

# EULER (FINITE DEPENDENCE) OPERATORS FOR SOLVING DYNAMIC DISCRETE CHOICE MODELS

## LECTURE 6

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in Dynamic Structural Econometrics

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## EULER EQUATIONS & FINITE DEPENDENCE

- Euler equations & Finite Dependence equations are representations of the optimal conditions of optimality.
  - a. They are always necessary conditions of optimality, and often necessary & sufficient.
  - b. **They do not involve computing Present Values** but only expected payoffs at two consecutive periods.
- In this lecture, I will review **methods for the solution of DDC models** that use Euler Equations (EE) / Finite Dependence (FD).
- We will see that, for a class of DDC models, **fixed-point iteration in EE-mapping** is computationally more efficient than standard solution methods like value function or policy function iterations.

## OUTLINE

1. Dynamic Discrete Choice models.
2. Finite Dependence & Euler Mapping
3. Analogy to Euler Equations in Continuous-Choice Models
  - Representation with CCPs as Continuous Decisions
4. Numerical Examples

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# 1. MODEL

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## FRAMEWORK: Payoffs, Actions, States

- Markov Decision Process (MDP) with a discrete action: Rust (1994).
- Each discrete time period  $t$ , an agent selects an action  $a_t$  to maximize expected intertemporal payoff:

$$\mathbb{E}_t \left[ \sum_{j=0}^{\infty} \beta^j \Pi_{a_{t+j}}(\mathbf{s}_{t+j}) \right], \text{ with } \Pi_{a_t}(\mathbf{s}_t) = \text{per-period payoff function}$$

- $\mathbf{s}_t \in \mathcal{S}$  = state vector, with transition probability:

$$\Pr(\mathbf{s}_{t+1} \mid a_t = a, \mathbf{s}_t) = f_a(\mathbf{s}_{t+1} \mid \mathbf{s}_t)$$

- The action  $a_t$  is drawn from a finite set  $\mathcal{A} = \{0, 1, \dots, J\}$ .

## Value Function, Bellman Equation, & Optimal Decision Rule

- The value function  $V(\mathbf{s}_t)$  satisfies the Bellman equation:

$$V(\mathbf{s}_t) = \max_{a_t \in \mathcal{A}} \left\{ \Pi_{a_t}(\mathbf{s}_t) + \beta \int V(\mathbf{s}_{t+1}) f_{a_t}(\mathbf{s}_{t+1} \mid \mathbf{s}_t) d\mathbf{s}_{t+1} \right\}.$$

- The optimal decision rule  $\alpha(\mathbf{s}_t) : \mathcal{S} \rightarrow \mathcal{A}$  selects the action that maximizes the right-hand side of the Bellman equation.

## Rust's Model: Assumptions & Integrated Bellman Equation

- We partition the state vector as  $\mathbf{s}_t = (\mathbf{x}_t, \varepsilon_t)$ , where:
  - $\mathbf{x}_t \in \mathcal{X}$  is observed by the researcher
  - $\varepsilon_t = \{\varepsilon_{a,t} : a \in \mathcal{A}\}$  are action-specific unobserved shocks.
- *Additive Separability*:  $\Pi_{a_t}(\mathbf{s}_t) = \pi_{a_t}(\mathbf{x}_t) + \varepsilon_{a_t,t}$
- *Conditional Independence*:  $f_{a_t}(\mathbf{s}_{t+1} \mid \mathbf{s}_t) = f_{a_t}(\mathbf{x}_{t+1} \mid \mathbf{x}_t) \cdot g(\varepsilon_{t+1})$
- $V^\sigma(\mathbf{x}_t) \equiv \int V(\mathbf{x}_t, \varepsilon_t) g(\varepsilon_t) d\varepsilon_t$  is the *integrated value function*.
- The *integrated Bellman equation* is then given by:

$$V^\sigma(\mathbf{x}_t) = \int \max_{a \in \mathcal{A}} \{v_a(\mathbf{x}_t) + \varepsilon_{a,t}\} g(\varepsilon_t) d\varepsilon_t,$$

where  $v_a(\mathbf{x}_t)$  denotes the *choice-specific value function*, defined as:

$$v_a(\mathbf{x}_t) \equiv \pi_a(\mathbf{x}_t) + \beta \sum_{\mathbf{x}_{t+1} \in \mathcal{X}} V^\sigma(\mathbf{x}_{t+1}) f_a(\mathbf{x}_{t+1} \mid \mathbf{x}_t).$$

