▶ Step 2: Solve

$$\max_{\alpha, \beta} \log \mathcal{L}(\Psi_a(\hat{p}_a, x_a; \alpha, \beta), \Psi_b(\hat{p}_b, x_b; \alpha, \beta); X)$$

# Nested Pseudo Likelihood (NPL)



Aguirregabiria and Mira (2007)

NPL iterates on the 2-step methods

1. Step 1: Estimate  $\hat{\rho}^0 = (\hat{\rho}_a^0, \hat{\rho}_b^0)$  from the data, set  $k = 0$
2. Step 2:

**REPEAT**

2.1 Solve

$$\alpha^{k+1}, \beta^{k+1} = \arg \max_{\alpha, \beta} \log \mathcal{L}(\Psi_a(\hat{\rho}_b^k, x_a; \alpha, \beta), \Psi_b(\hat{\rho}_a^k, x_b; \alpha, \beta); X)$$

2.2 One best-reply iteration on  $\hat{\rho}^k$

$$\begin{aligned}\hat{\rho}_a^{k+1} &= \Psi_a(\hat{\rho}_b^k, x_a; \alpha^{k+1}, \beta^{k+1}) \\ \hat{\rho}_b^{k+1} &= \Psi_b(\hat{\rho}_a^k, x_b; \alpha^{k+1}, \beta^{k+1})\end{aligned}$$

2.3 Let  $k := k+1$ ;

**UNTIL** convergence in  $(\alpha^k, \beta^k)$  and  $(\hat{\rho}_a^k, \hat{\rho}_b^k)$

## Some remarks

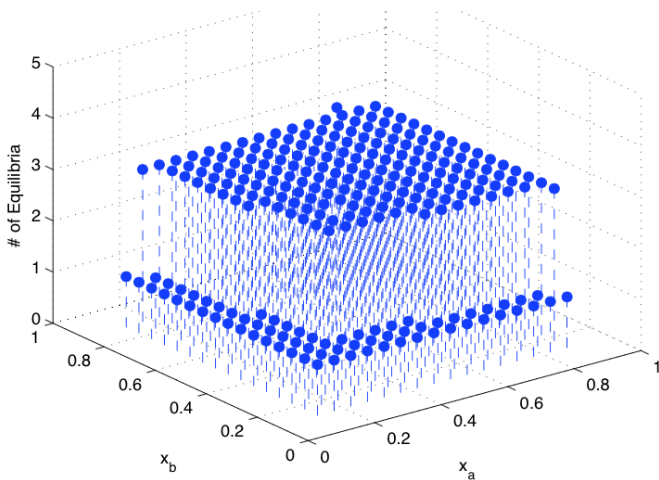
- ▶ NFXP/MPEC implementations of MLE is statistically efficient
  - ▶ NFXP computationally daunting
    - need to **compute ALL equilibria** for each  $\theta = (\alpha, \beta)$
    - find maxima of **discontinuous likelihood**
  - ▶ MPEC may be computational lighter
    - objective function and constraints are **smooth** in  $\alpha, \beta, p_a, p_b$
    - only need to satisfy equilibrium constraints in optimum
    - may get stuck in **local minima** (equilibria not played in the data).
  - ▶ MLE and Multiple equilibria
    - discontinuous likelihood function → non-standard inference
- ▶ Two step estimators
  - ▶ Computationally fast
  - ▶ Inefficient and biased in small samples (best response mapping not satisfied)
- ▶ NPL and EPL should bridge this gap
  - ▶ NPL can be unstable in games with multiple equilibria (NPL mapping not contraction)
  - ▶ EPL does not rely on contraction property, but still a iterative algorithm that depends on starting values and estimates of  $p_a, p_b$
- ▶ Problems amplified by orders of magnitude in dynamic games

## Estimation with Multiple Markets

- ▶ There are 25 different markets, i.e., 25 pairs of observed types  $(x_a^m, x_b^m)$ ,  $m = 1, \dots, 25$
- ▶ The grid on  $x_a$  has 5 points equally distributed between the interval  $[0.12, 0.87]$ , and similarly for  $x_b$
- ▶ Use the same true parameter values:  $(\alpha_0, \beta_0)$
- ▶ For each market with  $(x_a^m, x_b^m)$ , solve BNE conditions for  $(p_a^m, p_b^m)$ .
- ▶ There are multiple equilibria in most of 25 markets
- ▶ For each market, we (randomly) choose an equilibrium to generate  $T$  data points for that market
- ▶ The equilibrium used to generate data can be different in different markets - we flip a coin at each market.

