Pricing Convertible Bonds with Monte Carlo Simulation

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ABSTRACT

This paper proposes a pricing model that values convertible bonds with Monte Carlo simulation. The optimal exercise boundaries for the embedded American-style conversion, call, and put options are inferred from the conditional expected value of continuation which is obtained by least-squares regressions in combination with a backward-induction procedure. The simulation-based pricing method is more flexible than traditional valuation approaches based on finite differences and binomial trees. It allows to better model the dynamics of the underlying state variables and to account for the specifications of the instrument, such as the path dependencies inherent in many callable convertible bonds. Credit risk is accounted for directly by modeling the possibility of default.

Keywords: Convertible bonds; Monte Carlo simulation; Option pricing.
JEL classification: C15; C63; G13

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I. Introduction

Convertible bonds present several fundamental pricing challenges and, in spite of the name, they are in many respects much more similar to advanced derivative instruments than to conventional bonds. Depending on the individual specification, the price of a convertible bond is affected by the dynamics of the underlying stock, the credit risk of the issuing company, the term structure of interest rates, and, in the case of cross-currency issues, the evolution of exchange rates. In addition, the long-term American-style conversion and put options exercisable at the discretion of the investor, the early-redemption opportunities of the issuer, and various trigger conditions that may depend on the path of the conversion value make this instrument hard to price.

During the last decade, many hedge funds have emerged, trying to take advantage of situations of mispricing observed in the convertible-bond market. Yet, pricing models in the financial industry typically rely on binomial trees and finite difference methods, which are in general much less flexible than simulation-based approaches. In particular, these traditional models often disregard path dependencies and assume standard stochastic processes that are easy to handle, but represent loose approximations of the statistical reality of the data.

While Monte Carlo Simulation was traditionally considered unsuitable for solving the free-boundary problem associated with American options, in more recent years, several authors have presented valuable methods to deal with this kind of setting. In general, these approaches can be viewed as a combination of Monte Carlo Simulation and dynamic programming. The contributions of Li and Zhang (1996), Grant et al. (1996), Andersen (2000), and García (2003) build on the early work by Bossaerts (1989) and aim at describing the exercise behavior of straight American options parametrically with suitable functions. The specific exercise rule is obtained by maximizing the value of the option in the given parameter space. A different approach is chosen by Tilley (1993), Barraquand and Martineau (1995), and Raymar and Zwecher (1997), who use backward induction in combination with stratification techniques to approximate the transitional density function. Further pricing approaches that apply backward induction are pursued by Carrière (1996), Tsitsiklis and Van Roy (1999), Longstaff and Schwartz (2001), and Clément et al. (2002). Broadie and Glasserman (1997a) and Broadie et al. (1997) use simulated trees to generate upper an lower boundaries for the fair price of the derivative. Stochastic-mesh methods are used by Broadie and Glasserman (1997b), Avramidis and Hyden (1999), Broadie et al. (2000), and Boyle et al. (2000). Finally, Haugh and Kogan (2001) and Rogers (2001), who present methods to price Bermudan options with a duality approach, are only the last to be mentioned among many further contributions in the Monte Carlo Simulation pricing literature for American options.

Although models based on Monte Carlo Simulation are suitable for solving high-dimensional (many state variables) pricing problems that contain path-dependent features, the instruments covered in the financial literature are mainly straight options and relatively simple derivatives. This is not surprising as, at least with respect to convertible bonds, the American option feature does not present the only pricing challenge. Thus, instead, alternative approaches to value convertible bonds, i.e. closed-form solutions and numerical methods based on lattices, have so far been in the center of attention. Convertible-bond pricing research with closed-form solutions was initiated by Ingersoll (1977) and further enriched by Lewis (1991) and Benninga et al. (2002), among others. However, closed-form solutions have the disadvantage that they base on simplifying assumptions about the specifications of convertible bonds and the dynamics of the underlying state variables.

With regard to the field of numerical option pricing, the focus has so far been on lattice-based approaches. This line of research is elaborated by Brennan and Schwartz (1977), Brennan and Schwartz (1980), and McConnell and Schwartz (1986). Further studies that follow this line and that also aim at modeling credit risk in the lattice framework are performed by Bardhan et al. (1993) and Hung and Wang (2002). The basic disadvantages of lattice-based models are the exponentially increasing computing time with the number of state variables, the lack of flexibility in accurately modeling the dynamics of the state variables and certain path-dependencies present in convertible bonds.

Therefore, the recent advances in American option pricing with Monte Carlo Simulation provide new opportunities to improve the pricing methods for convertible bonds. In particular, the higher the complexity of the instrument and the more state variables need to be examined, the better do simulation-based pricing methods in comparison to alternative pricing approaches. Previous research in this area is performed by Buchan (1997, 1998) as well as Kind and Wilde (2003). Buchan (1997, 1998) follows the approach of Bossaerts (1989) and uses a parametric representation of the early-exercise rule. Kind and Wilde (2003) also model the early-exercise behavior of the investor and the issuer parametrically but perform the simulation in two stages, an optimization stage and a valuation stage.

This paper presents a pricing model for convertible bonds that builds on the least-squares regression approach proposed by Longstaff and Schwartz (2001). We adopt this regression approach because it is straightforward to implement for convertibles with call and put features and because it can cope, at acceptable computational cost, with several state variables and many exercise dates. The accurate consideration of credit risk when pricing convertible bonds poses theoretical challenges. Previous research has not yet come up with a completely satisfactory solution. This paper contributes to modeling credit risk as it shows ways to overcome the theoretical drawbacks in previous approaches to include credit risk. In particular, the chosen simulation framework allows great flexibility in modeling the specific evolution of credit risk in interaction with the other state variables.

After describing the setup of the convertible-bond pricing model, the pricing impact of several convertible-bond features is investigated with the implemented model. First, we provide general results of the performance of the backward-induction algorithm used to model the optimal exercise behavior. Second, we analyze the impact of convertible-bond provisions and provide results for features that are widely used in practice but that have not been covered yet in the convertible-bond literature.

The remainder of this paper is organized as follows. Section II introduces the characteristics of convertible bonds that are relevant for pricing, formally describes the American option pricing problem for convertible bonds, and presents the simulation method. Section III provides a solution to account for credit risk in the Monte Carlo Simulation pricing framework. In Section IV, a simple numerical example is provided to illustrate the main parts of the pricing algorithm. Section V examines the performance of the pricing model, addresses the impact of convertible bond specifications on pricing, and investigates several input models for the state variables. Finally, section VIII provides a summary and a conclusion.

II. Pricing Convertible Bonds with Least-Squares Monte Carlo Simulation

A. Convertible Bonds

Convertible-bond specifications vary widely depending on the market they are issued in, the syndicate of investment banks executing the issue, and the willingness of the issuer to develop innovative features. Nevertheless, convertible bonds contain distinctive features and common characteristics that are widely spread and can therefore be considered typical for convertible bonds.

In its most simple setting, a convertible bond is a regular bond that additionally offers the investor the possibility to convert it into a predetermined number of shares of a certain company. As soon as the investor chooses to exercise his/her option to redeem the convertible in exchange for stock, the convertible bond ceases to exist and all associated claims, such as future interest payments and the final redemption payment, become obsolete. Typically, convertible bonds contain additional embedded options such as the right to demand premature redemption of the convertible bond in exchange for a pre-specified amount of money. These options may be at the discretion of the investor (put option) as well as the issuer (call option). Since all options embedded in a convertible bond typically may be exercised before maturity, convertible bonds are of American character. The embedded options may be restricted to certain dates and may require certain conditions to be met for their exercise. A common restriction for the call option is a condition requiring the conversion value to exceed a pre-specified level, called *call trigger*. In some cases, convertible bond specifications give the investor further protection against a call by requiring the conversion value to exceed the call trigger for a certain period of time (often 20 out of the last 30 trading days) before the convertible becomes callable. Furthermore, the call commonly becomes effective only after a certain time period following the issuer's announcement to call the convertible. This period, called *call-notice period*, gives the investor the opportunity to convert the bond to receive the designated number of shares instead of the call price. Usually, the call-notice period has a duration of 15 to 30 days. The case in which the investor decides to convert the convertible bond in response to a preceding call is referred to as *forced conversion*, in contrast to voluntary conversion.

In the following, we consider a plain-vanilla convertible bond that offers at regular intervals constant coupon payments, is exchangeable into a certain number of shares at the discretion of the investor, is putable at the discretion of the investor, and is callable by the issuer. Very often, call prices C_t and put prices P_t change over time, and exercising the corresponding option is restricted to specific dates. Moreover, in practice, there is a stable relationship between the call price and the put price, which simplifies the analysis considerably: $P_t \leq C_t$.

