Computational Physics Assignment 1
PHYS39xx University of Sydney

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1. The motion of a T-handle can be modelled using the Euler equations, which represent free rotation of an arbitrary rigid body:

$$A\frac{d\omega_1}{dt} = (B - C)\omega_2\omega_3 \tag{1}$$

$$B\frac{d\omega_2}{dt} = (C - A)\omega_3\omega_1 \tag{2}$$

$$C\frac{d\omega_3}{dt} = (A - B)\omega_1\omega_2 \tag{3}$$

where ω_1, ω_2 and ω_3 are the angular velocities about the three principle axes of the body, and A, B and C are the moments of inertia for rotation about those axes. The principle axes are fixed in the body, so equations (1)-(3) describe the motion in the moving frame of reference.

It is possible to show from these equations that for an object with three different principle moments of inertia, $A < B < C$, pure rotation about the axis corresponding to the "intermediate" moment of inertia (B) is unstable. This instability is exhibited by the T-handle. In this question we will model the instability numerically.

a) The kinetic energy K and the magnitude of the angular momentum L of the body are:

$$K = \frac{1}{2}A\omega_1^2 + \frac{1}{2}B\omega_2^2 + \frac{1}{2}C\omega_3^2 \tag{4}$$

and

$$L^2 = A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 \tag{5}$$

Using equations (1)-(3), show that K and L are conserved.

Answer: Firstly, take the derivative of Kinetic Energy, equation (4), wrt to time using the chain rule and then sub in equations (1)-(3) which gives

$$\begin{aligned}
\frac{dK}{dt} &= \frac{1}{2}A\frac{d}{dt}\left(\omega_1^2\right) + \frac{1}{2}B\frac{d}{dt}\left(\omega_2^2\right) + \frac{1}{2}C\frac{d}{dt}\left(\omega_3^2\right) \\
&= \frac{1}{2}(2A\omega_1)\frac{d\omega_1}{dt} + \frac{1}{2}(2B\omega_2)\frac{d\omega_2}{dt} + \frac{1}{2}(2C\omega_3)\frac{d\omega_3}{dt} \\
&= \frac{A}{A}\omega_1(B - C)\omega_2\omega_3 + \frac{B}{B}\omega_2(C - A)\omega_3\omega_1 + \frac{C}{C}\omega_3(A - B)\omega_1\omega_2 \\
&= A(\omega_1\omega_2\omega_3 - \omega_1\omega_2\omega_3) + B(\omega_1\omega_2\omega_3 - \omega_1\omega_2\omega_3) + C(\omega_1\omega_2\omega_3 - \omega_1\omega_2\omega_3) \\
&= 0
\end{aligned}$$

Therefore, we can see that there is no change in Kinetic energy over time, hence it is conserved.

Now we will do the same for Angular momentum, equation (6), and sub in equations

(1)-(3) again

$$\begin{aligned}
\frac{d}{dt}(L^2) &= A^2\frac{d}{dt}\omega_1^2 + B^2\frac{d}{dt}\omega_2^2 + C^2\frac{d}{dt}\omega_3^2 \\
2L\frac{dL}{dt} &= 2A^2\omega_1\frac{d\omega_1}{dt} + 2B^2\omega_2\frac{d\omega_2}{dt} + 2C^2\omega_3\frac{d\omega_3}{dt} \\
L\frac{dL}{dt} &= A\omega_1(B-C)\omega_2\omega_3 + B\omega_2(C-A)\omega_3\omega_1 + C\omega_3(A-B)\omega_1\omega_2 \\
&= AB(A\omega_1\omega_2\omega_3 - B\omega_1\omega_2\omega_3) + AC(A\omega_1\omega_2\omega_3 - C\omega_1\omega_2\omega_3) + BC(B\omega_1\omega_2\omega_3 - C\omega_1\omega_2\omega_3) \\
&= 0
\end{aligned}$$

b) By introducing a scale factor t_s for time, show that the Euler equations can be written in the non-dimensional form

$$a\frac{d\bar{\omega}_1}{d\bar{t}} = (b-1)\bar{\omega}_2\bar{\omega}_3 \tag{6}$$

$$b\frac{d\bar{\omega}_2}{d\bar{t}} = (1-a)\bar{\omega}_3\bar{\omega}_1 \tag{7}$$

