

# Endogenous grid point methods (EGM, DCEGM and extensions)

DSE2025: Econometric Society Summer School in  
Dynamic Structural Econometrics

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# Discrete and continuous choice?

In economics discrete and continuous choice co-exist

- How much to work + when to retire/become an entrepreneur
- How much to save + when to buy a house/car/durables
- Which car to buy + how much to drive

Often modeled separately using traditional solution methods:

- Discrete choice → optimization over finite set
- Continuous choice → first order conditions + concavity(?)
- Dynamic → dynamic programming (VFI,policy,time iterations)

# Discrete and continuous choice?

In discrete-continuous choice models:

- Intrinsic non-concavity
- First order conditions not sufficient
- Kinks in value functions + discontinuities in policy functions

Traditional methods are not ideal

- Need global optimizer in each point of the state space
- Need to locate and keep track of kinks and discontinuities
- Need special numerical procedures for non-smooth objects

⇒ **Endogenous grid point methods**

# Plan for the lecture

- ➊ Original EGM for continuous choice **only**  
*Only for particular (yet interesting and important) models  
(stochastic growth models, consumption-savings (buffer stock)  
models)*
- ➋ DC-EGM for discrete-continuous choice **without taste shocks**  
*For models with one continuous and additional discrete choices  
Nasty and scary*
- ➌ DC-EGM for discrete-continuous choice **with taste shocks**  
*For models with one continuous and additional discrete choices  
Structural taste shocks or logit smoothing  
Much better, possible to work with*
- ➍ Some words on multi-dimensional extensions and occasionally  
binding constraints

# What is EGM?

The Method of Endogenous Gridpoints — fast method for solving dynamic stochastic consumption/savings problems

- ❶ finite and infinite horizon
- ❷ Strictly concave monotone and differentiable utility function
- ❸ one continuous state variable (*wealth*) and  
one continuous choice (*consumption*)
- ❹ particular structure of the law of motion for state variables  
(*intertemporal budget constraint*)
- ❺ very well accommodate potentially binding borrowing constraints

# DC-EGM for Discrete-Continuous problems

Expand the class of problems to be solved:

- ① A1. Strictly concave monotone and differentiable utility function
- ② Continuous state  $M_t$  with a particular motion rule
- ③ Additional (discrete) state variables  $st_t$ 
  - A2. Transition probabilities of  $st_t$  are independent of  $M_t$
- ④ One continuous ( $c_t$ ) and one\* discrete choice variable  $d_t$

Two flavors:

- ① **Without taste shocks:** DC-EGM iterates on value function and policy function, produces exact solutions for the optimal thresholds for discrete decisions (discrete policy)
- ② **With taste shocks:** DC-EGM iterates on **discrete choice specific** value and policy functions, produces choice probabilities for discrete alternatives

# Learning outcomes = points to remember

- ➊ If your model has one continuous (consumption) choice and additional discrete choices → Use DC-EGM
- ➋ In regular cases DC-EGM avoids all root-finding operations
- ➌ If utility is separable in continuous and discrete choices, DC-EGM deals very easily with credit constraints
- ➍ Extreme value taste shocks → solution is much better behaved
- ➎ Faster and more accurate than traditional approaches

# EGM

# Simple consumption/savings model (Phelps)

$$V_t(M_t) = \max_{0 \leq c \leq M_t} [u(c) + \beta E V_{t+1} (\tilde{R}(M_t - c))]$$

$M_t$  cash-in-hand, all resources available at period  $t$

$A_t = M_t - c_t$  assets at the end of period  $t$  (savings)

$\tilde{R}$  deterministic or stochastic return on savings

$u(c)$  utility of current consumption

$$u(c) = \frac{c^\rho - 1}{\rho} \xrightarrow{\rho \rightarrow 0} \log(c)$$

# Analytic solution (Hakansson, 1970, Phelps, 1962)

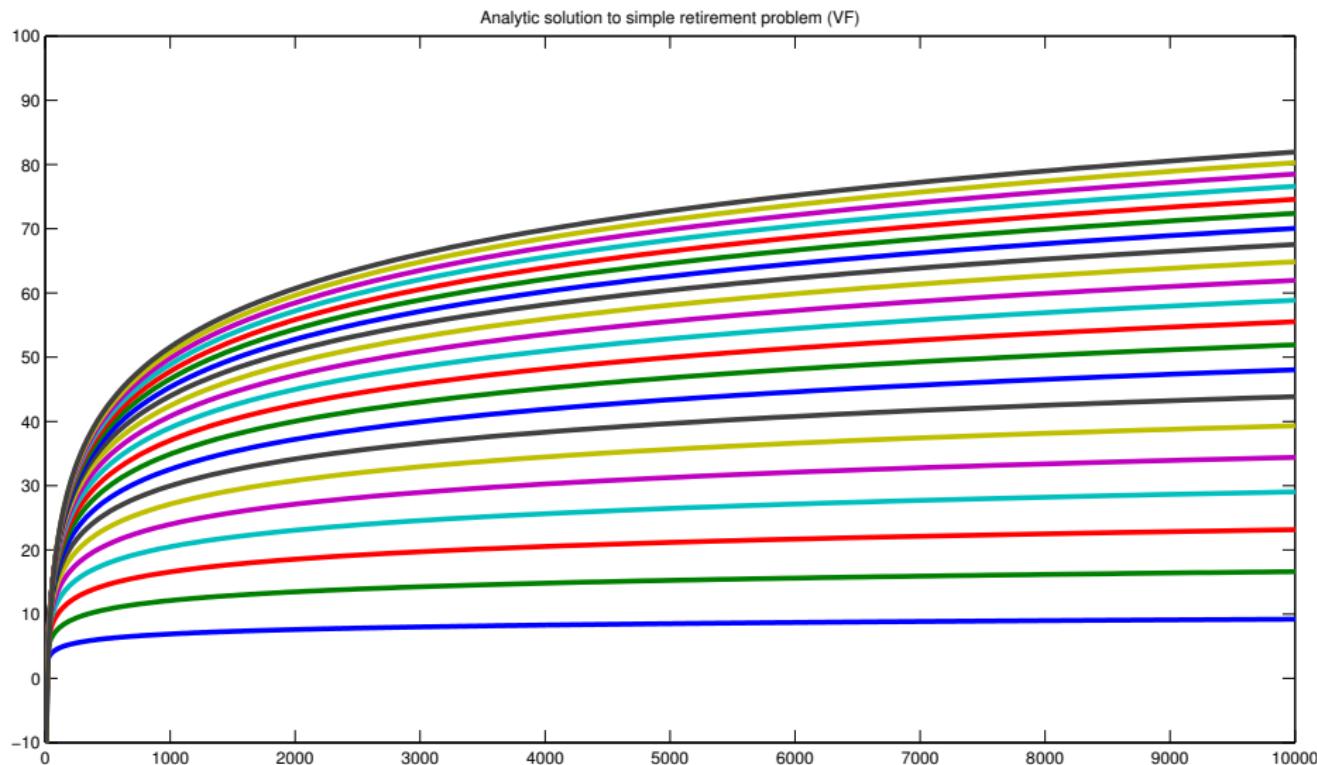
$$V_{T-t}(M) = \left[ \frac{M^\rho}{\rho} \right] \left( \sum_{i=0}^t K^i \right)^{(1-\rho)} - \frac{1}{\rho} \left( \sum_{i=0}^t \beta^i \right)$$

$$V_{T-t}(M) \xrightarrow{\rho \rightarrow 0} \log(M) \left( \sum_{i=0}^t \beta^i \right) + K_t$$

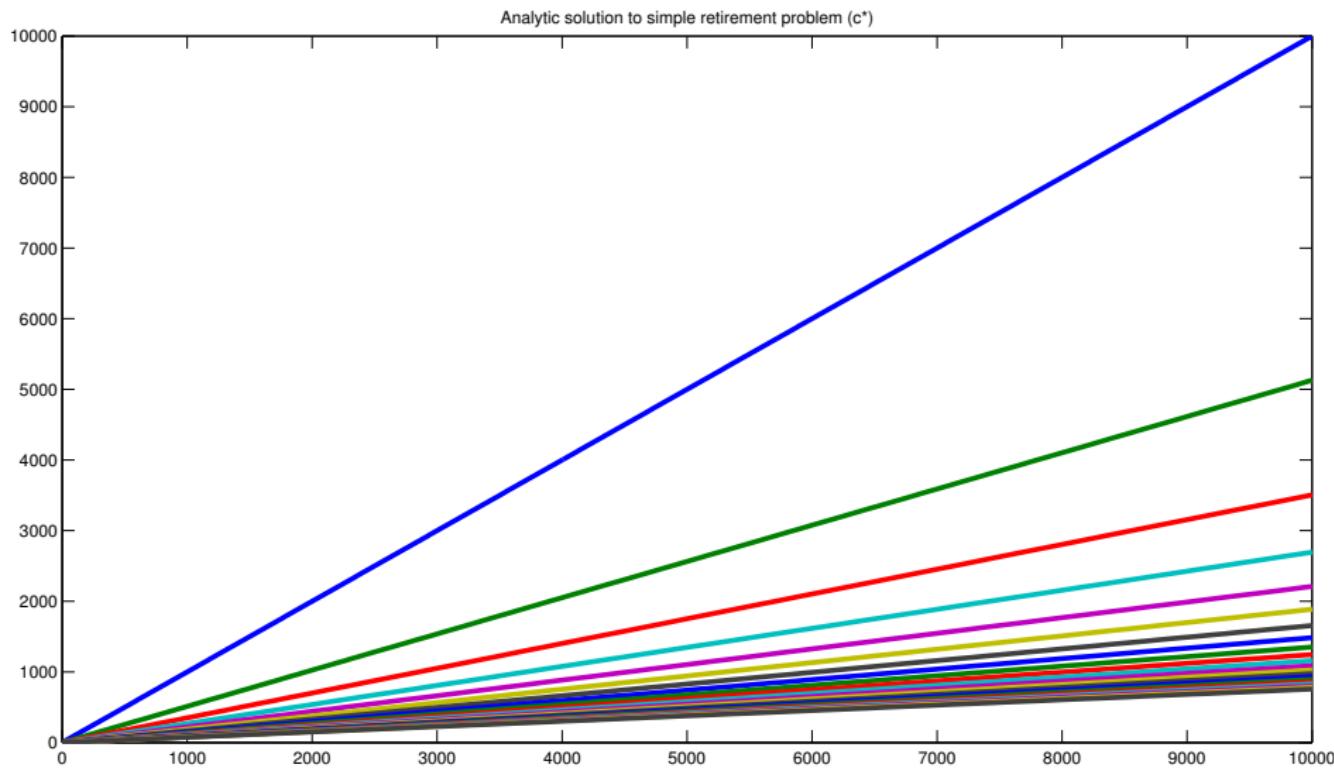
$$c_{T-t}(M) = M \left( \sum_{i=0}^t K^i \right)^{-1}$$

$K$  and  $K_t$  are functions of primitives,  $K \xrightarrow{\rho \rightarrow 0} \beta$

# Analytic solution : value functions



# Analytic solution : consumption rule



# Simple consumption/savings model (Deaton)

$$V_t(M_t) = \max_{0 \leq c \leq M_t} [u(c) + \beta E V_{t+1}(R(M_t - c) + \tilde{y})]$$

$M_t$  cash-in-hand, all resources available at period  $t$

$A_t = M_t - c_t$  assets at the end of period  $t$  (savings)

$R$  deterministic return on savings

$\tilde{y}$  stochastic income

$u(c)$  utility of current consumption

$$u(c) = \frac{c^\rho - 1}{\rho} \xrightarrow{\rho \rightarrow 0} \log(c)$$

No analytical solution!

# Traditional approach : value function iterations

① Fix grid over  $M_t$ . For every point on this grid:

② In the terminal period calculate

$$V_T(M_T) = \max_{0 \leq c_T \leq M_T} \{u(c_T)\} \text{ and}$$

$$c_T^* = \operatorname{argmax}_{0 \leq c_T \leq M_T} \{u(c_T)\}$$

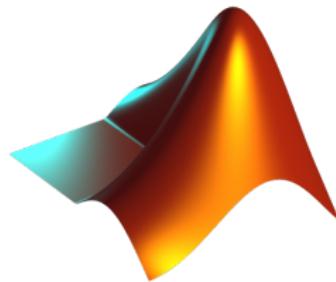
③ With  $t + 1$  value function at hand, proceed backward to period  $t$  and calculate

$$V_t(M_t) = \max_{0 \leq c_t \leq M_t} \left\{ u(c_t) + \beta EV_{t+1} \left( \tilde{R}(M_t - c_t) \right) \right\}$$

and

$$c_t^* = \operatorname{argmax}_{0 \leq c_t \leq M_t} \left\{ u(c_t) + \beta EV_{t+1} \left( \tilde{R}(M_t - c_t) \right) \right\}$$

using Bellman equation



- ➊ Phelps and Deaton models
- ➋ Run VFI solver
- ➌ See the code/python directory  
in the repository

# Euler equation

Bellman equation:  $V_t(M_t) = \max_{0 \leq c_t \leq M_t} [u(c_t) + \beta E V_{t+1} (\tilde{R}(M_t - c_t))]$

F.O.C. for Bellman equation:  $u'(c_t) = \beta E \left[ \frac{\partial V_{t+1}(M_{t+1})}{\partial M_{t+1}} \tilde{R} \right]$

Envelope theorem:

$$\begin{aligned} \frac{\partial V_t(M_t)}{\partial M_t} &= \beta E \left[ \frac{\partial V_{t+1}(M_{t+1})}{\partial M_{t+1}} \tilde{R} \right] \Rightarrow \frac{\partial V_t(M_t)}{\partial M_t} = u'(c_t) \Rightarrow \\ &\Rightarrow \frac{\partial V_{t+1}(M_{t+1})}{\partial M_{t+1}} = u'(c_{t+1}) \end{aligned}$$

Euler equation to characterize the **interior solutions**:  $u'(c_t) = \beta E [u'(c_{t+1}) \tilde{R}]$

# Traditional approach : solving Euler equation

- ➊ Fix grid over  $M_t$ . For every point on this grid:
- ➋ In the terminal period calculate  
 $c_T^* = \underset{0 \leq c_T \leq M_T}{\operatorname{argmax}} \{u(c_T)\}$
- ➌ With  $t + 1$  optimal consumption rule  $c_{t+1}^*(M_{t+1})$  at hand, proceed backward to period  $t$  and calculate  
 $c_t$  from equation  
$$u'(c_t) = \beta E \left[ u' \left( c_{t+1}^* \left( \tilde{R}(M_t - c_t) \right) \right) \tilde{R} \right]$$
to recover  $c_t^*(M_t)$
- ➍ When  $M_t$  is small enough so credit constraint binds, the Euler equation does not hold, and special provisions are necessary

# What if no root-finding is necessary?

## With numerical optimization

- Relatively slow: iterative numerical optimization in each point of state space!
- Hard to find global optimum in non-convex problems
- Loss of accuracy due to the absence of the point where credit constraint starts to bind on the fixed grid

Even when using state-of-the-art solvers!

## Without numerical optimization

- Much faster: no iterative methods in each point of the state space
- More accurate: using analytical structure of the problem

# Endogenous gridpoint method (EGM)



Carroll 2006 *Economics Letters*

The method of endogenous gridpoints for solving dynamic stochastic optimization problems.

## Idea

- Instead of searching for optimal decision in each point of the state space (traditional approaches)
- Look for the state variable (level of assets) where arbitrary chosen decision (consumption → savings) would be optimal (EGM)

# EGM algorithm

Start with  $c_T^* = M_T$ . In each period  $t = T, T - 1, \dots, 1$ :

## EGM step

- ➊ Take the next value  $A =$  current period savings ( $= M_t - c_t$ ) from fixed (or adaptive) grid
- ➋ Intertemporal budget constraint:  $A \rightarrow M_{t+1}$   
 $M_{t+1} = \tilde{R}(M_t - c_t) = \tilde{R} \cdot A$
- ➌ Policy function at period  $t + 1$ :  $M_{t+1} \rightarrow c_{t+1}$   
 $c_{t+1} = c_{t+1}^*(M_{t+1})$
- ➍ Inverted Euler equation:  $c_{t+1} \rightarrow c_t$   
 $c_t = (u')^{-1} \left( \beta E \left[ \tilde{R} \cdot u' (c_{t+1}^*(M_{t+1})) | A \right] \right)$
- ➎ Intratemporal budget constraint:  $c_t + A = M_t \rightarrow c_t(M_t)$   
 $M_t = c_t + A \rightarrow c_t^*(M_t)$

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# EGM step as parametric curve

$$u(c_t(M)) = \beta E \left[ \tilde{R} \cdot u' \left( c_{t+1}(\tilde{R}A) \right) | A \right]$$

Given any policy function  $c_0(M)$ , an updated policy function  $c(M)$  is given as a **parameterized curve**

$$\begin{cases} c = (u')^{-1} \left( \beta E \left[ \tilde{R} \cdot u' \left( c_0(\tilde{R}A) \right) | A \right] \right) \\ M = (u')^{-1} \left( \beta E \left[ \tilde{R} \cdot u' \left( c_0(\tilde{R}A) \right) | A \right] \right) + A \end{cases}$$

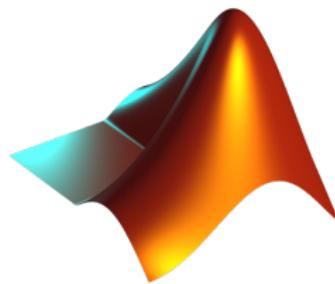
- $A$  is a parameter that takes non-negative values

# Matlab implementation (minimal.m)

```
o [quadp quadw]=quadpoints(EXPN,0,1);
quadstnorm=norminv(quadp,0,1);
sgrid=linspace(0,MMAX,NM);
policy{TBAR}.w=[0 MMAX];
policy{TBAR}.c=[0 MMAX];
5   for it=TBAR-1:-1:1
    w1=Y+exp(quadstnorm*SIGMA)*(1+R)*sgrid;
    c1=interp1(policy{it+1}.w,policy{it+1}.c,w1,'linear',
    rhs=quadw'*(1./c1);
    policy{it}.c=[0 1./(DF*(1+R)*rhs)];
10   policy{it}.w=[0 sgrid+policy{it}.c(2:end)];
end
```

# Accuracy and speed of EGM

	Traditional Euler	EGM
Running time	37 sec.	0.11 sec.
Max abs error, $c_t^*$	5e-9	4e-14
Mean abs error, $c_t^*$	1.4e-12	1.5e-14
Max abs error, $V_t(M)$	39.466	15.163
Mean abs error, $V_t(M)$	2.5e-02	3.2e-02



- ➊ Compare speed of VFI and EGM solvers
- ➋ Simulate flat consumption path using VFI and EGM solutions

