# General Relativity
# Assignment 2

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## Question 1

In general relativity, the dot product between two vectors is given by

$$\mathbf{a} \cdot \mathbf{b} = g_{\mu\nu}(x) a^{\mu}(x) b^{\nu}(x)$$

Suppose that both of the vectors of interest as 4-velocities, such that

$$a^{\mu}(x_a) = \frac{dx_a^{\mu}}{d\tau}, \qquad b^{\nu}(x_b) = \frac{dx_b^{\nu}}{d\tau}$$

and that we wish to make a change of coordinates to $x'^{\mu} = x'^{\mu}(x)$.

a) Express the 4-velocities in the new coordinates (i.e. find $a'^{\mu}(x') = \frac{dx'^{\mu}}{d\tau}$ )

**Answer:**

$$\begin{aligned} \mathbf{a} = a^{\alpha} \frac{\partial}{\partial x^{\alpha}} = \left( a^{\alpha} \frac{\partial x'^{\beta}}{\partial x^{\alpha}} \right) \frac{\partial}{\partial x'^{\beta}} &\equiv a'^{\beta} \frac{\partial}{\partial x'^{\beta}} \\ \implies a'^{\mu} &= \left( \frac{\partial x'^{\mu}}{\partial x^{\mu}} \right) a^{\mu} \end{aligned}$$

Therefore, we can see that if $a^{\mu}(x_a) = \frac{dx_a^{\mu}}{d\tau}$ then $a'^{\mu}(x') = \frac{dx'^{\mu}}{d\tau}$ and the same can be done for $b'^{\nu}$.

b) Show that the dot product does not change under the coordinate transformation, such that

$$\mathbf{a} \cdot \mathbf{b} = g_{\mu\nu}(x) a^{\mu}(x) b^{\nu}(x) = g'_{\mu\nu}(x') a'_{\mu\nu}(x') a'^{\mu}(x') b'^{\nu}(x')$$

**Answer:**

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= g_{\mu\nu}(x) a^{\mu}(x) b^{\nu}(x) \\ &= g_{\mu\nu}(x) \left( a'^{\mu}(x') \frac{\partial x^{\mu}}{\partial x'^{\mu}} \right) \left( b'^{\nu}(x') \frac{\partial x^{\nu}}{\partial x'^{\nu}} \right) \\ &= \left[ g'_{\mu\nu}(x') \frac{\partial x'^{\mu}}{\partial x^{\mu}} \frac{\partial x'^{\nu}}{\partial x^{\nu}} \right] \left( a'^{\mu}(x') \frac{\partial x^{\mu}}{\partial x'^{\mu}} \right) \left( b'^{\nu}(x') \frac{\partial x^{\nu}}{\partial x'^{\nu}} \right) \\ &= g'_{\mu\nu}(x') a'^{\mu}(x') b'^{\nu}(x') \end{aligned}$$

The covariant derivative for a rank-2 tensor is given by

$$\nabla_{\gamma} t_{\alpha\beta} = \frac{\partial t_{\alpha\beta}}{\partial x^{\gamma}} - \Gamma^{\mu}_{\gamma\alpha} t_{\mu\beta} - \Gamma^{\mu}_{\gamma\beta} t_{\alpha\mu}$$

c) Show that for any tensor $X_{\alpha\beta}$, the covariant derivative $\nabla_\gamma X_{\alpha\beta}$ is also a tensor if the Christoffel symbol transforms as

$$\Gamma^{\rho}_{\mu\nu} = \frac{\partial x^{\rho}}{\partial x^{\rho'}}\frac{\partial x^{\mu'}}{\partial x^{\mu}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma_{\mu'\nu'} - \frac{\partial x^{\mu'}}{\partial x^{\mu}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\frac{\partial^2 x^{\rho}}{\partial x^{\mu'}\partial x^{\nu'}}$$

