Lecture 12: Estimating STATIC discrete-choice games of incomplete information and Multiplicity of equilibria

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Estimating Games of Incomplete Information

Pakes, Ostrovsky and Berry, 2007
Various 2-Step CCP estimators: pseudo-ML, MoM, min χ^2

Aguirregabiria and Mira, 2007
Recursive 2-Step CCP estimator, Nested Pseudo-Likelihood (NPL)

Pesendorfer and Schmidt-Dengler, 2008
2-Step least squares

Kasahara and Shimotsu, 2012
Modified (dampened) NPL

Su, 2013 and Egesdal, Lai and Su, 2015
Constrained optimization (MPEC)

Blevins and Dearing, 2025
Efficient Pseudo-Likelihood (EPL)

Example: Static Entry Game

- Two firms: a and b
- Actions: each firm has two possible actions:

$$d_a = \begin{cases} 1, & \text{if firm } \underline{\text{a}} \text{ choose to enter the market} \\ 0, & \text{if firm } \underline{\text{a}} \text{ choose not to enter the market} \end{cases} \tag{1}$$

$$d_b = \begin{cases} 1, & \text{if firm } \underline{\text{b}} \text{ choose to enter the market} \\ 0, & \text{if firm } \underline{\text{b}} \text{ choose not to enter the market} \end{cases} \tag{2}$$

Example: Static Entry Game of Incomplete Information

Utility: Ex-post payoff to firms

$$u_a(d_a, d_b, x_a, \epsilon_a) = \begin{cases} [\alpha + d_b * (\beta - \alpha)]x_a + \epsilon_{a1}, & \text{if } d_a = 1 \\ 0 + \epsilon_{a0}, & \text{if } d_a = 0 \end{cases}$$

$$u_b(d_a, d_b, x_a, \epsilon_b) = \begin{cases} [\alpha + d_a * (\beta - \alpha)]x_b + \epsilon_{b1}, & \text{if } d_b = 1 \\ 0 + \epsilon_{b0}, & \text{if } d_b = 0 \end{cases}$$

- (α, β): structural parameters to be estimated
- (x_a, x_b): firms' observed types; **common knowledge**
- $\epsilon_a = (\epsilon_{a0}, \epsilon_{a1}), \epsilon_b = (\epsilon_{b0}, \epsilon_{b1})$: firms' unobserved types, **private information**
- (ϵ_a, ϵ_b) are observed only by each firm, but not by their opponent firm nor by the econometrician

Example: Static Entry Game of Incomplete Information

- Assume the error terms (ϵ_a, ϵ_b) have EV Type I distribution
- A Bayesian Nash equilibrium (p_a, p_b) satisfies

$$
\begin{aligned}
p_a &= \frac{\exp[p_b \beta x_a + (1 - p_b)\alpha x_a]}{1 + \exp[p_b \beta x_a + (1 - p_b)\alpha x_a]} \\
&= \frac{1}{1 + \exp[-\alpha x_a + p_b x_a(\alpha - \beta)]} \\
&\equiv \Psi_a(p_b, x_a; \alpha, \beta)
\end{aligned}
$$

$$
\begin{aligned}
p_b &= \frac{1}{1 + \exp[-\alpha x_b + p_a x_b(\alpha - \beta)]} \\
&\equiv \Psi_b(p_a, x_b; \alpha, \beta)
\end{aligned}
$$

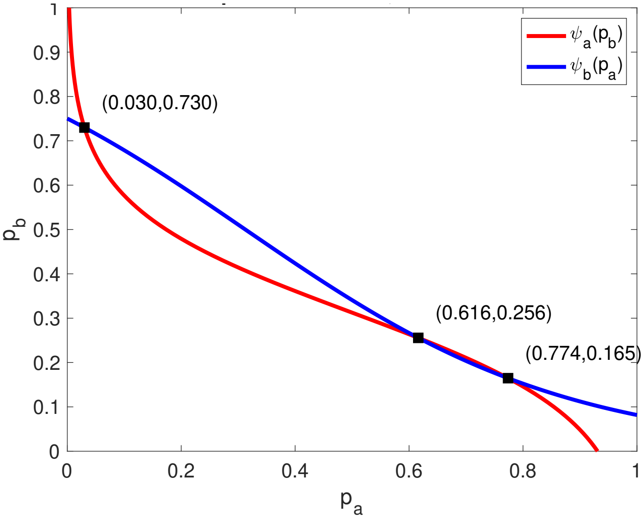
Static Game Example: Parameters

We consider a very contestable game throughout

- Monopoly profits: $\alpha * x_j = 5 * x_j$
- Duopoly profits: $\beta * x_j = -11 * x_j$
- Firm types: $(x_a, x_b) = (0.52, 0.22)$