# # of Equilibria with Different $(x_a^m, x_b^m)$

Figure: Number of equilibria



# NFXP - Estimation with Multiple Markets

Outer loop:

$$\max_{\alpha, \beta} \log \mathcal{L}(p_a^m(\alpha, \beta), p_b^m(\alpha, \beta); X)$$

Inner loop: For a given values of  $(\alpha, \beta)$  solve BNE equations for ALL equilibria,  $k = 1, \dots, K$  at each market,  $m = 1, \dots, M$ : That is,  $p_a^{m,k}(\alpha, \beta)$  and  $p_b^{m,k}(\alpha, \beta)$  are the solutions to

$$\begin{aligned} p_a^m &= \Psi_a(p_b^m, x_a^m; \alpha, \beta) \\ p_b^m &= \Psi_b(p_a^m, x_b^m; \alpha, \beta) \\ m &= 1, \dots, M \end{aligned}$$

where we again choose the equilibrium, that gives the highest likelihood value at each market  $m$

$$k^* = \arg \max_{k=1, \dots, K} \log \mathcal{L}(p_a^{m,k}(\alpha, \beta), p_b^{m,k}(\alpha, \beta); X)$$

such that

$$(p_a^m(\alpha, \beta), p_b^m(\alpha, \beta)) = (p_a^{m,k^*}(\alpha, \beta), p_b^{m,k^*}(\alpha, \beta))$$



# Estimation with Multiple Markets - MPEC

Constrained optimization formulation

$$\max_{\alpha, \beta, p_a^m, p_b^m} \log \mathcal{L}(p_a^m, p_b^m; X)$$

subject to

$$p_a^m = \Psi_a(p_b^m, x_a^m; \alpha, \beta)$$

$$p_b^m = \Psi_b(p_a^m, x_b^m; \alpha, \beta)$$

$$0 \leq p_a^m, p_b^m \leq 1,$$

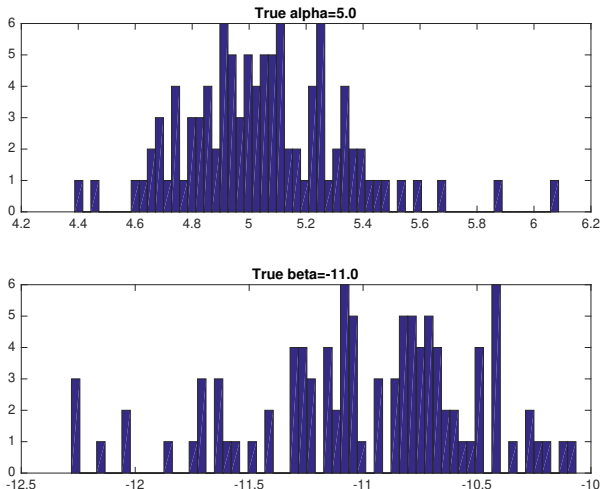
$$m = 1, \dots, M$$

- ▶ MPEC does not explicitly solve the BNE equations to find ALL equilibria at each market - for every trial value of parameters.
- ▶ But the number of parameters is much larger.
- ▶ Both MPEC and NFXP are based on Full Information Maximum Likelihood (FIML) estimators.

# NFXP: Monte Carlo - Multiple Markets (M=25, T=50)

Starting values  $\alpha_0 = \alpha$ ,  $\beta_0 = \beta$

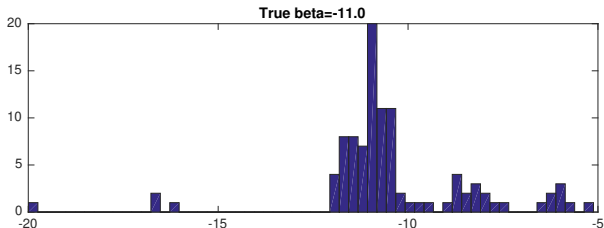
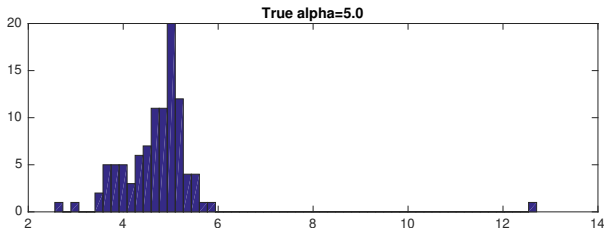
Random equilibrium selection in different markets



# MPEC: Monte Carlo - Multiple Markets (M=25, T=50)

Starting values  $\alpha_0 = \alpha$ ,  $\beta_0 = \beta$

Random equilibrium selection in different markets



# MPEC and NFXP: multiple markets

## NFXP:

- ▶ 2 parameters in optimization problem
- ▶ Needs to find ALL equilibria at each market (very hard in more complex problems)
- ▶ Good full solution methods required

## MPEC:

- ▶  $2 + 2M$  parameters in optimization problem
- ▶ Local minima with many markets:  
MPEC equilibrium constraints are satisfied and gradients of objective function is within tolerance - but wrong equilibrium selected