- ➌ See the code/python directory in the repository

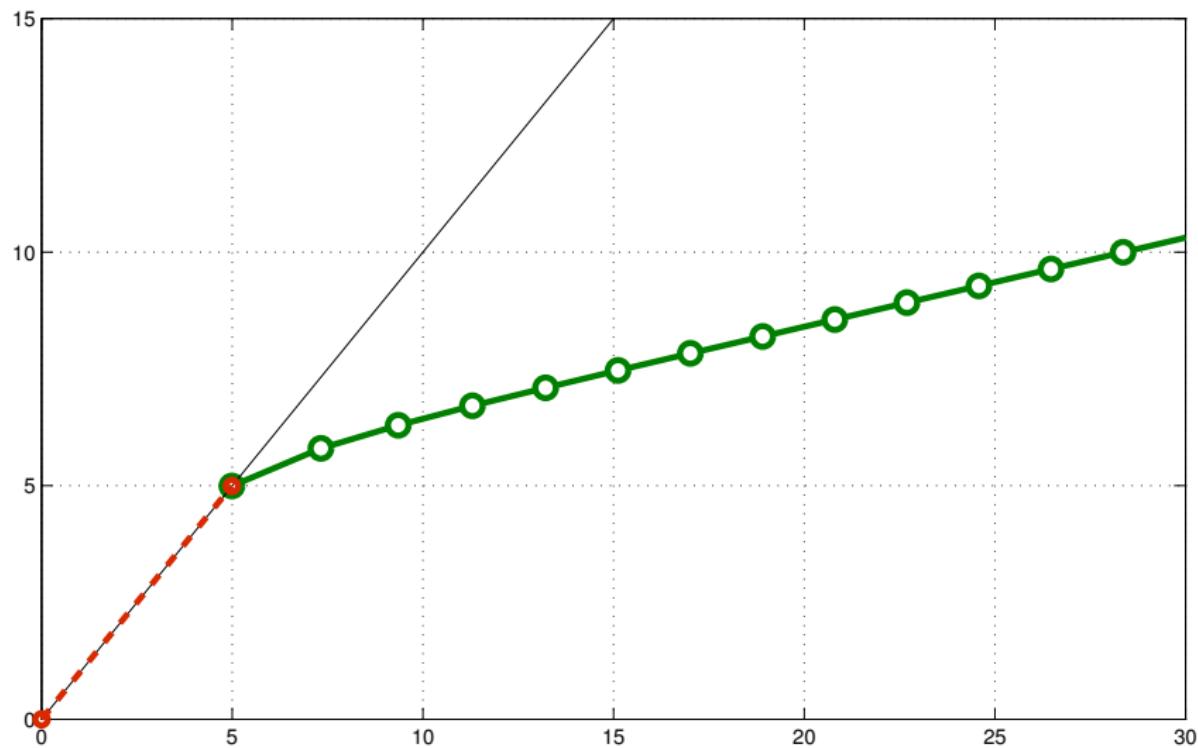
# EGM and credit constraint

Theorem: Monotonicity of savings

Monotone and concave utility function  $\Rightarrow$   
end-of-period assets  $A_t = M_t - c_t$  are non-decreasing in  $M_t$

- With  $A = 0$  the EGM loop recovers the value of cash-in-hand  $M_t^{cc}$  that bounds the credit constrained region
- For all  $M_t < M_t^{cc}$  credit constrained binds  $\Rightarrow c_t = M_t$
- Consumption rule in the credit constrained region is  $45^\circ$  line between  $(0, 0)$  and  $(M_t^{cc}, M_t^{cc})$
- As simple as “connect the dots”  $(0, 0)$  and  $(M_t^{cc}, M_t^{cc})$

# EGM and credit constraint



# Credit constraints and value function

- Credit constraints are handled so well by EGM because it is never necessary to compute utility of nearly zero consumption
- Inevitable when value functions have to be computed, but..

$$M_t < M_t^{cc}$$

$$V_t(M) = u(M) + \beta EV_{t+1}(0)$$

$EV_{t+1}(0)$  — expected value of ending period  $t$  with  $A_t = 0$

- Value function has analytic form for  $M_t < M_t^{cc}$ !

# DC-EGM

# Generalization of EGM



Iskhakov, Jørgensen, Rust, Schjerning, QE 2017

The Endogenous Grid Method for Discrete-Continuous Dynamic  
Choice Models with (or without) Taste Shocks

- The DC-EGM paper
- Two flavors: with and without EV taste shocks
- Solution method made for empirical applications



Giulio Fella, RED 2014

A Generalized Endogenous Grid Method for Non-Smooth and  
Non-Concave Problems

- Identify the regions of the problem where Euler equation is not sufficient for optimality
- Use global optimization methods inside (VFI) and EGM outside
- Similar to DC-EGM without taste shocks

# Simple retirement model

$$V_t(M_t, \mathbb{W}) = \max \left\{ \begin{array}{l} \max_{0 \leq c \leq M_t} u(c, \mathbb{R}) + \beta EV_{t+1} \left( \tilde{R}(M_t - c), \mathbb{R} \right) \\ \max_{0 \leq c \leq M_t} u(c, \mathbb{W}) + \beta EV_{t+1} \left( \tilde{R}(M_t - c) + y, \mathbb{W} \right) \end{array} \right\}$$

$$V_t(M_t, \mathbb{R}) = \max_{0 \leq c \leq M_t} \left[ u(c, \mathbb{R}) + \beta EV_{t+1} \left( \tilde{R}(M_t - c), \mathbb{R} \right) \right]$$

$\mathbb{R}, \mathbb{W}$  retirement and working **states  $st_t$**  that evolve according to **discrete choices  $d \in \{\mathbb{R}, \mathbb{W}\}$**

$y$  deterministic wage income

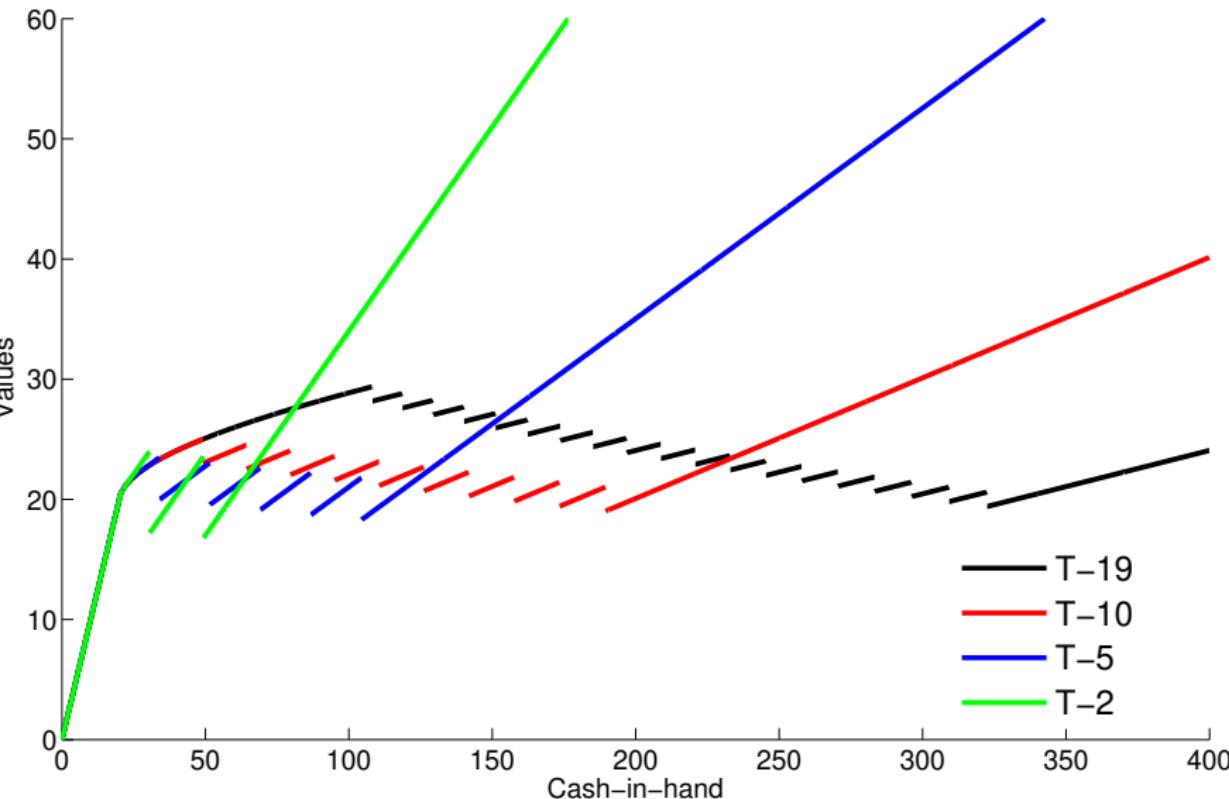
$$u(c, d) = \frac{c^\rho - 1}{\rho} - 1(d = \mathbb{W}) \xrightarrow{\rho \rightarrow 0} \log(c) - 1(d = \mathbb{W})$$

# Analytic solution

$$u(c) = \log(c), R = 1 \Rightarrow c_{T-t}^*(M, \mathbb{W}) =$$

$$\left\{ \begin{array}{ll} M & \text{if } M \leq y/\beta \\ (y + M)/(1 + \beta) & \text{if } y/\beta \leq M \leq \overline{M}_{T-t}^{l_1} \\ (2y + M)/(1 + \beta + \beta^2) & \text{if } \overline{M}_{T-t}^{l_1} \leq M \leq \overline{M}_{T-t}^{l_2} \\ \dots & \dots \\ ((t-1)y + M) \left( \sum_{i=0}^{t-1} \beta^i \right)^{-1} & \text{if } \overline{M}_{T-t}^{l_{t-2}} \leq M \leq \overline{M}_{T-t}^{l_{t-1}} \\ (ty + M) \left( \sum_{i=0}^t \beta^i \right)^{-1} & \text{if } \overline{M}_{T-t}^{l_{t-1}} \leq M \leq \overline{M}_{T-t}^{r_1} \\ [(t-1)y + M] \left( \sum_{i=0}^t \beta^i \right)^{-1} & \text{if } \overline{M}_{T-t}^{r_1} \leq M \leq \overline{M}_{T-t}^{r_2} \\ \dots & \dots \\ (2y + M) \left( \sum_{i=0}^t \beta^i \right)^{-1} & \text{if } \overline{M}_{T-t}^{r_{t-2}} \leq M \leq \overline{M}_{T-t}^{r_{t-1}} \\ (y + M) \left( \sum_{i=0}^t \beta^i \right)^{-1} & \text{if } \overline{M}_{T-t}^{r_{t-1}} \leq M \leq \overline{M}_{T-t} \\ M \left( \sum_{i=0}^t \beta^i \right)^{-1} & \text{if } \overline{M}_{T-t} < M \end{array} \right.$$

# Analytic solution



# How to approach discrete/continuous choice

The goal:

- Avoid root finding
- Keep efficient treatment of credit constraints

## DC-EGM ver. 1.0

- ① EGM step for each discrete choice  $d$  and every state  $st$
- ② Compute  $d$ -specific value functions and consumption rules
- ③ Compare the  $d$ -specific value functions to find optimal switching points (compute upper envelope)
- ④ Reconstruct overall consumption rule and value function from optimal switching points

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- No root finding!
- Efficient treatment of credit constraints (to be shown)
- Need to compute value functions
- Need to compute upper envelope

# Is Euler equation still a necessary condition?

## DC-EGM ver. 1.0

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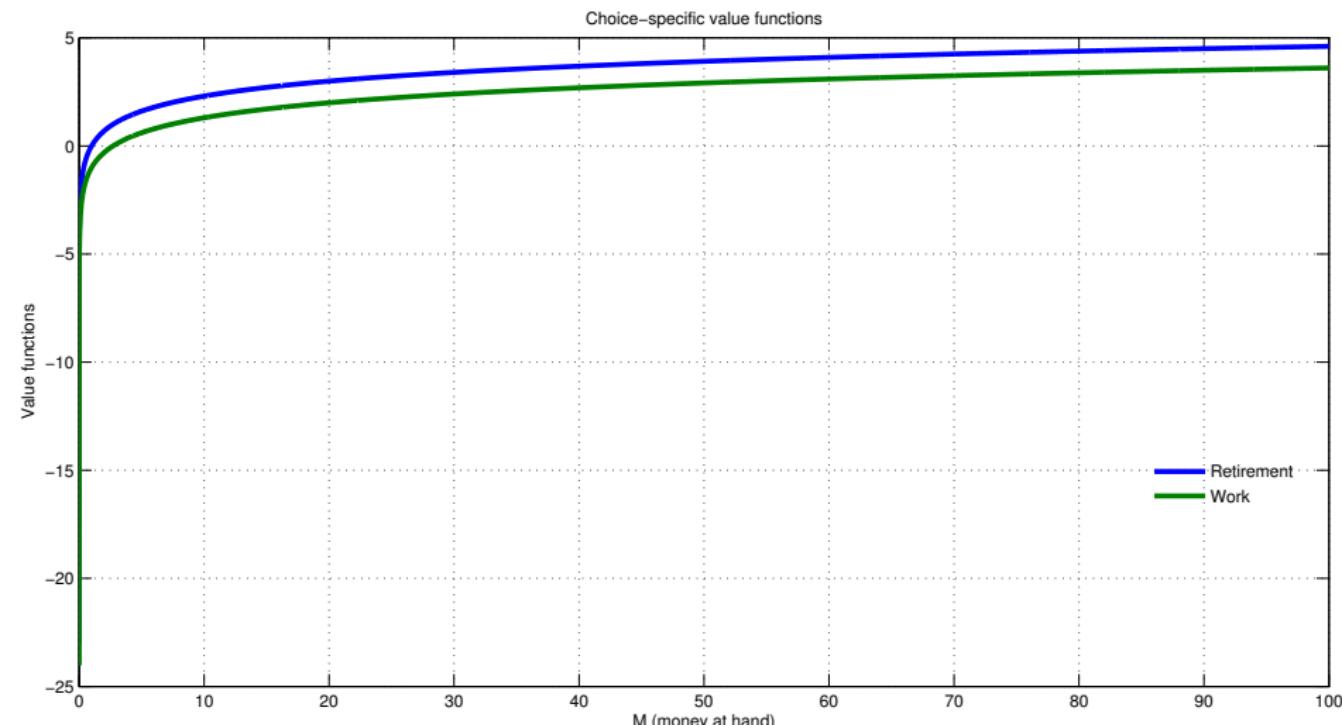


Clausen & Strub, 2010-2016

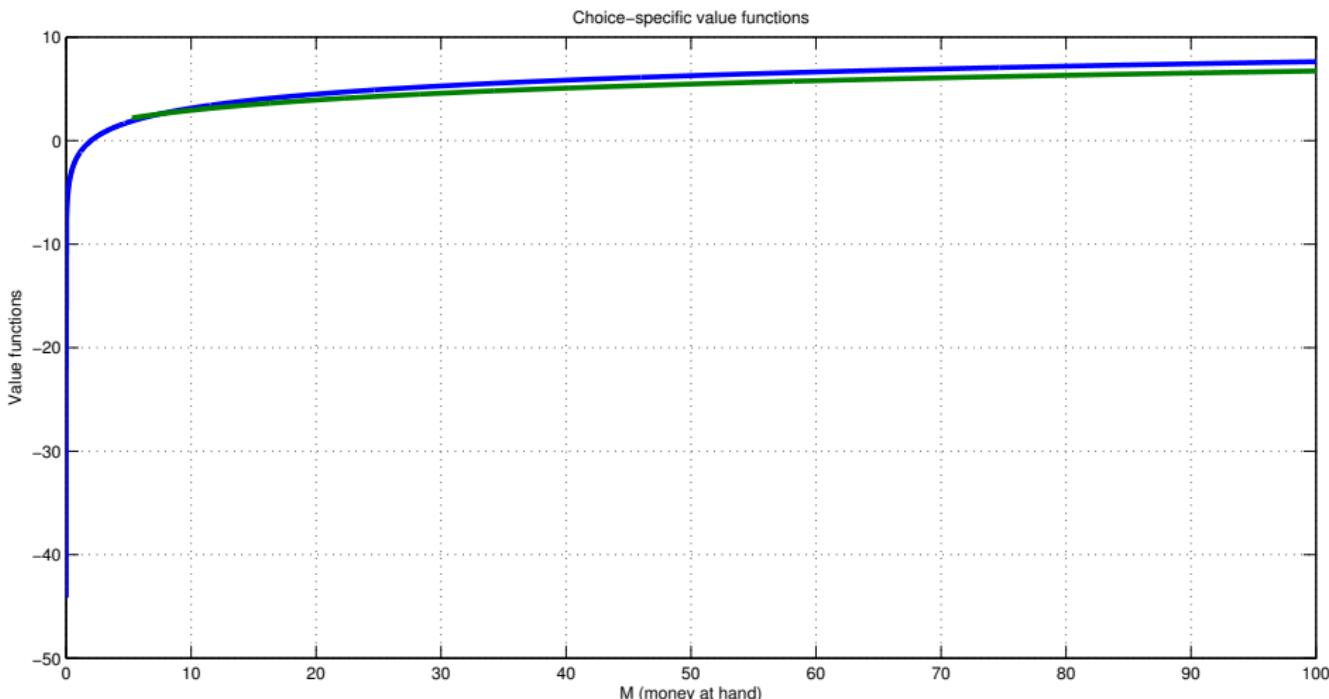
A General and Intuitive Envelope Theorem.

Show that Euler equation remains a necessary condition for the optimal continuous consumption.

