Random Permutation Codes
Lossless Source Coding of Non-Sequential Data

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October 7, 2024

Outline

Papers

(NeurIPS 2024) **Severo**, Khisti, Makhzani. *Random Cycle Coding: Lossless Compression of Cluster Assignments via Bits-Back Coding*.

(Under Review) **Severo**, Su, Liu, Johnson, Karrer, Van den Broeck, Muckley, Ullrich. *Enhancing and Evaluating Probabilistic Circuits for High-Resolution Lossless Image Compression*.

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(ICLR 2024) **Severo**, Theis, Ballé. *The Unreasonable Effectiveness of Linear Prediction as a Perceptual Metric* . `https://arxiv.org/abs/2310.05986`

(ICLR 2024) Kunze, **Severo**, Zani, van de Meent, Townsend. *Entropy Coding of Unordered Data Structures.* **Oral (top 12% of accepted papers at ICML NCW Workshop).** `https://openreview.net/forum?id=afQuNt3Ruh`

(ICML 2023) Neklyudov, Brekelmans, **Severo**, Makhzani. *Action Matching: A Variational Method for Learning Stochastic Dynamics from Samples.* `https://arxiv.org/abs/2210.06662`

(ICML 2023) **Severo**, Townsend, Khisti, Makhzani. *Random Edge Coding: One-Shot Bits-Back Coding of Large Labeled Graphs*. `https://arxiv.org/abs/2305.09705`

(JSAIT 2023) **Severo**, Townsend, Khisti, Makhzani, Ullrich. *Compressing Multisets with Large Alphabets using Bits-Back Coding*. **Best Paper Award at NeurIPS DGM Workshop 2021.** `https://arxiv.org/abs/2107.09202`

(ICASSP 2022) Domanovitz, **Severo**, Khisti, Yu. *Data-Driven Optimization for Zero-Delay Lossy Source Coding with Side Information.* `https://ieeexplore.ieee.org/document/9747823`

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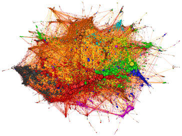
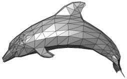
Motivation

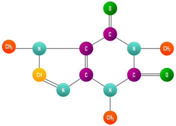
Non-sequential data is everywhere



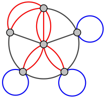
Social networks



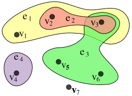
3D meshes



Molecules



Graphs



Hypergraphs

$$\{\{a, b\}, \{c, \{\ldots\}\}\}$$

Problem setting

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such that the **single-letter** code-length is close to the information content for all $x \in \mathcal{X}$

$$|C(x)| \approx -\log P_X(x).$$

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Later, X will be a non-sequential data type (e.g., set, graph). Assume $\mathcal{X}$ is too large to be held in memory.

Random Order Coding

Random Order Coding: Problem Statement

Given a sequence of i.i.d. discrete random variables

$$Z^n = (Z_1, \dots, Z_n),$$

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$$C(\mathcal{M}) \approx -\log P_{\mathcal{M}}(\mathcal{M}) = -\log P_{Z^n}(z^n) - \log M, \qquad (1)$$

where the constant M is known as the *multinomial coefficient* of $\mathcal{M}$

$$M = \frac{n!}{\prod_{z \in \mathcal{Z}} \mathcal{M}(z)!} \leq n!. \qquad (2)$$

Random Order Coding: encode $\mathcal{M}$ w/ $-\log P_{\mathcal{M}}(\mathcal{M})$ bits

Construct order information iteratively by "sampling without replacement" from $\mathcal{M}$. Alternate:

1. Decode sample (w.o. replacement) from $\mathcal{M}$
2. Encode sampled element using P_Z

until $\mathcal{M}$ is depleted.