B. Pricing Framework

We consider the finite time horizon $[0,T]$ and the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the space of all possible paths ω of the state variables relevant for pricing the convertible bond, $\mathcal{F}$ is the sigma field of disjoint events at time T, and $\mathbb{P}$ is the probability measure corresponding to $\mathcal{F}$. Moreover, in accordance with the no-arbitrage paradigm, the existence of an equivalent Martingale measure $\mathbb{Q}$ is postulated. We consider discrete stopping times, $t_0 \leq t_1 \leq t_2 \leq ... \leq t_K$, with $t_0 = 0$ and $t_K = T$. Since the conversion value of convertibles is defined on a daily basis (the closing price of the stock times the conversion ratio, i.e. $\gamma_t S_t$), the model takes into consideration the exact number of stopping times that is available in reality to both the issuer and the investor by choosing K in such a way that the time elapsing between one stopping time and the next is equal to one day. We closely follow the notation of Longstaff and Schwartz (2001) and define $CF(\omega, s; t, T)$ as the cash flows deriving from the convertible bond at time s, when the state path ω is realized, given that the convertible bond is not terminated at or before time t, and that both the issuer and the investor follow optimal exercise strategies for all stopping times s between time t and maturity T, i.e. $t < s \leq T$.

A special characteristic of convertible bonds is the strategic interaction between the investor and the issuer, which arises from the embedded bilateral options – conversion, call, and put options. The optimal exercise behavior of the issuer depends on the exercise strategy adopted by the investor and vice versa. Identifying the option-exercise behavior of both actors is essential for the pricing task, since the exercise dates ultimately determine the cash flows obtained by the convertible-bond holder. According to the paradigm of rational option pricing, both actors act in an optimal way and expect the counterpart to do the same. As in the case of most numerical techniques used to obtain prices of derivatives, also the least-squares approach starts with the price of the convertible bond at maturity and then works backwards until time zero is reached.

At maturity, the price of a convertible bond is equal to either the conversion value $\gamma_T S_T$ or the final redemption value κN of the bond, whichever is larger, i.e. $CF(\omega, s = T; t, T) = max(\kappa N, \gamma_T S_T)$. Since at maturity, the put price is always lower and the call price is always larger than the nominal value, it is not necessary to check whether or not it is optimal to

Table I
Optimal option exercise behavior

This table presents the optimal exercise decisions of the options embedded in a convertible bond and the corresponding payoffs. Besides the conversion option, also call and put options are included.

Case	Payoff	Condition	Exercise Restrictions
Conversion	$\gamma_{t_k} S_{t_k}$	if $\gamma_{t_k} S_{t_k} > F(\omega; t_k)$	for $t_k \in \Omega_{conv}$
		and $P_{t_k} \leq \gamma_{t_k} S_{t_k}$	for $t_k \in \Omega_{put} \cap \Omega_{conv}$
Put	P_{t_k}	if $P_{t_k} > F(\omega; t_k)$	for $t_k \in \Omega_{put}$
		and $\gamma_{t_k} S_{t_k} \leq P_{t_k}$	for $t_k \in \Omega_{put} \cap \Omega_{conv}$
Call	C_{t_k}	if $F(\omega; t_k) > C_{t_k}$	for $t_k \in \Omega_{call}$
		and $C_{t_k} \geq \gamma_{t_k} S_{t_k}$	for $t_k \in \Omega_{call} \cap \Omega_{conv}$
Forced Conversion	$\gamma_{t_k} S_{t_k}$	if $F(\omega; t_k) > C_{t_k}$	for $t_k \in \Omega_{call}$
		and $C_{t_k} < \gamma_{t_k} S_{t_k}$	for $t_k \in \Omega_{call} \cap \Omega_{conv}$
Continuation	0	otherwise	

exercise the embedded put and call options at maturity. Note that the cash flows received at maturity, after adjusting for coupon payments, serve to calculate the continuation value $F(\omega; t_{K-1})$ for one period before maturity, which is needed for the next steps. More explicitly, the continuation value is the value of holding the convertible bond for one more period instead of exercising immediately.

At any exercise date t_k before maturity, the strategic interaction gets more complicated and has several possible outcomes. The key to proceeding is the knowledge of the continuation value $F(\omega; t_k)$ of the convertible bond, i.e. the cash flows that can be expected from the convertible-bond investment given that no option is exercised at time t_k. The payoffs $CF(\omega, t_k; t_k, T)$ in state ω at time t_k resulting from the interaction between the issuer and the holder of the convertible bond until time t_k are obtained by backward induction. For each induction step, Table I shows the possible outcomes and the corresponding cash flows, where C_{t_k} is the call price, P_{t_k} is the put price, and $\gamma_{t_k} S_{t_k}$ denotes the conversion value of the convertible bond, given as the closing price of the stock times the conversion ratio. κN is the final redemption payment, given as the final redemption ratio κ times the nominal value N of the convertible bond. Ω_{conv}, Ω_{call}, and Ω_{put} are sets of times in which conversion, call, and put are allowed according to the offering contract. In the case of the call option, the set of possible exercise dates comprises both the call period as stated in the offering circulars and times at which the call trigger condition does not prohibit a call. Note that the presence of a call trigger condition may cause Ω_{call} to differ across paths because the call option might be triggered in certain paths only, due to the evolution of the conversion value in these paths.

As usual, the convertible-bond holder aims at maximizing at any given time the value of his/her asset, while the issuer minimizes the value of his/her liability. The investor always knows the value of immediately exercising the conversion option or put option and, since the investor cannot exercise both options at the same time, he/she will consider only the most valuable of the two alternatives: the put option, if $P_{t_k} > \gamma_{t_k} S_{t_k}$, and the conversion option, if $P_{t_k} < \gamma_{t_k} S_{t_k}$. The investor decides to exercise one of the two options if the cash flows of immediate exercise are higher than the continuation value $F(\omega;t_k)$ of the convertible bond. In a setting in which the continuation value is lower than either the put price or the conversion value, the investor exercises the most valuable of the two options – conversion or put – regardless of the behavior of the issuer. The issuer, on the other hand, will call back the convertible bond as soon as the continuation value $F(\omega;t_k)$ is higher than the call price C_{t_k}. However, even after the bond is called, the investor is still entitled to exercise the conversion option. Thus, in response to a call, the investor might want to convert the bond to receive the conversion value instead of the call price. This case is referred to as *forced conversion* because the investor is induced to convert by the call decision of the issuer. If none of these cases apply and neither the investor nor the issuer decide to execute their options, i.e. $max(\gamma_{t_k} S_{t_k}; P_{t_k}) \leq F(\omega;t_k) \leq C_{t_k}$, the convertible bond is kept alive. The point in time at which the exercise strategies demand for premature exercise of at least one option is called *stopping time* τ_i, which is specific for each path i. As soon as any of the embedded options is exercised, the convertible bond ceases to exist and all uncertainty regarding its payoffs disappears. Correspondingly, as soon as a new optimal stopping time is identified, the algorithm proceeds by setting to zero all cash flows occurring after this stopping time, i.e. $CF(\omega,t;t_k,T) = 0$ for all $t \in [\tau+1,T]$. In this way, the algorithm works backwards and determines for a given continuation value $F(\omega;t_k)$ the cash flows occurring at time t_k, which subsequently serve to determine the continuation value of one period earlier, i.e. $F(\omega;t_{k-1})$. The backward induction, applying the conditions in Table I, is performed for each individual path of the simulation set because the specific realizations of the state variables in these paths lead to different optimal exercise decisions and payoffs. We define the optimal stopping time τ_i^* as the stopping time obtained for each path i at the end of the backward-induction procedure.

Besides the payoffs as mentioned in Table I, the investor receives all regular coupon payments made until τ_i^* as well as accrued interest payments as provided in most offering circulars of existing convertible bonds. Correspondingly, the total cash-flows obtained from a convertible-bond investment, adjusted for coupon payments, are given by

$$CF_{tot}(\omega,\tau^*;t_k,T) = CF(\omega,\tau^*;t_k,T) + c(\tau^*), \tag{1}$$

where $CF(\omega,\tau^*;t_k,T)$ is the payoff at time τ^* according to Table I and $c(\tau^*)$ is the present value at time τ^* of all coupon payments and accrued interest arising during the period $[t_0,\tau^*]$.

The value of a convertible bond can be obtained by discounting all future cash flows with respect to the risk-neutral pricing measure $\mathbb{Q}$. Once the optimal exercise decisions and the corresponding payoffs are determined for each path, the price V_0 of a convertible bond at time t_0 is given by the average of the discounted payoffs over all simulated paths:

$$V_0 = \frac{1}{n}\sum_{i=1}^{n} e^{-\int_{t_0}^{\tau_i^*} r(\omega_i,s)ds} CF_{tot}\left(\omega_i, \tau_i^*; t_0, T\right), \tag{2}$$

where τ_i^* are the optimal stopping times for each path i, $CF_{tot}(\cdot)$ are the corresponding payoffs, adjusted for coupon payments and accrued interest until τ_i^*, and $r(\omega_i, s)$ is the instantaneous risk-free interest rate applicable during the period $[t_0, \tau^*]$ in path i.