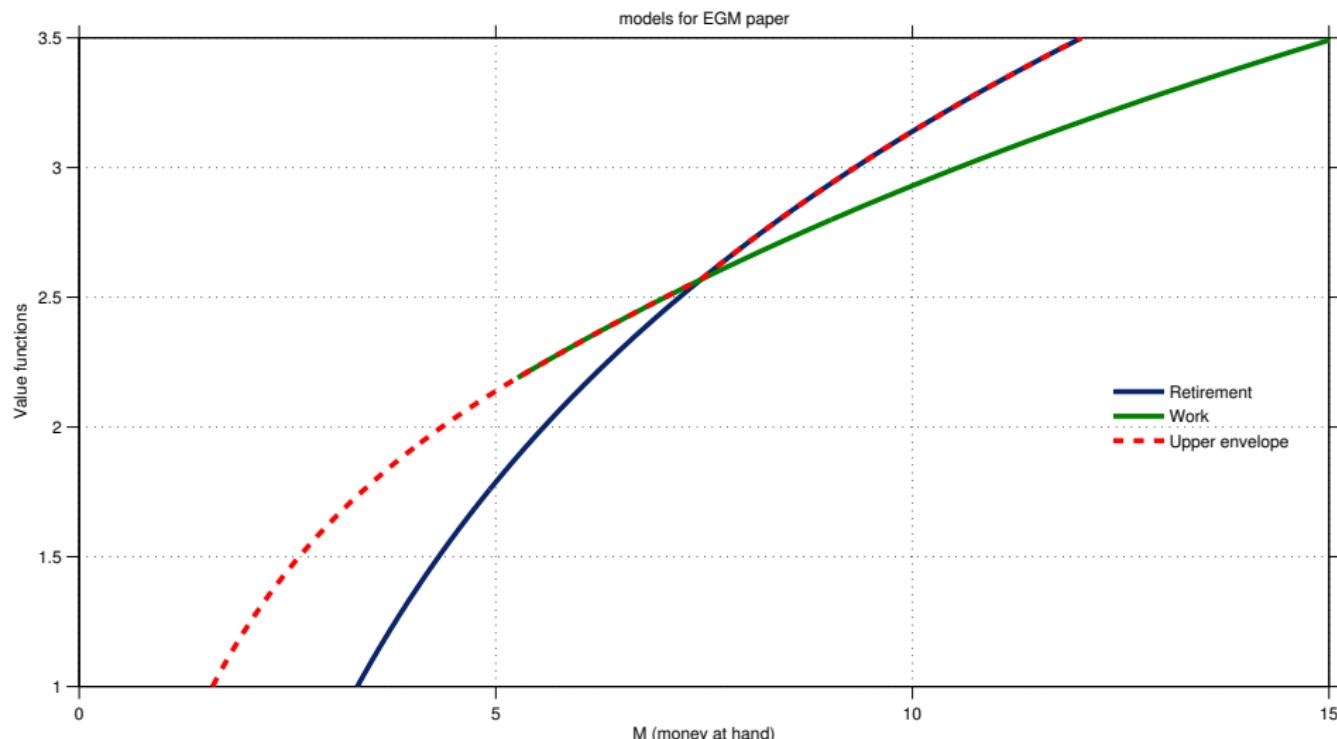
# Period $T$ : choice specific value functions



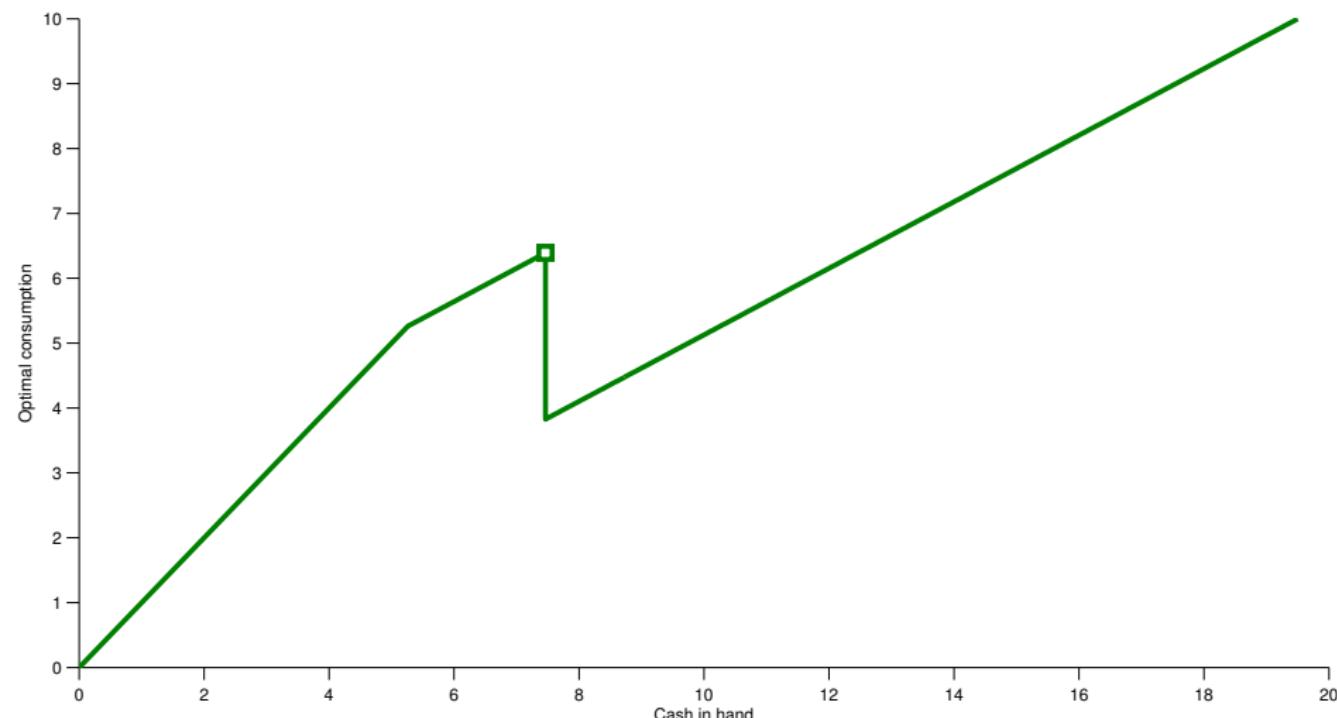
# Period $T - 1$ : Choice specific VF



# Period $T - 1$ : Choice specific VF



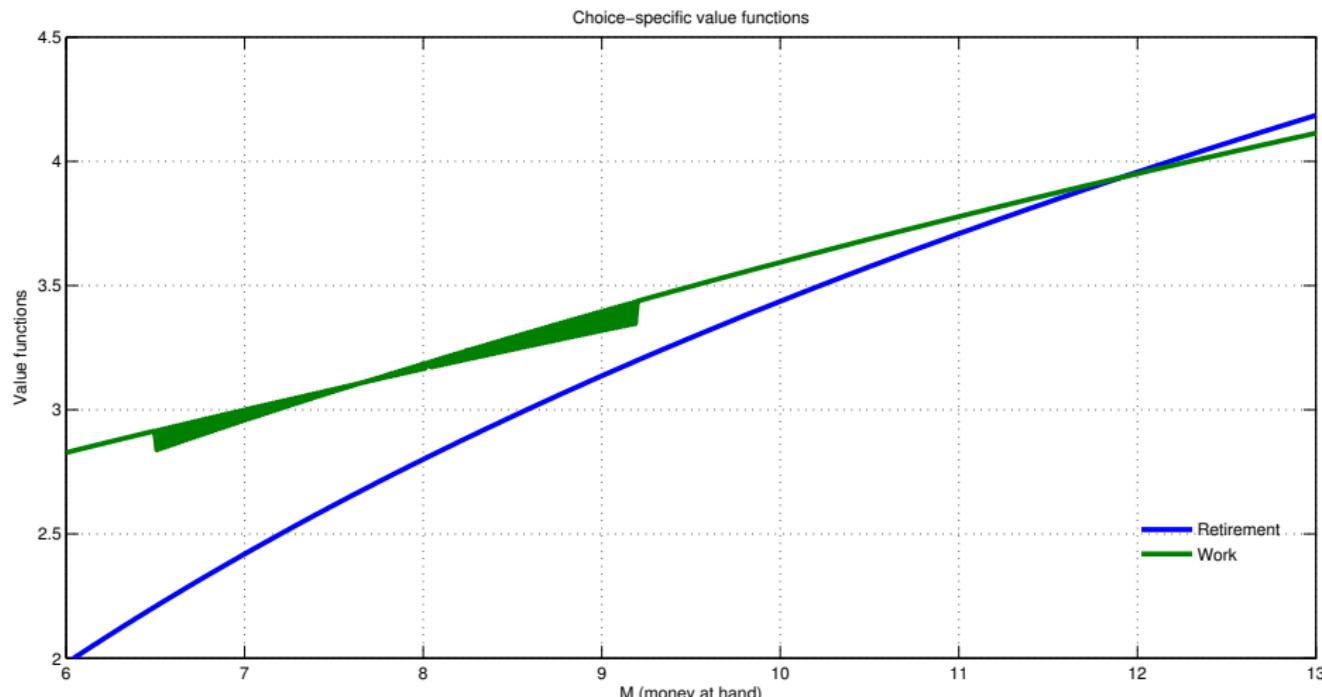
# Period $T - 1$ : Optimal consumption



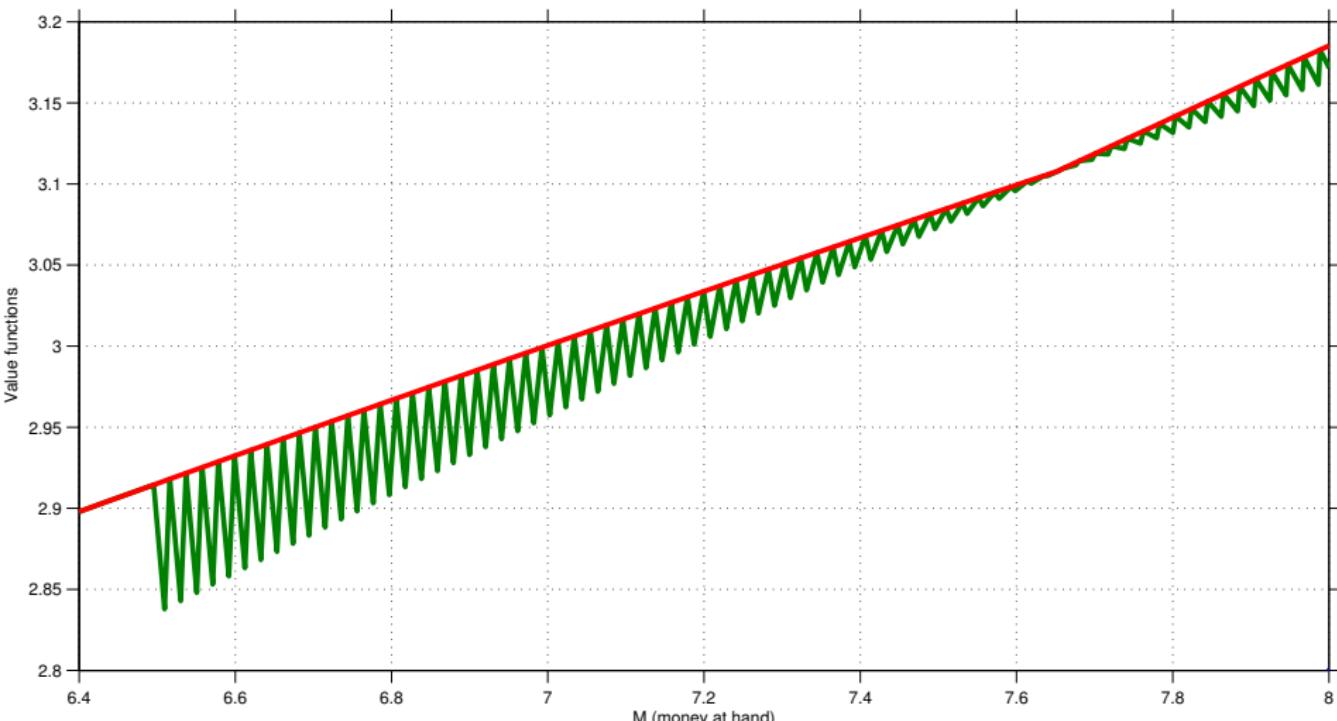
# So, what is going on

- ➊  $d$ -specific value functions intersect  
(due to trade-off between income and disutility of work)  
↓
- ➋ The **upper envelope** of the value functions has a kink  
and combined consumption function has a discontinuity

# Period $T - 2$ : Choice specific VF



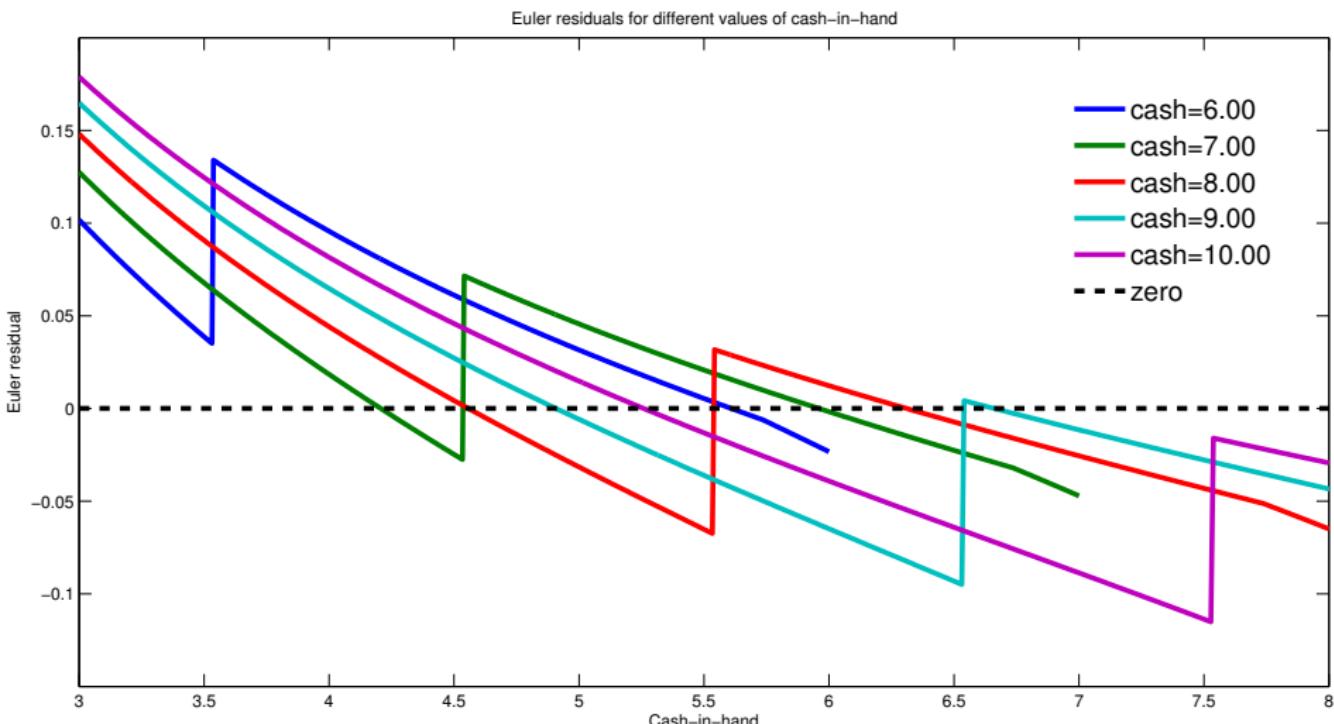
# Period $T - 2$ : Secondary upper envelope



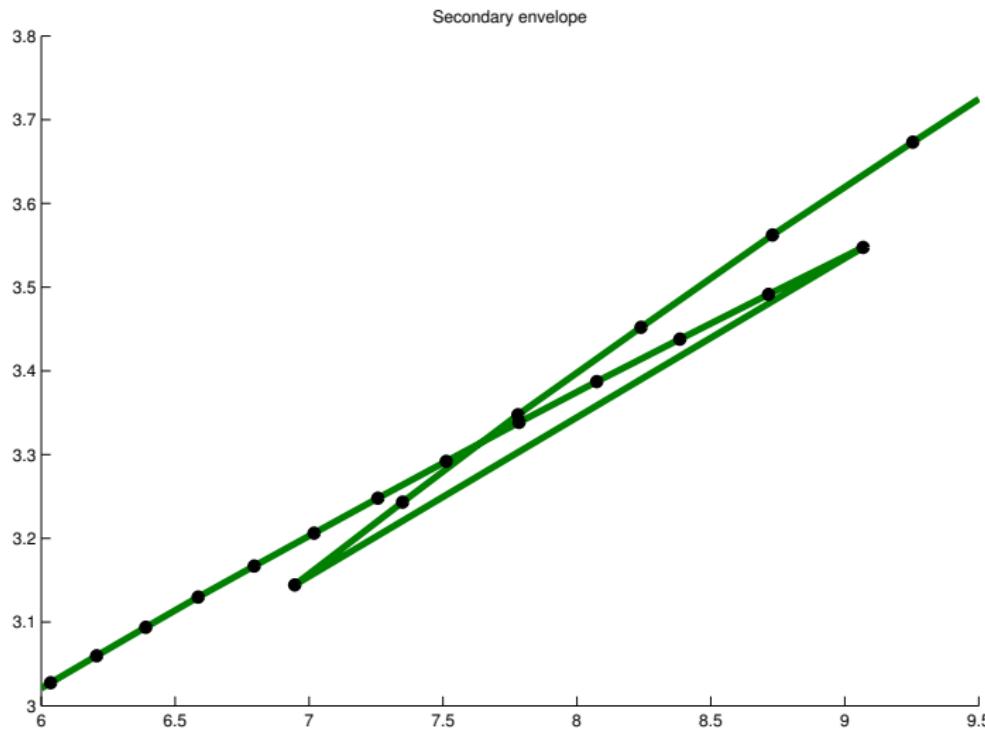
# So, what is going on

- ➊  $d$ -specific value functions intersect  
(due to trade-off between income and disutility of work)  
↓
- ➋ The **upper envelope** of the value functions has a kink  
and combined consumption function has a discontinuity  
↓
- ➌ Derivative of the value function has a discontinuity  
at the kink  
↓
- ➍ For some values of wealth (on endogenous grid) Euler equation has  
two solutions!  
If endogenous grid points are sorted → **zigzag region**

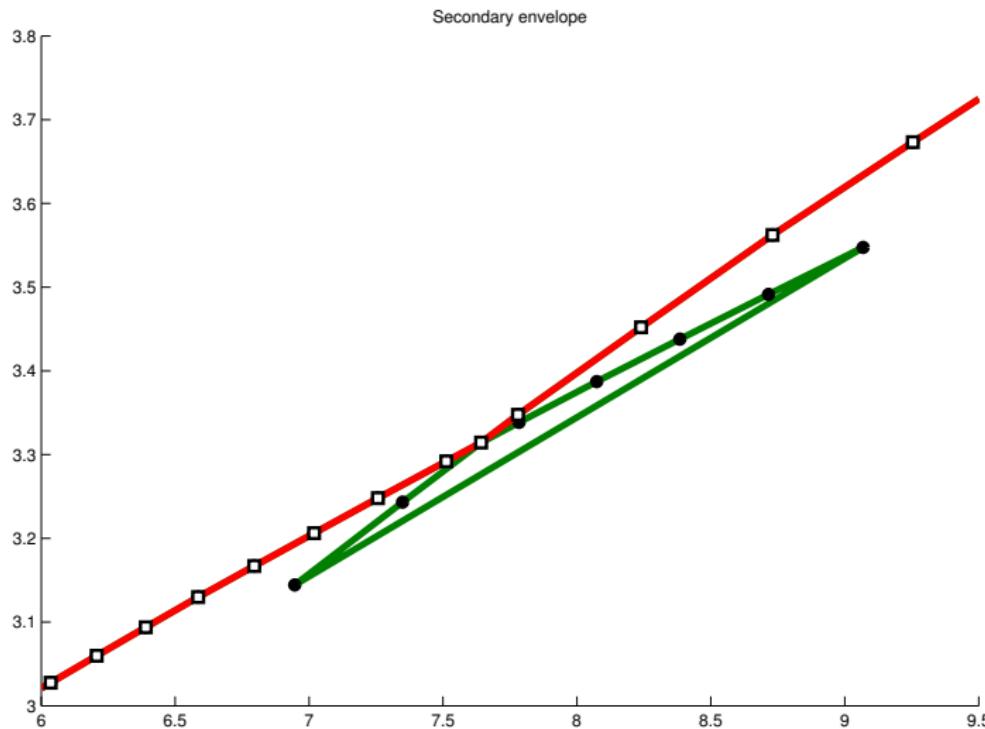
# Multiple zeros of Euler residuals



# Period $T - 2$ : Secondary upper envelope: detect



# Period $T - 2$ : Secondary upper envelope: result



# How to algorithmically detect “zigzag” regions?

Theorem: monotonicity

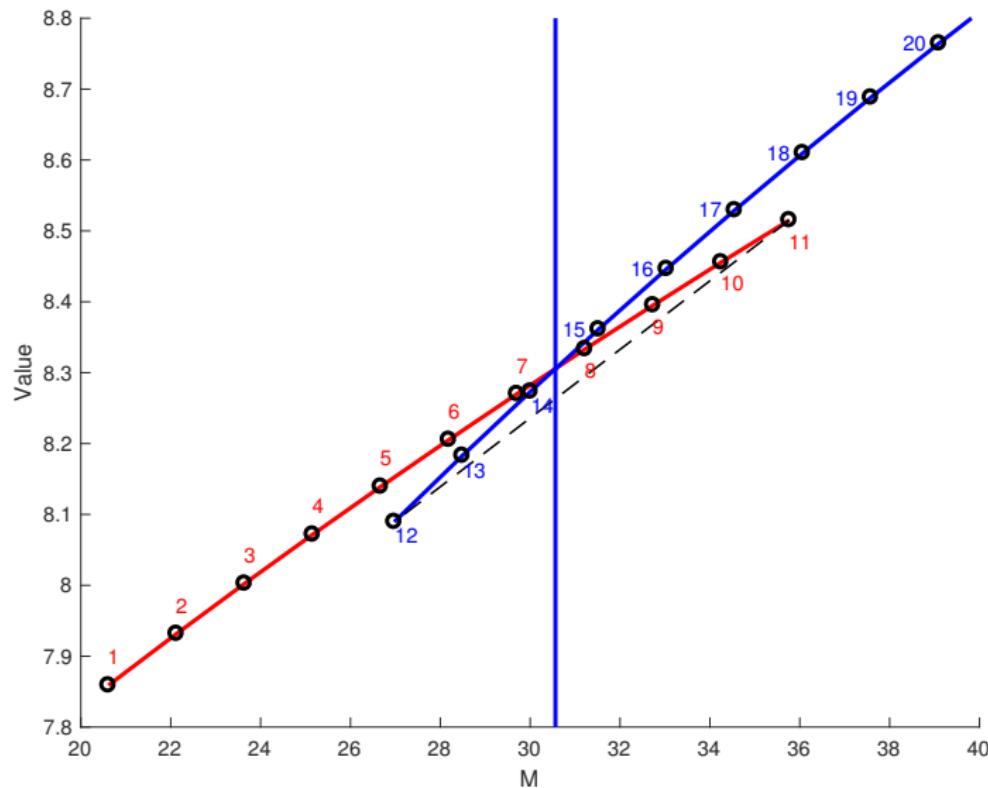
Under weak regularity conditions on the utility function and intertemporal budget constraint, savings function is weakly increasing.

