SEQUENTIAL ESTIMATION OF DYNAMIC DISCRETE CHOICE GAMES

LECTURE 10

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OUTLINE

1. **Structure of empirical dynamic games**

2. **Markov Perfect Equilibrium**

3. **Dynamic Games with Incomplete Information**

4. **Sequential Estimation**

   4.1. **NPL Estimator**

   4.2. **Alternative algorithms to compute NPL estimator**

      a. Fixed point iterations.

      b. Newton's method.

      c. Spectral method.

1. Structure of Dynamic Games

BASIC STRUCTURE

- Time is discrete and indexed by t.
- The game is played by N firms that we index by i.
- The action is taken to maximize the expected and discounted flow of profits in the market,

$$E_t \left(\sum_{s=0}^{\infty} \delta_i^s \; \pi_{it+s} \right)$$

  $\delta_i \in (0,1)$ is the discount factor, and π_{it} is firm i's profit at period t.

- Every period t, firms make a investment/dynamic decision: a_{it}.
- Here I focus on discrete choice games: $a_{it} \in \mathcal{A} = \{0, 1, ..., J\}$

DECISIONS, STATES, and PROFITS

- Current profit π_{it} depends on the firms's own action a_{it}, other firms' actions, $\boldsymbol{a}_{-it} = \{a_{jt} : j \neq i\}$, and a vector of state variables $\boldsymbol{x}_t$.

$$\pi_{it} = \pi_i \left(a_{it}, \boldsymbol{a}_{-it}, \boldsymbol{x}_t \right)$$

- $\boldsymbol{x}_t$ includes:

  a. Endogenous state variables that depend on the firms' investment decisions in previous periods, e.g., capital stocks.

  b. Exogenous state variables affecting costs and consumer demand.

EXAMPLE: DYNAMIC COMPETITION IN PRODUCT QUALITY

- Each firm has a **differentiated product**. Consumer demand depends on products' qualities (k_{it}) and prices (p_{it}).
- State $\boldsymbol{x}_t$ consists of:
  a. Endogenous product qualities: $\boldsymbol{k}_t = (k_{1t}, k_{2t}, ..., k_{Nt})$.
  b. Exogenous variables affecting demand or costs: $\boldsymbol{z}_t = (z_{1t}, z_{2t}, ..., z_{Nt})$.
- Given $\boldsymbol{x}_t$, firms' compete in prices a la Bertrand, and this determines **Bertrand equilibrium** variable profits for each firm: $r_i(\boldsymbol{x}_t)$.
- The total profit, π_{it}, consists on $r_i(\boldsymbol{x}_t)$ minus the cost of investing in quality improvement: $IC_i(a_{it}, k_{it})$:

$$\pi_{it} = r_i(\boldsymbol{x}_t) - IC_i(a_{it}, k_{it})$$

- Quality stock evolves endogeneously according to the transition rule:

$$k_{i,t+1} = k_{it} + a_{it}$$

EVOLUTION OF THE STATE VARIABLES

- **Exogenous common knowledge state variables:** follow an exogenous Markov process with transition probability function $f_z(\mathbf{z}_{t+1}|\mathbf{z}_t)$.

- **Endogenous state variables:** The form of the transition rule depends on the application:
  - Market entry: $k_{it} = a_{it-1}$, such that $k_{i,t+1} = a_{it}$
  - Investment without depreciation: $k_{i,t+1} = k_{it} + a_{it}$.
  - Investment - deterministic depreciation: $k_{i,t+1} = \lambda(k_{it} + a_{it})$
  - Investment - stochastic depreciation: $k_{i,t+1} = k_{it} + a_{it} - \xi_{i,t+1}$
- In a compact way, we use $f_x(\mathbf{x}_{t+1}|\mathbf{a}_t, \mathbf{x}_t)$ to represent the transition probability function of all the state variables.

2. Markov Perfect Equilibrium

OPTIMAL DECISION RULE IN THE DYNAMIC GAME

- Suppose that firm i believes that the other firms in the market behave – now and in the future – according to the strategy function:

$$\alpha_j(\mathcal{I}_t) \text{ for any } j \neq i$$

where $\mathcal{I}_t$ is a particular specification of the information used by firms at period t. More specifically:

$$\mathcal{I}_t = \left(\mathbf{x}_t,\ \boldsymbol{a}_{t-1},\ \mathbf{x}_{t-1}, ...,\ \boldsymbol{a}_{t-p},\ \mathbf{x}_{t-p}\right)$$

- Given these beliefs $\boldsymbol{\alpha}_{-i}$, firm i has the following the payoff:

$$\pi_i^{\boldsymbol{\alpha}}(a_{it}, \mathbf{x}_t) \quad = \quad \pi_i\left(a_{it},\ \boldsymbol{\alpha}_{-i}(\mathbf{x}_t),\ \mathbf{x}_t\right)$$