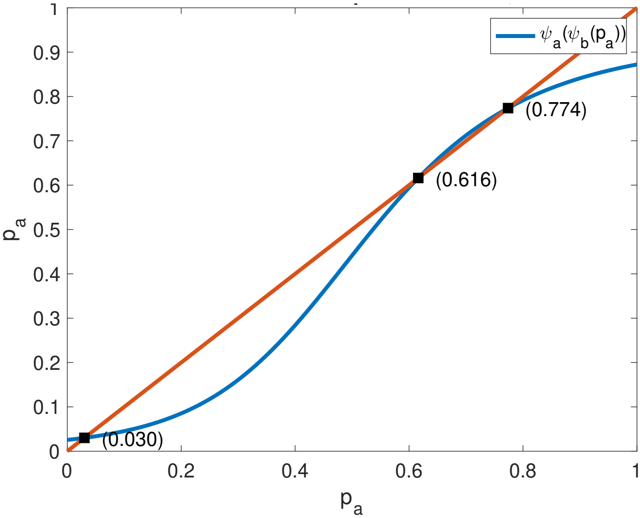
Static Game Example: Three Bayesian Nash Equilibria

Figure: Equilibria at intersections of best response functions



Static Game Example: Solving for Equilibria

Figure: Fixed points on second order best response function



Static Game Example: Solving for Equilibria

Solution method: Combination of succesive approximations and bisection algorithm

Succesive approximations (SA)

- Converge to nearest stable equilibrium.
- Start SA at $p_a = 0$ and $p_a = 1$.
- Unique equilibrium (K=1): SA will converge to it from anywhere.
- Three equilibria (K=3): Two will be stable, and one will be unstable.
- More equilibria (K>3): Not in this model.

Bisection method

- Use this to find the unstable equilibrium (if K=3).
- The bisection method that repeatedly bisects an interval and then selects a subinterval in which the fixed point (or root) must lie.
- The two stable equilibria, defines the initial interval to search over.
- The bisection method is a very simple and robust method, but it is also relatively slow.

Static Game Example: Data Generation and Identification

- Data Generating Process (DGP): the data are generated by a single equilibrium
- The two players use the same equilibrium to play 1000 times
- Data: $X = \{(d_a^i, d_b^i)_{i=1}^{1000}, (x_a, x_b) = (0.52, 0.22)\}$
- Given data X, we want to recover structural parameters α and β

Static Game Example: Maximum Likelihood Estimation

- Maximize the likelihood function

$$\begin{aligned}\max_{\alpha,\beta} \quad \log \quad & \mathcal{L}(p_a(\alpha,\beta);X) \\ = \quad & \sum_{i=1}^{N} \left(d_a^i * \log(p_a(\alpha,\beta)) + (1 - d_a^i *) \log(1 - p_a(\alpha,\beta))\right) \\ + \quad & \sum_{i=1}^{N} \left(d_b^i * \log(p_b(\alpha,\beta)) + (1 - d_b^i *) \log(1 - p_b(\alpha,\beta))\right)\end{aligned}$$

- $p_a(\alpha,\beta)$ and $p_a(\alpha,\beta)$ are the solutions of the Bayesian-Nash Equilibrium equations