## (INTEGRATED) VALUE FUNCTION OPERATOR

- The integrated Bellman equation defines a fixed-point mapping in the space of value functions: the *value function (VF) operator*:

$$\mathbf{V}^\sigma = \Gamma_{VF}(\mathbf{V}^\sigma) \quad (1)$$

where  $\mathbf{V}^\sigma = \{V^\sigma(\mathbf{x}_t) : \mathbf{x}_t \in \mathcal{X}\}$ .

- This mapping is a contraction with Lipschitz constant equal to the discount factor  $\beta$ .



## McFadden's Social Surplus function

- Given a vector of choice-specific values  $\mathbf{v} = \{v_a : a \in \mathcal{A}\} \in \mathbb{R}^{J+1}$ , the social surplus is defined as:

$$S(\mathbf{v}) \equiv \int \max_{a \in \mathcal{A}} \{v_a + \varepsilon_{a,t}\} g(\varepsilon_t) d\varepsilon_t.$$

- Using Social Surplus, the integrated Bellman equation can be written:

$$V^\sigma(\mathbf{x}_t) = S(\mathbf{v}(\mathbf{x}_t))$$

where  $\mathbf{v}(\mathbf{x}_t) = \{v_a(\mathbf{x}_t) : a \in \mathcal{A}\}$  = vector of choice-specific values.

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## 2. FINITE DEPENDENCE & EULER MAPPING

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## CHOICE-SPECIFIC VALUE DIFFERENCES

- Define the value difference  $\tilde{v}_a(\mathbf{x}_t) \equiv v_a(\mathbf{x}_t) - v_0(\mathbf{x}_t)$ , where alternative 0 is chosen as the baseline without loss of generality.

$$\tilde{\mathbf{v}}(\mathbf{x}_t) = \{\tilde{v}_a(\mathbf{x}_t) : a \in \mathcal{A} \setminus \{0\}\}$$

- Using value differences, we have:

$$V^\sigma(\mathbf{x}_t) = v_0(\mathbf{x}_t) + S([0, \tilde{\mathbf{v}}(\mathbf{x}_t)])$$

where  $[0, \tilde{\mathbf{v}}]$  is vector of choice-specific values with  $v_0(\mathbf{x}_t)$  set to zero.

- Using the definition of value difference  $\tilde{v}_a(\mathbf{x}_t)$ , we have:

$$\tilde{v}_a(\mathbf{x}_t) = \tilde{\pi}_a(\mathbf{x}_t) + \beta \sum_{\mathbf{x}_{t+1}} \left[ v_0(\mathbf{x}_{t+1}) + S([0, \tilde{\mathbf{v}}(\mathbf{x}_{t+1})]) \right] \tilde{f}_a(\mathbf{x}_{t+1} \mid \mathbf{x}_t)$$

with:

- $\tilde{\pi}_a(\mathbf{x}_t) \equiv \pi_a(\mathbf{x}_t) - \pi_0(\mathbf{x}_t).$

- $\tilde{f}_a(\mathbf{x}_{t+1} \mid \mathbf{x}_t) \equiv f_a(\mathbf{x}_{t+1} \mid \mathbf{x}_t) - f_0(\mathbf{x}_{t+1} \mid \mathbf{x}_t)$