**Answer:**
We can rewrite, the derivatives in terms of the Jocabian matrix $\frac{\partial x^{\alpha'}}{\partial x^{\alpha}} = J_{\alpha'\alpha}$ so the covariant derivative as

$$\begin{aligned}
\nabla_\gamma t_{\alpha\beta} &= \frac{\partial t_{\alpha\beta}}{\partial x^{\gamma}} - \Gamma^{\mu}_{\gamma\alpha}t_{\mu\beta} - \Gamma^{\mu}_{\gamma\beta}t_{\alpha\mu} \\
&= J_{\alpha'\alpha}J_{\beta'\beta}J_{\gamma'\gamma}\frac{\partial t_{\alpha'\beta'}}{\partial x^{\gamma'}} - J_{\mu\mu'}J_{\gamma'\gamma}\left[J_{\alpha'\alpha}\Gamma^{\mu'}_{\gamma'\alpha'}t_{\mu'\beta'}J_{\mu'\mu}J_{\beta'\beta} + J_{\beta\beta'}\Gamma^{\mu'}_{\gamma'\beta'}t_{\alpha'\mu'}J_{\alpha'\alpha}J_{\mu'\mu}\right] \\
&+ J_{\gamma'\gamma}\left[J_{\alpha'\alpha}\frac{\partial^2 x^{\mu}}{\partial x^{\gamma'}\partial x^{\alpha'}}t_{\mu'\beta'}J_{\mu'\mu}J_{\beta'\beta} + J_{\beta\beta'}\frac{\partial^2 x^{\mu}}{\partial x^{\gamma'}\partial x^{\beta'}}t_{\alpha'\mu'}J_{\alpha'\alpha}J_{\mu'\mu}\right] \\
&= J_{\alpha'\alpha}J_{\beta'\beta}J_{\gamma'\gamma}\left[\frac{\partial t_{\alpha'\beta'}}{\partial x^{\gamma'}} + J_{\mu'\mu}\frac{\partial^2 x^{\mu}}{\partial x^{\gamma'}\partial x^{\alpha'}}t_{\mu'\beta'} + J_{\mu'\mu}\frac{\partial^2 x^{\mu}}{\partial x^{\gamma'}\partial x^{\beta'}}t_{\alpha'\mu'} - \Gamma^{\mu'}_{\gamma'\alpha'}t_{\mu'\beta'} - \Gamma^{\mu'}_{\gamma'\beta'}t_{\alpha'\mu'}\right] \\
&= J_{\alpha'\alpha}J_{\beta'\beta}J_{\gamma'\gamma}\left[\frac{\partial t_{\alpha'\beta'}}{\partial x^{\gamma'}} + t_{\mu'\beta'}\frac{\partial x^{\mu'}}{\partial x^{\mu}}\frac{\partial x^{\mu}}{\partial x^{\gamma'}}\frac{\partial x^{\mu}}{\partial x^{\beta'}} + t_{\alpha'\mu'}\frac{\partial x^{\mu'}}{\partial x^{\mu}}\frac{\partial x^{\mu}}{\partial x^{\gamma'}}\frac{\partial x^{\mu}}{\partial x^{\alpha'}} - \Gamma^{\mu'}_{\gamma'\beta'}t_{\mu'\beta'} - \Gamma^{\mu'}_{\gamma'\beta'}t_{\alpha'\mu'}\right] \\
&= J_{\alpha'\alpha}J_{\beta'\beta}J_{\gamma'\gamma}\left[\frac{\partial t_{\alpha'\beta'}}{\partial x^{\gamma'}} + t_{\alpha'\beta'}\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\frac{\partial x^{\alpha}}{\partial x^{\gamma'}}\frac{\partial x^{\alpha}}{\partial x^{\beta'}} + t_{\alpha'\beta'}\frac{\partial x^{\beta'}}{\partial x^{\beta}}\frac{\partial x^{\beta}}{\partial x^{\gamma'}}\frac{\partial x^{\beta}}{\partial x^{\alpha'}} - \Gamma^{\mu'}_{\gamma'\beta'}t_{\mu'\beta'} - \Gamma^{\mu'}_{\gamma'\beta'}t_{\alpha'\mu'}\right] \\
&= J_{\alpha'\alpha}J_{\beta'\beta}J_{\gamma'\gamma}\left[\frac{\partial t_{\alpha'\beta'}}{\partial x^{\gamma'}} + t_{\alpha'\beta'}\frac{\partial x^{\alpha'}}{\partial x^{\gamma'}}\frac{\partial x^{\alpha}}{\partial x^{\beta'}} + t_{\alpha'\beta'}\frac{\partial x^{\beta'}}{\partial x^{\gamma'}}\frac{\partial x^{\beta}}{\partial x^{\alpha'}} - \Gamma^{\mu'}_{\gamma'\beta'}t_{\mu'\beta'} - \Gamma^{\mu'}_{\gamma'\beta'}t_{\alpha'\mu'}\right] \\
&= \frac{\partial x^{\gamma'}}{\partial x^{\gamma}}\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\frac{\partial x^{\beta'}}{\partial x^{\beta}}\nabla_{\gamma'}t_{\alpha'\beta'}
\end{aligned}$$

where we have contracted due to the chain rule.