$$\frac{d\bar{\omega}_3}{d\bar{t}} = (a-b)\bar{\omega}_1\bar{\omega}_2 \tag{8}$$

where the bars denote dimensionless quantities and where $a = A/C$, and $b = B/C$. Show also that the conserved quantities may be rewritten in non-dimensional form as

$$\bar{K} = \frac{1}{2}a\bar{\omega}_1^2 + \frac{1}{2}b\bar{\omega}_2^2 + \frac{1}{2}\bar{\omega}_3^2 \tag{9}$$

and

$$\bar{L}^2 = a^2\bar{\omega}_1^2 + b^2\bar{\omega}_2^2 + c^2\bar{\omega}_3^2 \tag{10}$$

Answer: Let $\bar{t} = t/t_s$ and $\bar{\omega}_i = \omega/\omega_{is}$, then rearrange to $t = \bar{t}t_s$ and $\omega_i = \bar{\omega}_i\omega_{is}$. Now we introduce these back into the Euler equations. Consider equation (1) first

$$\begin{aligned}
A\frac{d(\bar{\omega}_1\omega_{1s})}{d(\bar{t}t_s)} &= (B-C)\bar{\omega}_2\omega_{2s}\bar{\omega}_3\omega_{3s} \\
\implies A\frac{\omega_{1s}}{t_s}\frac{d\bar{\omega}_1}{d\bar{t}} &= \omega_{2s}\omega_{3s}(B-C)\bar{\omega}_2\bar{\omega}_3 \\
\implies \frac{A}{C}\frac{d\bar{\omega}_1}{d\bar{t}} &= \frac{t_s\omega_{2s}\omega_{3s}}{\omega_{1s}}\left(\frac{B}{C}-1\right)\bar{\omega}_2\bar{\omega}_3 \\
\implies a\frac{d\bar{\omega}_1}{d\bar{t}} &= \frac{t_s\omega_{2s}\omega_{3s}}{\omega_{1s}}(b-1)\bar{\omega}_2\bar{\omega}_3
\end{aligned}$$

Now we see that

$$\frac{t_s\omega_{2s}\omega_{3s}}{\omega_{1s}} = 1$$

The same can be done for equation (2)

$$\begin{aligned}
B\frac{d(\bar{\omega}_2\omega_{2s})}{d(\bar{t}t_s)} &= (C-A)\bar{\omega}_3\omega_{3s}\bar{\omega}_1\omega_{1s} \\
\implies B\frac{\omega_{2s}}{t_s}\frac{d\bar{\omega}_1}{d\bar{t}} &= \omega_{3s}\omega_{1s}(C-A)\bar{\omega}_3\bar{\omega}_1 \\
\implies \frac{B}{C}\frac{d\bar{\omega}_2}{d\bar{t}} &= \frac{t_s\omega_{3s}\omega_{1s}}{\omega_{2s}}\left(1-\frac{A}{C}\right)\bar{\omega}_3\bar{\omega}_1 \\
\implies b\frac{d\bar{\omega}_2}{d\bar{t}} &= \frac{t_s\omega_{3s}\omega_{1s}}{\omega_{2s}}(1-a)\,\bar{\omega}_3\bar{\omega}_1
\end{aligned}$$

Which gave another constant

$$\frac{t_s\omega_{3s}\omega_{1s}}{\omega_{2s}} = 1$$

Finally, the same is done for equation (3)

$$\begin{aligned}
C\frac{d(\bar{\omega}_3\omega_{3s})}{d(\bar{t}t_s)} &= (A-B)\bar{\omega}_1\omega_{1s}\bar{\omega}_2\omega_{2s} \\
\implies C\frac{\omega_{3s}}{t_s}\frac{d\bar{\omega}_3}{d\bar{t}} &= \omega_{1s}\omega_{2s}(A-B)\bar{\omega}_1\bar{\omega}_2 \\
\implies \frac{d\bar{\omega}_3}{d\bar{t}} &= \frac{t_s\omega_{1s}\omega_{2s}}{\omega_{3s}}\left(\frac{A}{C}-\frac{B}{C}\right)\bar{\omega}_1\bar{\omega}_2
\end{aligned}$$