## CHOICE SPECIFIC VALUE DIFFERENCES [2/2]

- Replacing  $v_0(\mathbf{x}_{t+1})$  with its expression

$$v_0(\mathbf{x}_{t+1}) = \pi_0(\mathbf{x}_{t+1}) + \beta \sum_{\mathbf{x}_{t+2}} V^\sigma(\mathbf{x}_{t+2}) f_0(\mathbf{x}_{t+2} \mid \mathbf{x}_{t+1})$$

we have:

$$\begin{aligned} \tilde{v}_a(\mathbf{x}_t) &= \tilde{\pi}_a(\mathbf{x}_t) + \beta \sum_{\mathbf{x}_{t+1}} \left[ \pi_0(\mathbf{x}_{t+1}) + S([0, \tilde{\mathbf{v}}(\mathbf{x}_{t+1})]) \right] \tilde{f}_a(\mathbf{x}_{t+1} \mid \mathbf{x}_t) \\ &+ \beta^2 \sum_{\mathbf{x}_{t+1}} \sum_{\mathbf{x}_{t+2}} V^\sigma(\mathbf{x}_{t+2}) f_0(\mathbf{x}_{t+2} \mid \mathbf{x}_{t+1}) \tilde{f}_a(\mathbf{x}_{t+1} \mid \mathbf{x}_t) \end{aligned}$$

## FINITE DEPENDENCE

- **DEFINITION. Two-Period Finite Dependence.**
- For arbitrary choices  $a_t$  and  $a_{t+1}$ , define the two-periods-forward transition probability of the state variables:

$$f_{a_t, a_{t+1}}^{(2)}(\mathbf{x}_{t+2} \mid \mathbf{x}_t) \equiv \sum_{\mathbf{x}_{t+1}} f_{a_{t+1}}(\mathbf{x}_{t+2} \mid \mathbf{x}_{t+1}) f_{a_t}(\mathbf{x}_{t+1} \mid \mathbf{x}_t)$$

- A controlled transition probability  $f_{a_t}(\mathbf{x}_{t+1} \mid \mathbf{x}_t)$  has the (global) two-period finite dependence property if for every state  $\mathbf{x}_t \in \mathcal{X}$  and choice  $a \in \mathcal{A}$  we have:

$$f_{a_t=a, a_{t+1}}^{(2)}(\mathbf{x}_{t+2} \mid \mathbf{x}_t) = f_{a_t=0, a_{t+1}}^{(2)}(\mathbf{x}_{t+2} \mid \mathbf{x}_t), \text{ for any } \mathbf{x}_{t+2} \in \mathcal{X}$$

## EXAMPLE: Multi-Armed Bandit Models

- A multi-armed bandit model is a MDP in which the only endogenous state variable is the decision made in the previous period.
- Formally, the state vector is given by  $\mathbf{x}_t = (a_{t-1}, \mathbf{z}_t)$ , and  $\mathbf{z}_t$  is a vector of exogenous state variables with transition density  $f_z(\mathbf{z}_{t+1} \mid \mathbf{z}_t)$ .
- This class of models encompasses a wide range of important economic applications, including market entry and exit, demand with switching costs, occupational choice, migration, and store location decisions.
- In multi-armed bandit models, the distribution of  $\mathbf{x}_{t+2}$  conditional on  $(\mathbf{x}_t, a_t, a_{t+1})$  does not depend on  $a_t$ .
- This implies that this class of models exhibits a global two-period finite dependence property.

## FINITE DEPENDENCE &amp; EULER MAPPING

- Remember the general expression:

$$\begin{aligned}\tilde{v}_a(\mathbf{x}_t) &= \tilde{\pi}_a(\mathbf{x}_t) + \beta \sum_{\mathbf{x}_{t+1}} \left[ \pi_0(\mathbf{x}_{t+1}) + S([0, \tilde{\mathbf{v}}(\mathbf{x}_{t+1}))] \right] \tilde{f}_a(\mathbf{x}_{t+1} \mid \mathbf{x}_t) \\ &+ \beta^2 \sum_{\mathbf{x}_{t+1}} \sum_{\mathbf{x}_{t+2}} V^\sigma(\mathbf{x}_{t+2}) f_0(\mathbf{x}_{t+2} \mid \mathbf{x}_{t+1}) \tilde{f}_a(\mathbf{x}_{t+1} \mid \mathbf{x}_t)\end{aligned}$$