d) Show explicitly that $\nabla_\gamma g_{\alpha\beta} = 0$

**Answer:**

$$\begin{aligned}
\nabla_\gamma g_{\alpha\beta} &= \frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} - \Gamma^{\mu}_{\gamma\alpha}g_{\mu\beta} - \Gamma^{\mu}_{\gamma\beta}g_{\alpha\mu} \\
&= \partial_\gamma g_{\alpha\beta} - \frac{1}{2}g_{\mu\beta}g^{\mu\delta}\left(\partial_\alpha g_{\gamma\delta} + \partial_\gamma g_{\alpha\delta} - \partial_\delta g_{\gamma\alpha}\right) - \frac{1}{2}g_{\alpha\mu}g^{\mu\delta}\left(\partial_\beta g_{\gamma\delta} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\gamma\beta}\right)
\end{aligned}$$

We know that $g_{\mu\beta}g^{\mu\delta} = \delta^{\delta}_{\beta}$ and $g_{\alpha\mu}g^{\mu\delta} = \delta^{\delta}_{\alpha}$, giving

$$\begin{aligned}
\nabla_\gamma g_{\alpha\beta} &= \partial_\gamma g_{\alpha\beta} - \frac{1}{2}g_{\mu\beta}g^{\mu\beta}\left(\partial_\alpha g_{\gamma\beta} + \partial_\gamma g_{\alpha\beta} - \partial_\beta g_{\gamma\alpha}\right) - \frac{1}{2}g_{\alpha\mu}g^{\mu\alpha}\left(\partial_\beta g_{\gamma\alpha} + \partial_\gamma g_{\beta\alpha} - \partial_\alpha g_{\gamma\beta}\right) \\
&= \partial_\gamma g_{\alpha\beta} - \frac{1}{2}\left(\partial_\alpha g_{\gamma\beta} - \partial_\alpha g_{\gamma\beta} + \partial_\beta g_{\gamma\alpha} - \partial_\beta g_{\gamma\alpha} + \partial_\gamma g_{\alpha\beta} + \partial_\gamma g_{\beta\alpha}\right) \\
&= 0
\end{aligned}$$

## Question 2

Consider an observer who falls from rest at the event horizon ($r = 2M$) in the Schwarzschild metric. By considering the effective potential described in the lectures:

a) Show that this corresponds to an effective energy of $\epsilon = -\frac{1}{2}$.

**Answer:**

The Schwarzschild metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

and the conserved quantity of energy is given by

$$\epsilon = \frac{e^2 - 1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r)$$

As the observer is at rest at $r = 2M \implies \frac{dr}{d\tau} = 0$ and $V_{eff}(r) = -\frac{M}{r}$, which gives $\epsilon = -\frac{1}{2}$.

b) Determine the equation of motion for the falling observer and show that the duration of the proper time experienced in the journey from the event horizon to the origin is

$$\Delta\tau = \pi M$$

**Answer:**

So we know the initial condition at $r = 2M$, $\epsilon = -\frac{1}{2}$, and it is a radial plunge orbit. First we can see that

$$\frac{e^2 - 1}{2} = -\frac{1}{2} \implies e = \frac{dt}{d\tau} = 0$$

We also know that

$$\begin{aligned}
\epsilon = -\frac{1}{2} &= \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} \\
\frac{dr}{d\tau} &= \left(\frac{2M}{r} - 1\right)^{1/2} \\
\implies u^\alpha &= \left(0, \left(\frac{2M}{r} - 1\right)^{1/2}, 0, 0\right)
\end{aligned}$$

Separating and integrating the radial velocity equation from 0 to $2M$ using an online integral calculator gives

$$\begin{aligned}
\Delta t &= \int_0^{2M} \left(\frac{2M}{r} - 1\right)^{-1/2} dr \\
&= \pi M
\end{aligned}$$

c) In the following, consider a black hole with $M = 1$. Numerically integrate the equation of motion as per b) but consider a range of effective energies ranging between $\epsilon = -\frac{1}{2}$ and 0, plotting the experienced proper time as a function of the effective energy. In terms of free-falling observers, what do the different values of $\epsilon$ correspond to, and what does this imply for who experiences the largest proper time below the event horizon.