$$A_t(M'_t) \geq A_t(M''_t) \text{ for every } M'_t \geq M''_t \text{ for all } t.$$

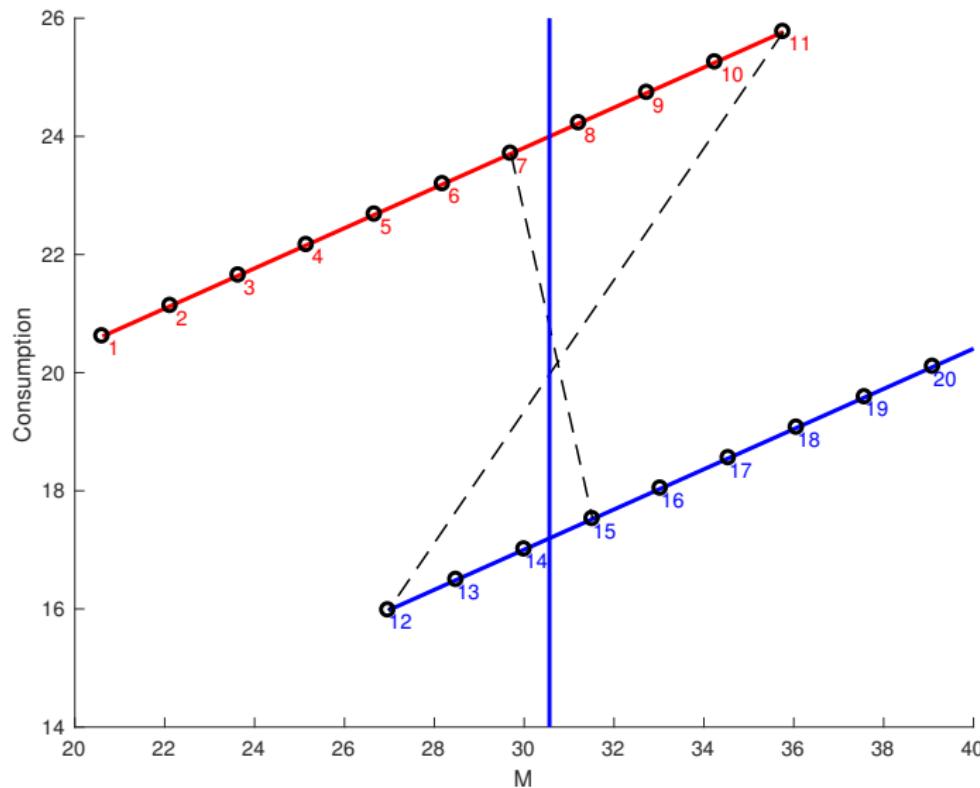
Note: savings function may still have “upward” jumps

- ➊ Sort the exogenous grid over  $A$  in **ascending order**
- ➋ Then the sequence of endogenous grid points over  $M$  has to be in **ascending order as well** as long as Euler equation is sufficient
- ➌ Every time the endogenous grid “**bends back**” the endogenous grid is separated into subsets of points
- ➍ Calculate the **Upper envelope** on the segments over the subsets
- ➎ **Delete suboptimal endogenous points**
- ➏ Find and add a kink point to the endogenous grid

# What happens to optimal consumption?



# What happens to optimal consumption?



## Alternative: fast upper envelope scan (FUES)



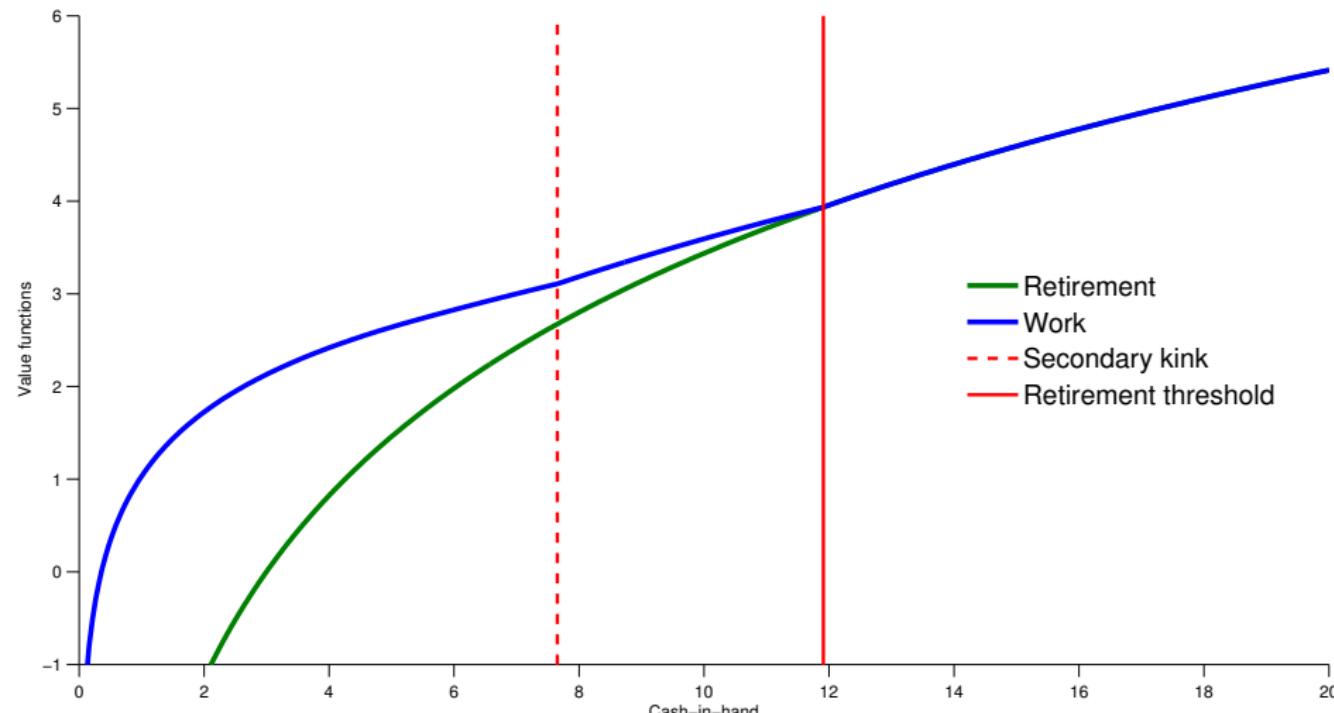
Loretti Dobrescu, Akshay Shanker (2022)

Fast Upper-Envelope Scan for Discrete-Continuous Dynamic Programming

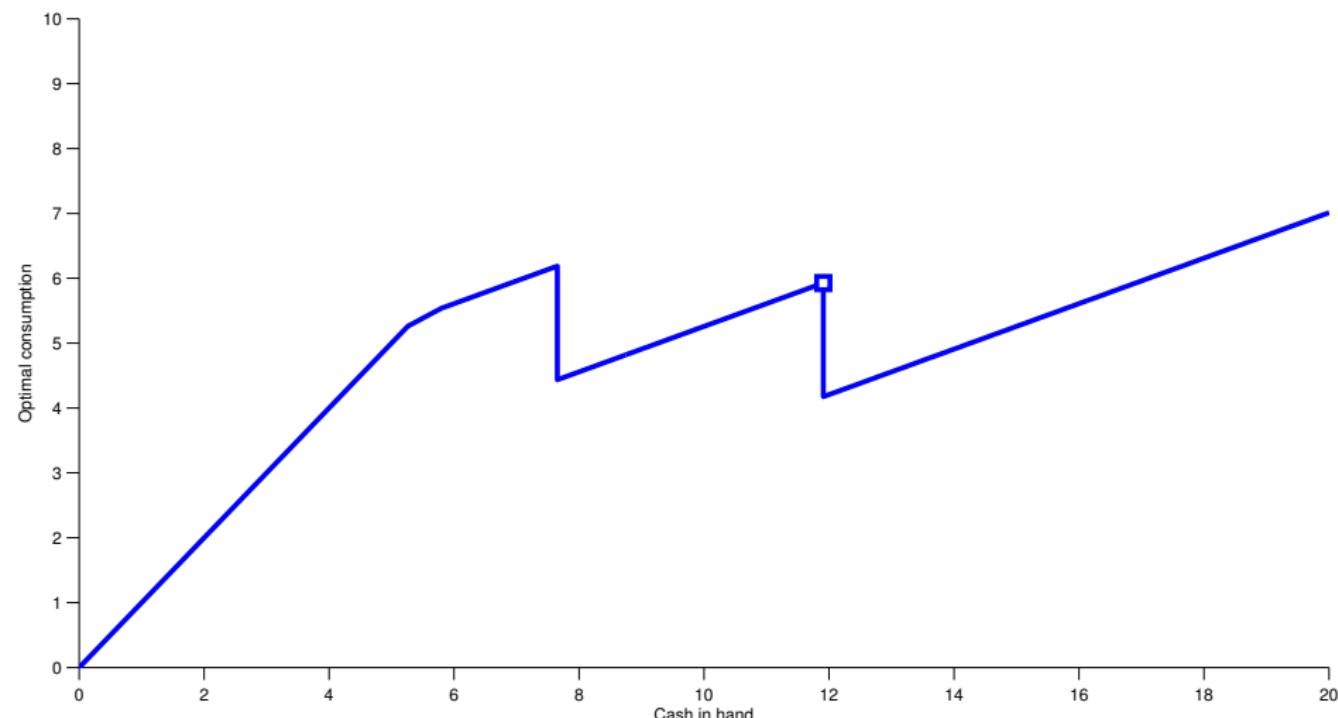
Given the endogenous points of the value function and the corresponding optimal consumption:

- ① Sort the endogenous points in ascending order
- ② Choose a jump threshold level for change in consumption
- ③ Given two consecutive points on the endogenous grid, if the value "function" decreases and in the same time consumption makes a jump, delete the point as inferior
- ④ Continue to the next point on the sorted endogenous grid
  - Does not rely on the monotonicity of the savings function
  - Can be generalized to higher dimensions
  - But depends on a pre-set threshold which potentially leads to inaccuracies of the computed model solution

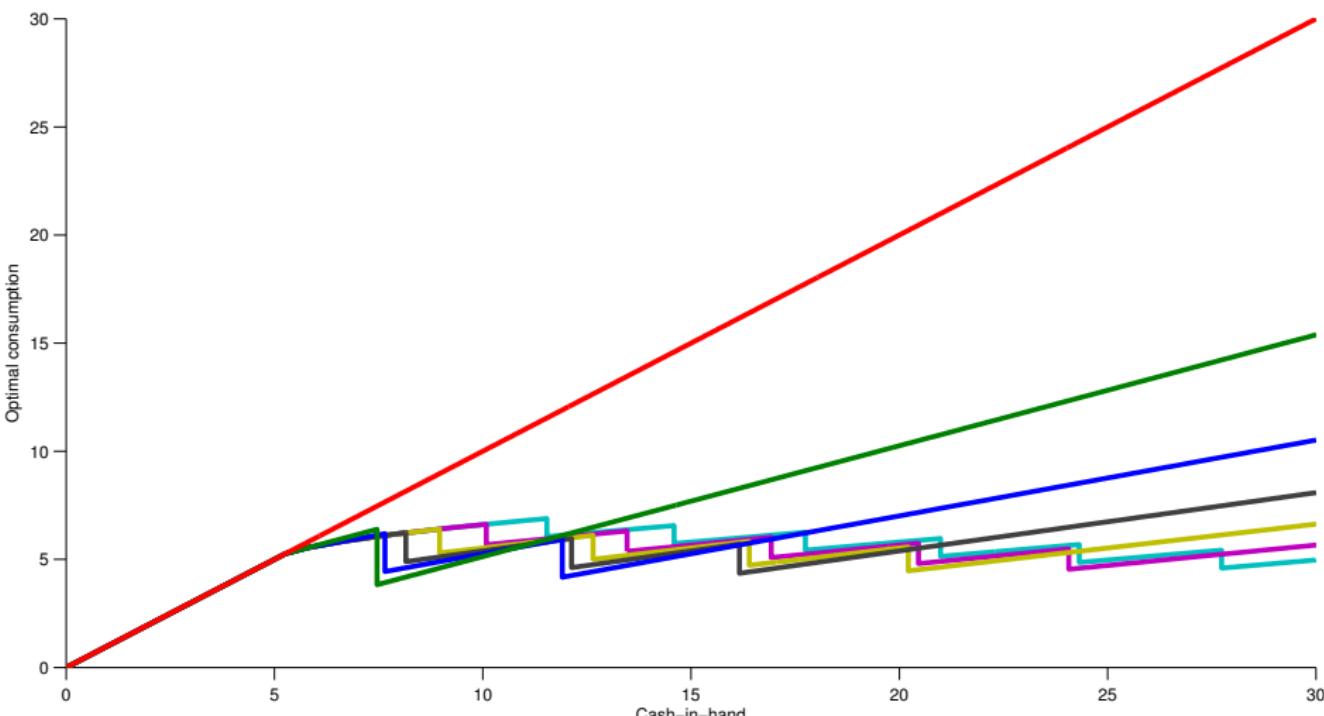
# Period $T - 2$ : VF, primary and secondary kinks



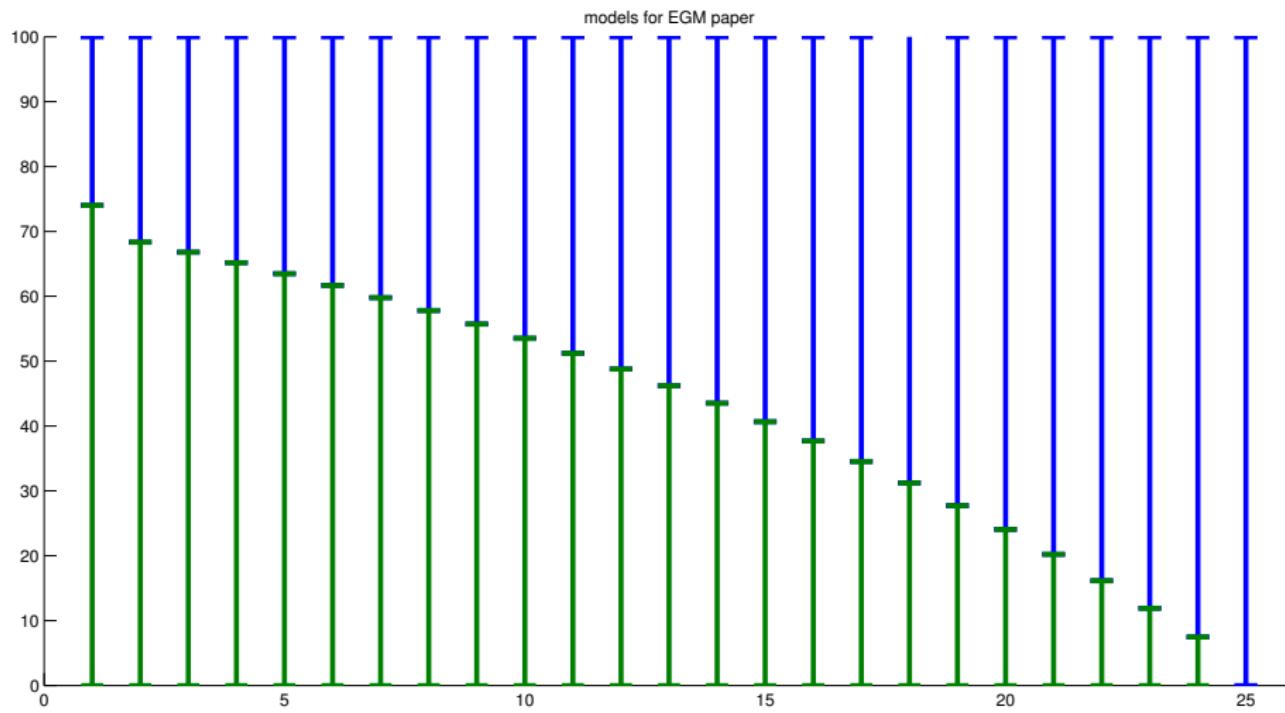
# Period $T - 2$ : Optimal consumption



# Optimal consumption (many periods)



# Optimal retirement (many periods)



# DC-EGM full algorithm

## DC-EGM ver. 2.0

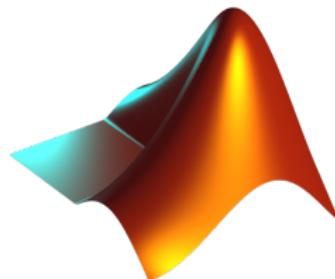
- ❶ Start from terminal period, compute optimal consumption rule and value function. Loop backwards over time:
- ❷ EGM step for each discrete choice  $d$  and every state  $st$
- ❸ Compute  $d$ -specific value functions and consumption rules
- ❹ Compute the “secondary” upper envelope over the “zig-zag” regions of the  $d$ -specific value functions and update the corresponding consumption rules
- ❺ Compare the  $d$ -specific value functions to find optimal switching points (compute upper envelope)
- ❻ Reconstruct overall consumption rule and value function from optimal switching points

# Properties of the full solution

- ➊ Value functions are non-concave and have **kinks**
- ➋ Consumption functions have **discontinuities**
- ➌ Discontinuities/kinks **propagate** through time and **accumulate**

This properties are attributes of the model itself.  
Any solution method has to deal with these complexities.