- Under the two-period FD property, the last term on this equation equals zero, and we have:

$$\tilde{v}_a(\mathbf{x}_t) = c_a(\mathbf{x}_t) + \beta \sum_{\mathbf{x}_{t+1}} S([0, \tilde{\mathbf{v}}(\mathbf{x}_{t+1}))] \tilde{f}_a(\mathbf{x}_{t+1} \mid \mathbf{x}_t)$$

where  $c_a(\mathbf{x}_t)$  is closed-form of primitives:

$$c_a(\mathbf{x}_t) = \tilde{\pi}_a(\mathbf{x}_t) + \beta \sum_{\mathbf{x}_{t+1}} \pi_0(\mathbf{x}_{t+1}) \tilde{f}_a(\mathbf{x}_{t+1} \mid \mathbf{x}_t)$$

# EULER MAPPING

- Equation

$$\tilde{v}_a(\mathbf{x}_t) = c_a(\mathbf{x}_t) + \beta \sum_{\mathbf{x}_{t+1}} S([0, \tilde{\mathbf{v}}(\mathbf{x}_{t+1})]) \tilde{f}_a(\mathbf{x}_{t+1} \mid \mathbf{x}_t)$$

defines a fixed-point mapping in the space of value differences.

- This mapping is the *Euler mapping* ( $EE$ ). In vector form:

$$\tilde{\mathbf{v}} = \Gamma_{EE}(\tilde{\mathbf{v}})$$

where  $\tilde{\mathbf{v}} = \{v_a(\mathbf{x}_t) : a \in \mathcal{A} \setminus \{0\}, \mathbf{x}_t \in \mathcal{X}\}$ .

- Given the vector  $\tilde{\mathbf{v}}$  that solves this mapping, we have a complete solution to the dynamic programming problem.



## EULER MAPPING IS A STRONGER CONTRACTION

- **PROPOSITION.** Consider a discrete choice MDP with the two-period FD property. The Euler mapping  $\Gamma_{EE}$  is a contraction in the complete metric space  $(\mathcal{V}^R, \|\cdot\|_\infty)$ , and its Lipschitz constant  $\delta$  is strictly smaller than  $\beta$ .
- That is, there is a constant  $\delta \in (0, 1)$  with  $\delta < \beta$  such that for any pair  $\tilde{\mathbf{v}}$  and  $\tilde{\mathbf{w}}$  in  $\mathcal{V}^R$ ,

$$\|\Gamma_{EE}(\tilde{\mathbf{v}}) - \Gamma_{EE}(\tilde{\mathbf{w}})\|_\infty \leq \delta \|\tilde{\mathbf{v}} - \tilde{\mathbf{w}}\|_\infty \quad \blacksquare$$

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# 3. ANALOGY TO EULER EQUATIONS IN CONTINUOUS-CHOICE MODELS

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# CONTINUOUS CHOICE MDP

- Agent chooses  $a_t \in \mathbb{R}$  to maximize  $\mathbb{E}_t \left[ \sum_{j=0}^T \beta^j \pi_t(a_{t+j}, \mathbf{s}_{t+j}) \right]$ .
- $\mathbf{s}_t$  has two components:  $\mathbf{s}_t = (k_t, \mathbf{z}_t)$ .
- $\mathbf{z}_t$  = vector of exogenous state variables following a Markov process that can be non-homogeneous over time.
- $k_t$  is an endogenous state variable, with transition rule:

$$k_{t+1} = f(a_t, \mathbf{s}_t) + \zeta_{t+1}$$

$\zeta_{t+1}$  is an innovation shock that does not depend on  $a_t$  or  $k_t$ .

- The value function  $V_t(\mathbf{s}_t)$  solves the Bellman equation:

$$V_t(\mathbf{s}_t) = \max_{a_t \in \mathbb{R}} \{ \pi_t(a_t, \mathbf{s}_t) + \beta \mathbb{E} [V_{t+1}(\mathbf{s}_{t+1}) \mid a_t, \mathbf{s}_t] \}$$

# EULER EQUATION (EE)

- In this model, an EE is the combination of the **F.O.C. of optimality** at time  $t$  and an **Envelope Condition** at time  $t$ .