- and the transition probability for the state variables:

$$\boldsymbol{\alpha}_{-i}(\mathcal{I}_t) \ \&\ f_x(\mathbf{x}_{t+1}|a_{it}, \boldsymbol{\alpha}_{-i}(\mathcal{I}_t), \mathbf{x}_t) \quad \Rightarrow \quad f_{\mathcal{I},i}^{\boldsymbol{\alpha}}(\mathcal{I}_{t+1}|a_{it}, \mathcal{I}_t)$$

BEST RESPONSE & NASH EQUILIBRIUM

- Given $\pi_i^{\boldsymbol{\alpha}}(a_{it}, \mathbf{x}_t)$ and $f_{\mathcal{I},i}^{\boldsymbol{\alpha}}(\mathcal{I}_{t+1}|a_{it}, \mathcal{I}_t)$, we can define the best response of firm i as the solution to the single-agent DP problem defined by this Bellman equation:

$$V_i^{\boldsymbol{\alpha}}(\mathcal{I}_t) = \max_{a_{it}} \left\{ \pi_i^{\boldsymbol{\alpha}}(a_{it}, \mathbf{x}_t) + \delta_i \int V_i^{\boldsymbol{\alpha}}(\mathcal{I}_{t+1}) \ f_{\mathcal{I},i}^{\boldsymbol{\alpha}}(\mathcal{I}_{t+1}|a_{it}, \mathcal{I}_t) \right\}$$

- Let $BR_i(\boldsymbol{\alpha}_{-i})$ be the optimal strategy function that solves this DP problem. It is a best response to the beliefs $\boldsymbol{\alpha}_{-i}$.

- A **Nash Equilibrium** of this dynamic game consists of an N-tuple of strategy functions $\{\alpha_i(\mathcal{I}_t) : i = 1, 2, ..., N\}$ such that, for every firm i:

$$\alpha_i = BR_i(\boldsymbol{\alpha}_{-i})$$

That is:

1. Every firm behaves according to its best response strategy.
2. Beliefs are rational, i.e., the actual firms' strategies in equilibrium.

MARKOV PERFECT EQUILIBRIUM

- The previous definition of Nash Equilibrium depends on the choice of the information set $\mathcal{I}_t$. We have as many types of NE as possible selections of $\mathcal{I}_t$.

- Most dynamic IO models assume Markov Perfect Equilibrium (MPE), (Maskin & Tirole, ECMA 1988; Ericson & Pakes, REStud 1995).

- This solution concept corresponds to NE when **players' strategies are functions of only payoff-relevant state variables**, $\mathcal{I}_t = \mathbf{x}_t$.

- **Why this restriction?**:
  - **Rationality (Maskin & Tirole):** if other players use this type of strategies, a player cannot make higher payoff by conditioning its behavior on non-payoff relevant information (e.g., lagged values of the state variables)

  - **Dimensionality:** It is convenient because it reduces the dimensionality of the state space.

MARKOV PERFECT EQUILIBRIUM – DEFINITION

- Let $\boldsymbol{\alpha} = \{\alpha_i(\mathbf{x}_t) : i = 1, 2, ..., N\}$ be a set of strategy functions.
- A MPE is an N-tuple of strategy functions $\boldsymbol{\alpha}$ such that every firm is maximizing its value given the strategies of the other players.
- For given strategies of the other firms, the decision problem of a firm is a single-agent dynamic programming (DP) problem.

MARKOV PERFECT EQUILIBRIUM: Best Response DP

- Let $V_i^{\boldsymbol{\alpha}}(\mathbf{x}_t)$ be the value function of the DP problem that describes the best response of firm i to the strategies of the other firms in $\boldsymbol{\alpha}$.

- This value function is the unique solution to the Bellman equation:

$$V_i^{\boldsymbol{\alpha}}(\mathbf{x}_t) = \max_{a_{it}} \left\{ \pi_i^{\boldsymbol{\alpha}}(a_{it}, \mathbf{x}_t) + \delta_i \int V_i^{\boldsymbol{\alpha}}(\mathbf{x}_{t+1}) \; f_{x,i}^{\boldsymbol{\alpha}}(\mathbf{x}_{t+1} | a_{it}, \mathbf{x}_t) \; d\mathbf{x}_{t+1} \right\}$$

- with (here, I consider there is no time-to-build):

$$\pi_i^{\boldsymbol{\alpha}}(a_{it}, \mathbf{x}_t) = \pi_i \left(a_{it}, \; \boldsymbol{\alpha}_{-i}(\mathbf{x}_t), \; \mathbf{x}_t \right)$$

- and:

$$f_{x,i}^{\boldsymbol{\alpha}}(\mathbf{x}_{t+1} | a_{it}, \mathbf{x}_t) = f_x(\mathbf{x}_{t+1} \mid a_{it}, \; \boldsymbol{\alpha}_{-i}(\mathbf{x}_t), \; \mathbf{x}_t)$$

MPE — EXISTENCE

- Doraszelski & Satterhwaite (RAND, 2010) show that **existence** of a MPE in pure strategies **is not guaranteed** in this model **when the choice set for a_{it} is discrete**.

- A possible approach to guarantee existence is to allow for mixed strategies. However, computing a MPE in mixed strategies poses important computational challenges.

- To establish existence, Doraszelski & Satterhwaite (RAND, 2010) propose **incorporating private information state variables**.