DC-EGM matches the analytical solution perfectly!



- ➊ Replicate the solution using `model_retirement.m`
- ➋ Simulate the consumption path for  $\beta R = 1$  and discuss the accuracy of the solutions

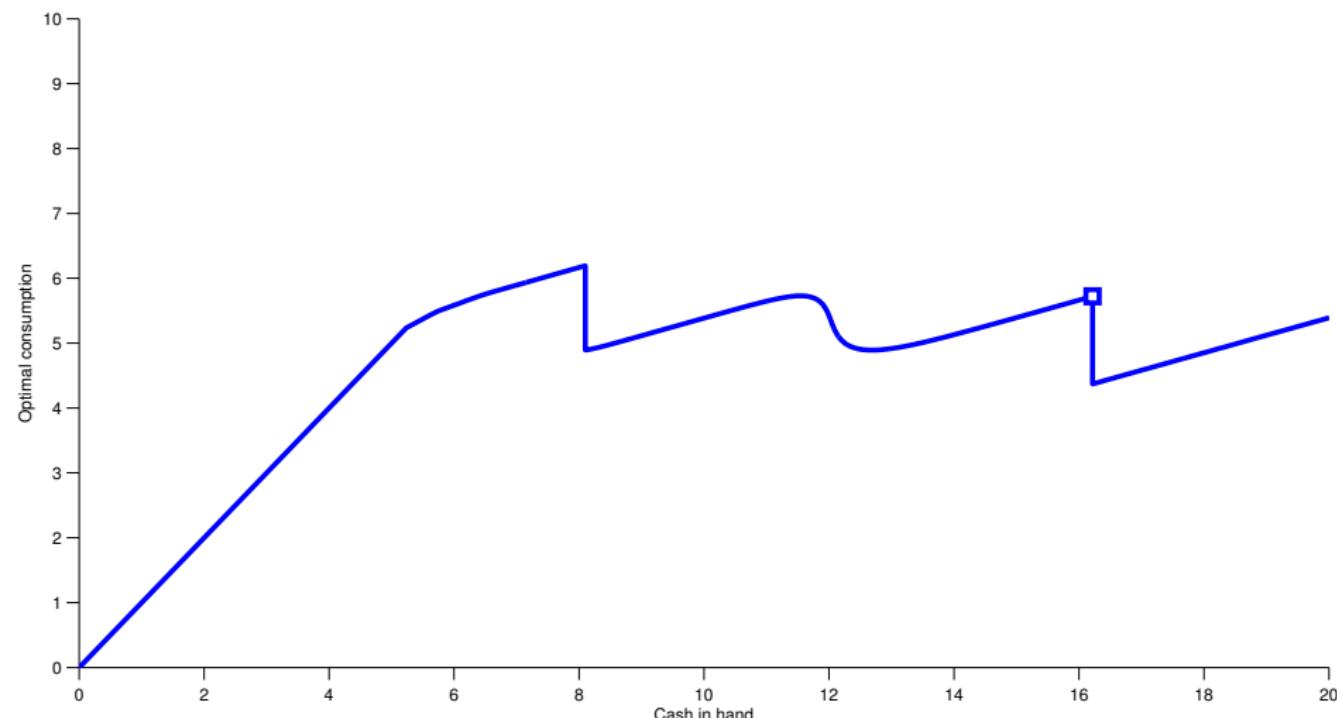
- ➌ See the code/python directory in the repository

# Random returns $\tilde{R}$

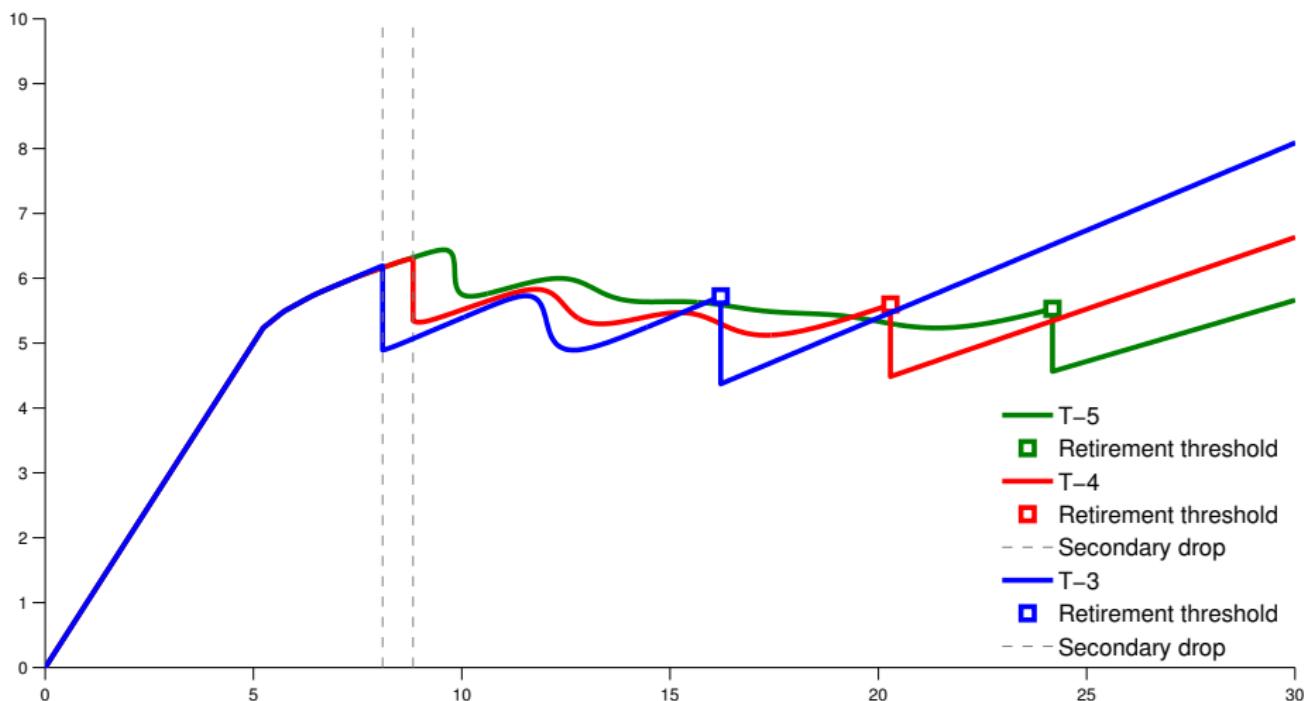
Random shocks do help, however:

- Smooth out secondary kinks only
- Primary kinks (switching between discrete options) remain
- May not smooth out all kinks: continuous but sharp declines in optimal consumption at  $t$  may lead to a discontinuity/kink at  $t - 1$
- Expectations in Euler equation have to be taken over discontinuous functions
  - More kinks/discontinuities from sloppy computation
  - Need to integrate over “continuous” intervals separately

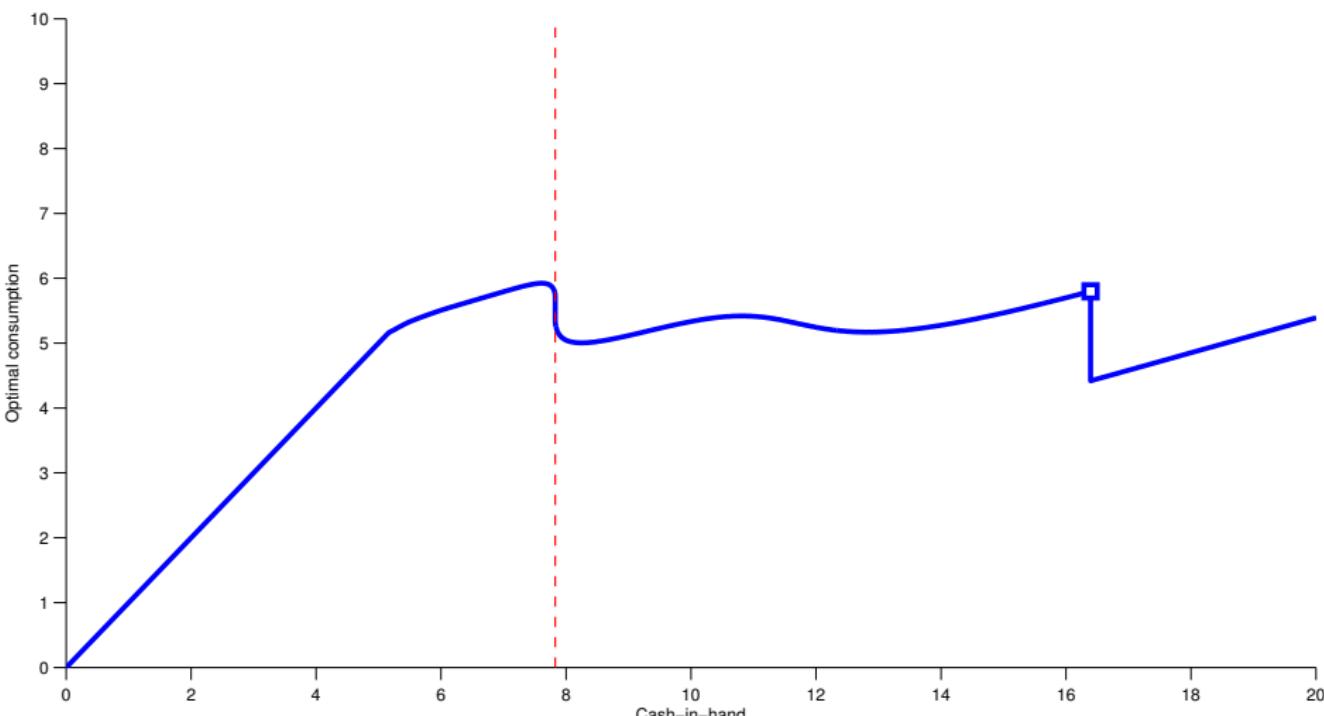
# Period $T - 3$ : Optimal consumption with $\sigma = 0.1$



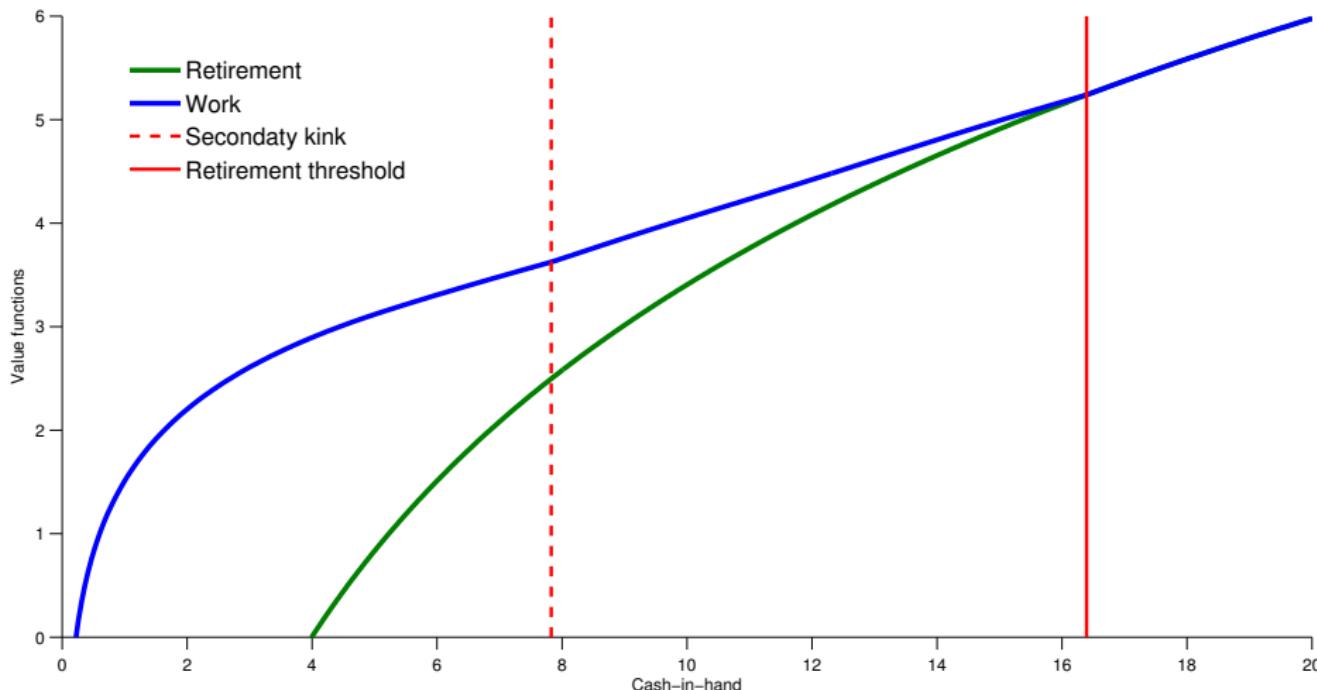
# Before $T - 3$ : Optimal consumption with $\sigma = 0.1$