C. Determining the Continuation Value

Having shown that the optimal exercise behavior of the issuing company and the party holding the convertible critically depends on the continuation value of the convertible bond, we now describe how to determine the continuation value. Estimating the continuation value is the core of the presented simulation algorithm. At any time t_k, according to no-arbitrage valuation theory, the continuation value $F(\omega; t_k)$ of the bond can be expressed as the expected value of its future cash flows under the risk-neutral measure $\mathbb{Q}$:

$$F(\omega; t_k) = E^{\mathbb{Q}}\left[\sum_{j=k+1}^{K} \exp\left(-\int_{t_k}^{t_j} r(\omega, s)\, ds\right) CF_{tot}\left(\omega, t_j; t_k, T\right) \mid \mathcal{F}_{t_k}\right], \tag{3}$$

where $r(\omega, t)$ is the instantaneous risk-free interest rate in state ω at time t, and $CF_{tot}\left(\omega, t_j; t_k, T\right)$ are the total cash flows obtained under the condition that option exercise is only possible after t_k. While it is straightforward to determine the discounted cash flows given a set of simulated conversion-value paths, the critical task involves determining the expected value conditional on the information set available. To accomplish this, the presented simulation algorithm mainly follows the least-squares regression technique by Longstaff and Schwartz (2001). We assume that the continuation value of the convertible bond at time t_k can be expressed as a linear combination of a set of basis functions of the state variables that relate the information available at time t_k to the discounted future cash flows:

$$F(\omega; t_k) = \sum_{i=1}^{\infty} a_i f_i(X). \tag{4}$$

X are the state variables at time t_k in path ω, the a_i are constant coefficients, and the $f_i(X)$ are basis functions. By taking a finite number M of basis functions $f_i(X)$, it is possible to approximate the continuation value of the convertible bond as

$$F^{*}(\omega;t_k)=\sum_{i=1}^{M}a_i f_i(X). \tag{5}$$

It is worth noting that, for multidimensional problems, $f_i(X)$ can also include cross products of the state variables X_i to capture their interaction. The conditional expectation function $F^{*}(\omega;t_k)$ is estimated by regressing the discounted values of $CF_{tot}(\omega,s;t_{k+1},T)$ on the basis functions $f_i(X)$. By substituting the values of the state variables into the conditional expectation function, we obtain, for each path, an estimate of the continuation value in t_k. As shown in the previous subsection, the continuation value enables us to determine the optimal exercise behavior and thus also the cash flows $CF_{tot}(\omega,s;t_k,T)$ arising in t_k or thereafter, given that the bond is not terminated at a previous date. Since any decision to exercise the embedded options at time t_k has implications on the future cash-flows, a backward-inducting procedure, starting at maturity, is applied to determine for each stopping time the expected future cash-flows. Thus, to identify the price of the convertible bond, the above calculation is repeated until t_0 is reached and the optimal exercise behavior of the two parties is determined for all paths and exercise dates. At this point, the fair price of the convertible bond is determined by identifying the first stopping time for each path and averaging the corresponding discounted cash flows over all paths.

III. Integrating Credit Risk

As for straight corporate bonds, credit risk is an important component of convertible bonds. This is even more so as convertible bonds, particularly in the US market, are increasingly issued by medium-size growth firms as a means of funding their growth. In these cases, the augmented business risk of the issuing companies leads to a high default risk of the convertible-bond investment. For this reason, much attention is currently paid to methods that integrate credit risk into valuation models for convertible bonds. Since firm-value models have proved to be very difficult to calibrate, modern valuation approaches consider the evolution of the stock price and integrate credit risk via alternative – and often more simple – credit-risk models. McConnell and Schwartz (1986) discount all cash flows with a grossed-up interest rate to account for the possibility of default. Tsiveriotis and Fernandes (1998) present a more sophisticated way of integrating credit risk. They argue that the issuing company is always able to deliver its own stock while it may default on cash payments such as the principal at maturity, or the call price and the put price at earlier exercise dates. For this reason, they suggest to split all cash flows deriving from the convertible bond into a default-free part to be discounted with the risk-free interest rate and a component subject to counterparty risk to be discounted with a risk-augmented rate. Although this method is widely used (e.g. in Hull (2000) and Ammann et al. (2003)), it still presents some serious flaws. Ayache et al. (2003) provide some examples of inconsistencies in the Tsiveriotis and Fernandes (1998) approach. However, there are further inconsistencies as the theoretically optimal conversion strategy does not maximize the value of the convertible bond in the model.

Figure 1. Sub-optimal exercise behavior in the Tsiveriotis and Fernandes credit risk model

This figure shows the consequences of splitting the cash-flows in risk-free parts and parts subject to default risk. Payoff indicates the cash-flow obtained at maturity. Stock price is the value of the stock realized at maturity. The time of maturity is denoted by T. The thick lines are the payoffs received at time T, determined according to $max(\gamma_T S_T, \kappa N)$, and their discounted values.

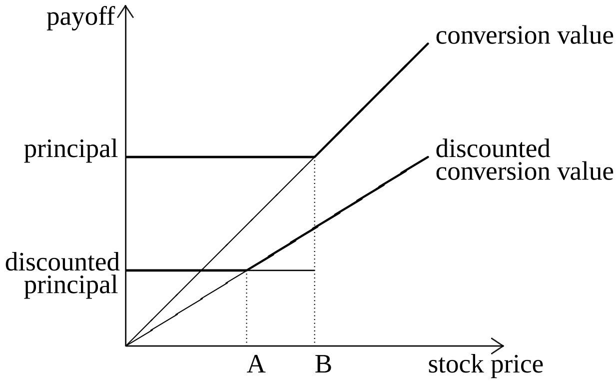


Figure 1 illustrates, for the case of the investor's exercise decision at maturity, the distorting effects of assigning different discount rates to cash-flows belonging to different credit risk categories. At maturity, the investor's optimal exercise strategy implies receiving either the conversion value or the redemption value, whichever is higher. However, discounting the cash-flows with different interest rates leads to a jump in the payoff function. Consequently, the discounted cash-flows of the convertible bond would be higher if the investor decided to convert even at lower conversion values, between the points A and B in Figure 1. This, however, is not economically meaningful. Even before maturity, splitting cash-flows into different credit risk classes can lead to distorting results. This may be seen from a simple example. Consider a convertible bond with a constant call price of 130$. In a scenario with a conversion value of 125$ and a continuation value of 135$ three years from now ($t_3 = 3$), the issuer would optimally call the convertible because the call price is lower than the continuation value. The investor would not subsequently convert the bond because the conversion value (125$) is lower than the call price (130$). If we assume that the risk-free interest rate is 0.1% and the relevant credit spread is 0.05%, the present value of this path is equal to $130 \cdot exp(-(0.1+0.05)\cdot 3) = 82.9$. However, since the present value of conversion would be $125 \cdot exp(-0.1 \cdot 3) = 92.6$, not converting the bond upon call is evidently not the value-maximizing strategy.

We propose an alternative approach for integrating credit risk in the simulation framework that is theoretically and practically more appealing. Credit risk is modeled more explicitly by allowing for the possibility of default, which is governed by a company-specific default probability P^d and an issue-specific recovery rate ρ.[1] Although it is straightforward to let these variables vary over time or follow some dynamics, to keep a basic setting, we think of P^d and ρ as constant over time and for all paths. In case of bankruptcy the equity holders do not receive any cash flow and thus the recovery rate of stocks is equal to zero. In the presented pricing algorithm, we superimpose a specific default process, which sets the stock price to zero in case of default. Therefore, we have to distinguish between the price of the stock without default, S_t, and the price of the stock after accounting for the possibility of default, S_t^*. At any time t, the stock value after default is $S_t^* = S_t \cdot 1_{\{\tau > t\}}$ or equivalently

$$S_t^* = \begin{cases} S_t & if \tau > t \\ 0 & if \tau \leq t \end{cases} \tag{6}$$

where τ is the time of default and $1_{\{\tau > t\}}$ is an indicator function taking the value 1 if default already has occurred and 0 otherwise. Similarly to standard option pricing theory, we assume that the dynamics of the stock price before default can be characterized by a drift term and a diffusion:

$$dS_t = (r_t + g) S_t dt + \sigma S_t dW \tag{7}$$

r_t is the risk-free interest rate at time t, σ is the diffusion term, and dW is a Wiener process. g is an additional drift component included to compensate for the possibility of bankruptcy. To account for default risk, g is chosen in a way that the risk-free discounted expected value of S_t^* is equal to the current stock price S_0:

$$S_0 = \exp(-rt) \cdot E\left[1_{\{\tau > t\}} \cdot S_t\right], \tag{8}$$

or, equivalently,

$$S_0 = \exp(-rt) \cdot E\left[P_t^s \cdot S_t\right], \tag{9}$$

where P_t^s is the probability that the stock will not default until time t.