- This incomplete information version of Ericson-Pakes model has been the one adopted in most **empirical applications**.
  - The main reason is that – as we illustrate below – i.i.d. private information shocks are very **convenient type of unobservables from an econometric point of view**.

3. Dynamic Games with Incomplete Information

PRIVATE INFORMATION SHOCKS

- State variables in $\mathbf{x}_t$ are known to all the firms in the market at period t (common knowledge).

- In addition, a firm's investment cost function $IC_i(.)$ depends on a vector of state variables ε_{it} with two properties:

  1. ε_{it} is **private information of firm i**. It is unknown to the other firms.

  2. ε_{it} is **i.i.d. over time and independent across firms** with CDF G_i that has full support on $\mathbb{R}^{|A|}$.

- Strategy functions are now $\alpha_i(\mathbf{x}_t, \varepsilon_{it})$.

- MPE has the same definition as above but with strategies $\alpha_i(\mathbf{x}_t, \varepsilon_{it})$.

CONDITIONAL CHOICE PROBABILITIES

- It is very convenient to represent a firm's strategy using **Conditional Choice Probability (CCP) function**. For any value $(a, \mathbf{x})$:

$$P_i(a|\mathbf{x}) \equiv \Pr\left(\alpha_i(\mathbf{x}_t, \varepsilon_{it}) = a \mid \mathbf{x}_t = \mathbf{x}\right)$$

- Since function P_i results from integrating function α_i over the continuous variables in ε_{it}, P_i is a lower dimensional object than α_i.

- In discrete choice games with $\varepsilon_{it}(a_{it})$ entering additively in the profit function, there is a **one-to-one relationship** between best-response strategy functions $\alpha_i(\mathbf{x}_t, \varepsilon_{it})$ and its CCP function $P_i(.|\mathbf{x}_t)$.

- It is obvious that given $\alpha_i(\mathbf{x}_t, \varepsilon_{it})$ there is a unique $P_i(.|\mathbf{x}_t)$.

- The inverse relationship – given $P_i(.|\mathbf{x}_t)$ there is a unique best response function $\alpha_i(\mathbf{x}_t, \varepsilon_{it})$ – is a corollary of **Hotz-Miller inversion Theorem**.

MPE as FIXED POINT OF a MAPPING IN CCPs

- Given strategy functions described by CCP functions $\mathbf{P}$, we can define **expected profit** $\pi_i^{\mathbf{P}}$ and **expected transition** $f_i^{\mathbf{P}}$ as:

$$\pi_i^{\mathbf{P}}(a_{it}, \mathbf{x}_t) = \sum_{a_{-it}} \left[\prod_{j \neq i} P_j \left(a_{jt} \mid \mathbf{x}_t \right) \right] \pi_i \left(a_{it}, \boldsymbol{a}_{-it}, \mathbf{x}_t \right)$$

$$f_i^{\mathbf{P}}(\mathbf{x}_{t+1} | a_{it}, \mathbf{x}_t) = \sum_{a_{-it}} \left[\prod_{j \neq i} P_j \left(a_{jt} \mid \mathbf{x}_t \right) \right] f_x(\mathbf{x}_{t+1} | a_{it}, \boldsymbol{a}_{-it}, \mathbf{x}_t)$$

- We also define **expected conditional-choice values**:

$$v_i^{\mathbf{P}}(a_{it}, \mathbf{x}_t) \equiv \pi_i^{\mathbf{P}}(a_{it}, \mathbf{x}_t) + \delta \int V_i^{\mathbf{P}}(\mathbf{x}_{t+1}) \; f_i^{\mathbf{P}}(\mathbf{x}_{t+1} | a_{it}, \mathbf{x}_t) \; d\mathbf{x}_{t+1}$$

- with:

$$V_i^{\mathbf{P}}(\mathbf{x}_t) = \int \max_{a_{it}} \left\{ v_i^{\mathbf{P}}(a_{it}, \mathbf{x}_t) + \varepsilon_{it}(a_{it}) \right\} dG_i(\varepsilon_{it})$$

MPE as FIXED POINT OF a MAPPING IN CCPs [2]

- A MPE is a vector of CCPs, $\mathbf{P} \equiv \{P_i(a_i|\mathbf{x}) : \text{for any } (i, a_i, \mathbf{x})\}$, such that, for any $(i, a, \mathbf{x})$:

$$P_i(a_i|\mathbf{x}) = \Pr\left(a_i = \arg\max_{a'}\left\{v_i^{\mathbf{P}}(a', \mathbf{x}) + \varepsilon_i(a')\right\} \mid \mathbf{x}\right)$$

- This system of equations defines a Fixed Point mapping from the space of CCPs $\mathbf{P}$ into itself:

$$\mathbf{P} = \Psi(\mathbf{P})$$

- Mapping $\Psi(.)$ is continuous. Therefore, **by Brower's Fixed Point Theorem an equilibrium exists**.

- In general, this model has multiple equilibria.

MPE IN TERMS OF CCPs: AN EXAMPLE

- Suppose that vector ε_{it}'s are iid Extreme Value Type I.