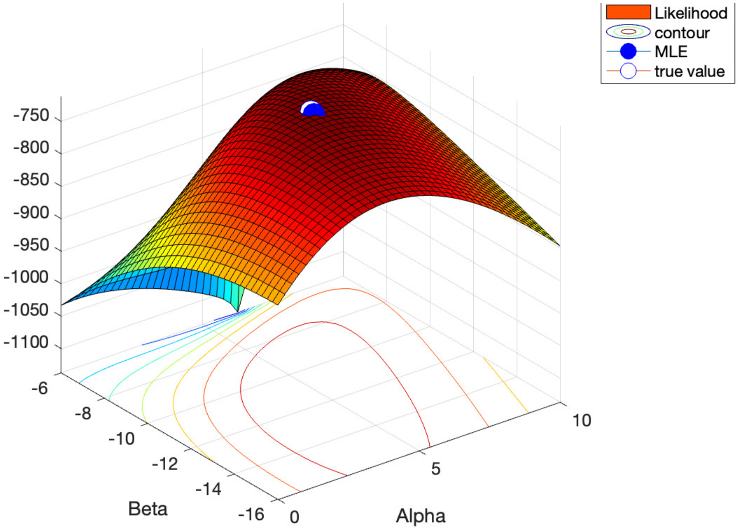
$$\begin{aligned}p_a \quad &= \quad \frac{1}{1+\exp[-\alpha x_a + p_b x_a(\alpha-\beta)]} \equiv \Psi_a(p_b, x_a; \alpha, \beta) \\ p_b \quad &= \quad \frac{1}{1+\exp[-\alpha x_b + p_a x_b(\alpha-\beta)]} \equiv \Psi_b(p_a, x_b; \alpha, \beta)\end{aligned}$$

Static Game Example: MLE via NFXP

- Outer Loop
  - Choose (α, β) to maximize the likelihood function
    $$\log \mathcal{L}(p_a(\alpha, \beta), p_b(\alpha, \beta); X)$$
- Inner loop:
  - For a given (α, β), solve the BNE equations for **ALL** equilibria: $(p_a^k(\alpha, \beta), p_b^k(\alpha, \beta)), k = 1, ..., K$
  - Choose the equilibrium that gives the highest likelihood value:
    $$k^* = \arg \max_{k=1,\dots,K} \log \mathcal{L}(p_a^k(\alpha, \beta), p_b^k(\alpha, \beta); X)$$
    such that
    $$(p_a(\alpha, \beta), p_b(\alpha, \beta)) = (p_a^{k*}(\alpha, \beta), p_b^{k*}(\alpha, \beta))$$

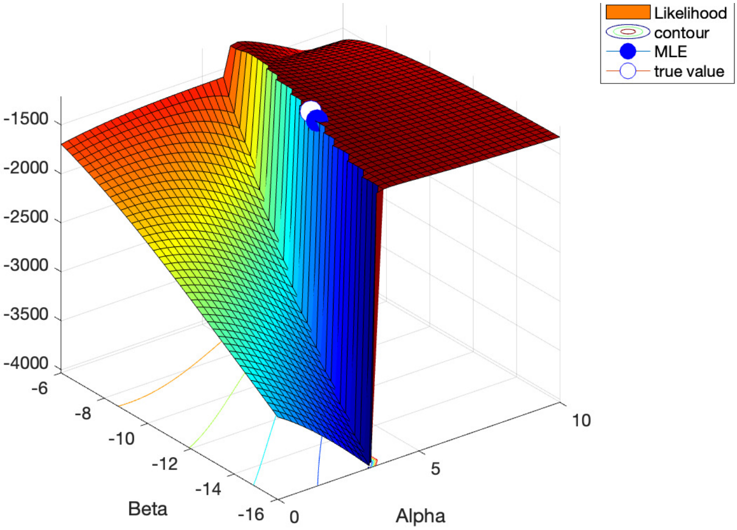
NFXP's Likelihood as a Function of (α, β) - Eq 1

Figure: Data generated from equilibrium 1



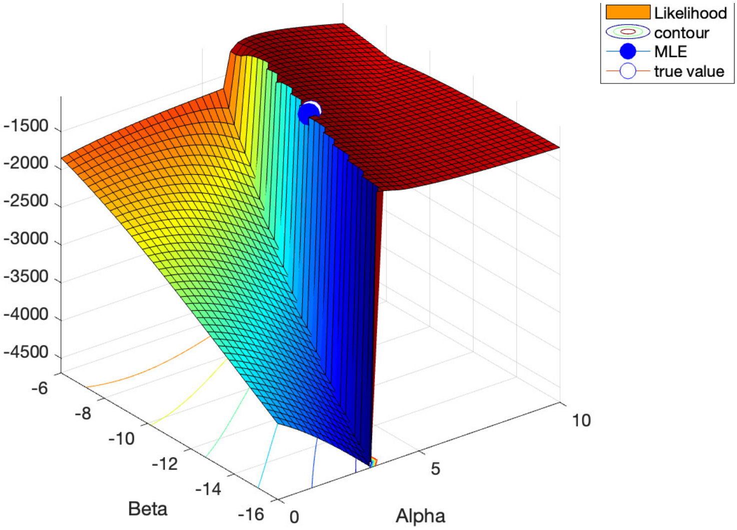
NFXP's Likelihood as a Function of (α, β) - Eq 2