Which gave another constant

$$\frac{t_s\omega_{1s}\omega_{2s}}{\omega_{3s}} = 1$$

All three constants can be solved simultaneously to determine what the constants (ω_{is}'s) equal to satisfy the non-dimensional forms. It is quite clear from the layout of each constant equation that there is only one solution and all three constants equal one another.

$$\omega_{1s} = \omega_{2s} = \omega_{3s} = \frac{1}{t_s}$$

Therefore, the Euler equations can be written in non-dimensional form as

$$\begin{aligned}
a\frac{d\bar{\omega}_1}{d\bar{t}} &= (b-1)\bar{\omega}_2\bar{\omega}_3 \\
b\frac{d\bar{\omega}_2}{d\bar{t}} &= (1-a)\bar{\omega}_3\bar{\omega}_1 \\
\frac{d\bar{\omega}_3}{d\bar{t}} &= (a-b)\bar{\omega}_1\bar{\omega}_2
\end{aligned}$$

Let $\bar{K} = K/K_s$ and using the solution from above, $\bar{\omega}_i = \omega_i t_s$, we see

$$\begin{aligned}
\bar{K}K_s &= \frac{1}{2}A\left(\frac{\bar{\omega}_1}{t_s}\right)^2 + \frac{1}{2}B\left(\frac{\bar{\omega}_2}{t_s}\right)^2 + \frac{1}{2}C\left(\frac{\bar{\omega}_3}{t_s}\right)^2 \\
\bar{K} &= \frac{C}{K_s t_s^2}\left(\frac{1}{2}\frac{A}{C}\bar{\omega}_1^2 + \frac{1}{2}\frac{B}{C}\bar{\omega}_2^2 + \frac{1}{2}\bar{\omega}_3^2\right) \\
\bar{K} &= \frac{1}{2}a\bar{\omega}_1^2 + \frac{1}{2}b\bar{\omega}_2^2 + \frac{1}{2}\bar{\omega}_3^2
\end{aligned}$$

where $K_s = C/t_s^2$. We can do the same for Angular momentum, Let $\bar{L}^2 = L^2/L_s^2$ and $\bar{\omega}_i = \omega_i t_s$ gives

$$\begin{aligned}
\bar{L}^2 L_s^2 &= A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 \\
&= A^2\left(\frac{\bar{\omega}_1}{t_s}\right)^2 + B^2\left(\frac{\bar{\omega}_2}{t_s}\right)^2 + C^2\left(\frac{\bar{\omega}_3}{t_s}\right)^2 \\
\bar{L}^2 &= \frac{C^2}{L_s^2 t_s^2}\left(\frac{A^2}{C^2}\bar{\omega}_1^2 + \frac{B^2}{C^2}\bar{\omega}_2^2 + \bar{\omega}_3^2\right) \\
\bar{L}^2 &= a^2\bar{\omega}_1^2 + b^2\bar{\omega}_2^2 + \bar{\omega}_3^2
\end{aligned}$$

where $L_s^2 = C^2/t_s^2$.

Hereafter we will use the non-dimensional equations (6)-(10), but omit the bars.

Consider numerical solution of the non-dimensional Euler equations (6)-(8), using the fourth order Runge-Kutta (RK4) method presented in Weeks 4 and 5.

c) Implement a right-hand side function to represent the coupled system of ODEs (6)-(8) and a main code which uses the right-hand side function to numerically solve the ODEs using RK4. You are encouraged to use the rk4step.m routine provided in the lectures.

Your code should:

- use a non-dimensional time step $\tau = 0.05$;
- integrate for a total non-dimensional time 20;
- use $a = 0.1$ and $b = 0.5$;
- produce a single plot showing ω_1, ω_2, and ω_3 versus time;
- produce a plot showin K versus time, and a separate plot showing L versus time. (The two plots can be two panels in one figure.)

Apply your code to solve the following intitial-value problems. One case should show stable oscillation, and the second case should show instability demonstrated by the T-handle. Include the plots produced by your code in your assignment submission.

(i) At $t = 0 : \omega_1 = 1.0, \omega_2 = 0.5, \omega_3 = 0.1$.