- Then, a MPE is a vector $\mathbf{P} \equiv \{P_i(a|\mathbf{x}) : \text{for any } (i, a, \mathbf{x})\}$, such that:

$$P_i(a|\mathbf{x}) = \frac{\exp\left\{v_i^{\mathbf{P}}(a, \mathbf{x})\right\}}{\sum_{a'} \exp\left\{v_i^{\mathbf{P}}(a', \mathbf{x})\right\}}$$

- where

$$v_i^{\mathbf{P}}(a_{it}, \mathbf{x}_t) \equiv \pi_i^{\mathbf{P}}(a_{it}, \mathbf{x}_t) + \delta \sum_{\mathbf{x}_{t+1}} V_i^{\mathbf{P}}(\mathbf{x}_{t+1})\ f_i^{\mathbf{P}}(\mathbf{x}_{t+1}|a_{it}, \mathbf{x}_t)$$

- and $V_i^{\mathbf{P}}$ is the unique solution to the Bellman equation:

$$V_i^{\mathbf{P}}(\mathbf{x}_t) = \ln\left(\sum_{a_i} \exp\left\{\pi_i^{\mathbf{P}}(a_{it}, \mathbf{x}_t) + \delta \sum_{\mathbf{x}_{t+1}} V_i^{\mathbf{P}}(\mathbf{x}_{t+1})\ f_i^{\mathbf{P}}(\mathbf{x}_{t+1}|a_{it}, \mathbf{x}_t)\right\}\right)$$

4. SEQUENTIAL ESTIMATION

ESTIMATION – PRELIMINARIES

- Primitives of the model: $\{\pi_i,\ \beta_i,\ f_x,\ G_\varepsilon\}$, can be described in terms of a vector of parameters $\boldsymbol{\theta}$ that is unknown to the researcher.

- It is convenient to distinguish four sub-vectors in $\boldsymbol{\theta}$, $(\boldsymbol{\theta}_\pi,\ \boldsymbol{\theta}_f,\ \boldsymbol{\beta},\ ,\boldsymbol{\theta}_\varepsilon)$.

- In most empirical applications, the main challenge is in the **estimation of "dynamic parameters" in $\boldsymbol{\theta}_\pi$**:

  - $\boldsymbol{\theta}_f$ can be estimated "outside" of the dynamic decision model.

  - Consumer demand and firms' variable costs – which are part of $\boldsymbol{\theta}_\pi$ – can be estimated "outside" of the dynamic decision model.

  - Most applications assume that $\boldsymbol{\theta}_\varepsilon$ (distribution of ε) and $\boldsymbol{\beta}$ are known.

  - Often, the focus in the estimation of the dynamic game is parameters capturing dynamics, i.e., investment costs, entry/exit costs, fixed costs.

OUTLINE ON ESTIMATION

1. Maximum Likelihood Est. (MLE) of models with **unique equilibrium**
    - Rust's **Nested Fixed Point (NFXP)** algorithm.
2. Maximum Likelihood Est. (MLE) of models with **multiple equilibria**
3. Sequential **CCP methods**

4.1. MLE WITH UNIQUE EQUILIBRIUM

MLE: MODELS WITH UNIQUE EQUILIBRIUM

- There exist sufficient conditions implying that a dynamic game has a unique equilibrium for every possible value of the parameters $\boldsymbol{\theta}$.
- An example of sufficient conditions for equilibrium uniqueness are:
  - i. Finite horizon T.
  - ii. Within every period t, firms make decisions sequentially: firm 1 first, firm 2 second, ..., firm N last. These decisions become common knowledge to the firms later in the sequence.
- Let $P_{it}(a_{it} \mid \mathbf{x}_t, \boldsymbol{\theta})$ be the equilibrium CCP function for firm i at period t when the vector of parameters is $\boldsymbol{\theta}$.
- The **full log-likelihood function** is: $\ell(\boldsymbol{\theta}) = \sum_{m=1}^{M} \ell_m(\boldsymbol{\theta})$, where $\ell_m(\boldsymbol{\theta})$ is the contribution of market m:

$$\ell_m(\boldsymbol{\theta}) = \sum_{i=1}^{N}\sum_{t=1}^{T} \log P_{it}(a_{imt}|\mathbf{x}_{mt}, \boldsymbol{\theta}) + \log f_x(\mathbf{x}_{m,t+1}|\boldsymbol{a}_{mt}, \mathbf{x}_{mt}, \boldsymbol{\theta}_f)$$

NESTED FIXED POINT (NFXP) ALGORITHM

- The MLE is: $\widehat{\boldsymbol{\theta}} = argmax_{\boldsymbol{\theta}} \; \ell(\boldsymbol{\theta})$.

- Rust's NFXP algorithm is a method to compute the MLE. It combines BHHH iterations (**outer algorithm**) with equilibrium solution algorithm (**inner algorithm**) for each trial value $\boldsymbol{\theta}$.