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$\{\mathtt{a}, \mathtt{b}, \mathtt{b}\}$

$L(\mathcal{M}) = \varepsilon$

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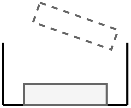

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$$L(\mathcal{M}) = \varepsilon - \log \frac{1}{2/3}$$

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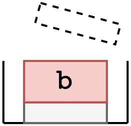

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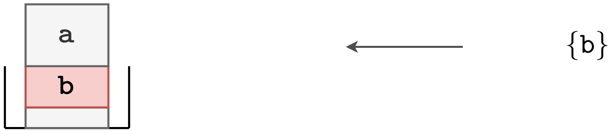

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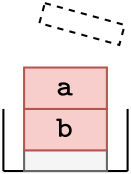

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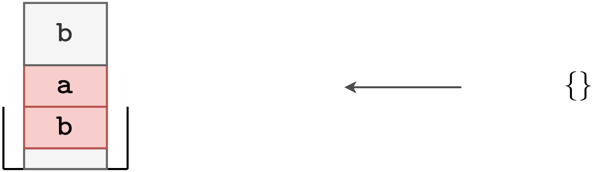

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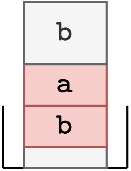

$\{\}$

$$L(\mathcal{M}) = \varepsilon + \log \frac{1}{P_Z(\mathtt{b})^2 P_Z(\mathtt{a})} - \log \frac{1}{(2/3)(1/2)(1/1)}$$

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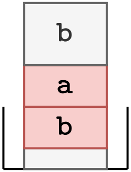

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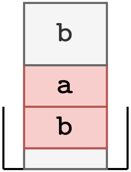

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$$L(\mathcal{M}) = \varepsilon + \log \frac{1}{P_{\mathcal{M}}(\{\mathtt{a}, \mathtt{b}, \mathtt{b}\})}$$

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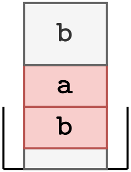

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$$L(\mathcal{M}) = \varepsilon + \log \frac{1}{P_{\mathcal{M}}(\{\mathtt{a}, \mathtt{b}, \mathtt{b}\})}$$

Complexity: $\mathcal{O}(n \cdot P_Z + n \cdot \log m)$

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Problem: the stack can deplete at any decoding/sampling step!

However, the average increase at any step is positive:

$$\begin{aligned}
\mathbb{E}\left[\Delta_i \,|\, \mathcal{M}\right] &= \mathbb{E}\left[\log P_{Z_i \,|\, \overline{\mathcal{M}}_i}(Z_i \,|\, \overline{\mathcal{M}}_i) - \log P_Z(Z_i) \middle| \mathcal{M}\right] && (3)\\
&= D_{\mathrm{KL}}(P_{Z_i \,|\, \overline{\mathcal{M}}_i}(\cdot \,|\, \overline{\mathcal{M}}_i) \,\|\, P_Z) && (4)\\
&\geq 0 && (5)
\end{aligned}$$

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Partition $\mathcal{Z}^n$ into subsets $x \subset \mathcal{Z}^n$ of equivalent sequences.

Each x can be mapped uniquely to some $\mathcal{M}$ such that

$$P_{\mathcal{M}}(\mathcal{M}) = P_X(x).$$

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In the equivalence class view, a multiset $X = \mathcal{M}$ is a random variable with alphabet equal to the **quotient set**: $\mathcal{X} = \mathcal{Z}^n / \sim$.

Multisets as Equivalence Classes: Example

Let $\mathscr{Z} = \{\blacktriangle, \bigstar, \square\}, n = 3$.

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Multiset	Equivalence class in $\mathscr{Z}^3/\sim$
$\{\blacktriangle, \blacktriangle, \blacktriangle\}$	$\{\blacktriangle\blacktriangle\blacktriangle\}$
$\{\bigstar, \bigstar, \bigstar\}$	$\{\bigstar\bigstar\bigstar\}$
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$\{\blacktriangle, \blacktriangle, \bigstar\}$	$\{\blacktriangle\blacktriangle\bigstar, \blacktriangle\bigstar\blacktriangle, \bigstar\blacktriangle\blacktriangle\}$

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The information content of the multiset $\mathcal{M}$,

$$-\log P_{\mathcal{M}}(\mathcal{M}) = -\log P_{Z^n}(z^n) - \log M,$$

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The information content of the multiset $\mathcal{M}$,

$$-\log P_{\mathcal{M}}(\mathcal{M}) = -\log P_{Z^n}(z^n) - \log M,$$

can be rewritten as,

$$-\log P_X([z^n]) = -\log P_{Z^n}(z^n) - \log|[z^n]|.$$

Combinatorial Random Variables

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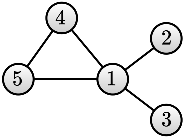
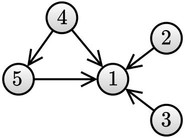
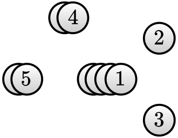
	Undirected Graphs	Directed Graphs	Multisets
X			
$[Z^n]$	$21, 31, 41, 51, 45$	$21, 31, 41, 51, 45$	2131415145
	$21, 14, 13, 15, 45$	$21, 31, 51, 41, 45$	1111234455
	$\ldots$	$\ldots$	$\ldots$