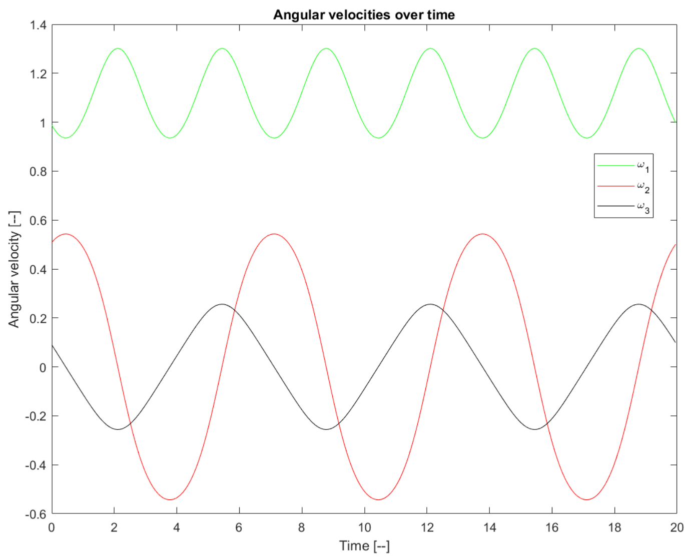

Figure 1: Angular Velocity over Time

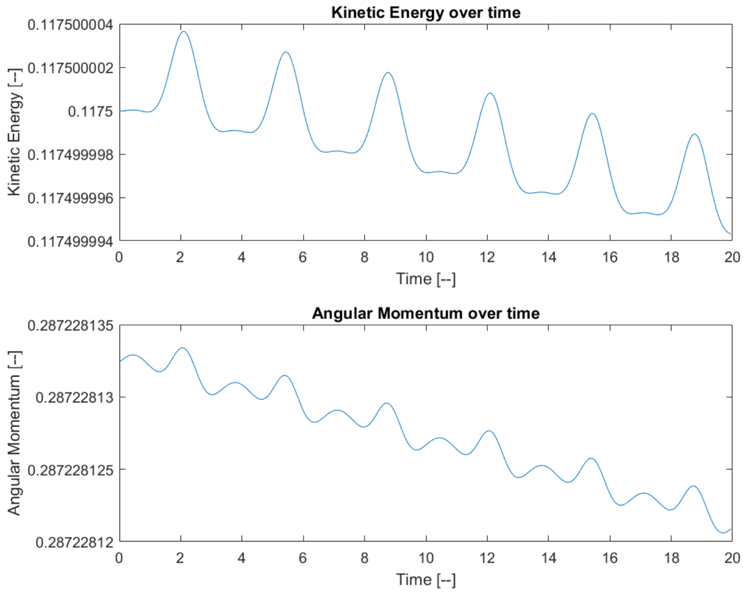

Figure 2: Angular Momentum and Kinetic Energy over Time

(ii) At $t = 0 : \omega_1 = 10^{-3}, \omega_2 = 2.0, \omega_3 = 0.0$.