1. Start at an initial guess: $\widehat{\boldsymbol{\theta}}_0$.

2. At every **outer iteration** k, apply a BHHH iteration:

$$\widehat{\boldsymbol{\theta}}_{k+1} = \widehat{\boldsymbol{\theta}}_k + \left(\sum_{m=1}^{M} \frac{\partial \ell_m(\widehat{\boldsymbol{\theta}}_k)}{\partial \boldsymbol{\theta}} \frac{\partial \ell_m(\widehat{\boldsymbol{\theta}}_k)}{\partial \boldsymbol{\theta}'} \right)^{-1} \left(\sum_{m=1}^{M} \frac{\partial \ell_m(\widehat{\boldsymbol{\theta}}_k)}{\partial \boldsymbol{\theta}} \right)$$

3. The score vector $\partial \ell_m(\widehat{\boldsymbol{\theta}}_k)/\partial \boldsymbol{\theta}$ depends on $\partial \log P_i(a_{imt}|\mathbf{x}_{mt}, \widehat{\boldsymbol{\theta}}_k)/\partial \boldsymbol{\theta}$. To obtain these derivatives, the **inner algorithm** solves for the equilibrium CCPs given $\widehat{\boldsymbol{\theta}}_k$ using fixed point iterations.

4. Outer BHHH iterations until $||\widehat{\boldsymbol{\theta}}_{k+1} - \widehat{\boldsymbol{\theta}}_k|| <$ small constant

4.2. MLE WITH MULTIPLE EQUILIBRIA

(a) NFXP with Multiple Equilibria

(b) MPEC

NFXP WITH MULTIPLE EQUILIBRIA

- Let $\mathcal{P}(\boldsymbol{\theta})$ the set of regular MPE associated with a value $\boldsymbol{\theta}$ of the structural parameters.

- Doraszelski & Satterwaite (2011) show that for every value of $\boldsymbol{\theta}$ the set $\mathcal{P}(\boldsymbol{\theta})$ is discrete and finite.

$$\mathcal{P}(\boldsymbol{\theta}) = \{\boldsymbol{P}^{\tau}(\boldsymbol{\theta}) : \ \tau = 1, 2, ..., \mathcal{T}\}$$

- Suppose that the model has a structure such that we have an algorithm to compute the equilibrium set $\mathcal{P}(\boldsymbol{\theta})$ for any trial value of $\boldsymbol{\theta}$.

- For instance, the Recursive Lexicographic Search (RLS) algorithm in Iskhakov, Rust, & Schjerning (2016).

- Then, the MLE is defined as:

$$(\widehat{\boldsymbol{\theta}}_{MLE}, \hat{\tau}_{MLE}) = argmax_{\boldsymbol{\theta}, \tau \in \mathcal{P}(\boldsymbol{\theta})} \sum_{i=1}^{N} \sum_{t=1}^{T} \log P_{it}^{\tau}(a_{imt} | \mathbf{x}_{mt}, \boldsymbol{\theta})$$

NFXP WITH MULTIPLE EQUILIBRIA (2)

- The NFXP algorithm proceeds as follows.

1. Start at initial guess, $\boldsymbol{\theta}^0$.
2. **Inner Iteration-S - Solution:** At iteration $n+1$, given $\boldsymbol{\theta}^n$, apply RLS algorithm to find the set of equilibria $\mathcal{P}(\boldsymbol{\theta}^n)$.
3. **Inner Iteration-M - Max in τ:** Maximize in τ: Select the equilibrium type $\tau^*(\boldsymbol{\theta}^n)$ with the largest value of the likelihood given $\boldsymbol{\theta}^n$.
4. **Outer Iteration - BHHH:** Given the equilibrium-specific log-likelihood function $\ell^{\tau^*(\boldsymbol{\theta}^n)}(\boldsymbol{\theta})$, apply BHHH algorithm to obtain new $\boldsymbol{\theta}^{n+1}$.
5. Iterate until $||\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}^n|| < cconv$.

MPEC WITH MULTIPLE EQUILIBRIA

- With Multiple Equilibria, $\ell(\boldsymbol{\theta})$ is not a function but a correspondence. The MLE cannot be defined as the $argmax$ of $\ell(\boldsymbol{\theta})$.

- To define the MLE in a model with multiple equilibria, it is convenient to define an *extended* or **Pseudo Likelihood function**.

- For arbitrary values of $\boldsymbol{\theta}$ and firms' CCPs $\mathbf{P}$, define:

$$Q(\boldsymbol{\theta}, \mathbf{P}) = \sum_{m=1}^{M} \sum_{i=1}^{N} \sum_{t=1}^{T} \log \Psi_i(a_{imt} \mid \mathbf{x}_{mt}, \boldsymbol{\theta}, \mathbf{P})$$

where Ψ_i is the *best response probability function*.