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Restricting the class of $\sim$ allows for efficient algorithms! (see thesis)

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A CRV is a random variable with alphabet equal to the quotient set $\mathcal{Z}^n/\!\sim$, where the equivalence relation is *finer* than that of multisets.

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Graphs, hyper-graphs, multigraphs $\mapsto$ Random Edge Coding (REC)

Partitions and cluster assignments $\mapsto$ Random Cycle Coding (RCC)

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Between 15% and 70% savings in realistic use cases!

Thank you!

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$$\log 1/P(G)$$

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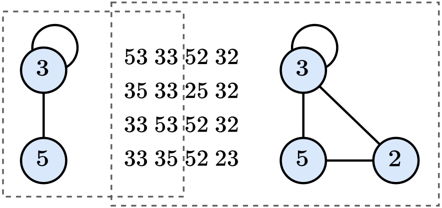

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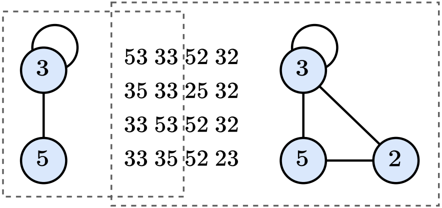

Equivalent sequences $\mathbf{v} \sim \mathbf{w}$ map to the same graph

$$53\ 33 \sim 33\ 53 \sim 33\ 35 \sim 35\ 33$$

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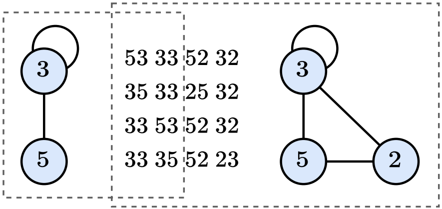

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For $\mathbf{v} = (v_1, v_2, \ldots, v_{2k})$

$$P(v_{i+1} \,|\, v^i) \propto d_{v^i}(v_{i+1}) + 1, \tag{6}$$

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The joint distribution can be expressed as

$$P(\mathbf{v}) = \frac{1}{n^{\uparrow k}} \prod_{v \in [n]} d_{v^k}(v)!, \tag{7}$$

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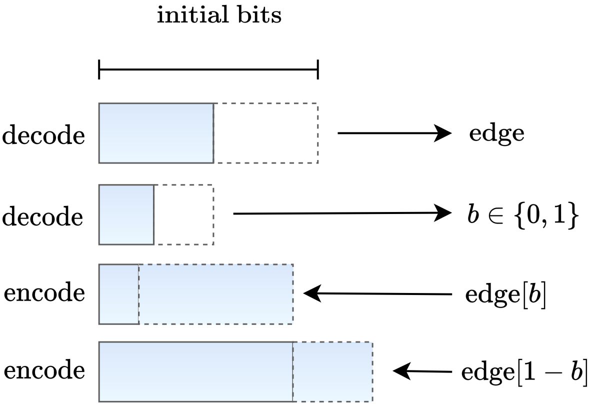

Random Edge Coding

	Social Networks				Others	
	YouTube	FourSq.	Digg	Gowalla	Skitter	DBLP
# Nodes	3,223,585	639,014	770,799	196,591	1,696,415	317,080
# Edges	9,375,374	3,214,986	5,907,132	950,327	11,095,298	1,049,866
$10^6 \times$Density	1.8	15.8	19.8	50.2	7.7	20.9
(Ours) PU w/ REC	**15.19**	9.96	10.62	12.19	14.26	15.92
Pool Comp.	15.38	**9.23**	11.59	**11.73**	7.45	**8.78**
Slashburn	17.03	10.67	**9.82**	11.83	12.75	12.62
Backlinks	17.98	11.69	12.56	15.56	11.49	10.79
List Merging	15.80	9.95	11.92	14.88	**8.87**	14.13