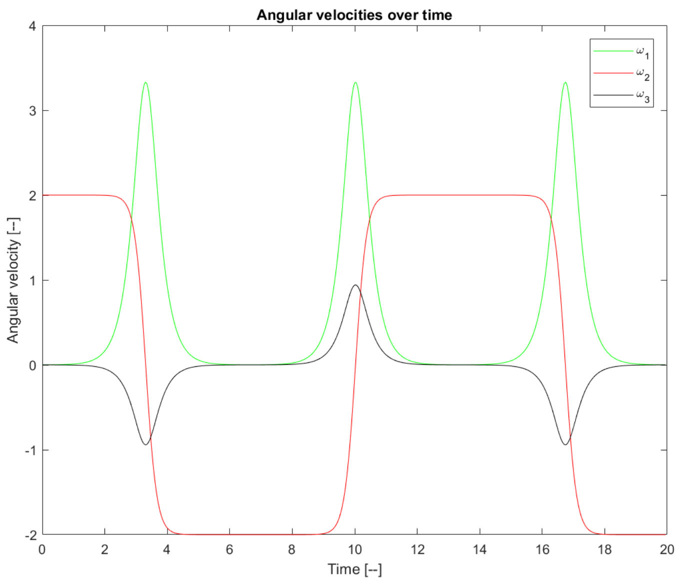

Figure 3: Angular Velocity over Time

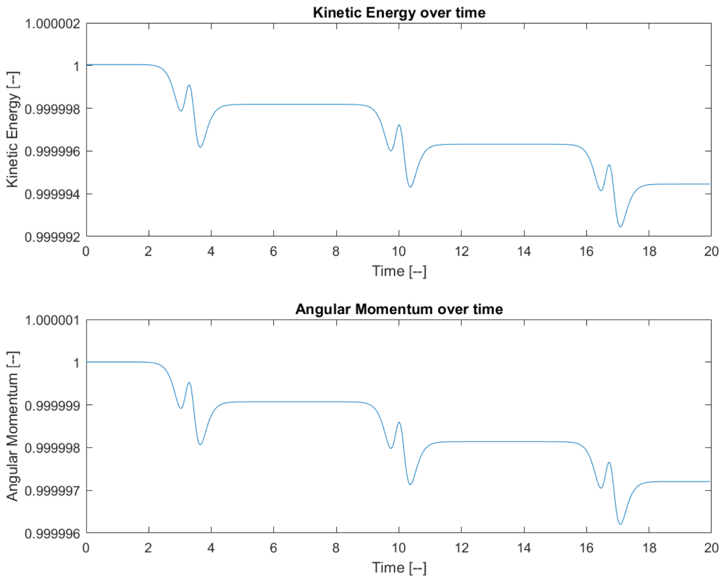

Figure 4: Angular Momentum and Kinetic Energy over Time

d) Briefly explain which case represents the flipping T-handle, and why. How does the solution match what is seen in the YouTube video? Also, briefly explain whether your solutions are accurate.

Answer: Comparing this to the YouTube video, it is clear that the section rotating around the screw is at first rotating anticlockwise and then changes direction to rotate clockwise after the flip (when following the direction of the orange sticker). This rotation must be over axis 2, the red line in figure 3, as it goes from positive to negative and back, but stays positive for some time and also stays negative for some time. This appears to match what is in the video. There is nothing in figure 1 that shows this behaviour. This can also be understood by the initial conditions for the two cases. In this second case the $d\bar{\omega}_1/d\bar{t} = 0$, $d\bar{\omega}_2/d\bar{t} = 0$ and $d\bar{\omega}_3/d\bar{t} = -2 \times 10^{-3}$, which is very small. Over time $d\bar{\omega}_3/d\bar{t}$ starts to increases in magnitude and this increase in magnitudemakes the angular velocity of the the other two axes become non-zero based on equations 6 and 7; depending on the sign of equation 8. Then the non-zero angular velocities of axis 1 and 2 change equation 8 giving us this unstable evolution seen in the youtube video. Because the initial conditions in the first case are all non-zero, this gives the T-handle stability where the angular velocities calculated in equations 6,7 and 8 always have the same direction (sign) at the same part of its rotation, period after period. So there is no case where small changes in one angular velocity have significant effects on the other two angular velocities, it is perfectly stable.

The solutions are relatively accurate, they portray the unstable effects observed but as is evident in both Figure 2 and 4, there is still no conservation of energy or angular momentum over time which is due to error from the integration steps accumulating over time.

2. The linear harmonic oscillator is described by the (non-dimensional) 1-D dynamics ODEs

$$\frac{dx}{dt} = v \text{ and } \frac{dv}{dt} = -x \tag{11}$$

The total energy

$$E = \frac{1}{2}(x^2 + v^2) \tag{12}$$

of the linear oscillator is conserved.

(a) Analytically apply Euler's method to the linear harmonic oscillator, and show that the energy increases in one step according to

$$E_{n+1} = (1 + \tau^2)E_n \tag{13}$$

Briefly explain why this result means that the error in energy will always eventually be large.

Answer:

$$\begin{aligned}
E_{n+1} &= \frac{1}{2}(x_{n+1}^2 + v_{n+1}^2) \\
&= \frac{1}{2}\left((x_n + \tau v_n)^2 + (v_n - \tau x_n)^2\right) \\
&= \frac{1}{2}\left(x_n^2 + 2\tau x_n v_n + \tau^2 v_n^2 + v_n^2 - 2\tau x_n v_n + \tau^2 x_n^2\right) \\
&= \frac{1}{2}\left(x_n^2 + v_n^2 + \tau^2(v_n^2 + x_n^2)\right) \\
&= \left(1 + \tau^2\right)\frac{1}{2}(x_n^2 + v_n^2) \\
&= \left(1 + \tau^2\right)E_n
\end{aligned}$$

The error in energy will always eventually be large because energy of the LHO is conserved but we see above, that the next timestep is not equal to the current timestep. Thus, at each timestep there will be an error which will sum to a large value using Euler's method.