MPEC WITH MULTIPLE EQUILIBRIA [2]

- The MLE is the pair $(\widehat{\boldsymbol{\theta}}_{MLE}, \widehat{\mathbf{P}}_{MLE})$ that **maximizes Q subject to the constraint that CCPs are equilibrium strategies**:

$$(\widehat{\boldsymbol{\theta}}_{MLE}, \widehat{\mathbf{P}}_{MLE}) = \begin{cases} \arg\max\limits_{(\boldsymbol{\theta},\mathbf{P})} & Q(\boldsymbol{\theta}, \mathbf{P}) \\ & \\ \text{subject to:} & \mathbf{P} = \Psi(\boldsymbol{\theta}, \mathbf{P}) \end{cases}$$

- Or using the Lagrangian function:

$$(\widehat{\boldsymbol{\theta}}_{MLE}, \widehat{\mathbf{P}}_{MLE}, \widehat{\boldsymbol{\lambda}}_{MLE}) = \arg\max_{(\boldsymbol{\theta},\mathbf{P},\boldsymbol{\lambda})} \; Q(\boldsymbol{\theta}, \mathbf{P}) + \boldsymbol{\lambda}' \left[\mathbf{P} - \Psi(\boldsymbol{\theta}, \mathbf{P}) \right]$$

- The F.O.C. are the Lagrangian equations:

$$\begin{cases} \widehat{\mathbf{P}}_{MLE} - \Psi(\widehat{\boldsymbol{\theta}}_{MLE}, \widehat{\mathbf{P}}_{MLE}) & = & \mathbf{0} \\ \nabla_{\boldsymbol{\theta}} Q(\widehat{\boldsymbol{\theta}}_{MLE}, \widehat{\mathbf{P}}_{MLE}) - \widehat{\boldsymbol{\lambda}}'_{MLE} \; \nabla_{\boldsymbol{\theta}} \Psi(\widehat{\boldsymbol{\theta}}_{MLE}, \widehat{\mathbf{P}}_{MLE}) & = & \mathbf{0} \\ \nabla_{\mathbf{P}} Q(\widehat{\boldsymbol{\theta}}_{MLE}, \widehat{\mathbf{P}}_{MLE}) - \widehat{\boldsymbol{\lambda}}'_{MLE} \; \nabla_{\mathbf{P}} \Psi(\widehat{\boldsymbol{\theta}}_{MLE}, \widehat{\mathbf{P}}_{MLE}) & = & \mathbf{0} \end{cases}$$

MPEC WITH MULTIPLE EQUILIBRIA [3]

- A Newton method can be used to obtain a root of this system of Lagrangian equations.
- A key computational problem is the **very high dimensionality of this system of equations**.
- The most costly part of this algorithm is the **calculation of the Jacobian matrix $\nabla_{\mathbf{P}}\Psi(\widehat{\theta},\widehat{\mathbf{P}})$**. In dynamic games, in general, this is not a sparse matrix, and can contain billions or trillions of elements.
- The evaluation of the best response mapping $\Psi(\theta,\mathbf{P})$ for a new value of $\mathbf{P}$ requires solving for a valuation operator and solving a system of equations with the same dimension as $\mathbf{P}$.

4.3. SEQUENTIAL CCP METHODS

TWO-STEP CCP METHODS

- Methods that avoid solving for firms' best responses or an equilibrium, even once.

- **Hotz & Miller (REStud, 1993)** was a seminal contribution on this class of methods. They show that the conditional choice values can be written as known functions of CCPs, transition probabilities, and $\boldsymbol{\theta}$.

- Suppose that one-period profit is linear-in-parameters:

$$\pi_i(a_{it}, \boldsymbol{a}_{-it}, \mathbf{x}_t) \;=\; h(a_{it}, \boldsymbol{a}_{-it}, \mathbf{x}_t)' \; \boldsymbol{\theta}_{\pi,i}$$

where $h(a_{it}, \boldsymbol{a}_{-it}, \mathbf{x}_t)$ is a vector of known functions to the researcher.

- The conditional-choice value function $v_i^{\mathbf{P}}(a_{it}, \mathbf{x}_t)$ is:

$$v_i^{\mathbf{P}}(a_{it}, \mathbf{x}_t) = \mathbb{E}\left(\sum_{j=0}^{\infty} \beta^j \; h(\boldsymbol{a}_{t+j}, \mathbf{x}_{t+j})' \; \boldsymbol{\theta}_{\pi,i} + \varepsilon_{i,t+j}(a_{i,t+j}) \mid a_{it}, \mathbf{x}_t \right)$$

where future actions, $\boldsymbol{a}_{t+j}$, are taken according to equilibrium CCPs.

TWO-STEP CCP METHODS [2]

- We can write:

$$v_i^{\mathbf{P}}(a_{it}, \mathbf{x}_t) \;=\; \widetilde{h}_i^{\mathbf{P}}(a_{it}, \mathbf{x}_t)\; \boldsymbol{\theta}_{\pi,i} \;+\; \widetilde{e}_i^{\mathbf{P}}(a_{it}, \mathbf{x}_t)$$

with:

$$\widetilde{h}_i^{\mathbf{P}}(a_{it}, \mathbf{x}_t) \;=\; \mathbb{E}\left(\sum_{j=0}^{\infty} \beta^j \; h(\boldsymbol{a}_{t+j}, \mathbf{x}_{t+j}) \mid a_{it}, \mathbf{x}_t \right)$$

$$\widetilde{e}_i^{\mathbf{P}}(a_{it}, \mathbf{x}_t) \;=\; \mathbb{E}\left(\sum_{j=0}^{\infty} \beta^j \; [\gamma - \ln P_i(a_{i,t+j} | \mathbf{x}_{t+j})] \mid a_{it}, \mathbf{x}_t \right)$$

- Given firms' equilibrium CCPs, P, β, and the transition probability of $\mathbf{x}$, we can calculate these present values using, for instance, forward Monte Carlo Simulation.