Figure: Data generated from equilibrium 2



NFXP's Likelihood as a Function of (α, β) - Eq 3

Figure: Data generated from equilibrium 3



Static Game Example: MLE via MPEC

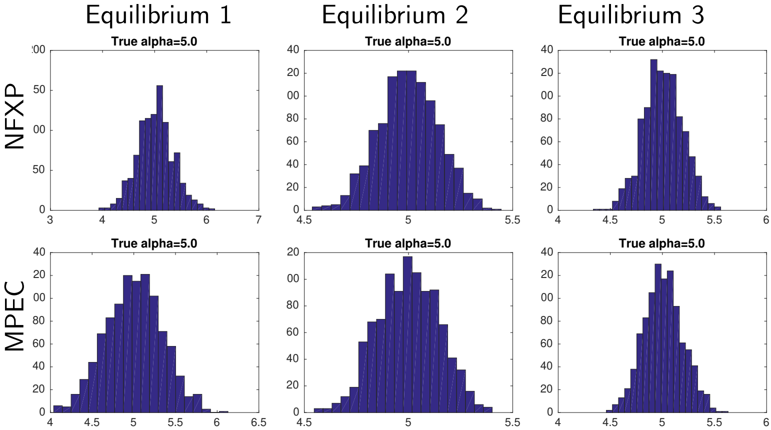
- Maximize the likelihood function

$$\begin{aligned}
\max_{\alpha,\beta,p_a,p_b} \quad \log \quad & \mathcal{L}(p_a; X) \\
= \quad & \sum_{i=1}^{N} \left(d_a^i * \log(p_a) + (1 - d_a^i *) \log(1 - p_a) \right) \\
+ \quad & \sum_{i=1}^{N} \left(d_b^i * \log(p_b) + (1 - d_b^i *) \log(1 - p_b) \right)
\end{aligned}$$

- Subject to p_a and p_a are the solutions of the Bayesian-Nash Equilibrium equations

$$\begin{aligned}
p_a &= \frac{1}{1 + \exp[-\alpha x_a + p_b x_a(\alpha - \beta)]} \\
p_b &= \frac{1}{1 + \exp[-\alpha x_b + p_a x_b(\alpha - \beta)]} \\
0 &\le p_a, p_b \le 1
\end{aligned}$$

Monte Carlo Results



Estimates of parameter α
Data generated form each of the three equilibria

Least Square Estimators

Pesendofer and Schmidt-Dengler (2008)

- Step 1: Estimate $\hat{p} = (\hat{p}_a, \hat{p}_b)$ from the data
- Step 2: Solve

$$\min_{\alpha,\beta} \left\{ (\hat{p}_a - \Psi_a(\hat{p}_b, x_a; \alpha, \beta))^2 + (\hat{p}_b - \Psi_b(\hat{p}_a, x_b; \alpha, \beta); X))^2 \right\}$$

For dynamic games, Markov perfect equilibrium conditions are characterized by

$$p = \Psi(p, \theta)$$

- Step 1: Estimate $\hat{p}$ from the data
- Step 2: Solve

$$\min_{\alpha,\beta} [\hat{p} - \Psi(\hat{p}; \theta)]' W [\hat{p} - \Psi(\hat{p}; \theta)]'$$

2-Step Methods: Pseudo Maximum Likelihood

In 2-step methods

- Step 1: Estimate $\hat{p} = (\hat{p}_a, \hat{p}_b)$ from the data
- Step 2: Solve

$$\max_{\alpha,\beta,p_a,p_b} \quad \log \mathcal{L}(p_a, p_b; X)$$

subject to

$$\begin{array}{rcl} p_a & = & \Psi_a(\hat{p}_a, x_a; \alpha, \beta) \\ p_b & = & \Psi_b(\hat{p}_b, x_b; \alpha, \beta) \\ 0 & \leq & p_a^m, p_b^m \leq 1, m = 1, ..., M \end{array}$$

Or equivalently

- Step 1: Estimate $\hat{p} = (\hat{p}_a, \hat{p}_b)$ from the data
- Step 2: Solve

$$\max_{\alpha,\beta} \quad \log \mathcal{L}(\Psi_a(\hat{p}_b, x_a; \alpha, \beta), \Psi_b(\hat{p}_a, x_b; \alpha, \beta); X)$$