(b) Symplectic integrators nearly conserve energy. The Verlet method is one example. A simpler example, applicable to a variety of dynamics problems, is the Euler-Cromer scheme:

$$\begin{aligned}
x_{n+1} &= x_n + \tau v_n \\
v_{n+1} &= v_n + \tau a_{n+1}
\end{aligned} \tag{14}$$

Apply this symplectic integrator analytically to the linear harmonic oscillator, and show that, for all values of n:

$$\frac{1}{2}(x_n^2 + v_n^2 + \tau x_n v_n) = C \tag{15}$$

where C is a constant.

Answer:

$$
\begin{aligned}
E_{n+1} &= \frac{1}{2}(x_{n+1}^2 + v_{n+1}^2) \\
&= \frac{1}{2}\left((x_n + \tau v_n)^2 + (v_n - \tau x_{n+1})^2\right) \\
&= \frac{1}{2}\left(x_n^2 + 2\tau x_n v_n + \tau^2 v_n^2 + v_n^2 - 2\tau x_{n+1} v_n + \tau^2 x_{n+1}^2\right) \\
&= \frac{1}{2}\left(x_n^2 + v_n^2 + \tau x_n v_n + \tau x_n v_n + \tau^2 v_n^2 - \tau x_{n+1} v_n - \tau x_{n+1} v_n + \tau^2 x_{n+1}^2\right) \\
&= \frac{1}{2}\left(x_n^2 + v_n^2 + \tau x_n v_n\right) + \frac{1}{2}\left(\tau v_n (x_n + \tau v_n) - \tau x_{n+1}(v_n - \tau x_{n+1}) - \tau x_{n+1} v_n\right) \\
&= \frac{1}{2}\left(x_n^2 + v_n^2 + \tau x_n v_n\right) + \frac{1}{2}\tau\left(v_n x_{n+1} - x_{n+1} v_{n+1} - x_{n+1} v_n\right) \\
&= \frac{1}{2}\left(x_n^2 + v_n^2 + \tau x_n v_n\right) - \frac{1}{2}\tau x_{n+1} v_{n+1}
\end{aligned}
$$

$$
\begin{aligned}
\implies E_{n+1} + \frac{1}{2}\tau x_{n+1} v_{n+1} &= \frac{1}{2}\left(x_n^2 + v_n^2 + \tau x_n v_n\right) \\
\frac{1}{2}(x_{n+1}^2 + v_{n+1}^2 + \tau x_{n+1} v_{n+1}) &= \frac{1}{2}\left(x_n^2 + v_n^2 + \tau x_n v_n\right) \\
&= C
\end{aligned}
$$

where C is a constant as the updated value at the next time step, LHS, is calculated the exact same way as the current value, RHS. Thus, it will always be the same.

(c) By introducing new variables (X_n, V_n) related to the original variables by

$$x_n = \frac{X_n - V_n}{\sqrt{2}} \quad \text{and} \quad v_n = \frac{X_n + V_n}{\sqrt{2}} \tag{16}$$

show analytically that the points described by equation 15 lie on the curve

$$\frac{X_n^2}{a^2} + \frac{V_n^2}{b^2} = 1 \tag{17}$$

where

$$a^2 = \frac{4C}{2+\tau} \quad \text{and} \quad b^2 = \frac{4C}{2-\tau} \tag{18}$$

Answer:

$$
\begin{aligned}
\left(\left(\frac{X_n - V_n}{\sqrt{2}}\right)^2 + \left(\frac{X_n + V_n}{\sqrt{2}}\right)^2 + \tau\left(\frac{X_n - V_n}{\sqrt{2}}\right)\left(\frac{X_n + V_n}{\sqrt{2}}\right)\right) &= C \\
\frac{1}{2}\left(X_n^2 - X_n V_n + V_n^2 + X_n^2 + X_n V_n + V_n^2 + \tau X_n^2 - \tau V_n^2\right) &= 2C \\
2X_n^2 + \tau X_n^2 + 2V_n^2 - \tau V_n^2 &= 4C \\
\frac{X_n^2(2+\tau)}{4C} + \frac{V_n^2(2-\tau)}{4C} &= 1 \\
\frac{X_n^2}{a^2} + \frac{V_n^2}{b^2} &= 1
\end{aligned}
$$

(d) Sketch the shape of equation 17 in the $x_n - v_n$ plane, for the case $C = \frac{1}{2}$. Sketch also the shape of an exact solution to the linear harmonic oscillator with $E = \frac{1}{2}$. Based on the two curves, explain why total energy is nearly conserved by the Euler-Cromer method in application to this problem.