TWO-STEP CCP METHODS [3]

- Given this representation of conditional choice values, the pseudo likelihood function $Q(\boldsymbol{\theta}, \mathbf{P})$ has practically the same structure as in a static or reduced form discrete choice model.

- Best response probabilities that enter in $Q(\boldsymbol{\theta}, \mathbf{P})$ can be seen as the choice probabilities in a standard random utility model:

$$\Psi_i(a_{imt} = j | \mathbf{x}_{mt}, \boldsymbol{\theta}, \mathbf{P}) = \frac{\exp\{\widetilde{h}_i^{\mathbf{P}}(j, \mathbf{x}_{mt}) \ \boldsymbol{\theta}_i + \widetilde{e}_i^{\mathbf{P}}(j, \mathbf{x}_{mt})\}}{\sum_{k=0}^{J} \exp\{\widetilde{h}_i^{\mathbf{P}}(k, \mathbf{x}_{mt}) \ \boldsymbol{\theta}_i + \widetilde{e}_i^{\mathbf{P}}(k, \mathbf{x}_{mt})\}}$$

- Given $\widetilde{h}_i^{\mathbf{P}}(., \mathbf{x}_{mt})$ and $\widetilde{e}_i^{\mathbf{P}}(., \mathbf{x}_{mt})$ and a parametric specification for the distribution of ε (e.g., logit, probit), the vector of parameters $\boldsymbol{\theta}_i$ can be estimated as in a standard logit or probit model.

TWO-STEP CCP METHODS [3]

- The method proceeds in two steps.
- Let $\widehat{\mathbf{P}}^0$ be a consistent nonparametric estimator of true $\mathbf{P}^0$. The two-step estimator of $\boldsymbol{\theta}$ is defined as:

$$\widehat{\boldsymbol{\theta}}_{2S} = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, \widehat{\mathbf{P}}^0)$$

- Under standard regularity conditions, this two-step estimator is root-M consistent and asymptotically normal.
- It can be extended to incorporate market unobserved heterogeneity (e.g., Aguirregabiria & Mira (2007); Arcidiacono & Miller (2011)).
- Monte Carlo Simulation can be used to compute present values: Bajari, Benkard, & Levin (2007).
- Limitation: Finite sample bias due to imprecise estimates of CCPs in the first step.

Nested Pseudo Likelihood (NPL)

- Imposes equilibrium restrictions but does NOT require:

  - Repeatedly solving for MPE for each trial value of $\boldsymbol{\theta}$ (as NFXP)
  - Computing $\nabla_{\mathbf{P}} \Psi(\widehat{\boldsymbol{\theta}}, \widehat{\mathbf{P}})$ (as NFXP and MPEC)

- A NPL $(\widehat{\boldsymbol{\theta}}_{NPL}, \widehat{\mathbf{P}}_{NPL})$, that satisfy two conditions:

  (1) given $\widehat{\mathbf{P}}_{NPL}$, we have that: $\widehat{\boldsymbol{\theta}}_{NPL} = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, \widehat{\mathbf{P}}_{NPL})$

  (2) given $\widehat{\boldsymbol{\theta}}_{NPL}$, we have that: $\widehat{\mathbf{P}}_{NPL} = \Psi(\widehat{\boldsymbol{\theta}}_{NPL}, \widehat{\mathbf{P}}_{NPL})$

- The NPL estimator is consistent and asymptotically normal under the same regularity conditions as the MLE. For dynamic games, the NPL estimator has larger asymptotic variance than the MLE.

Nested Pseudo Likelihood (NPL) [2]

- An algorithm to compute the NPL is the **NPL fixed point algorithm**.
- Starting with an initial $\widehat{\mathbf{P}}_0$, at iteration $k \geq 1$:

  (Step 1) given $\widehat{\mathbf{P}}_{k-1}$, $\widehat{\boldsymbol{\theta}}_k = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}, \widehat{\mathbf{P}}_{k-1})$;

  (Step 2) given $\widehat{\boldsymbol{\theta}}_k$, $\widehat{\mathbf{P}}_k = \Psi(\widehat{\boldsymbol{\theta}}_k, \widehat{\mathbf{P}}_{k-1})$.
- A natural choice for the initial $\widehat{\mathbf{P}}_0$ is a frequency estimator of CCPs using the data.
- Step 1 is very simple in most applications. It has the same comp. cost as obtaining the MLE in a static single-agent discrete choice model.
- Step 2 is equivalent to solving once a system of linear equations with the same dimension as $\mathbf{P}$.
- A limitation of this fixed point algorithm is that **convergence is not guaranteed**. An alternative algorithm that has been used to compute NPL is a **Spectral Residual algorithm**.