Nested Pseudo Likelihood (NPL)

Aguirregabiria and Mira (2007)

NPL iterates on the 2-step methods

1. Step 1: Estimate $\hat{p}^0 = (\hat{p}_a^0, \hat{p}_b^0)$ from the data, set $k = 0$
2. Step 2:
   REPEAT
   2.1 Solve

   $$\alpha^{k+1}, \beta^{k+1} = \arg\max_{\alpha,\beta} \quad \log \mathcal{L}(\Psi_a(\hat{p}_b^k, x_a; \alpha, \beta), \Psi_b(\hat{p}_a^k, x_b; \alpha, \beta); X)$$

   2.2 One best-reply iteration on $\hat{p}^k$

   $$\begin{aligned} \hat{p}_a^{k+1} &= \Psi_a(\hat{p}_a^k, x_a; \alpha^{k+1}, \beta^{k+1}) \\ \hat{p}_a^{k+1} &= \Psi_b(\hat{p}_b^k, x_b; \alpha^{k+1}, \beta^{k+1}) \end{aligned}$$

   2.3 Let k:=k+1;

   UNTIL convergence in (α^k, β^k) and $(\hat{p}_a^k, \hat{p}_b^k)$

Some remarks

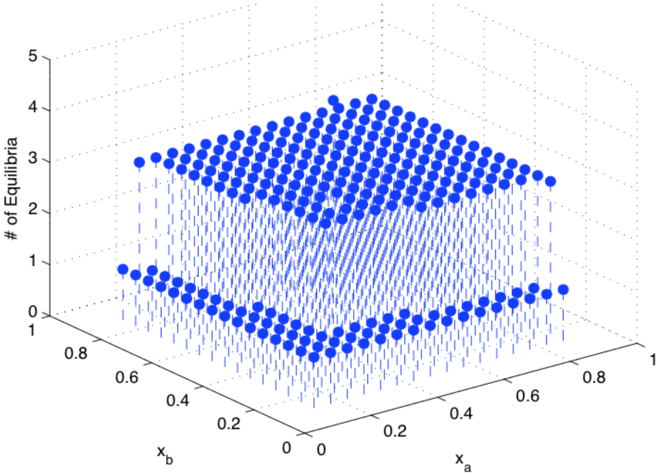
- NFXP/MPEC implementations of MLE is statistically efficient
  - NFXP computationally daunting
    → need to compute ALL equilibria for each $\theta = (\alpha, \beta)$
    → find maxima of discontinuous likelihood
  - MPEC may be computational lighter
    → objective function and constraints are smooth in α, β, p_a, p_b
    → only need to satisfy equilibrium constraints in optimum
    → may get stuck in local minima (equilibria not played in the data).
  - MLE and Multiple equilibria
    → discontinuous likelihood function → non-standard inference
- Two step estimators
  - Computationally fast
  - Inefficient and biased in small samples
    (best response mapping not satisfied)
- NPL and EPL should bridge this gab
  - NLP can be unstable in games with multiple equilibria
    (NPL mapping not contraction)
  - EPL does not reply on contraction property, but still a iterative algorithm that depends on starting values and estimates of p_a, p_b
- Problems amplified by orders of magnitude in dynamic games

Estimation with Multiple Markets

- There 25 different markets, i.e., 25 pairs of observed types $(x_a^m, x_b^m), m = 1, ..., 25$
- The grid on x_a has 5 points equally distributed between the interval $[0.12, 0.87]$, and similarly for x_b
- Use the same true parameter values: (α_0, β_0)
- For each market with (x_a^m, x_b^m), solve BNE conditions for (p_a^m, p_b^m).
- There are multiple equilibria in most of 25 markets
- For each market, we (randomly) choose an equilibrium to generate T data points for that market
- The equilibrium used to generate data can be different in different markets - we flip a coin at each market.