Answer:

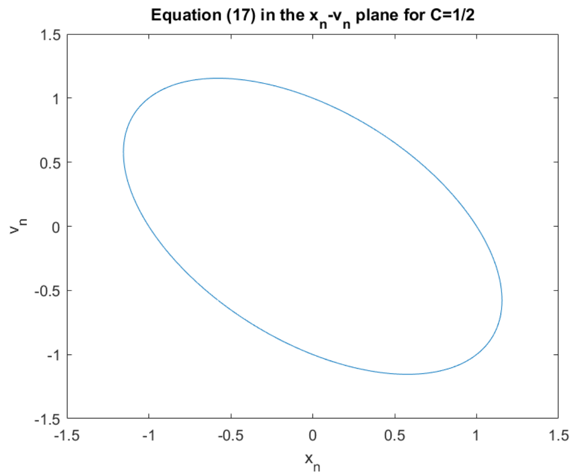

Figure 5: Equation (17) in the $x_n - v_n$ plane for $C = 1/2$

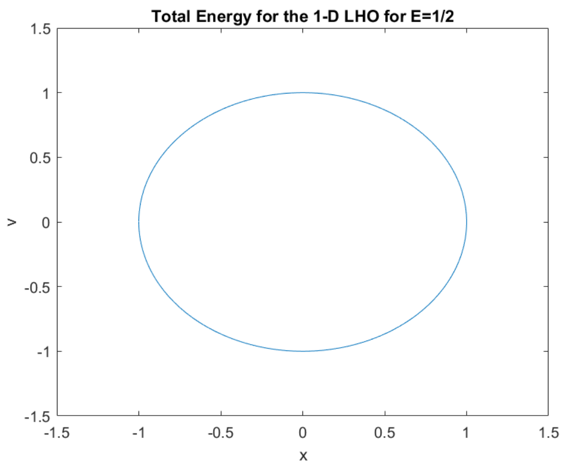

Figure 6: Exact Solution to the LHO with $E = 1/2$

Figure 6 shows that Energy is conserved for all values of x and v for the total Energy, the radius remains constant where $r = \sqrt{2E}$. The solution of Euler-Cromer is not conserved due to the extra term in Equation 15, $\tau x_n v_n$. This term ensures that the solution of Energy varies with x and v in the shape of an ellipse, so in this case the radius does not remain constant for all values of x and v, so the total Energy which is $E = (1/2)r^2$ varies with x and v.

1 Appendix:

Listings

Listing 1: Rigid Body RK4 Main Code

```
% rigbod_rk4mod.m
% This solves the Coupled non-dimensionalised Euler equations of free-
% rotation of an arbitrary rigid body, with a modular implementation
% of an RK4 step

% Clear memory and show only a few digits
close all,clear all; format short;

% Name of function evaluating RHS of ODEs
rhs_string='rhs_rigbod';

% Define non-dimensional moments of inertia
a = 0.1;
b = 0.5;

% Time step, total integration time and number of time steps
tau=0.05; T=20;
nsteps=ceil(T/tau);

% Initial conditions
omega1=10^(-3); omega2 = 2.0; omega3=0; % Initial angles in radians

x=[omega1 omega2 omega3];

% Fourth-order Runge-Kutta integration
for n=1:nsteps

  % Time
  time(n)=(n-1)*tau;

  % Runge-Kutta step
  x=rk4step(x,time(n),tau,a,b,rhs_string);

  % angular velocities of the three principle axes of the rigid body
  omega1out(n)=x(1);
  omega2out(n)=x(2);
  omega3out(n)=x(3);
```

```

  % Calculate Kinetic Energy
  K(n)=(1/2)*a*x(1)^2+(1/2)*b*x(2)^2+(1/2)*x(3)^2;

  %Calculate Angular momentum
  L(n)=sqrt(a^2*x(1)^2+b^2*x(2)^2+x(3)^2);

end
figure
plot(time,omega1out,'g',time,omega2out,'r',time,omega3out,'k')
title(['Angular velocities over time'])
xlabel('Time [--]')
ylabel('Angular velocity [--]')
legend('omega_1','omega_2','omega_3')
figure
ax1 = subplot(2,1,1);
plot(ax1,time,K)
xlabel('Time [--]')
ylabel('Kinetic Energy [--]')
title(ax1,'Kinetic Energy over time')
ax2 = subplot(2,1,2);
plot(ax2,time,L)
xlabel('Time [--]')
ylabel('Angular Momentum [--]')
title(ax2,'Angular Momentum over time')
```

