1 Taylorreihe

a) Following Def. from Ex. 5.2.

$$f'(x) = \frac{d\, e^{x \log a}}{dx} = \log a \cdot f(x)$$

$$\Rightarrow a_n = \frac{(\log a)^n}{n!}.$$

b) $$\frac{d^n}{dx^n} \frac{1}{(1+x)^2} = (-2) \frac{d^{n-1}}{dx^{n-1}} \frac{1}{(1+x)^3}$$

$$= \dots = (-1)^n (n+1)! \frac{1}{(1+x)^{n+2}}$$

$$\Rightarrow f^{(n)}(0) = (-1) \cdot (n+1) \cdot f^{(n-1)}(0)$$
$$= (-1)^n (n+1)!$$

$$\Rightarrow a_n = (-1)^n (n+1)$$

c) $$e^{-x^2} = \sum_k \frac{1}{k!} y^k \Big|_{y=-x^2} = \sum_k \frac{(-1)^k}{k!} x^{2k}$$

$$\Rightarrow a_n = \begin{cases} \frac{(-1)^{n/2}}{(n/2)!} & n \text{ even} \\ 0 & \text{else} \end{cases}$$

2. Partielle Integration

a) $\int_0^1 x e^{-2x}\, dx = -\frac{1}{2} x e^{-2x}\Big|_0^1 + \frac{1}{2}\int_0^1 e^{-2x} dx$

$= -\frac{1}{2} e^{-2} - \frac{1}{4} e^{-2} + \frac{1}{4}$

$= -3/4\, e^{-2} + 1/4$

b) $\int \sin x \cos x \, dx$

$= \sin^2 x - \int \cos x \cdot \sin x \, dx$

$\Leftrightarrow \int \sin x \cos x \, dx = \frac{\sin^2 x}{2}$

$\Rightarrow \int_0^{\pi/2} \sin x \cos x \, dx = 1/2$

d) $\int_1^2 \frac{\log x}{x^2}\,dx = -\frac{\log x}{x}\Big|_1^2 + \int_1^2 \frac{1}{x^2}\,dx$

$= \left(-\frac{\log x}{x} - \frac{1}{x}\right)_1^2 = \frac{1}{2} - \frac{\log 2}{2}$

e) $\int e^{ax}\cos bx = \frac{e^{ax}}{a}\cos bx + b \int \frac{e^{ax}}{a}\sin bx\,dx$

$= \frac{e^{ax}}{a}\cos bx + \frac{b\,e^{ax}}{a^2}\sin bx - \frac{b^2}{a^2}\int e^{ax}\cos bx$

$\Longleftrightarrow (a^2+b^2)\int e^{ax}\cos bx\,dx$

$= e^{ax}(a\cos bx + b\sin bx)$

$\Longleftrightarrow \int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$

3. Integral durch Subst.

a) $\int_0^1 x e^{x^2} dx$ $\quad \left(\begin{array}{l} y = x^2 \\ \Rightarrow dy = 2x\,dx \end{array} \right)$

$= \frac{1}{2}\int_0^1 e^y dy = \underline{\underline{\frac{1}{2}(e-1)}}$

b) $\int_0^2 (7-3x)^{-\frac{1}{2}} dx$ $\quad \left(\begin{array}{l} y = 7-3x \\ dy = -3dx \end{array} \right)$

$= -\frac{1}{3}\int_7^1 y^{-\frac{1}{2}} dy$

$= \frac{1}{3} \cdot 2 \cdot y^{\frac{1}{2}}\Big|_1^7 = \underline{\frac{2}{3}(\sqrt{7}-1)}$

c) $\int x^2 \sqrt{2x^3+4}$ $\quad \left(\begin{array}{l} y = 2x^3+4 \\ dy = 6x^2 \end{array} \right)$

$= \frac{1}{6}\int y^{\frac{1}{2}} dy = \frac{1}{9} y^{3/2} = \underline{\frac{1}{9}(2x^3+4)^{\frac{3}{2}}}$

d) $\int_a^b x \cdot \cos(x^2+1)\, dx$

$= \frac{1}{2} \int_{a^2+1}^{b^2+1} \cos y \, dy$

$= \frac{1}{2} \left(\sin(b^2+1) - \sin(a^2+1) \right)$

4 Mehrfachintegrale

a) $$\int_0^a dx \int_0^b dy \int_0^h dz \; e^{-\alpha z}$$

$$= a \cdot b \cdot \left(-\frac{1}{\alpha}\right) \cdot e^{-az} \Big|_{z=0}^{h}$$

$$= \frac{ab}{\alpha}\left(1 - e^{-ah}\right)$$

b) $$\int_0^{\pi/2} dx \int_0^x dy \; \sin x \; \sin y = \frac{1}{2}$$

c) NR:

$$\int x^n \log x \, dx = \frac{x^{n+1}}{n+1} \log x - \frac{x^{n+1}}{(n+1)^2}$$

$$\Rightarrow \int_1^2 \int_1^{y^2} \log x+y \, dx \, dy$$

$$= \int_1^2 \int_{1+y}^{y+y^2} \log x \, dx \, dy$$

$$= \int_1^2 (y^2+y) \log y(1+y) - (y^2+y)$$
$$- (y+1) \log 1+y + (1+y) \, dy$$

$$= \int_1^2 (y^2-1) \log 1+y + (y^2+y) \log y$$
$$+ 1-y^2 \, dy$$

$$= \int_2^3 (y^2-2y) \log y \, dy + \int_1^2 (y^2+y) \log y \, dy$$
$$+ \left[y - \frac{y^3}{3} \right]_1^2$$

$$= \int_1^3 y^2 \log y \, dy - 2 \int_2^3 y \log y \, dy$$

$$+ \int_1^2 y \log y \, dy + \left(2 - \frac{8}{3} - 1 + \frac{1}{3}\right)$$

$$= \left(\frac{y^3}{3} \log y - \frac{y^3}{9}\right)_1^3 - 2\left(\frac{y^2}{2} \log y - \frac{y^2}{4}\right)_2^3$$

$$+ \left(\frac{y^2}{2} \log y - \frac{y^2}{4}\right)_1^2 - \frac{4}{3}$$

$$= \cancel{9 \log 3} - \cancel{3} + \frac{1}{9}$$

$$- 2\left(\cancel{\frac{9}{2} \log 3} - \frac{9}{4} - \cancel{2 \log 2} + \cancel{1}\right)$$

$$+ \cancel{2 \log 2} - \cancel{1} + \frac{1}{4} - \frac{4}{3}$$

$$= 6 \log 2 - 6 + \underbrace{\frac{1}{9} + \frac{19}{4} - \frac{4}{3}}_{\frac{4 + 171 - 48}{36}}$$

$$= \underline{\underline{-89/36 + 6 \log 2}}$$

5. Mehrfachintegrale

$$m = \frac{\rho_0}{a \cdot bc} \int_0^a dx \int_0^b dy \int_0^c dz \, (2x + 3y) z^2$$

$$= \frac{\rho_0}{abc} \int_0^a dx \int_0^b dy \left(\frac{2}{3}x + y\right) \cdot c^3$$

$$= \frac{c^2 \cdot \rho_0}{a \cdot b} \int_0^a dx \left(\frac{3}{3} x b + \frac{1}{2} b^2\right)$$

$$= \underline{\frac{1}{6} \frac{2a^2 + 3b^2}{ab} \cdot c^2 \cdot \rho_0}$$

$\Rightarrow$ for $\rho = \rho_0$ we have

$$m = \rho_0 \cdot V = \underline{a \cdot b \cdot c \cdot \rho_0}$$