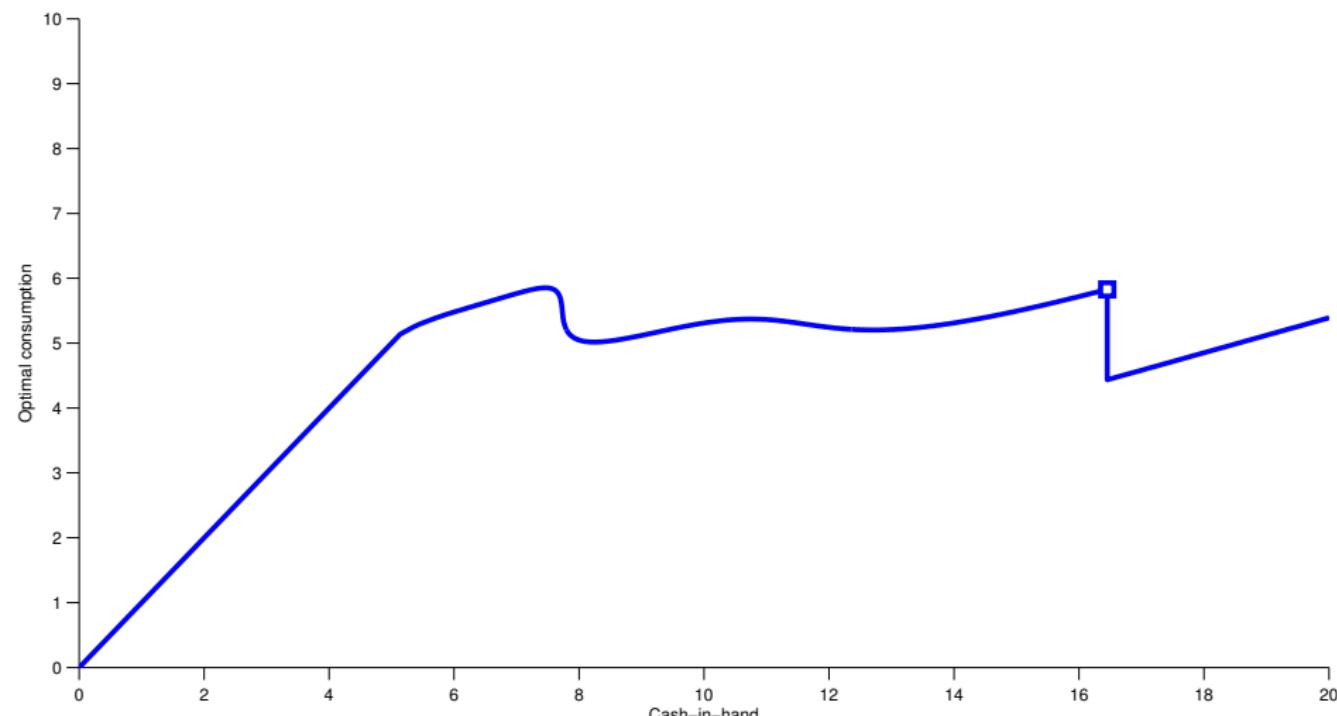


# Period $T - 3$ : Optimal consumption with $\sigma = .2$

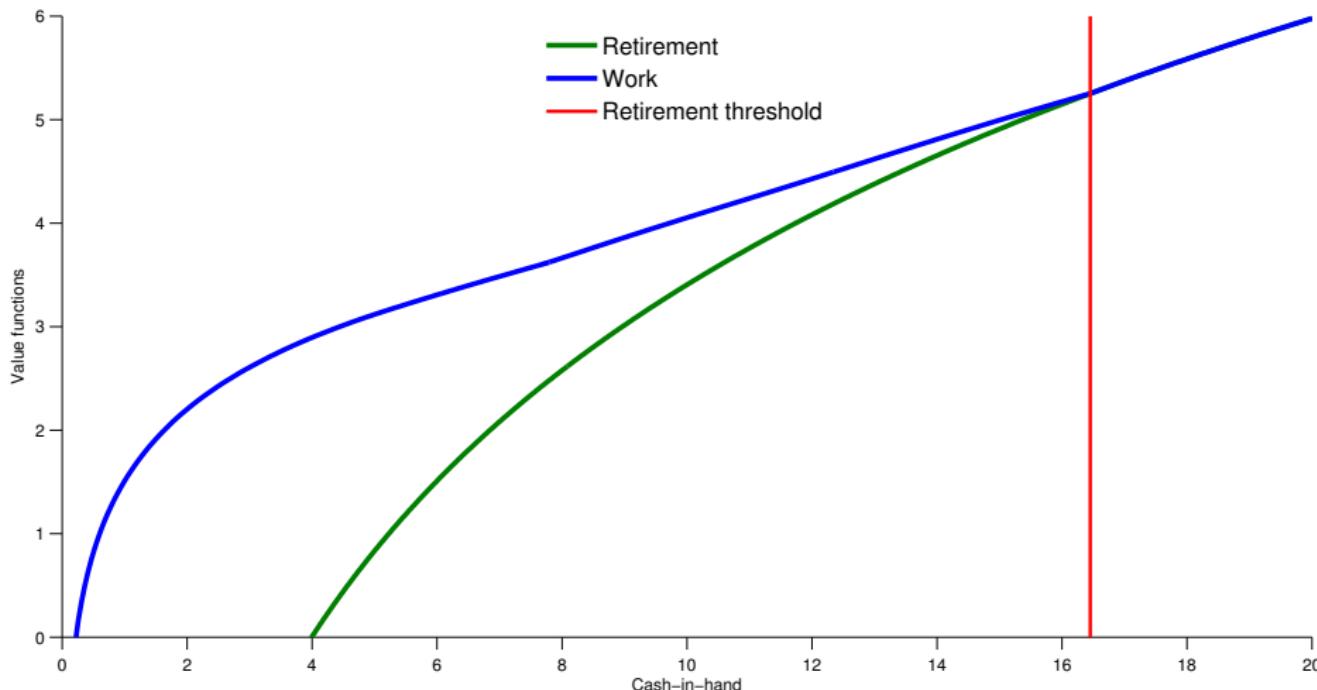


# Period $T - 3$ : VF with $\sigma = .2$



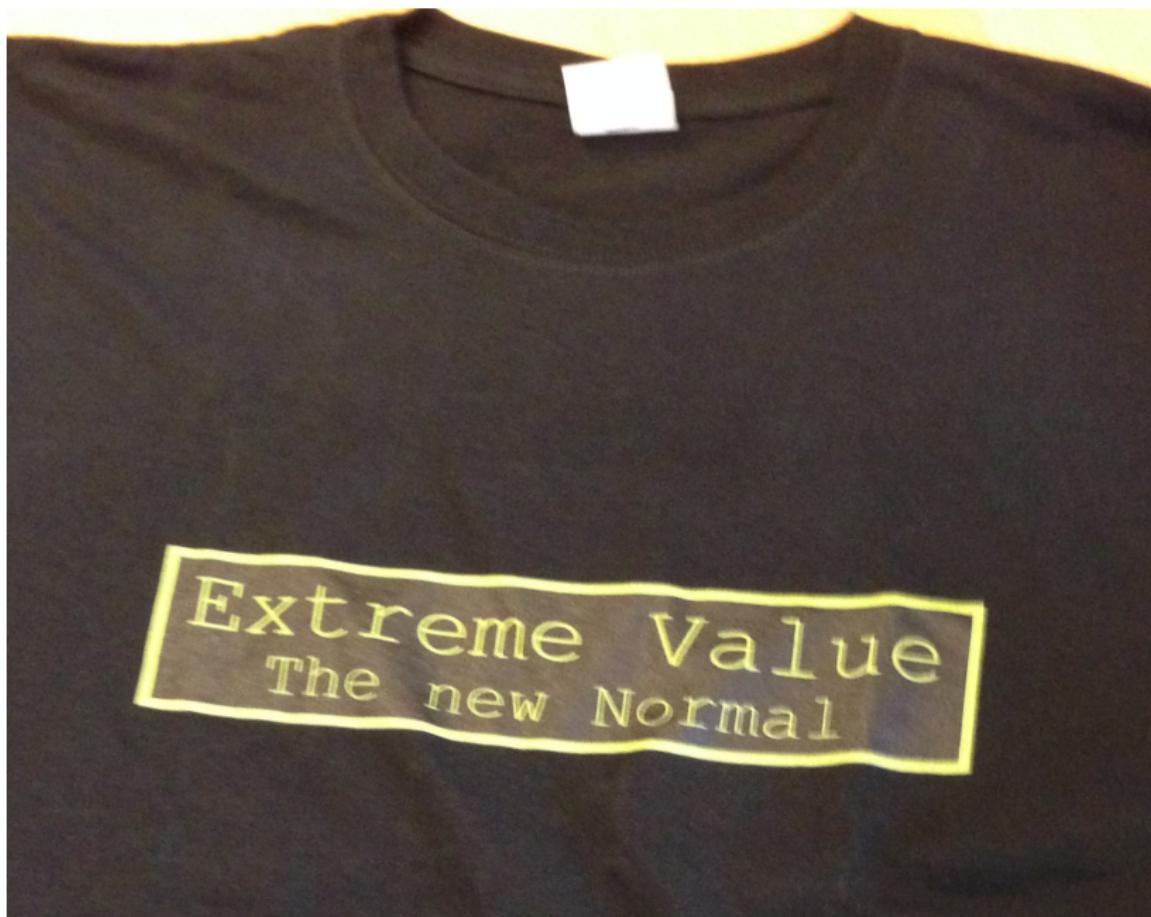
Period  $T - 3$  : Optimal consumption with  $\sigma = .22$ 

# Period $T - 3$ : VF with $\sigma = .22$



# Extreme value distributed taste shocks

- Smooth out primary kinks
- Extreme value distribution – closed form expectations and standard in empirical applications
- Two interchangeable interpretations
  - Structural: unobserved state variables
  - Logit smoothing: to streamline the solution
- Work together with other shocks in the model
  - EV taste shocks smooth out primary kinks
  - Random returns smooth out secondary kinks
- Complete smoothing is not guaranteed in general: secondary kinks may persist



# Retirement problem with taste shocks

Re-formulate in terms of **choice specific** value functions

$$V_t(M_t, \mathbb{W}) = \max \left\{ \begin{array}{l} v_t(M_t, \mathbb{W}, \mathbb{R}) + \sigma \epsilon_{\mathbb{R}} \\ v_t(M_t, \mathbb{W}, \mathbb{W}) + \sigma \epsilon_{\mathbb{W}} \end{array} \right\}$$

$$\begin{aligned} v_t(M_t, \mathbb{W}, \mathbb{W}) &= \max_{0 \leq c \leq M_t} \left[ u(c) + \beta EV_{t+1} \left( \tilde{R}(M_t - c) + y, \mathbb{W} \right) \right] \\ v_t(M_t, \mathbb{W}, \mathbb{R}) &= \max_{0 \leq c \leq M_t} \left[ u(c) + \beta EV_{t+1} \left( \tilde{R}(M_t - c), \mathbb{R} \right) \right] \end{aligned}$$

$$EV_{t+1}(x, \mathbb{W}) = \sigma \log \left[ \exp \frac{v_{t+1}(x, \mathbb{W}, \mathbb{W})}{\sigma} + \exp \frac{v_{t+1}(x, \mathbb{W}, \mathbb{R})}{\sigma} \right]$$

$$V_t(M_t, \mathbb{R}) = \max_{0 \leq c \leq M_t} \left[ u(c) + \beta EV_{t+1} \left( \tilde{R}(M_t - c), \mathbb{R} \right) \right]$$

# Smoothed Euler equation

Without taste shocks – “**discontinuous**” Euler equation:

$$u'(c_t) = \beta E \left[ u' \left( c_{t+1}(\mathbb{W}/\mathbb{R}) \right) \tilde{R} \right]$$

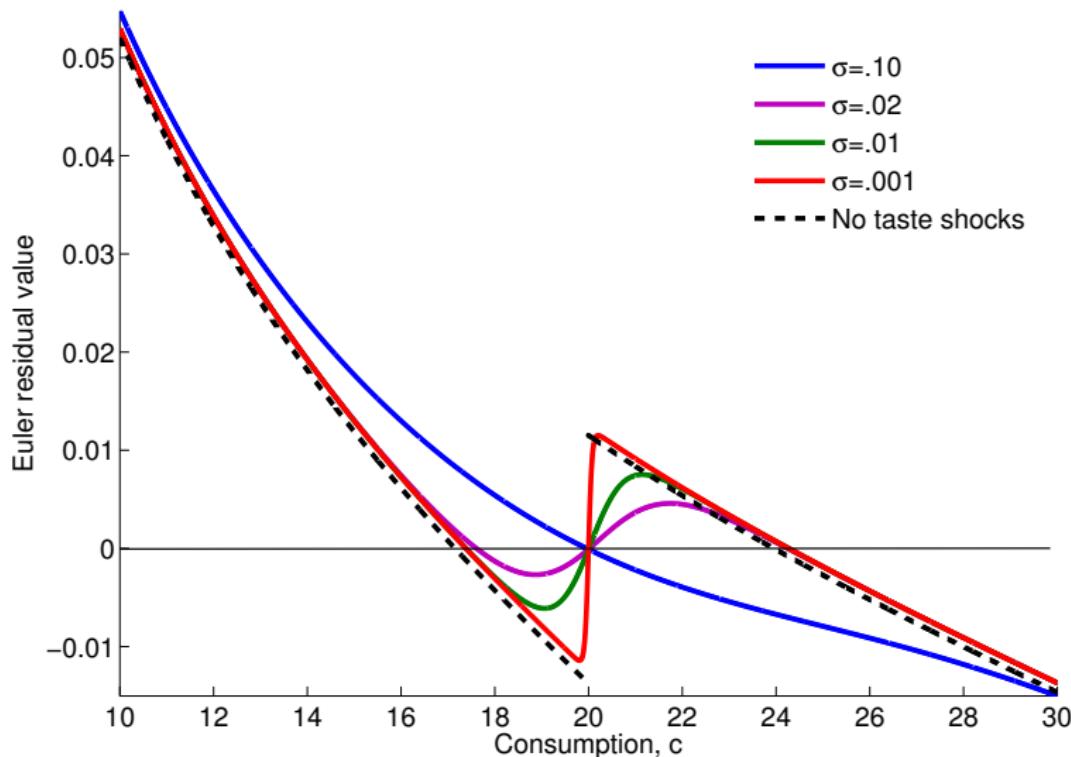
With EV taste shocks – **smoothed** Euler equation:

$$u'(c_t) = \beta E \left[ P_{t+1}(\mathbb{W}) u' \left( c_{t+1}(\mathbb{W}) \right) \tilde{R} + P_{t+1}(\mathbb{R}) u' \left( c_{t+1}(\mathbb{R}) \right) \tilde{R} \right]$$

Choice probability

$$P_{t+1}(\mathbb{W}) = \frac{\exp \frac{v_{t+1}(M_{t+1}, \mathbb{W}, \mathbb{W})}{\sigma}}{\exp \frac{v_{t+1}(M_{t+1}, \mathbb{W}, \mathbb{W})}{\sigma} + \exp \frac{v_{t+1}(M_{t+1}, \mathbb{W}, \mathbb{R})}{\sigma}}$$

# Smoothed Euler equation



# DC-EGM with taste shocks

DC-EGM ver. 3.0

- ① Start from terminal period, compute optimal consumption rule and value function. Loop backwards over time:
- ② EGM step for each discrete choice  $d$  and every state  $st$
- ③ Compute  $d$ -specific value functions and consumption rules
- ④ Compute the “secondary” upper envelope over the “zig-zag” regions of the  $d$ -specific value functions and update the corresponding consumption rules
- ⑤ Compare the  $d$ -specific value functions to find optimal switching points (compute upper envelope)
- ⑥ Reconstruct overall consumption rule and value function from optimal switching points

# DC-EGM with taste shocks

- ① With EV taste shocks DC-EGM becomes **simpler**
- ② The problem is re-formulated in terms of **choice specific value functions**
- ③ Calculation of *primary* upper envelope is replaced by calculation of **logsum**
- ④ Easier computation of expectations (due to less discontinuities)
- ⑤ More memory is required to store choice specific value functions

# Extreme value Homotopy

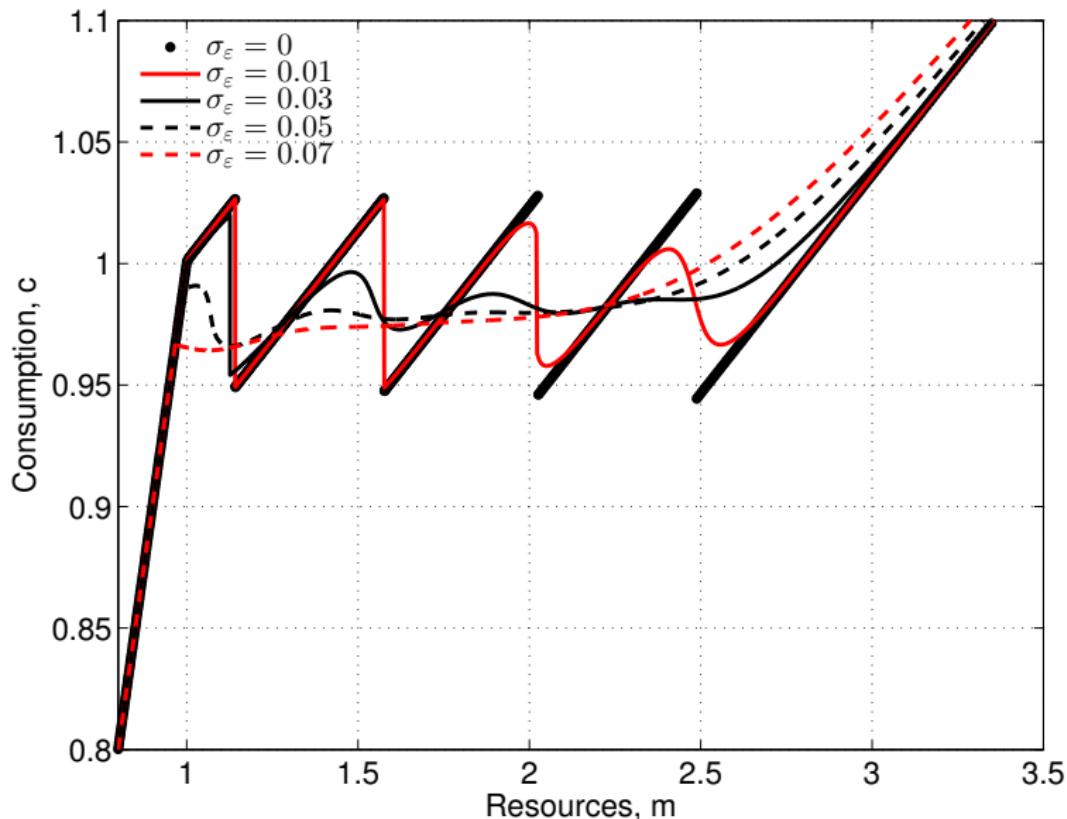
Theorem: approximation with logit smoother

Let  $\sigma$  be the scale of Type 1 extreme value taste shocks for the discrete choices in a DC problem with  $D$  choices. Then we have the following bound

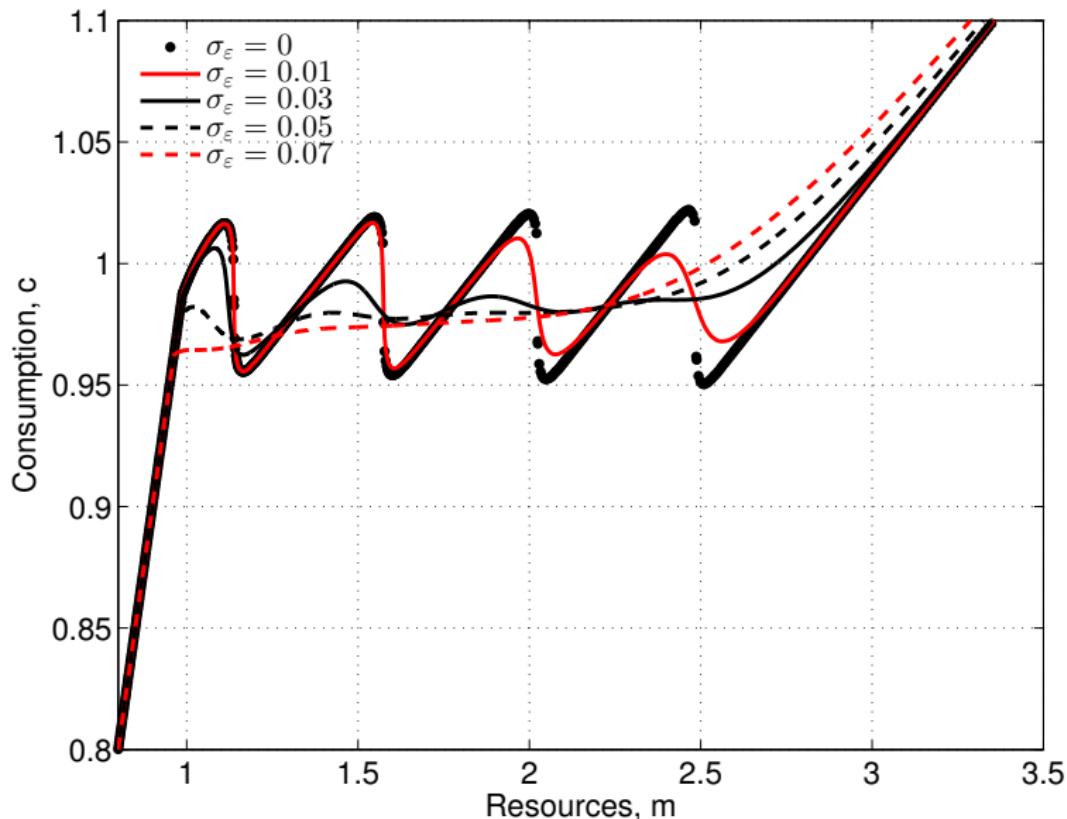
$$|EV_{\sigma,t}(s) - V_t(s)| \leq \sigma \left[ \sum_{j=0}^{T-t} \beta^j \right] \log(D)$$

This implies that the extreme-value perturbed policy functions  $c_{\sigma,t}(s, \epsilon)$  and  $\delta_{\sigma,t}(s, \epsilon)$  converge pointwise to  $c_t(s)$  and  $\delta_t(s)$ , the optimal continuous and discrete decision rules to a DP problem without any taste shocks as  $\sigma \rightarrow 0$ .

# Optimal consumption with taste shocks only



# Optimal consumption with random returns



# Credit constraints

- Credit constraints are handled so well by EGM because it is never necessary to compute utility of nearly zero consumption
- Instead we “connect the dots”  $(0, 0)$  and  $(M_t^{cc}, M_t^{cc})$

$M_t^{cc}$  — level of wealth corresponding to  $A_t = 0$

- Inevitable when value functions have to be computed
- If utility is additively separable in consumption and discrete choices (AS), the problem can be avoided entirely!