Listing 2: Right Hand Side Function for Rigid Body ODEs

```
% rhs_rigbod.m
% Function to evaluate the right hand side of the coupled (non-
% dimensional) ODEs describing the nonlinear pendulum
%
% Inputs:
% x - the current value of the dependent variable. For the pendulum
%   ODEs x = [theta omega] where theta is the angle and omega is the
%   angular velocity.
% t - the current value of the independent variable.
% Outputs:
% rhs - a row vector representing the value of the right hand side
%   of the ODEs.

function rhs=rhs_rigbod(x,t,a,b)

omega1=x(1);
omega2=x(2);
omega3=x(3);
rhs(1)=(1/a)*(b-1)*omega2*omega3;
rhs(2)=(1/b)*(1-a)*omega3*omega1;
```

```
rhs(3)=(a-b)*omega1*omega2;
```

Listing 3: Modularized Version of the RK4 Step Function

```
function xout=rk4step(x,t,tau,a,b,rhs_string)

% Perform one step of RK4 for the ODE dx/dt=f(x,t). The RHS of the ODE
% is described by the function matching the supplied string rhs_string.
%
% Routine adapted from Garcia, Numerical Methods in Physics.
%
% Inputs:
% x - current values of dependent variables
% t - current value of independent variable
% tau - step size
% rhs_string - name of function specifying the RHS of the ODE
%
% Outputs:
% xout - new value of x after a step of size tau

htau=0.5*tau;
f1=feval(rhs_string,x,t,a,b);
th=t+htau;
xtemp=x+htau*f1;
f2=feval(rhs_string,xtemp,th,a,b);
xtemp=x+htau*f2;
f3=feval(rhs_string,xtemp,th,a,b);
tf=t+tau;
xtemp=x+tau*f3;
f4=feval(rhs_string,xtemp,tf,a,b);
xout=x+tau*(f1+2*f2+2*f3+f4)/6;

return;
```

Listing 4: Plotting of the $x_n - v_n$ plane for $C = 1/2$

```
%Ellipse_LHO_ODE.m
%Plot the x_n and v_n variables based on LHO from introducing new
%variables.
clear; close all;
C=1/2; % constant value
tau = 1;
a=sqrt((4*C)/(2+tau)); % horizontal radius
b=sqrt((4*C)/(2-tau)); % vertical radius
t=-pi:0.01:pi; %dependent variable
X_n=a*cos(t); % New variable X_n
V_n=b*sin(t); % New variable Y_n
x_n = (X_n-V_n)/sqrt(2); %Calculation of old variable x_n
v_n = (X_n+V_n)/sqrt(2); %Calculation of old variable y_n
```

```
plot(x_n,v_n)
xlabel('x_n')
ylabel('v_n')
title('Equation (17) in the x_n-v_n plane for C=1/2')
axis([-1.5 1.5 -1.5 1.5])
```

Listing 5: Plotting the $x - v$ plane for $E = 1/2$

```
%Exact_LHO_ODE.m
%Plot the x and v variables based on LHO with E=1/2
clear; close all;
E=1/2; % Total Energy
r = sqrt(1); %radius of circle
t=-pi:0.01:pi; %dependent variable
x=r*cos(t); % New variable X_n
v=r*sin(t); % New variable Y_n
plot(x,v)
xlabel('x')
ylabel('v')
title('Total Energy for the 1-D LHO for E=1/2')
axis([-1.5 1.5 -1.5 1.5])
```