- **F.O.C. of optimality:**

$$\frac{\partial \pi_t}{\partial a_t} + \beta \mathbb{E}_t \left[ \frac{\partial V_{t+1}}{\partial k_{t+1}} \right] \frac{\partial f_t}{\partial a_t} = 0$$

- **Envelope Condition:**

$$\frac{\partial V_t}{\partial k_t} = \frac{\partial \pi_t}{\partial k_t} + \beta \mathbb{E}_t \left[ \frac{\partial V_{t+1}}{\partial k_{t+1}} \right] \frac{\partial f_t}{\partial k_t}$$

- Combining these two conditions, we obtain the **Euler Equation:**

$$\frac{\partial \pi_t}{\partial a_t} + \beta \mathbb{E}_t \left[ \frac{\partial \pi_{t+1}}{\partial k_{t+1}} - \frac{\partial \pi_{t+1}}{\partial a_{t+1}} \frac{\frac{\partial f_{t+1}}{\partial k_{t+1}}}{\frac{\partial f_{t+1}}{\partial a_{t+1}}} \right] \frac{\partial f_t}{\partial a_t} = 0$$

## EXAMPLE: CONSUMPTION MODEL

- $a_t$  = consumption,  $k_t$  = wealth. Payoff function (CRRA):

$$\pi_t(a_t, \mathbf{s}_t) = \frac{1}{\gamma + 1} a_t^{\gamma+1}$$

- Transition rule for wealth:

$$k_{t+1} = f(a_t, \mathbf{s}_t) = R_t k_t + y_t - a_t$$

- The well-known EE is:

$$a_t^\gamma - \beta \mathbb{E}_t [a_{t+1}^\gamma R_{t+1}] = 0$$

# PROBABILITY-CHOICE REPRESENTATION OF DDC MODELS

- Let  $\mathbf{P}_t \in [0, 1]^J$  be an arbitrary vector of CCPs (not necessarily optimal) for every action  $a$ .
- Given primitive  $f_x$  and CCPs  $\mathbf{P}_t$ , define transition probabilities:

$$f^P(\mathbf{x}_{t+1} | \mathbf{P}_t, \mathbf{x}_t) \equiv \sum_{a=0}^J P_t(a | \mathbf{x}_t) f_x(\mathbf{x}_{t+1} | a, \mathbf{x}_t)$$

- Given primitives  $\pi$  and  $f_\varepsilon$  and CCPs  $\mathbf{P}_t$ , define payoff function:

$$\Pi^P(\mathbf{P}_t, \mathbf{x}_t) \equiv \sum_{a=0}^J P_t(d) [\pi(a, \mathbf{x}_t) + e(a, \mathbf{P}_t)],$$

with

$$e(a, \mathbf{P}_t) = \mathbb{E} [\varepsilon_{a,t} \mid \Lambda^{-1}(a, \mathbf{P}_t) + \varepsilon_{a,t} \geq \Lambda^{-1}(j, \mathbf{P}_t) + \varepsilon_{j,t} \quad \forall j]$$

## PROBABILITY-CHOICE REPRESENTATION OF DDC [2]

- Given  $\{\beta, \Pi^P, f^P\}$ , we can define a DP problem where the decision variable is the vector of CCPs,  $\mathbf{P}_t$ , the state variables are  $\mathbf{x}_t$  and Bellman equation is:

$$V_t^P(\mathbf{x}_t) = \max_{\mathbf{P}_t \in [0,1]^J} \left\{ \Pi^P(\mathbf{P}_t, \mathbf{x}_t) + \beta \sum_{\mathbf{x}_{t+1}} V^P(\mathbf{x}_{t+1}) f^P(\mathbf{x}_{t+1} | \mathbf{P}_t, \mathbf{x}_t) \right\}$$

# EQUIVALENCE OF PROB-CHOICE & DISCRETE-CHOICE

1. The optimal decision in probability-choice problem is equal to the optimal choice probability in the corresponding discrete choice problem:
2. Let  $v^P(\mathbf{P}_t, \mathbf{x}_t)$  be the analog of the conditional choice value function but in the probability-choice DP. That is:

$$v^P(\mathbf{P}_t, \mathbf{x}_t) \equiv \Pi^P(\mathbf{P}_t, \mathbf{x}_t) + \beta \sum_{\mathbf{x}_{t+1}} V^P(\mathbf{x}_{t+1}) f^P(\mathbf{x}_{t+1} | \mathbf{P}_t, \mathbf{x}_t)$$