By applying the formula for the expected value of a product of stochastic variables ($E[X \cdot Y] = E[X] \cdot E[Y] + Cov[X,Y]$) and substituting $S_0 \cdot \exp((r_t + g - 0.5\sigma^2)t + \sigma W_t)$ for S_t, Equation 9 can be rewritten as

$$S_0 = \exp(-rt) \cdot \{E[P_t^s] S_0 \exp((r+g)t) + Cov[P_t^s, S_t]\}, \tag{10}$$

This equation can subsequently be solved for g:

$$g = \frac{1}{t} \ln \left\{ \frac{S_0 \exp(rt) - \rho_{S,P^s} \sigma_S \sigma_{P^s}}{S_0 E[P_t^s]} \right\} - r. \tag{11}$$

At this point, it is possible to price the convertible bond by using the stock drift g in combination with the company-specific default probability P_t^d, and the issue-specific recovery rate χ. First, the stock prices S_t are simulated with the appropriate risk-neutral drift as in Equation 7. Second, the optimal exercise strategies of the investor and the issuer at any possible exercise date are determined as described in Section II. Finally, credit risk is taken into account when discounting the resulting cash flows to a previous date (e.g. for determining the continuation value at an earlier possible exercise date). Instead of actually letting individual paths default at a certain time, all cash-flows are adjusted to reflect the probability of default. This procedure substantially increases the efficiency of the valuation algorithm. More precisely, the original cash flows CF at time t are transformed into discounted, credit-risk adjusted cash flows CF^* at time s (with $s < t$) according to the following formula:

$$CF^* = \left[\left(1 - P_{s,t}^d\right) \cdot CF + P_{s,t}^d \cdot \chi N \right] \cdot \exp[-rt \cdot (t-s)], \tag{12}$$

where χ is the recovery rate, N is the nominal value of the convertible, and $P_{s,t}^d$ is the default probability from time s to time t. Thus, the payoff considered for the valuation of the convertible bond is equal to the weighted average of the default-free cash flows and the recovery payment, discounted with the appropriate risk-free interest rates. The corresponding weights are the probability of survival and the default probability, respectively. The following example will clarify the methodology.

IV. Numerical Example

A simple example will illustrate the presented least-squares simulation method for convertible bonds and demonstrate how the continuation value is estimated and credit risk is integrated. In the first step, a European-style convertible bond is priced. The valuation is then extended to a convertible with identical characteristics, but additional American-style conversion, call, and put options. The convertible bond considered has the following specifications: a maturity T of 2 years, a redemption value N of 100. The initial conversion value is assumed to be 90. The stock is assumed to provide a single dividend payment until maturity which occurs right before time t_1 and takes the form of a dividend yield amounting to 10 percent of the conversion value. While discrete dividend payments are likely to have a major impact on the early exercise behavior, discrete coupon payments are not because existing convertible bonds commonly provide accrued coupon payments covering the period between the last regular coupon payment date and the time of early exercise. Thus, for simplicity, the convertible bond examined in the example is assumed to provide no coupon payments. Moreover, to keep the example sparse, the interest rate is assumed to be non-stochastic at a constant level of 0.08. The two state variables are the conversion value $\gamma_t S_t$ and the default probability P^d, which are assumed to have a correlation of -0.5. To compensate for the possibility of default, the risk-neutral drift of the stock price has to be adjusted by adding an additional drift term g as introduced in the previous section. By assuming that $E[P_1^d] = 0.1$ and applying equation (12), g gets equal to 0.105. Table II illustrates the first step of the pricing model, which consists of simulating under the risk-neutral measure a finite number of paths for the state variables. In particular, eight example paths for the drift-adjusted stock price and the default probability are provided. Panel A displays the conversion-value paths. As shown in Panel B, the paths are adjusted for dividend payments, where the dividend yield has the effect of lowering each path at and after the dividend payment date, i.e. beginning at time t_1, to 90 percent of the original value. Panel C describes the evolution of the annualized default probabilities for each path. For the first period, i.e. $[t_0, t_1]$, the anticipated default probability is the same for all paths because the starting conditions are identical. Only for period $[t_1, t_2]$ does the default probability differ across paths. Since the default probability after maturity has no meaning for the current pricing task, no default probabilities are displayed for t_2. In Panel D, the corresponding cumulated survival probabilities of the company are reported. The initial survival probability is equal to one at t_0 and decreases over time taking different values for each path after t_1.

The evaluation of the convertible bond is cash-flow based and starts at the final exercise date, which is maturity ($T = 2$). At this time, the investor chooses to convert the convertible bond if the conversion value is higher than the redemption value, and thus receives a cash flow equal to $max(N, \gamma_t S_t)$. The cash flows received at maturity from the convertible bond are shown in Table III. While in the first, second, and sixth path the investor prefers to receive the redemption value, in the remaining paths, he/she will decide to exercise the conversion option. In fact, these cash flows are realized only if the options embedded in the convertible bond are not exercised before time t_2 and if no default occurs.

For obtaining the price of the European-style equivalent of this convertible bond, it is necessary to consider only the exercise decisions and the corresponding payoffs at maturity. These values are displayed in Table III. However, for each path, there are three possible events related to credit risk that have to be considered: bankruptcy in t_1, bankruptcy in t_2, and no bankruptcy. In case of default, the investor receives a fraction of the principal (the recovery rate), which is assumed to be 30 percent for the convertible bond in this example.[2] Only if there is no default does the investor receive the payoff exhibited in Table III. Hence the value of the convertible is determined by weighting with the corresponding probability the discounted cash flows arising from each event and averaging over all paths. Table IV summarizes this calculation by reporting the cash flows, the discounted cash flows (in parentheses), and the probability of each event. The resulting price for the European-style convertible bond is 102.856.

In the case of a standard convertible bond, however, the possibility of early exercise has to be taken into account. The American-style convertible bond considered now has the following additional characteristics: a call price C of 120, a put price P of 90, and a conversion option that can be exercised prematurely by the investor. The exercise decisions at maturity are unaffected by this extension and remain as shown in Table III. Prior to maturity, it is optimal for the investor and the issuer to base their option exercise behavior on a comparison between cash flows occurring at immediate option exercise and the continuation value of the convertible bond $F(\omega, t_k)$, i.e. the expected cash flows (under the risk-neutral measure) from not exercising immediately. While the investor acts to maximize his/her payoffs, the issuer does exactly the opposite. Thus, the conditional expected value of continuation has to be determined backwards for each possible exercise date. For the decision period immediately preceding maturity, the continuation value is obtained by regressing the known cash flows (conditional on the fact that the bond is not terminated previously), discounted to t_1, on a set of basis functions of the relevant state variables at t_1. The cash flows occurring at maturity are discounted with the risk-free interest rate, amounting to 8 percent in this example, and the possibility of default is taken into account by weighting the cash flows with the specific one-period survival probability applicable to each path. More explicitly, the cash flows in Table V are obtained with the following computation: $Y = P^s_{1,2} \cdot CF_2 \cdot \exp(-r \cdot \Delta t) + P^d_{1,2} \cdot \chi \cdot N \cdot \exp(-r \cdot \Delta t)$. For the first path, for instance, this leads to a value of $0.9 \cdot 100 \cdot \exp(-0.08) + 0.1 \cdot 0.3 \cdot 100 \cdot \exp(-0.08) = 85.850$. With this procedure, the continuation value of the convertible bond is determined by using and the cross-sectional information in the conversion-value paths and the possibility of default is accounted for. For simplicity, in this example, the expected discounted cash flows from not exercising the embedded options immediately are estimated with a very simple least-squares regression, linearly relating the obtained cash flows and the conversion value: $Y_i = a + bX_i + \varepsilon_i$. The setup and the results of this regression are displayed in Table V. For a full-size numerical application, a more advanced regression function including the default probability and also cross terms as regressors would be more appropriate.

The parameters from the linear regression are estimated as $a = 51.015$ and $b = 0.646$. Applying these estimates in the conditional expectation equation leads to the continuation value of the convertible bond: $F(\omega, t_1) = E[Y \mid X] = 51.015 + 0.646 \cdot X$. The continuation value is subsequently used to determine in each path at time t_1 the optimal exercise strategies and the resulting cash flows. According to Table I, five outcomes are possible. Depending on the variables conversion value, continuation value, call price, and put price, the investor may choose to put or voluntarily convert the bond, the issuer may call for early redemption, the investor may prefer to convert the bond in response to a call (forced conversion), or no action may be taken by the two parties. As soon as either the investor or the issuer decides to exercise an option, the convertible bond ceases to exist. Table VI presents a detailed description of the optimal early exercise decisions at time t_1 for the convertible bond examined in the example. Applying the rules for the early-exercise strategies shows that the embedded options will be

Table II
Example paths for various state variables

This table shows eight example paths of the conversion value, $\gamma_t S_t$, and the default probability, P^d, for three dates t_0, t_1, t_2. Panel B gives the conversion-value paths, adjusted to account for the 10 percent dividend yield paid out after t_1. Panel C displays paths for the annualized default probability for the periods [t_0,t_1] and [t_1,t_2]. Panel D shows the corresponding cumulated survival probability.