# of Equilibria with Different (x_a^m, x_b^m)

Figure: Number of equilibria



NFXP - Estimation with Multiple Markets

Outer loop:

$$\max_{\alpha,\beta} \qquad \log \mathcal{L}(p_a^m(\alpha,\beta), p_b^m(\alpha,\beta); X)$$

Inner loop: For a given values of (α, β) solve BNE equations for ALL equilibria, $k = 1, .., K$ at each market, $m = 1, .., M$: That is, $p_a^{m,k}(\alpha, \beta)$ and $p_a^{m,k}(\alpha, \beta)$ are the solutions to

$$\begin{aligned} p_a^m &= \Psi_a(p_b^m, x_a^m; \alpha, \beta) \\ p_b^m &= \Psi_b(p_a^m, x_b^m; \alpha, \beta) \\ m &= 1, ..., M \end{aligned}$$

where we again choose the equilibrium, that gives the highest likelihood value at each market m

$$k^* = \arg \max_{k=1,\dots,K} \log \mathcal{L}(p_a^{m,k}(\alpha,\beta), p_b^{m,k}(\alpha,\beta); X)$$

such that

$$(p_a^m(\alpha,\beta), p_b^m(\alpha,\beta)) = (p_a^{m,k*}(\alpha,\beta), p_b^{m,k*}(\alpha,\beta))$$

Estimation with Multiple Markets - MPEC

Constrained optimization formulation

$$\max_{\alpha,\beta,p_a^m,p_b^m} \quad \log \mathcal{L}(p_a^m, p_b^m; X)$$

subject to

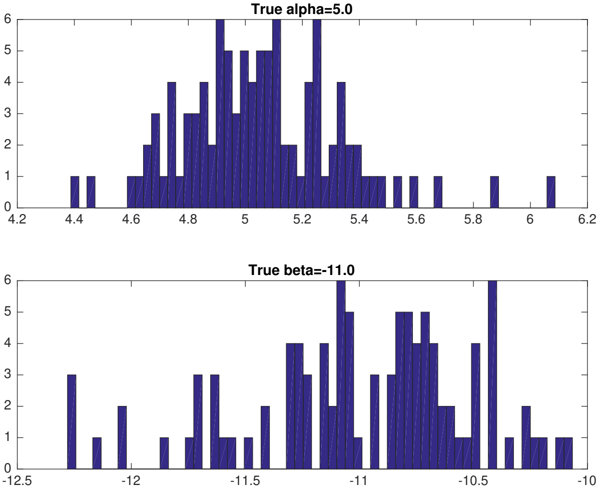
$$\begin{aligned} p_a^m &= \Psi_a(p_b^m, x_a^m; \alpha, \beta) \\ p_b^m &= \Psi_b(p_a^m, x_b^m; \alpha, \beta) \\ 0 &\leq p_a^m, p_b^m \leq 1, \\ m &= 1, ..., M \end{aligned}$$

- MPEC does not explicitly solve the BNE equations to find ALL equilibria at each market - for every trial value of parameters.
- But the number of parameters is much larger.
- Both MPEC and NFXP are based on Full Information Maximum Likelihood (FIML) estimators.

NFXP: Monte Carlo - Multiple Markets (M=25, T=50)

Starting values $\alpha_0 = \alpha$, $\beta_0 = \beta$

Random equilibrium selection in different markets



MPEC: Monte Carlo - Multiple Markets (M=25, T=50)

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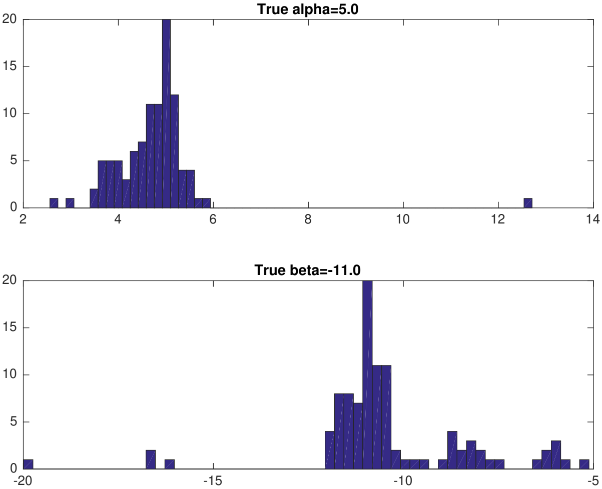

MPEC and NFXP: multiple markets

NFXP:

- 2 parameters in optimization problem
- Needs to find ALL equilibria at each market
  (very hard in more complex problems)
- Good full solution methods required

MPEC:

- $2 + 2M$ parameters in optimization problem
- Local minima with many markets:
  MPEC equilibrium constraints are satisfied and gradients of objective function is within tolerance - but wrong equilibrium selected