# Credit constraints

## Dealing with credit constraints

- For each  $d_t$  compute  $M_{t,d_t}^{cc}$  correspond to zero savings  
EGM loop can be started from  $A = 0$

$$M_{t,d_t}^{cc} : \forall M < M_{t,d_t}^{cc} \quad c_t^* = M$$

- Value function for  $M < M_{t,d_t}^{cc}$  has analytic form

$$V_t^{d_t}(M) = u(M, d_t) + \beta EV_{t+1}^0(d_t)$$

$EV_{t+1}^0(d_t)$  — expected value of ending period  $t$  with  $A_t = 0$

- (AS)  $\Rightarrow V_t^{d_t}(M) = u(M) + v(d_t) + \beta EV_{t+1}^0(d_t)$

- $V_t^{d_t}(M)$  do not intersect when  $M < \min_{d_t} \{ M_{t,d_t}^{cc} \}$

$\Rightarrow$  No need to search for intersection points at nearly zero wealth  
 $\Rightarrow$  Choice probabilities do not change

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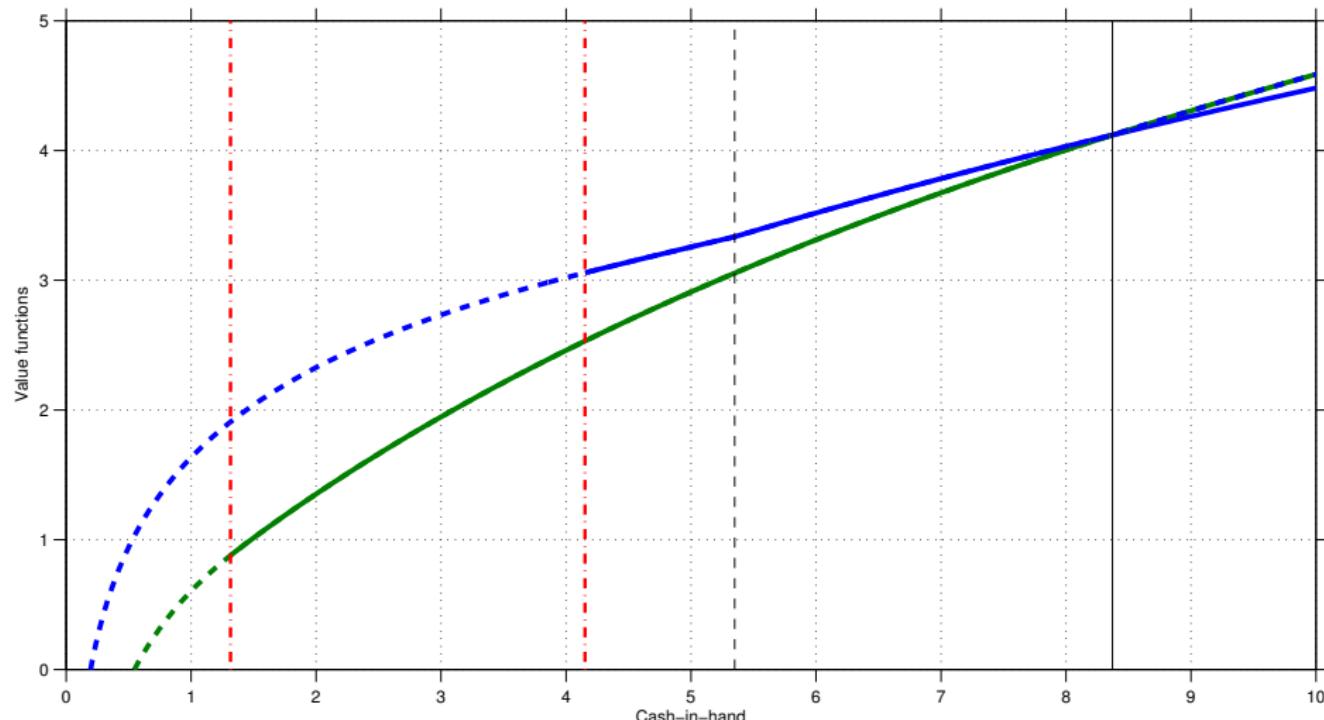
$EV_{t+1}^0(d_t)$  — expected value of ending period  $t$  with  $A_t = 0$

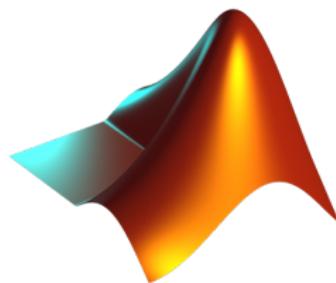
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 $\Rightarrow$  Choice probabilities do not change

# Pension benefit .25y





- ➊ Solve the `model_retirement.m` with taste shocks
- ➋ Simulate some consumption paths and distributions of retirement age
- ➌ See the `code/python` directory in the repository

# Multi-dimensional and Generalizations



Giulio Fella, 2025

Endogenous grid method (review, downloadable)

# EGM + VFI



Barillas & Fernandez-Villaverde, JEDC 2007

A Generalization of the Endogenous Grid Method

- ① Run EGM w.r.t. *one* choice keeping other controls fixed
- ② Perform a VFI w.r.t. the rest of decision variables

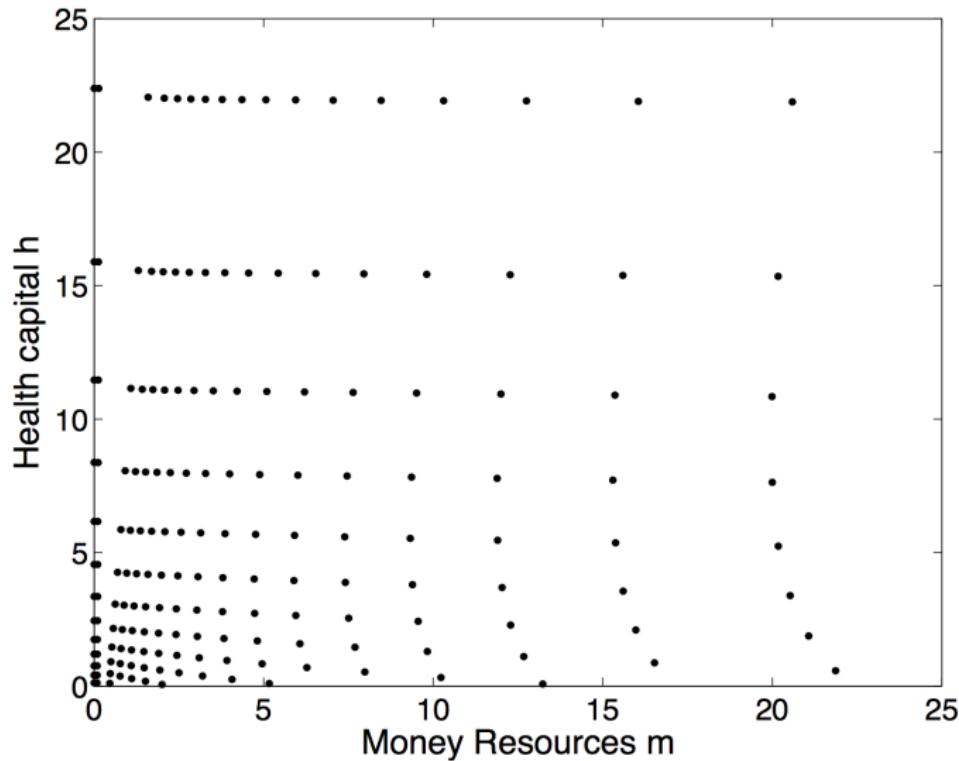


Ludwig & Schön, Computational Economics, 2018

Endogenous Grids in Higher Dimensions: Delaunay Interpolation and Hybrid Methods

- ① Solve the model of human capital investment + consumption/savings
- ② Compare three approaches which differ by the interpolation method
- ③ Need to interpolate on irregular multidimensional grid

# Multidimensional endogenous grid



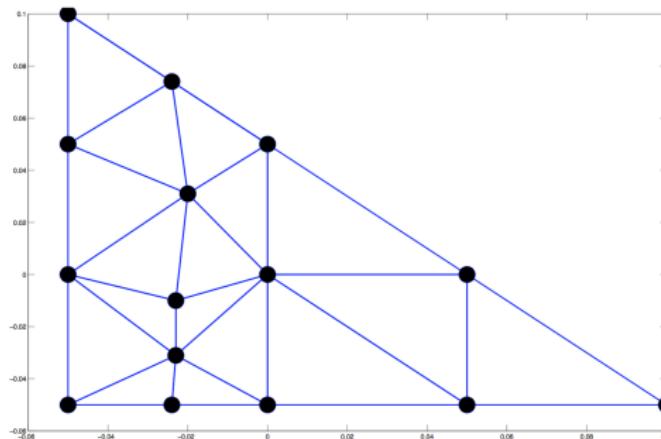
# Interpolation on the irregular grid



Johannes Brumm, Michael Grill, JEDC 2014

Computing equilibria in dynamic models with occasionally binding constraints

- Delaunay triangulation based interpolation



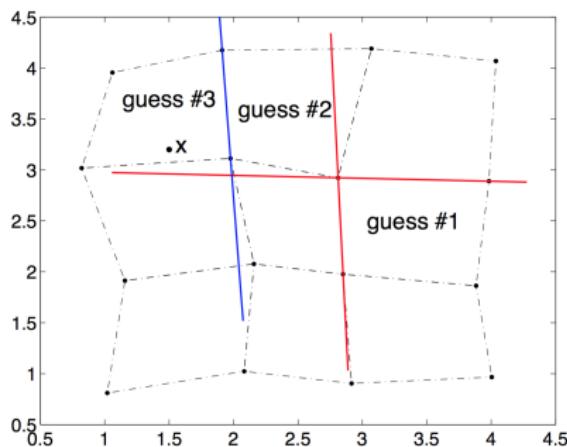
# Interpolation on the irregular grid



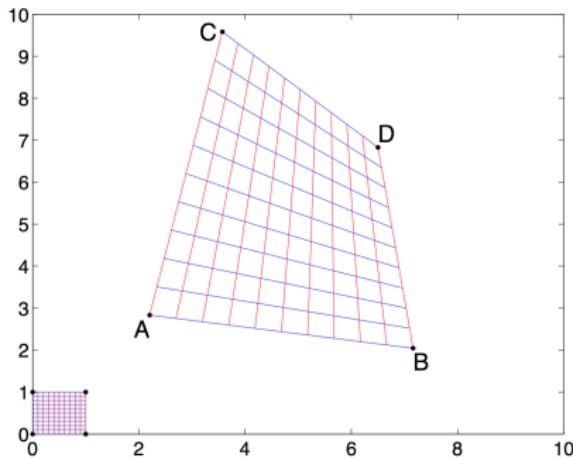
Matthew White, JEDC 2015

The Method of Endogenous Gridpoints in Theory and Practice

- “Visibility walk” approach to allocate the non-linear rectangle containing the interpolation point
- Map non-linear rectangles into regular ones



(a) Identifying the sector by visibility walk



(b) Identifying relative coordinates

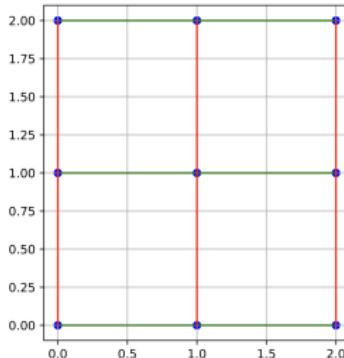
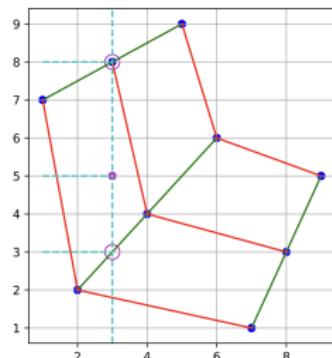
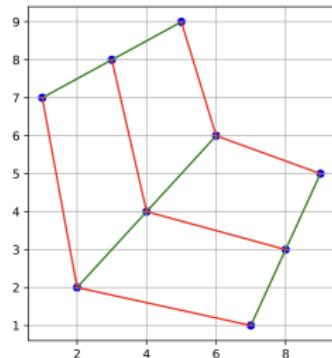
# Interpolation on the irregular grid



Alan Lujan, 2024

EGM<sup>n</sup>: The Sequential Endogenous Grid Method

- Improves on the interpolation scheme in White(2015)
- Interpolate using “index coordinate points” of the matrices of endogenous grids
- Multi-linear interpolation on non-linear grids afterwards



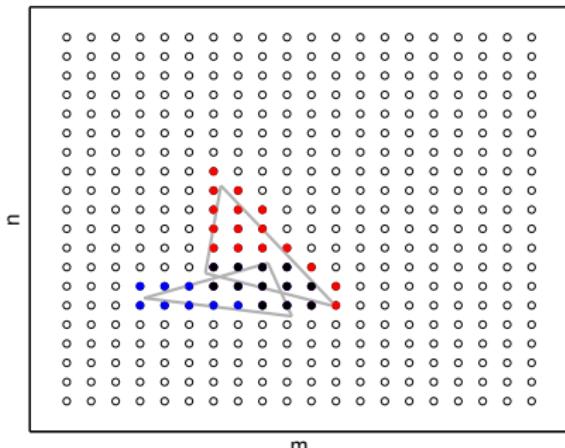
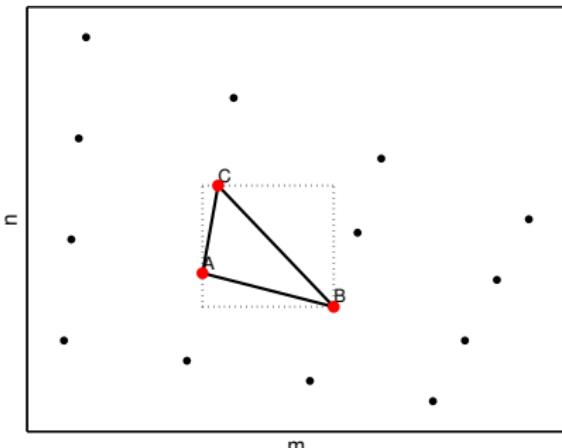
[github.com/alanlujan91/multinterp](https://github.com/alanlujan91/multinterp)

# Interpolation on the irregular grid



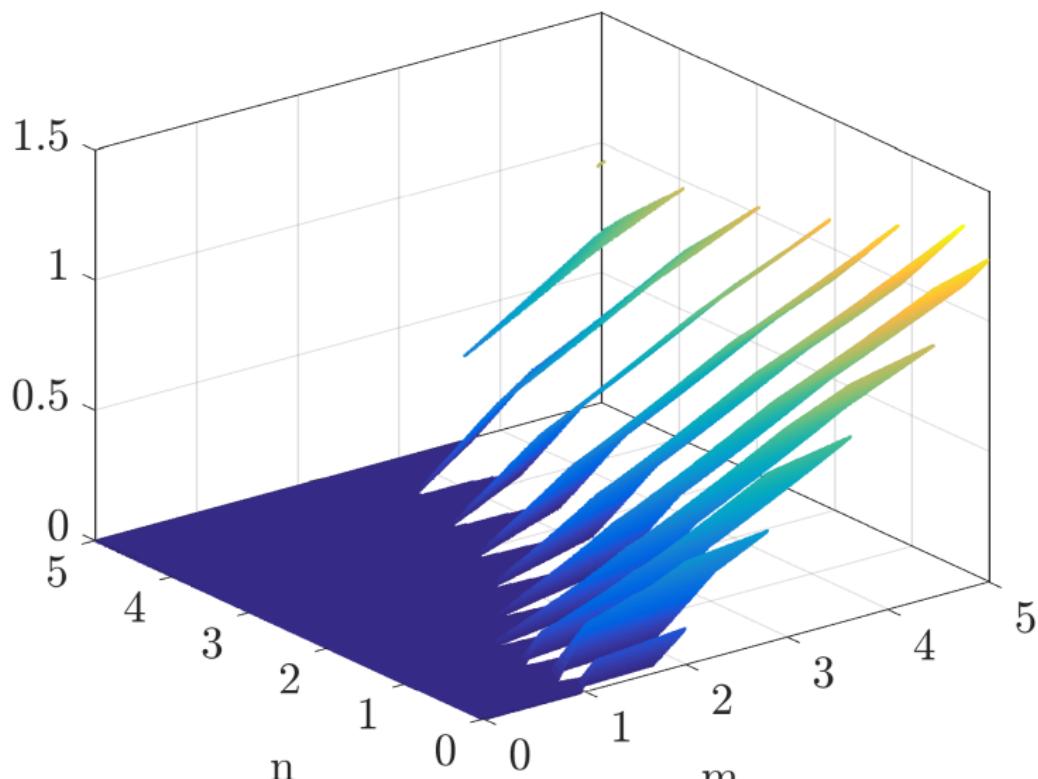
Jeppe Druedahl, Thomas Jørgensen, JEDC 2017  
A General Endogenous Grid Method for Multi-Dimensional Models  
with Non-Convexities and Constraints

- Focus on occasionally binding constraints and for non-convexities
- Re-interpolate on regular grid while performing upper envelope



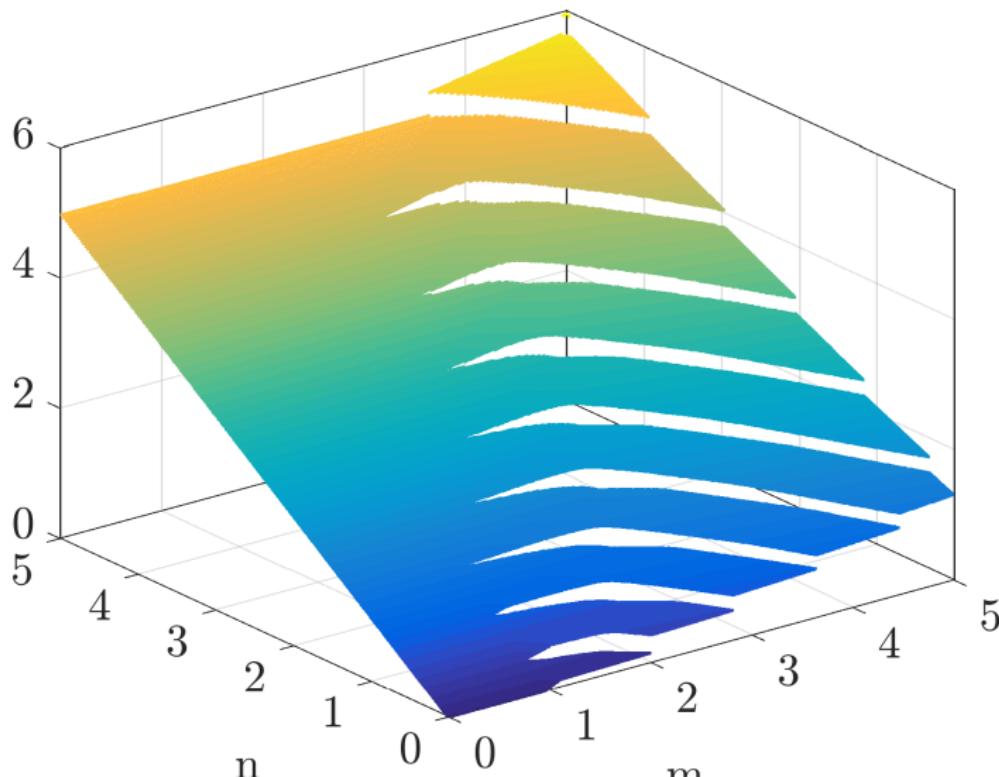
# Consumption + pension contributions model

## Pension fund contributions policy function



# Consumption + pension contributions model

Next period pension wealth  $n$



# General theory on multidimensional EGM



Matthew White, JEDC 2015

The Method of Endogenous Gridpoints in Theory and Practice

- Invertibility condition for the system of non-linear equations



Jeppe Druedahl, Thomas Jørgensen, JEDC 2017

A General Endogenous Grid Method for Multi-Dimensional Models with Non-Convexities and Constraints

- Formulate the sufficient condition,  
i.e. particular mapping has to be an injection (one-to-one mapping)



Iskhakov, Econ Letters 2015

Multidimensional endogenous gridpoint method: solving triangular dynamic stochastic optimization problems without root-finding operations + Corrigendum

- Focus on analytical invertibility to avoid root-finding operations

# Triangular example in $\mathbb{R}^3$

$$V_t(M_t) = \max_c \left[ u(c) + \beta E\{V_{t+1}(M_{t+1}) | M_t, c\} \right]$$

$M_t \in \mathbb{R}^3$  vector of resources at time  $t$

$c_t \in \mathbb{R}^3$  utilization of resources

$A_t \in \mathbb{R}^3$  resources at the end of  $t$  (**post-decision states**)

$A_t = g(M_t, c_t)$  intra-temporal evolution

$M_{t+1} = f(A_t, \tilde{R})$  inter-temporal evolution (budget constraint)