Algorithms to Compute the NPL Estimator

- The NPL estimator can be described as a fixed point in the space of the vector of CCPs:

$$\widehat{\mathbf{P}} = \phi(\widehat{\mathbf{P}})$$

where $\phi\left(\widehat{\mathbf{P}}\right)$ is the **NPL mapping**:

$$\phi(\widehat{\mathbf{P}}) \equiv \Psi\left(\widehat{\mathbf{P}}, \widehat{\theta}(\widehat{\mathbf{P}})\right)$$

$\Psi\left(\mathbf{P}, \boldsymbol{\theta}\right)$ is the equilibrium mapping. $\widehat{\theta}(\widehat{\mathbf{P}})$ is Pseudo MLE mapping.

- We study 3 algorithms to compute the NPL estimator.
    1. Fixed point iterations in the NPL mapping ϕ.
    2. Newton's method to solve system of equations $\mathbf{P} - \phi(\mathbf{P}) = 0$
    3. Spectral residual method to solve system of equations $\mathbf{P} - \phi(\mathbf{P}) = 0$

- (1) does not guarantee convergence. (2) does, but it is impractical in most applications. (3) has advantages relative to (1) &(2).

Fixed Point NPL Iterations

- Let $\mathbf{P}^0 \equiv \{\mathbf{P}_i^0 : \text{for any } i\}$ be arbitrary vector of CCPs.

- At iteration n:

$$\mathbf{P}^n \; = \; \phi(\mathbf{P}^{n-1}) \; = \; \Psi\left(\mathbf{P}^{n-1}, \, \widehat{\theta}(\mathbf{P}^{n-1})\right)$$

- We check for convergence:

$$\begin{cases} \text{if } \left\|\mathbf{P}^n - \mathbf{P}^{n-1}\right\| \leq \kappa \quad \text{then} \quad \mathbf{P}^n \text{ and } \boldsymbol{\theta}^n = \widehat{\theta}(\mathbf{P}^{n-1}) \text{ is the NPL} \\ \text{if } \left\|\mathbf{P}^n - \mathbf{P}^{n-1}\right\| > \kappa \quad \text{then} \quad \text{Proceed to iteration } n+1 \end{cases}$$

where κ is a small positive constant, e.g., $\kappa = 10^{-6}$.

- **Convergence is NOT guaranteed**. This is a serious limitation.

Newton's Method

- Define the function $f(\mathbf{P}) \equiv \mathbf{P} - \Psi\left(\mathbf{P}, \widehat{\theta}(\mathbf{P})\right)$.
- Finding the NPL estimator is equivalent to finding a zero (root) of f.
- We can use Newton's method to find a root of f.
- At iteration n: $(\nabla f(\mathbf{P})$ is the Jacobian matrix)

$$\mathbf{P}^{n} = \mathbf{P}^{n-1} + \left[\nabla f(\mathbf{P}^{n-1})\right]^{-1} f(\mathbf{P}^{n-1})$$

- We check for convergence: $\left\|\mathbf{P}^{n} - \mathbf{P}^{n-1}\right\| \leq \kappa$
- **Convergence is guaranteed** (to one of the multiple equilibria).

Newton's Method [2]

- The main computational cost of a Newton's iteration comes from the computation of Jacobian matrix $\nabla f(\mathbf{P})$.

- There is not a closed-form expression for the derivatives in this matrix. And in this class of models, this matrix is not sparse.

- This matrix is of dimension $N|\mathcal{A}||\mathcal{X}| \times N|\mathcal{A}||\mathcal{X}|$, and the computation of one single element in this matrix involves solving many single-agent dynamic programming problems, each of them with a complexity $O(|\mathcal{X}|^3)$.

- In summary, Newton's method is not practical in most empirical applications, in which $|\mathcal{X}|$ is greater than 10^5.

Spectral Residual Method

- It is a general method for solving high-dimension systems of nonlinear equations, $f(\mathbf{P}) = 0$.

- It has two very attractive features:

1. It is derivative free, and the cost of one iteration is equivalent to evaluation $f(\mathbf{P})$ – the same cost as one fixed point iteration.

2. It converges to a solution under mild regularity conditions – similar good convergence properties to Newton's.

Spectral Residual Method [2]

- Spectral methods propose the following updating rule/iteration:

$$\mathbf{P}_{n+1} = \mathbf{P}_n - \alpha_n \; f\left(\mathbf{P}_n\right)$$

where α_n is the spectral steplength, which is a scalar.

- Different updating rules have been proposed in the literature. Barzilai and Borwein (1988) is commonly used:

$$\alpha_n = \frac{[\mathbf{P}_n - \mathbf{P}_{n-1}]'[f(\mathbf{P}_n) - f(\mathbf{P}_{n-1})]}{[f(\mathbf{P}_n) - f(\mathbf{P}_{n-1})]'[f(\mathbf{P}_n) - f(\mathbf{P}_{n-1})]}$$

- The intuition for the convergence of the Spectral Residual method is that the updating of α_n can guarantee the right direction to convergence.