Path	t_0	t_1	t_2
		Panel A: Conversion value	
1	90	40	70
2	90	140	90
3	90	180	230
4	90	90	120
5	90	130	200
6	90	60	40
7	90	150	220
8	90	125	150
		Panel B: Dividend adjusted conversion value	
1	90	36	63
2	90	126	81
3	90	162	207
4	90	81	108
5	90	117	180
6	90	54	36
7	90	135	198
8	90	112.5	135
		Panel C: Default probability paths	
1	0.1	0.10	–
2	0.1	0.13	–
3	0.1	0.09	–
4	0.1	0.12	–
5	0.1	0.08	–
6	0.1	0.11	–
7	0.1	0.08	–
8	0.1	0.09	–
		Panel D: Cumulated survival probability	
1	1	0.9	0.81
2	1	0.9	0.783
3	1	0.9	0.819
4	1	0.9	0.792
5	1	0.9	0.828
6	1	0.9	0.801
7	1	0.9	0.828
8	1	0.9	0.819

Table III
Cash-flow matrix at maturity

This table shows the cash flows at maturity (time t_2), conditional on the fact that the convertible bond is not terminated at any prior date. The values are determined as $max(N; \gamma_t S_t)$. 'Action in t_2' describes the action optimally taken by the investor at time t_2 given that the convertible bond is still alive.

Path	t_1	t_2	Action in t_2
1	–	100	redemption
2	–	100	redemption
3	–	207	conversion
4	–	108	conversion
5	–	180	conversion
6	–	100	redemption
7	–	198	conversion
8	–	135	conversion

Table IV
Pricing a European-style convertible with credit risk

This table identifies for each path three possible outcomes with respect to default: default in t_1, default in t_2, and no default. For each of these events, the table reports the resulting cash flow (first row), the cash flow discounted to t_0 (second row in parentheses), and the probability of the occurrence of this event (third row). The last column shows the expected discounted payoff per path. The price of the European-style convertible bond is displayed at the bottom of the table.

Path	Default in t_1	Default in t_2	No Default	Sum of probability-weighted discounted cash flows
	30	30	100	
1	(27.69)	(25.56)	(85.214)	74.094
	0.1	0.09	0.81	
	30	30	100	
2	(27.69)	(25.56)	(85.214)	72.483
	0.1	0.117	0.783	
	30	30	207	
3	(27.69)	(25.56)	(176.394)	149.307
	0.1	0.081	0.819	
	30	30	108	
4	(27.69)	(25.56)	(92.032)	78.419
	0.1	0.108	0.792	
	30	30	180	
5	(27.69)	(25.56)	(153.386)	131.613
	0.1	0.072	0.828	
	30	30	100	
6	(27.69)	(25.56)	(85.214)	73.557
	0.1	0.099	0.801	
	30	30	198	
7	(27.69)	(25.56)	(168.724)	144.314
	0.1	0.072	0.828	
	30	30	135	
8	(27.69)	(25.56)	(115.039)	99.057
	0.1	0.081	0.819	
				102.856

Table V
Regression in t_1

This table shows the results from regressing linearly the discounted, credit-risk adjusted cash flows from time t_2, denoted by Y, on the conversion value at t_1, represented as X: $Y_i = a + bX_i + \varepsilon_i$. Credit risk is accounted for by weighting future cash flows in case of no default by the survival probability, $P^s_{1,2}$, and cash flows deriving from default by the default probability, $P^d_{1,2} : Y = P^s_{1,2} CF_2 exp(-r\delta t) + P^d_{1,2} \chi N exp(-r\delta t)$.

Path	Y	X
1	85.85	36
2	83.911	126
3	176.38	162
4	91.056	81
5	155.084	117
6	85.204	54
7	170.37	135
8	115.897	112.5
Intercept	51.015	
Slope	0.646	

Table VI
Optimal early exercise decision in t_1

This table shows the resulting optimal early exercise decision in t_1. $F(\omega, t_1)$ represents the calculated continuation value of the convertible bond. 'Payoff' is the present value of the cash flow the investor receives from the convertible bond under the condition that both investor and issuer pursue their optimal exercise strategies. 'Action in t_1' describes the action taken by investor and issuer in t_1 according to their optimal exercise strategies given that the convertible bond is still alive at that time.

Path	$F(\omega, t_1)$	$\gamma_\tau S_\tau + d_\tau$	C_τ	P_τ	Payoff	Action in t_1
1	73.580	36	120	90	90	Put
2	132.482	126	120	90	126	Forced conversion
3	156.043	162	120	90	162	Voluntary conversion
4	103.031	81	120	90	-	No action
5	126.592	117	120	90	120	Call
6	83.36	54	120	90	90	Put
7	138.372	135	120	90	135	Forced conversion
8	123.647	112.5	120	90	120	Call

exercised prematurely in seven out of the eight paths and the convertible bond will cease to exist after t_1 in these cases. Only in path four does neither the investor nor the issuer take any action to exercise their options. In the first and sixth path, the investor decides to exercise his/her put option because the put price is higher than both the conversion value and the expected cash flows from continuation. In path three, the decision to convert the convertible is taken voluntarily by the investor because the conversion value is higher than the continuation value. In path two, five, seven, and eight, the issuer calls back the convertible bond because the call price is lower than the continuation value. However, in response to the call in path two and seven, the investor will subsequently convert the convertible bond to receive the conversion value (126 and 135) instead of the call price (120).

For each path, the optimal exercise behavior at the times t_1 and t_2 leads to the stopping rule and the payoffs displayed in Table VI. Again, credit risk is accounted for by weighting the identified cash-flows with the path-specific survival probability of the convertible bond up to the time the cash-flow would occur, i.e. the cumulated survival probabilities as presented in Panel D of Table II. For each period preceding that date, the recovery payment in case of default is weighted with the default probability during that period, as is shown in Panel C of Table II. Since the cash flows arising from the convertible bond at all dates are known, the current value of the convertible bond can be calculated as the sum of all discounted, credit-risk adjusted cash flows, averaged over the paths. The obtained value is $F_0 = 798.173/8 = 99.772$. This value is smaller than the one of the European-style convertible bond as calculated before demonstrating that in this case, the American-style conversion and put options of the investor are worth less than the right of the issuer to call back the convertible bond.

In the case of more potential stopping times, the presented algorithm is applied recursively, rolling back from maturity until time t_0: For each stopping time, under the condition that the convertible bond is still alive and none of the options is exercised this period, the payoffs are applied to generate an estimate of the continuation value of the convertible bond. This value is compared to the cash flows occurring at immediate option exercise. Once applied, the resulting optimal exercise strategies lead to a new set of conditional cash flows for one period earlier. The procedure is repeated recursively to finally determine the optimal exercise decisions as well as the corresponding cash flows and the time of their occurrence.