- Then,  $v^P(\mathbf{P}_t, \mathbf{x}_t)$  is twice continuously differentiable, globally concave in  $\mathbf{P}_t$ , and the optimal CCP  $\mathbf{P}_t$  is uniquely characterized by the marginal condition:

$$\frac{\partial v^P(\mathbf{P}_t, \mathbf{x}_t)}{\partial \mathbf{P}_t} = 0$$



## EULER EQUATIONS IN DDC MODELS

- We derive EE of the probability choice DP problem by using the Lagrange conditions of a **Constrained optimization problem**.

$$\max_{\{\mathbf{P}_t, \mathbf{P}_{t+1}\}} \Pi^P(\mathbf{P}_t, \mathbf{x}_t) + \beta \sum_{\mathbf{x}_{t+1}} \Pi^P(\mathbf{P}_{t+1}, \mathbf{x}_{t+1}) f^P(\mathbf{x}_{t+1} \mid \mathbf{P}_t, \mathbf{x}_t)$$

$$\text{subject to: } f_{(2)}^P(\mathbf{x}_{t+2} \mid \mathbf{P}_t, \mathbf{P}_{t+1}, \mathbf{x}_t) = f^P(\mathbf{x}_{t+2} \mid \mathbf{P}_t^*, \mathbf{P}_{t+1}^*, \mathbf{x}_t)$$

- By one-period deviation principle, the optimal  $\mathbf{P}(\cdot)$  of the DP problem is the unique solution to this constrained optimization problem.
- We obtain EEs by combining Lagrange conditions in such a way that we get rid of Lagrange multipliers.

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## 4. NUMERICAL EXAMPLES

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## NUMERICAL EXAMPLES

- We present numerical examples that illustrate large computational advantages of the EE operator for:
- **[1] The exact solution of the DP problem** (relative to Value function iteration (VFI), Relative Value function iterations (RVFI), and Policy function iterations (PFI)).
- **[2] The estimation of structural parameters** (relative to Hotz-Miller, and to Nested Pseudo Likelihood methods).
- **[3] The approximation of counterfactuals in applications where the model cannot be solved exactly** (relative to Approximate VFI, and Approximate PFI).

## DATA GENERATING PROCESS

- Binary choice model of market entry-exit.
- Inactive firms get profit  $\pi(0, \mathbf{x}_t) + \varepsilon_t(0)$ , with  $\pi(0, \mathbf{x}_t) = 0$ , and active firms earn profit  $\pi(1, \mathbf{x}_t) + \varepsilon_t(1)$ , with:

$$\pi(1, \mathbf{x}_t) = VP_t - FC_t - EC_t * (1 - d_{t-1})$$

and

$$VP_t = [\theta_0^{VP} + \theta_1^{VP} z_{1t} + \theta_2^{VP} z_{2t}] \exp(\omega_t)$$

$$FC_t = \theta_0^{FC} + \theta_1^{FC} z_{3t}$$

$$EC_t = \theta_0^{EC} + \theta_1^{EC} z_{4t}$$

# DATA GENERATING PROCESS

**Table 1**  
**Parameters in the DGP**

Payoff Parameters:	$\theta_0^{VP} = 0.5;$ $\theta_1^{VP} = 1.0;$ $\theta_2^{VP} = -1.0$ $\theta_0^{FC} = 0.5;$ $\theta_1^{FC} = 1.0$ $\theta_0^{EC} = 1.0;$ $\theta_1^{EC} = 1.0$
Each $z_j$ state variable:	$z_{jt}$ is AR(1), $\gamma_0^j = 0.0;$ $\gamma_1^j = 0.6$
Productivity :	$\omega_t$ is AR(1), $\gamma_0^\omega = 0.2;$ $\gamma_1^\omega = 0.9$
Low persistence model:	$\sigma_e^\omega = \sigma_e = 1$
High persistence model:	$\sigma_e^\omega = \sigma_e = 0.01$
Discount factor	$\beta = 0.95$

