MATH3063 Assignment 2
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The relevant system of nonlinear ordinary differential equations is

$$\frac{dR}{dt} = -aR - dR + rP$$
$$\frac{dP}{dt} = 2aR - rP - fP^2$$

where the constants a, d, f, and r are all positive, and $a > d$.

1. Show that the ODEs are equivalent to the nondimensionalised system

$$\frac{dx}{d\tau} = -\left(\frac{a+d}{r}\right)x + y$$
$$\frac{dy}{d\tau} = \left(\frac{2a}{r}\right)x - y - y^2$$

and give expressions for the dimensionless time τ and the dimensionless variables x and y.
Answer: The first ODE, dR/dt was nondimensionlised by

$$\frac{dR}{dt} = -aR - dR + rP$$
$$\frac{1}{K}\frac{dR}{dt} = -a\frac{R}{K} - d\frac{R}{K} + r\frac{P}{K}$$
$$\frac{1}{Kr}\frac{dR}{dt} = -a\frac{R}{rK} - d\frac{R}{rK} + r\frac{P}{rK}$$
$$\frac{dx}{d\tau} = -x\frac{a}{r} - x\frac{d}{r} + y$$
$$\frac{dx}{d\tau} = -\left(\frac{a+d}{r}\right)x + y$$

where $x = R/K$, $y = P/K$, and $\tau = rt$.

Now that we know these values, we can sub them in to nondimensionalise the second ODE, dP/dt

$$\frac{dP}{dt} = 2aR - rP - fP^2$$
$$\frac{d(Ky)}{d(\tau/r)} = 2aKx - rKy - fK^2y^2$$
$$\frac{dy}{d\tau} = \frac{2aKx}{rK} - \frac{rKy}{rK} - \frac{fK^2y^2}{rK}$$
$$\frac{dy}{d\tau} = \left(\frac{2a}{r}\right)x - y - \frac{fKy^2}{r}$$
$$\frac{dy}{d\tau} = \left(\frac{2a}{r}\right)x - y - y^2$$

where, $fK/r = 1 \rightarrow f = r/K$.

2. From now on, let $a = 0.9$, $d = 0.1$, and $r = 1$. Find the **two** steady states of the nondimensionalised system above.

   Answer: The non dimensionalised system can be rewritten as

$$\frac{dx}{d\tau} = -x + y$$
$$\frac{dy}{d\tau} = 1.8x - y - y^2$$

   and the Jacobian is

$$J = \begin{bmatrix} -1 & 1 \\ 1.8 & -2y - 1 \end{bmatrix}$$

   The equilibria are

   - at $\dot{x} = 0$ is the line $y = x$
   - at $y = 0$ is the parabola $x = (1/1.8)(y^2 + y)$.

   The intersection of the two equilibria were found

$$1.8y = y^2 + y$$
$$y^2 - 0.8y = 0$$
$$y(y - 0.8) = 0$$

   Therefore, they are at $(0, 0)$ and $(0.8, 0.8)$.

3. Classify the **two** steady states using linear stability analysis.

   Answer: At $(0, 0)$ the Jacobian is

$$J = \begin{bmatrix} -1 & 1 \\ 1.8 & -1 \end{bmatrix}$$

   - $\det(J) = 1 - 1.8 = -0.8 < 0$. Therefore this point is a **saddle**.

   At $(0.8, 0.8)$ the Jacobian is

$$J = \begin{bmatrix} -1 & 1 \\ 1.8 & -2.6 \end{bmatrix}$$

   - $\det(J) = 2.6 - 1.8 = -0.8 < 0$. Therefore, this point is not a saddle.
   - $\text{tr}(J) = -3.6 < 0$. Therefore, this point is stable.
   - $\Delta = \text{tr}(J)^2 - 4\det(J) = 12.96 - 3.2 > 0$. Therefore, this point is a **stable node**.

4. Plot the nullclines as accurately as possible and then sketch some representative trajectories (include at least those trajectories starting in the corners of the phase diagram).