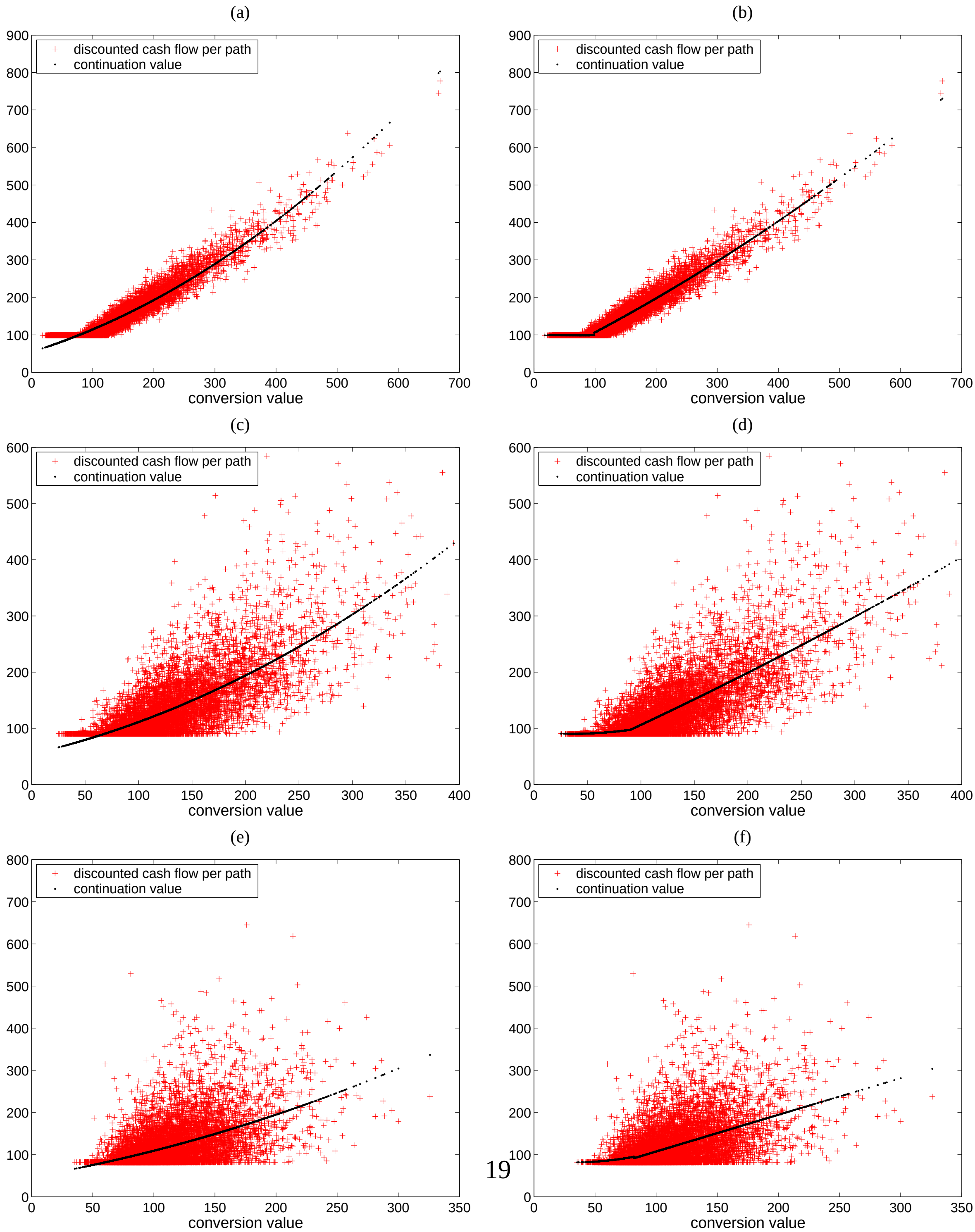
V. Performance of the Pricing Model

A. Estimating the Continuation Value

The core of the pricing algorithm for valuing American-style derivatives consists in modeling the optimal exercise decision of the actors holding an option. In general, an option is exercised prematurely if the payoffs from immediately exercising the option are higher than the discounted expected payoffs from not doing so. Since the payoff obtained from immediate ex-

Figure 2. Continuation value Function $F(\omega,t)$

This figure compares the results obtained from two types of regressions, performed for three different future points in time, respectively: 30 (plot a and b), 250 (c,d), and 500 (e,f) trading days before maturity. For each plot in the left column, two regression are performed to determine the continuation value, one for conversion values higher and one for values lower than the discounted redemption value. For the plots in the right column, one regression each is performed to fit through all the points. The number of simulated conversion value paths is 10’000 on a daily basis. The investigated convertible bond has a maturity of three years, provides no coupon payments, and is not subject to default risk. The initial conversion value is 100. The stock has an annual volatility of 0.3 and does not provide dividend payments. The convertible bond does not include any call and put options. The interest rate is assumed to be constantly ten percent until maturity. For the regressions, polynomials with a degree of up to two are considered for the conversion value.



ercise is a known quantity, the optimal exercise behavior can be determined by estimating the continuation value of the derivative. In the presented pricing model, the continuation value is estimated with a backward-induction procedure combined with least-squares regressions. In particular, at each possible exercise date, future cash flows from continuation are regressed on a set of basis functions of the state variables to obtain for all possible outcomes of the state variables an estimate of the conditional expected value of continuation. Thus, the crucial part of the pricing algorithm is the ability to accurately estimate the continuation value of the convertible bond which determines the optimal option exercise behavior of investor and issuer. Therefore, the method used for estimating the continuation value, i.e. the regression approach, is examined more in detail.

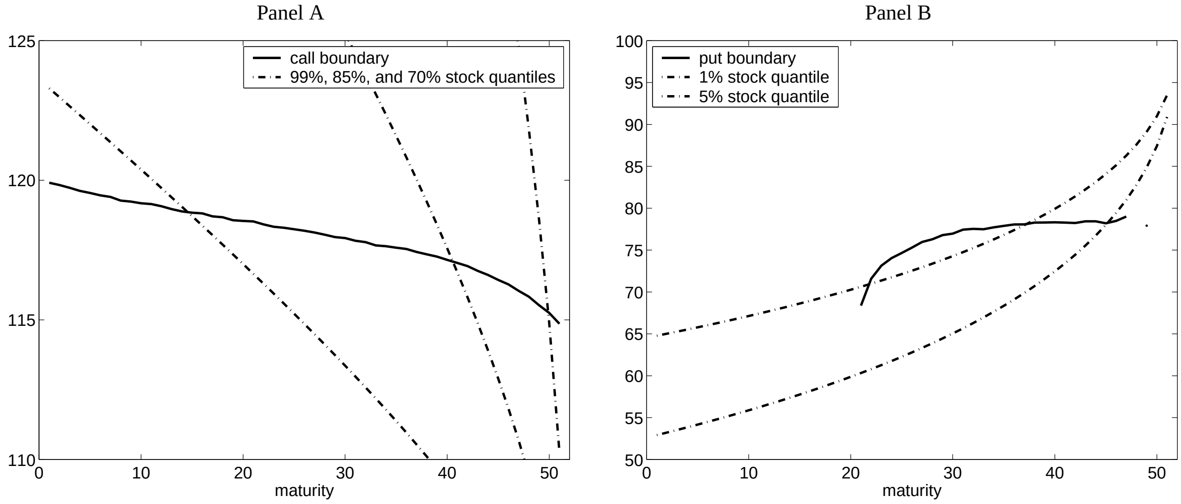
Figure 2 shows, as a simple example, the estimated quadratic continuation-value function for a three-year convertible bond at three different future dates: after one year, after two years, and one month before maturity. The investigated convertible bond has a maturity of three years, provides no coupon payments, does not include call and put options, and is not subject to default risk. The stock has an annual volatility of 0.3 and does not provide dividend payments. Moreover, the initial conversion value is 100, and the interest rate is assumed to be constantly ten percent until maturity. The number of simulated conversion-value paths is 10’000 on a daily basis.

The pattern of the points in the individual plots is characteristic because two payoff events are possible, given the chosen provisions: conversion and redemption. Paths with currently low conversion values tend to lead to regular redemption, while paths with high conversion values tend to end in conversion, in case there is no call option that could be exercised. Due to this dual pattern, a single polynomial regression applied to all data points, as given on the left side of the figure, is not able to provide an appropriate representation of the expected value of continuation. Instead, adjusting the approach of Longstaff and Schwartz (2001) by performing two separate polynomial regressions on paths with conversion values higher and lower than the discounted redemption value is found to give a better estimate of the continuation value. The corresponding plots are given on the right side of the figure.

Numerical tests confirm the result obtained by Longstaff and Schwartz (2001) that fairly simple specifications of the continuation-value function are sufficient to generate accurate estimates of the exercise boundaries and thus of derivative prices. The obtained prices turn out to be fairly robust and including stock regressors of degrees higher than two or three does not further improve the quality of the regression. In general, the regression fit for the time right before maturity is rather good as it is fairly easy to estimate the continuation value. The longer the time to maturity, the less information does the level of the state variables provide for determining the future payoff, but this does not indicate that the regression method is imprecise. On the contrary, the low R^2-values of these regressions fully reflect the information available to the convertible bond holder and the issuer.

Figure 3. Exercise Boundaries for Options embedded in Convertible Bonds

This figure displays the exercise boundary for call (Panel A) and put (Panel B) options embedded in a convertible bond with one-year maturity. The call price is 120.0 and the put price is 96.0. The number of simulated conversion-value paths is 100'000 and the simulation frequency is weekly (52 steps). The initial conversion price is 100. The stock is assumed to follow a standard geometric Brownian Motion with an annual volatility of 0.3. The stock does not provide dividend payments. The interest rate is assumed to be constantly ten percent until maturity. The regressions are performed with second order polynomials.



B. Pricing Results

At this point, some numerical examples are provided to determine the performance of the regression technique. Table VII shows pricing results for convertible bonds. Since the technique can be applied to various derivatives, pricing results for simple put and call options are displayed additionally as a reference. For these simple instruments, reference values are available, either in form of closed-form solutions or binomial trees. These reference values are compared to values obtained by the simulation-based pricing method to provide support for the pricing accuracy. In the last part of the table, prices for convertible bonds are displayed, generated with the simulation-based pricing method only because this is the only method able to capture the characteristics of the instrument.

C. Exercise Boundaries

Panel A of Figure 3 displays the exercise boundary for a convertible bond with an embedded call option. The call boundary indicates the stock level at which the continuation value is equal to the call price. The boundary is not constant over time but converges to the call price as the time to maturity approaches zero. The issuer will exercise his/her call option as soon as the conversion value exceeds the call boundary. Thus, the exercise region is given by the area above the call boundary. In addition to the call boundary, the plot contains the 70%, 85%, and 99% quantiles providing information about the distribution of the conversion-

Table VII
Pricing Performance of Least-Squares Approach

This table compares Monte Carlo prices with prices obtained with binomial trees and closed-form solutions (for the European case). The table presents results for put options (Panel A), call options (Panel B), and convertible bonds (Panel C). Standard errors are reported in parentheses.