$\tilde{R}$  idiosyncratic stochastic shocks

$u : \mathbb{R}^3 \rightarrow \mathbb{R}$  utility of current resource utilization

Plan:

- Apply EGM ideas to discover the necessary assumptions on the fundamental of the problem that make it possible
- Abstract away from the budget constraint and other feasibility restrictions

# Post-decision states

- Note, that we already assumed:

$$M_{t+1} = f(A_t, \tilde{R}) = f(g(M_t, c_t), \tilde{R})$$

- allowing to replace the condition in the expectation in the Bellman equation to

$$E\{V_{t+1}(M_{t+1})|M_t, c_t\} = E\{V_{t+1}(M_{t+1})|A_t\}$$

- $\Leftrightarrow A_t$  are **sufficient statistics** for  $(M_t, c_t)$

**(TRI1)** Assume that the vector of **post-decision state** variables follows

$$\begin{aligned} A_t^1 &= g^1(M_t, c_t) &= g^1(M_t^1, c_t^1) \\ A_t^2 &= g^2(M_t, c_t) &= g^2(M_t^1, c_t^1, M_t^2, c_t^2) \\ A_t^3 &= g^3(M_t, c_t) &= g^3(M_t^1, c_t^1, M_t^2, c_t^2, M_t^3, c_t^3) \end{aligned}$$

# Bellman optimization and FOCs

$$V_t(M_t) = \max_c \left[ u(c) + \beta E \left\{ V_{t+1} \left( \begin{array}{c} f^1(A_t^1, A_t^2, A_t^3, \tilde{R}) \\ f^2(A_t^1, A_t^2, A_t^3, \tilde{R}) \\ f^3(A_t^1, A_t^2, A_t^3, \tilde{R}) \end{array} \right) \middle| A_t \right\} \right]$$

The first order conditions are:

$$u'_1(c) + \beta E \sum_{i=1}^3 \sum_{j=1}^3 (V_{t+1})'_i (f^i)'_j (\partial g^j / \partial c^1) = 0$$

$$u'_2(c) + \beta E \sum_{i=1}^3 \sum_{j=2}^3 (V_{t+1})'_i (f^i)'_j (\partial g^j / \partial c^2) = 0$$

$$u'_3(c) + \beta E \sum_{i=1}^3 (V_{t+1})'_i (f^i)'_3 (\partial g^3 / \partial c^3) = 0$$

## Envelope conditions

$$(V_t)'_1 = \beta E \sum_{i=1}^3 \sum_{j=1}^3 (V_{t+1})'_i (f^i)'_j (\partial g^j / \partial M_t^1)$$

$$(V_t)'_2 = \beta E \sum_{i=1}^3 \sum_{j=2}^3 (V_{t+1})'_i (f^i)'_j (\partial g^j / \partial M_t^2)$$

$$(V_t)'_3 = \beta E \sum_{i=1}^3 (V_{t+1})'_i (f^i)'_3 (\partial g^3 / \partial M_t^3)$$

Combining these with the FOCs on the previous slide, we have

$$u'_3(c_t) = -\frac{\partial g^3 / \partial c^3}{\partial g^3 / \partial M_t^3} (V_t)'_3$$

$$u'_2(c_t) - u'_3(c_t) = -\frac{\partial g^2 / \partial c^2}{\partial g^2 / \partial M_t^2} [(V_t)'_2 - (V_t)'_3]$$

$$u'_1(c_t) - u'_2(c_t) = -\frac{\partial g^1 / \partial c^1}{\partial g^1 / \partial M_t^1} [(V_t)'_1 - (V_t)'_2]$$

# Euler equation

We have now expressed partial derivatives of the value function  $(V_t)'_i$  as a linear functions of the marginal utilities  $\mathcal{L}_{t+1}[u'(c_t)]$  with coefficients given by partial derivatives of  $g^i(\bullet)$  functions

The system of Euler equations

$$u'_1(c_t^1, c_t^2, c_t^3) = \beta E \sum_{i=1}^3 \sum_{j=1}^3 \mathcal{L}_{t+1}[u'(c_{t+1})] (f^i)'_j (\partial g^j / \partial c^1)$$

$$u'_2(c_t^1, c_t^2, c_t^3) = \beta E \sum_{i=1}^3 \sum_{j=2}^3 \mathcal{L}_{t+1}[u'(c_{t+1})] (f^i)'_j (\partial g^j / \partial c^2)$$

$$u'_3(c_t^1, c_t^2, c_t^3) = \beta E \sum_{i=1}^3 \mathcal{L}_{t+1}[u'(c_{t+1})] (f^i)'_3 (\partial g^3 / \partial c^3)$$

- interpolate  $c_{t+1}(M_{t+1})$  from previous backward induction iteration
- RHS known, need to solve for  $c_t^1, c_t^2, c_t^3$

# Sufficient conditions for EGM to be applicable

**(TRI2)** Hessian of the utility function  $u(c)$  is upper-triangular (up to permutations)

*(to allow for solution by back substitution)*

**(CON)** Utility function  $u(c)$  is strictly concave

*(to allow for inversion of each Euler equation)*

**(INV)** Intra-temporal transition functions  $g^i(M, c)$  have analytical inverses in the first argument  $M = (g^i)^{-1}(A, c)$

*(to allow for construction of endogenous grid)*

- Then the system of Euler equations can be solved by back substitution **without root-finding operations** by inversion
- Proof is constructive: follow the EGM step for a grid imposed over the post-decision state space

Conditions **TRI1**, **TRI2**, **CON**, **INV** are sufficient for the dynamic problem can be solvable (for interior solution) by the **Multidimensional EGM** algorithm

# When are choices separable?

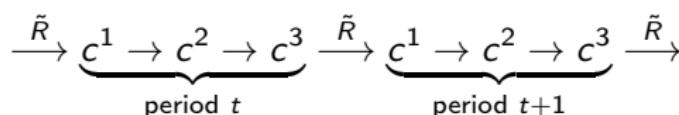


Alan Lujan, 2024

$\text{EGM}^n$ : The Sequential Endogenous Grid Method

A fruitful idea coming out of the previous analysis:

Represent multiple dimensions of choice as a sequence of choices



- ➊ each choice is made subject to a pre-decision state  $M^i$
- ➋ post-decision states are given by  $A^i = g^i(M^i, c^i)$  where  $g^i$  is a deterministic function
- ➌ pre- and post-decision states are chained such that  $M^{i+1} = A^i$
- ➍ random shocks are realized in transition from  $t$  to  $t + 1$

Then **TRI1** is satisfied: multidimensional EGM may apply

# What if the marginal utilities are not invertible analytically?



Hallengreen, Jørgensen, Olesen, 2024

The Endogenous Grid Method without Analytical Inverse Marginal Utility

A possible approach is:

- ① Precompute marginal utilities on a fixed grid
- ② Interpolate the function using any appropriate method
- ③ Use numerical inverse in the EGM step, effectively “swapping ”
  - The rest of the algorithm is unchanged
  - Can be applied to both one- and multi-dimensional problems

# Where can DC-EGM accuracy be improved?



Wending Liu, Iskhakov, 2025

Policy Tree Descendant Algorithm for Solving  
Consumption-Savings Problems with Safety Net

$$V_t(M_t) = \max_{0 \leq c \leq M_t} \left[ \log(c) + \beta E V_{t+1} \left( \max \left\{ M_0, R(M_t - c) + y_{t+1} \right\} \right) \right]$$

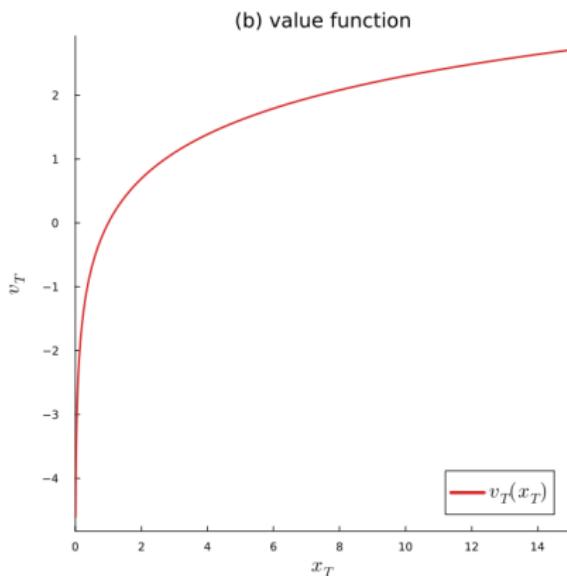
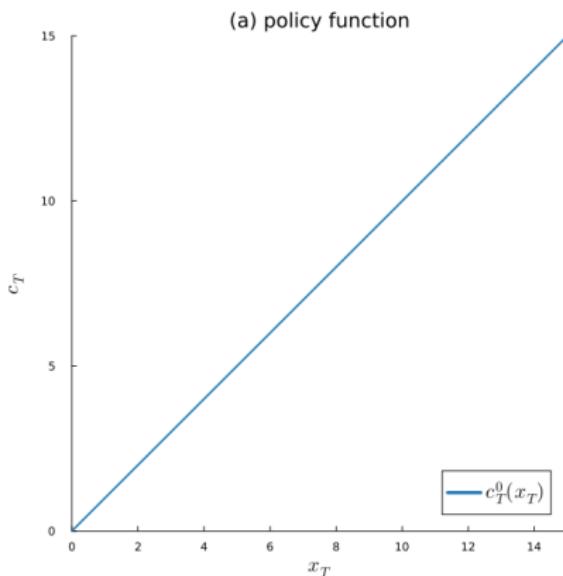
- Consumption floor introduces a kink in the value function
- Kink propagation in deterministic problems as in DCEGM

Source of inaccuracy in EGM/DCEGM:

- ❶ Interpolation of value functions
  - ❷ Location of the kinks/discontinuities is approximated
- Store linear policy function segments with their domains
  - Recognize the tree structure of policy segments descendance
  - Forward propagation to avoid interpolation

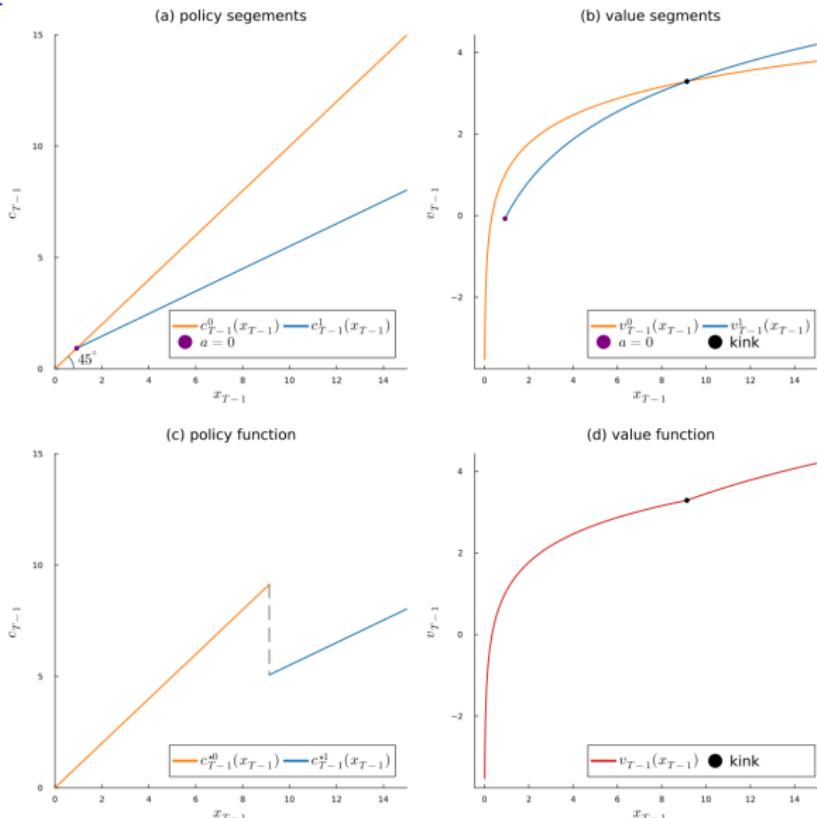
# Backward induction to build policy descending tree

The terminal period  $t = T$



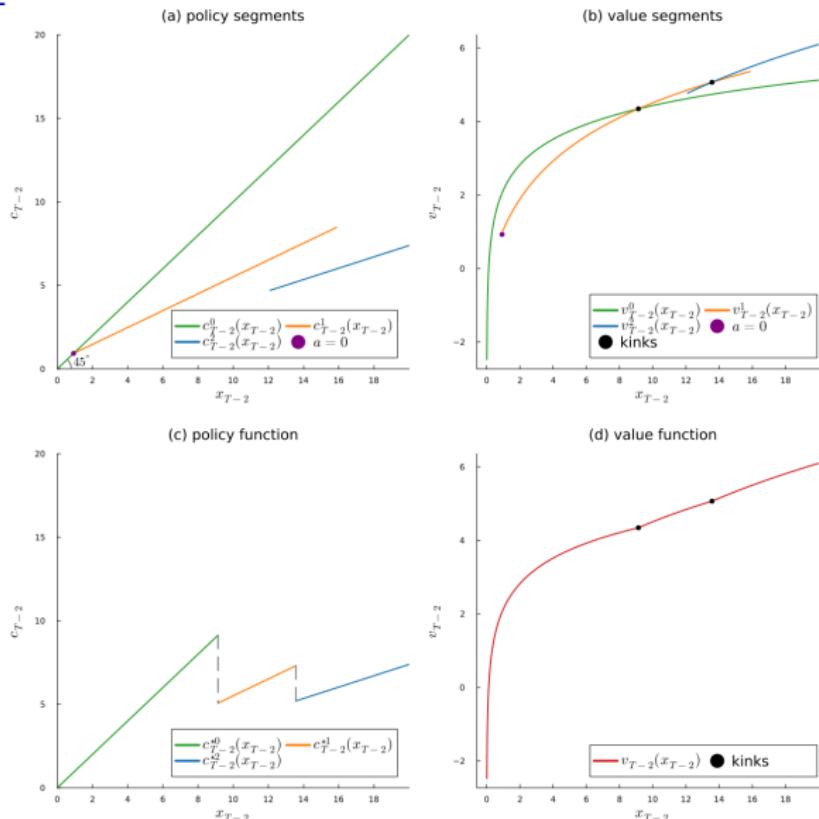
# Backward induction to build policy descending tree

Period  $t = T - 1$

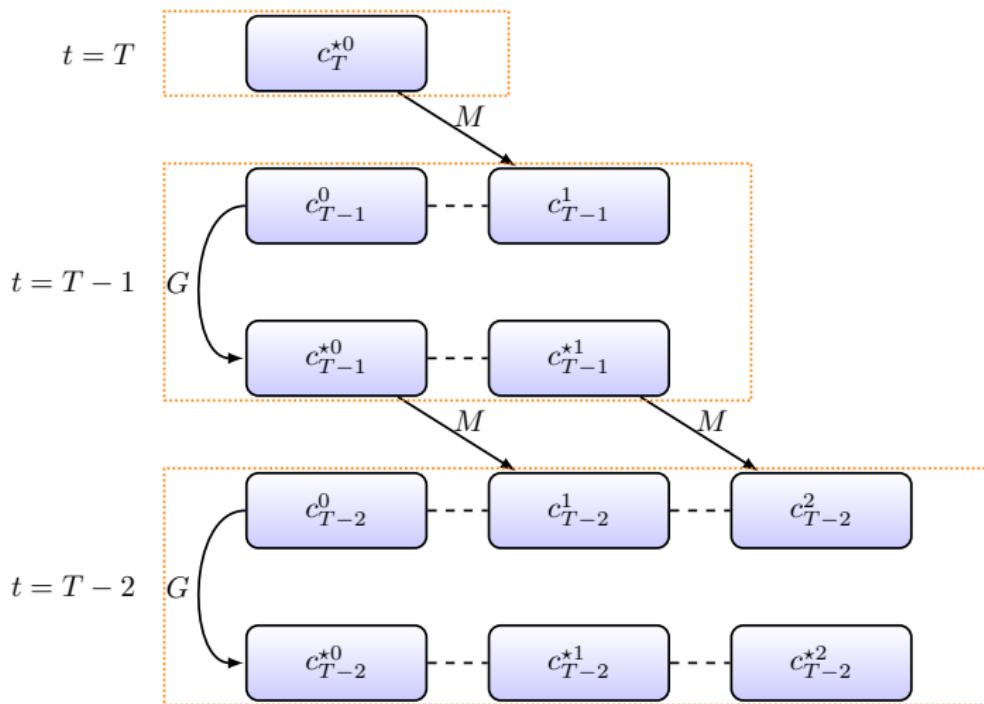


# Backward induction to build policy descending tree

Period  $t = T - 2$



# Policy descending tree



# Accuracy of the policy descending tree algorithm

- 5000 simulated lifecycles
- Simple comparison of attained life-time utility
- $\text{range}(x, y, n) = n$  points evenly spaced between  $x$  and  $y$
- runtime is averaged over 100 runs

Method	Grids of $a$	Time (ms)	# winners
PTD		1.75	5000
EGM1	range(0.0, 50.0, 100)	5.38	4821
EGM2	range(0.0, 100.0, 200)	10.64	4991
EGM3	range(0.0, 200.0, 400)	21.32	4993

- **Stochastic formulation** by Monte Carlo integration over draws of income streams  $(y_1, y_2, \dots, y_T)$  following



Cai, Judd, QE 2025

A simple but powerful simulated certainty equivalent approximation method for dynamic stochastic problems

# Estimating life cycle models using endogenous gridpoint methods

# What to do with EGM methods

We can solve **many** problems of this type ⇒

- ❶ Fast solver for important problems with discrete/continuous choice  
→
  - calibration
  - structural estimation with your favourite method
  - NFXP: inner loop to solve the model, outer loop to optimize the objective function
- ❷ Use the solver repeatedly in some “outer loop” →
  - individual heterogeneity : solve the model for each individual in the sample
  - unobserved heterogeneity : random effects
  - flexibility of distributional assumptions



Giulio Fella, 2025 (see for a list of applications)

Endogenous grid method

# EGM vs. MPEC



Jørgensen, 2012 *Economics Letters*

Structural Estimation of Continuous Choice Models: Evaluating EGM and MPEC.

Deaton consumption/savings model in infinite horizon, MC experiment with ML on synthetic data, 1 structural parameter

$\beta$		RMSE	Time
.70	EGM	0.002	0.1 sec.
	MPEC	0.049	112.4 sec.
.95	EGM	0.006	1.9 sec.
	MPEC	0.009	93.7 sec.
.99	EGM	0.000	5.0 sec.
	MPEC	0.000	30.9 sec.

## Points to take home

- ➊ EGM and DC-EGM is fast and accurate solution methods
- ➋ No root-finding operations in regular case
- ➌ Efficient with credit constraint
- ➍ Deterministic discrete-continuous problems are hard:
- ➎ Kinks in value functions, discontinuous policy functions
- ➏ Snowball effect in the accumulation of kinks over time
- ➐ With EV taste shocks the problem is alleviated
- ➑ EV taste shocks can be structural or added for smoothing
- ➒ Facilitate estimation using discrete choice data