# COMPARING CONTRACTION PROPERTIES (MODULUS)

Table 2						
Degree of Contraction (Lipschitz)						
$ \mathcal{X} $	Low Persistence			High Persistence		
	EE	VF	RVF	EE	VF	RVF
64	0.20	0.95	0.59	0.34	0.95	0.95
486	0.18	0.95	0.54	0.34	0.95	0.95
2,032	0.18	0.95	0.53	0.31	0.95	0.95
6,250	0.18	0.95	0.53	0.32	0.95	0.95
15,552	0.18	0.95	0.53	0.28	0.95	0.95
200,000	0.18	0.95	0.53	0.28	0.95	0.95

## COMPARING COMPUTING TIMES

**Table 3(a)**  
**Comparison of Solution Methods**  
 (Model with low persistence)

# states $ \mathcal{X} $	Number iters.				Time per iter. (secs)			
	EE	PF	VF	RVF	EE	PF	VF	RVF
64	13	5	351	36	< 0.001	0.01	< 0.001	< 0.001
486	13	5	246	31	< 0.001	0.60	< 0.001	< 0.001
2048	13	5	345	30	0.005	16.29	0.01	0.01
6250	13	5	344	30	0.04	150.0	0.08	0.08
15552	13	5	344	30	0.17	916.2	0.46	0.46
200,000	13	5	344	30	21.05	198,978	67.64	64.47

## COMPARING COMPUTING TIMES

**Table 3(b)**  
**Comparison of Solution Methods**  
 (Model with low persistence)

# states $ \mathcal{X} $	<b>Total Time (in seconds)</b>				<b>Time Ratios</b>			
	EE	PF	VF	RVF	$\frac{EE}{EE}$	$\frac{PF}{EE}$	$\frac{VF}{EE}$	$\frac{RVF}{EE}$
64	< 0.001	0.05	0.03	< 0.001	1	60	37.0	4.0
486	0.01	3.02	0.35	0.03	1	300	26.6	3.4
2048	0.05	81.44	2.76	0.27	1	1320	53.1	5.2
6250	0.47	750.0	26.14	2.37	1	1,595	55.9	5.1
15552	2.25	4,581	158.9	13.89	1	2,040	70.7	6.2
200,000	273.6	$\simeq 1M^*$	23,270	1,934	1	3,630	85.0	7.1



## COMPARING COMPUTING TIMES

**Table 4**  
**Comparison of EE and RVF Solution Methods**  
 (Model with high persistence)

Number of states $ \mathcal{X} $	<b>Euler</b>		<b>Relative value</b>		<b>Ratio total time RVF / EE-v</b>
	# iter.	Time-per-iter.	# iter.	Time-per-iter.	
64	24	< 0.001	378	< 0.001	24.5
486	24	< 0.001	350	< 0.001	30.4
2048	17	0.01	341	0.02	38.8
6250	18	0.06	322	0.12	38.0
15552	17	0.43	333	0.87	40.5
200,000	16	46.2	319	98.5	42.5

# PARAMETER ESTIMATION

**Table 5(c)**
**Monte Carlos: Estimation of Parameters**
 $N = 1,000$  &  $T = 2$ ; Monte Carlo rep. = 1,000

Parameter (True value)	Root Mean Squared Error			
	2-step Eff. HM	2-step EE	MLE (NPL)	K-step EE
Total RMSE	0.703	0.935	0.680	0.684
Ratio $\frac{RMSE\ HM}{RMSE\ EE}$	0.75		0.99	
Time (in secs)	1067.80	0.218	7345.79	3.261
Ratio $\frac{Time\ PF}{Time\ EE}$	4898		2252	

# PARAMETER ESTIMATION

**Table 8**  
**Monte Carlo: Counterfactual Estimates**

	Prob. Being Active	Entry Prob.	Exit Prob.	State Persist.	Total Output
True Policy Effect	-0.065	-0.117	-0.068	+0.116	-0.106
Root Mean Square Error					
VF iterations	67.5%	17.1%	30.1%	11.8%	82.8%
PF iterations	67.5%	17.1%	30.1%	11.8%	82.8%
EE-value iterations	37.1%	6.3%	14.8%	7.2%	38.6%