Panel A: Put Options

		European			American	
S_0	strike	closed form	tree	MC	tree	MC
40	36	6.711	6.712	6.713	7.102	7.081
				(0.023)		(0.019)
40	40	5.060	5.059	5.053	5.312	5.276
				(0.021)		(0.018)
40	44	3.783	3.783	3.764	3.948	3.951
				(0.018)		(0.016)

Panel B: Call Options

		European			American	
S_0	strike	closed form	tree	MC	tree	MC
100	80	0.654	0.654	0.649	0.670	0.668
				(0.003)		(0.003)
100	100	5.302	5.302	5.301	5.730	5.728
				(0.010)		(0.009)
100	110	10.155	10.155	10.161	11.341	11.359
				(0.015)		(0.011)

Panel C: Convertible Bonds

			European			American	
S_0	call price	put price	closed form	tree	MC	tree	MC
90	-	-	101.520	101.520	101.504	103.727	103.663
					(0.095)		(0.066)
90	-	99	101.520	101.520	101.589	105.683	105.700
					(0.095)		(0.061)
90	118	-	101.520	101.520	101.543	102.878	102.882
					(0.095)		(0.044)
90	118	99	101.520	101.520	101.412	104.745	104.737
					(0.095)		(0.037)
110	-	-	110.377	110.376	110.234	115.436	115.334
					(0.133)		(0.081)
110	-	99	110.377	110.376	110.386	116.428	116.367
					(0.132)		(0.079)
110	118	-	110.377	110.376	110.341	113.811	113.845
					(0.134)		(0.041)
110	118	99	110.377	110.376	110.429	114.433	114.442
					(0.134)		(0.036)

value paths. The exercise boundary is particularly accurate if it falls in a region with many conversion-value paths since the presence of many data points facilitates the estimation of the continuation function via least-squares regression. In this area, the exercise boundary also has a particularly large pricing impact. The exercise boundary of a put option embedded in a convertible bond is displayed in Panel B of Figure 3. Since the convertible in this example is assumed not to be subject to default risk, the payoffs at maturity are always equal to or larger than the redemption value, which is 100. For this reason, with a sufficiently short maturity, the put price is always lower than the discounted payoffs from continuation. Thus, for short maturities, it is not optimal for the convertible-bond holder to exercise the put option regardless of the actual conversion price. This phenomenon is documented in Panel B Figure 3 by the absence of the put boundary for maturities shorter than 22 weeks. In fact, the redemption value (100), discounted over 21 weeks with a risk-free rate of 0.05, equals 96.053, which is just slightly higher than the put price of 96. Convertible bonds may contain both a call and a put option, which introduces a strategic interaction between issuer and investor.

VI. Impact of Convertible Bond Specifications on Pricing

Real-world convertible bonds often contain specifications that differ in several respects from the standard specification that is usually assumed in academic papers. These additional specifications are often neglected because they are not analytically tractable and because traditional lattice-based models cannot deal with them.

A typical issue are the provisions appearing in the context of call options. For most real-world convertible bonds, the call option of the issuer is not active immediately after issuance but becomes active as soon as the conversion value reaches a certain threshold, the call trigger. Moreover, some contractual specifications of issued convertible bonds demand the conversion value to exceed the call trigger for at least twenty out of the last thirty trading days for the call to be triggered. The flexibility of the presented pricing tool permits to model these path-dependent features and to quantify their effect on the fair value of convertible bonds.

Table VIII presents prices of convertibles characterized by a simple call option, a call trigger, and a call trigger with qualifying period. The column named 'Values with Call' displays convertible-bond prices for instruments with embedded standard call options and different call prices. The fourth column, named 'Values with Call Trigger', presents prices for convertibles protected by call triggers of various levels. The fifth column, 'Values with Qualifying Period', presents prices, additionally taking into account the qualifying-period requirement: The call sets in as soon as the conversion value exceeds the call trigger by more than 20 trading days out of the last 30 trading days. The standard errors are given in parentheses below the corresponding convertible bond prices. The third number indicates the number of paths (out of 20'000), in which the convertible bond is called by the issuer. The fourth number is the percentage pricing impact measured with respect to the price obtained without modeling the examined

Table VIII
Pricing Convertible Bonds with Call Trigger and Qualifying Period

This table shows pricing results for different call specifications of the convertible bond. 'Values with Call' indicates convertible-bond prices with different embedded standard call options. 'Values with Call Trigger' presents prices for convertibles protected by different call triggers. 'Values with Qualifying Period' presents prices taking into account the qualifying-period requirement: The call sets in as soon as the conversion value exceeds the call trigger by more than 20 trading days out of the last 30 trading days. The standard errors are given in parentheses below the corresponding convertible bond prices. The third number indicates the number of paths (out of 20'000), in which the convertible bond is called by the issuer. The fourth number is the percentage pricing impact measured with respect to the price obtained without modeling the examined feature. The investigated convertible bond provides no coupon payments and has no credit risk. The initial conversion value is 100 and the volatility is 30%. The convertible bond does not include a put option and the interest rate is assumed to be constant at three percent. A quadratic polynomial is used for the estimation of the continuation value. All prices are generated with 20'000 simulations at daily frequency for convertible bonds with a maturity of one year (250 trading days). Each row stands for a different setting with respect to call trigger and call price.

call price			101		
			100.97		
value with call			(0.017%)		
			19301		
call trigger	150	130	115	105	101
	107.28	106.96	106.33	102.87	101.11
value with	(0.137%)	(0.109%)	(0.085%)	(0.035%)	(0.020%)
call trigger	4264	8514	11651	17466	19047
	+6.25%	+5.92%	+5.31%	+1.88%	+0.14%
	107.32	107.13	106.89	105.17	103.99
value with	(0.146%)	(0.127%)	(0.109%)	(0.071%)	(0.059%)
qualifying period	2739	6041	8834	14318	16102
	+0.04%	+0.17%	+0.52%	+2.24%	+2.84%
call price			105		
			102.76		
value with call			(0.032%)		
			17774		
call trigger	150	130	115	105	-
	107.28	106.96	106.33	102.87	-
value with	(0.137%)	(0.109%)	(0.085%)	(0.035%)	-
call trigger	4264	8514	11651	17466	-
	+6.25%	+5.92%	+5.31%	+1.88%	-
	107.32	107.13	106.62	105.17	-
value with	(0.146%)	(0.127%)	(0.098%)	(0.071%)	-
qualifying period	2739	6041	10499	14318	-
	+0.04%	+0.17%	+0.97%	+2.24%	-
call price			115		
			105.57		
value with call			(0.108%)		
			13717		
call trigger	150	130	115	-	-
	107.28	106.96	106.33	-	-
value with	(0.137%)	(0.109%)	(0.070%)	-	-
call trigger	4264	8514	13402	-	-
	+1.62%	+1.31%	+0.02%	-	-
	107.32	107.13	106.62	-	-
value with	(0.146%)	(0.127%)	(0.098%)	-	-
qualifying period	2739	6041	10499	-	-
	+0.04%	+0.17%	+0.97%	-	-

feature. Both the presence of the simple call trigger and the qualifying period decreases the number of paths in which the convertible bond is called by the issuer. Similarly, each additional restriction to the call option decreases the value of the call option and consequently increases the value of the convertible. The pricing impact of the trigger and the qualifying period is larger when the trigger is high. In the presented example, prices with and without the call trigger condition deviate by up to 6.25% in the case of a convertible bond with a call price of 101 and a call trigger equal to 150. The additional impact of the path-dependencies imposed by the qualifying period over a regular call trigger without qualifying period amounts to a maximum value of 2.84% for a convertible bond with call price and call trigger equal to 101 in the presented example. The highest deviation between prices with all trigger features combined and a simple call possibility amounts to 6.29%. It is obtained for a convertible bond with a call price of 101 and a call trigger of 150. In general, the presented results indicate that neglecting the detailed specifications occurring around the call option may cause substantial deviations from the true value.

VII. Investigation of Input Models

A. Interest Rate Dynamics

With respect to its value and influence on the price of a convertible bond, the regular bond part obtained at redemption is a major part of convertible bonds, especially when they are at- and out-of-the money. Thus, as driving factor behind the value of fixed-income securities, it is important to forecast interest rates and to adequately capture their future dynamics. There is a huge literature on interest rate processes confirming that the assumption of constant interest rates is inappropriate for modeling purposes. Although allowing for stochastic interest rates is a standard procedure for fixed-income securities, time-varying interest rates have so far been given a minor priority in the context of convertible bond pricing. In lattice-based pricing approaches, stochastic interest rates are often propagated as an additional state variable, however, due to the additional complexity and computational time they are almost never implemented. A pricing approach based on Monte Carlo Simulation, on the other hand, allows to relatively straightforward implement interest rates as additional state variable and model their interdependencies with the other state variables relevant for convertible bond pricing, i.e. mainly the conversion value and credit risk. Furthermore, since computational time grows only linearly with the number of state variables, stochastic interest rates may be included in the presented approach at relatively low cost for less extensive pricing applications. Therefore, stochastic interest rates are included in the analysis performed in this paper. The purpose is first, to determine the impact of a term-structure model on convertible-bond prices in comparison to non-stochastic approaches. Second, we examine how the pricing model reacts to stochastic interest rates and to which extent, in the presence of stochastic interest rates, current interest rates serve as additional explaining variable in the regressions that determine the continuation value of the convertible bond.

In the literature, many possibilities to model interest-rate dynamics have emerged. Although the presented simulation-based pricing method allows for virtually any model for stochastic interest rates, we employ a CIR term structure model (according to Cox et al., 1985), which is simple and straightforward to implement. In particular, interest rates are modeled to follow a CIR process given by:

$$dr_t = (\kappa\theta_r - \kappa r_t)\,dt + \sigma_r\sqrt{r_t}dW_{r,t}, \tag{13}$$

where κ is a constant mean-reversion velocity, θ_r is a constant long term interest rate, and σ_r is a constant volatility term. However, to avoid negative interest rates in the simulation, the natural logarithm of interest rates, $z_t = \ln(r_t)$, is modeled. Applying Ito's Lemma results in the following dynamics for z_t:

$$dz_t = \left(\frac{\kappa\theta_r - \kappa\exp(z_t)}{\exp(z_t)} - \frac{\sigma_r^2}{2\exp(z_t)}\right)dt + \frac{\sigma_r}{\sqrt{\exp(z_t)}}dW_{r,t}. \tag{14}$$

Since the simulations are performed in finite time, possible numerical approximation errors due to discretization have to be accounted for. Thus, instead of the simple Euler scheme, the more accurate discretization scheme proposed by Milstein (1978) is implemented:

$$\begin{aligned} z_{t+\triangle t} \quad = \quad & z_t + \left[\frac{\kappa\theta_r - \kappa\exp(z_t)}{\exp(z_t)} - \frac{\sigma_r^2}{2\cdot\exp(z_t)} + \frac{\sigma_r^2}{4\cdot\exp(2z_t)}\right]\triangle t \\ & + \frac{\sigma_r}{\exp(0.5\cdot z_t)}\varepsilon_{t+\triangle t} - \frac{\sigma_r}{4\cdot\exp(2z_t)}\varepsilon_{t+\triangle t}^2. \end{aligned} \tag{15}$$

All calculations for stochastic interest rates in this paper are performed with the given specification. Moreover, interest rates are allowed to correlate with the stock price. It is straightforward to extend the treatment of interest rates in the presented simulation-based pricing model as it is flexible enough to allow for more sophisticated interest rate processes that are correlated with further state variables.

Prices of convertible bonds typically are driven by several state variables such as the stock price, interest rates, default probabilities, recovery rates, and even exchange rates in the case of cross-currency convertible bonds. In the following, we examine the impact of stochastic interest rates on convertible-bond prices. Table IX shows several prices obtained with a term-structure model assuming different levels of the initial conversion value and correlations between stock and interest rates. These prices are compared with convertible-bond prices un-

Table IX
Pricing impact of stochastic interest rates

This table shows the percentage price impact of a term-structure model on prices of European-style convertible bonds for different initial stock prices and for different values of the correlation between stock and interest rate. Different initial stock prices imply different moneyness values for the convertible bonds. Moneyness ranges from 0.03 to 8.94 with corresponding stock prices ranging from S=1 to S=320. The number of paths for each simulation is 6000, with the same random-number series for each pricing. The first number indicates the absolute price of the convertible bond. The number in parentheses indicates the error of the simulated price computed as standard deviation of the mean of the simulated discounted payoffs. For prices calculated with stochastic interest rates the second number refers to the percentage change with respect to prices with constant interest rates. All convertible bonds have a face value F=100, maturity T=6 years, conversion ratio γ=1.0, coupon c=0, and constant credit spread cs=0.086. The issuing firm pays no dividends and is not entitled to call back the convertible bond at any time apart from maturity. The stock price follows a geometric Brownian motion, $\frac{dS_t}{S_t} = r_t dt + \sigma_S dW_{S,t}$, with volatility σ_S=0.3, and the instantaneous interest rate follows a one-factor CIR interest-rate process, $dr_t = \kappa_r(\theta_r - r_t)dt + \sigma_r\sqrt{r_t}dW_{r,t}$, with an initial short rate r=0.06, and parameters as estimated via GMM by Ait-Sahalia (1996): θ_r=0.090495, κ_r=0.89218, σ_r=0.180948. The correlations $\rho_{S,r}$ between dW_S and dW_r range from $\rho_{S,r}$=-0.5 to $\rho_{S,r}$=+0.5.

Stock price (moneyness)	1 (0.03)	20 (0.56)	30 (0.84)	35.78 (1.00)	40 (1.12)	80 (2.24)	320 (8.94)
Constant int. rates	35.66 (0.00)	38.12 (0.14)	42.74 (0.26)	46.25 (0.33)	49.09 (0.38)	82.97 (0.87)	321.64 (3.60)
Stochastic int. rates							
$\rho_{S,r} = -0.5$	35.66 -0.01% (0.05)	37.81 -0.81% (0.15)	42.25 -1.16% (0.26)	45.68 -1.23% (0.34)	48.51 -1.18% (0.39)	82.52 -0.54% (0.88)	321.89 0.00% (3.60)
$\rho_{S,r} = -0.2$	35.66 -0.01% (0.05)	38.03 -0.23% (0.14)	42.62 -0.29% (0.26)	46.10 -0.31% (0.33)	48.96 -0.27% (0.39)	82.87 -0.12% (0.88)	321.91 0.01% (3.60)
$\rho_{S,r} = 0.0$	35.66 -0.02% (0.05)	38.19 0.18% (0.14)	42.87 0.29% (0.26)	46.38 0.28% (0.33)	49.25 0.32% (0.38)	83.09 0.15% (0.89)	321.92 0.01% (3.60)
$\rho_{S,r} = +0.2$	35.65 -0.02% (0.05)	38.36 0.62% (0.14)	43.12 0.87% (0.26)	46.65 0.88% (0.33)	49.53 0.90% (0.38)	83.30 0.40% (0.87)	321.93 0.02% (3.60)
$\rho_{S,r} = +0.5$	35.67 0.01% (0.05)	38.62 1.32% (0.14)	43.49 1.75% (0.26)	47.07 1.78% (0.33)	49.95 1.76% (0.38)	83.61 0.78% (0.87)	321.95 0.03% (3.60)

der the assumption of constant interest rates. The investigated convertible bonds provide no coupon payments and have an annual default probability of one percent, with a recovery rate of 30 percent of the face value in case of default. Moreover, for simplicity, the convertible bonds are assumed not to include call and put options. While the maturity of the convertible bond is ten years in Panel A, it is three years in Panel B. The initial conversion values at which the convertible bonds are at-the-money are 42 in Panel A and 80 in Panel B.

The obtained results show that the impact of stochastic interest rates is largest for at-the-money convertibles and smallest for far in- and out-of-the-money convertibles. While far in-the-money convertibles behave like stock because they are almost surely to be converted, far out-of-the-money convertibles have similar characteristics to straight bonds because the conversion option is almost worthless. In both cases, the evolution of interest rates until maturity does not affect convertible bond prices, as long as the dynamics reflect the appropriate term structure. A further important factor, besides the moneyness, is the correlation between interest rate and stock price. In particular, high absolute values of correlation increase the value of the convertible significantly, although the actual size of the impact depends on the parameters of the term-structure model. Correspondingly, the maximum price impact displayed for stochastic interest rates is 3.99 percent, obtained for at-the-money convertibles and a correlation coefficient of 0.5 in the interest rate regime of Panel A. However, even for correlations around zero, a substantial price deviation can be observed, amounting to 2.57 percent for at-the-money convertibles in Panel A. For comparison, the maximum deviation recorded in Panel B, a scenario with shorter maturity and lower interest rates, is 1.05 percent.

VIII. Summary

This paper presents a model that prices convertible bonds with Monte Carlo Simulation. The pricing approach is computationally efficient, easy to implement, and suitable to handle the complexity of convertible bonds. In particular, the model provides enough flexibility to appropriately specify the dynamics of the state variables and their interaction. Furthermore, the method is able to capture the contractual specifications of today's convertible bonds as well as many emerging innovations of the instrument. In contrast to previous pricing approaches, default risk is modeled explicitly. This is found to be more suitable than applying an augmented interest rate to discount cash-flows subject to credit risk.

For the implemented model, results are presented that show the pricing performance of the model as well as the impact of various convertible bond specifications and input models for the state variables on pricing. The presented sensitivity results demonstrate the importance of accurately modeling the state variables and the contractual specifications as they can account for substantial price deviations. Therefore, a pricing framework based on Monte Carlo Simulation appears to be more promising for practical applications and future academic studies than previous closed-form solutions and lattice-based methods.

Although only convertible bonds are covered in this paper, it is straightforward to apply the pricing model to other derivatives with embedded options as well, with only minor adjustments. These instruments include convertible preferred stock, exchangeable bonds, and reverse convertible bonds, among a wide range of others.

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