**Answer:**

We can calculate the more general equation by solving for $\epsilon$ and not $\epsilon = -\frac{1}{2}$. This gives

$$\frac{dr}{d\tau} = \left(2\epsilon + \frac{2M}{r}\right)^{1/2}$$

Integrating this using the scipy integrator shown in the Appendix.

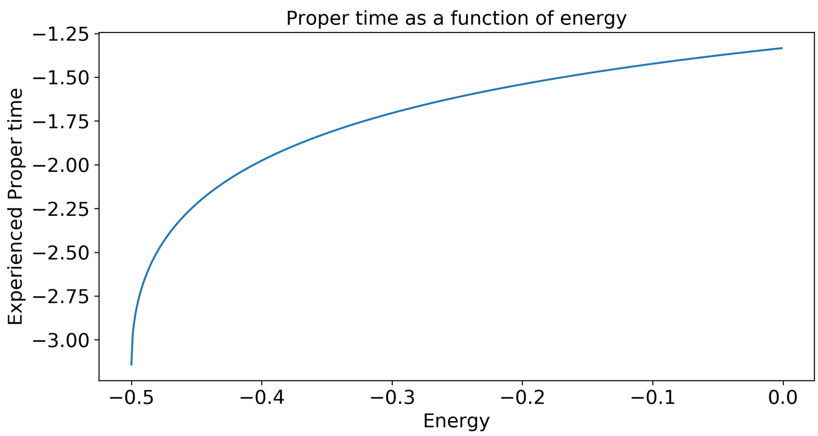


Figure 1: Proper time as a function of Energy

We can see that increasing the energy, that is increasing the effective potential towards zero because the person starts from rest, implies that $r$ is increasing. So as someone starts from rest further away from the event horizon, the negative time they experience is reduced. It could be understood such that with more initial velocity when entering the black hole, the observer will travel less into the past and experience less time until they reach the centre of the black hole. Also, the observer who is at rest at the event horizon will experience the longest amount of time before they reach the centre.

d) Consider an observer who falls from rest at $r = 2M$ who fires a rocket in an attempt slow their fall below the event horizon. Given the results seen in c), what is the impact of firing the rocket on the time experienced by the faller?

**Answer:**

As we saw in part c), starting from rest at the event horizon is the case where the observer will experience the longest amount of time before reaching the centre, this implies that any change in velocity that the observer is able to attain will increase their velocity towards the centre and reduce the amount of time experienced.

# Question 3

Consider a flat FRW universe whose metric is given by

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$$

and consider a particle fired from the origin at time $t_*$, with a speed in the $x$-direction of $V_*$ as measured by a co-moving observer (at constant $x$, $y$ and $z$), By considering an appropriate Killing vector;

a) Show that the components of the particle's 4-velocity are given by

$$u^x = \frac{q}{a^2(t)}, \qquad u^t = \sqrt{\left(1 + \frac{q^2}{a^2(t)}\right)}$$

where $q$ is a conserved quantity derived from the Killing vector.

**Answer:**

The Lagrangian is given by

$$L\left(\frac{dx^\alpha}{d\sigma}, t\right) = \left[\left(\frac{dt}{d\sigma}\right)^2 - a^2(t)\left(\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2 + \left(\frac{dz}{d\sigma}\right)^2\right)\right]^{1/2}$$

Because there is no $x$ dependence, we can find the killing vector and conserved quantity

$$\begin{aligned} \frac{d}{d\sigma}\left[\frac{\partial L}{\partial(dx/d\sigma)}\right] &= 0 \\ \implies \frac{\partial L}{\partial(dx/d\sigma)} &= a^2(t)\frac{1}{L}\frac{dx}{d\sigma} \\ q &= a^2(t)u^x \\ \implies u^x &= \frac{q}{a^2(t)} \end{aligned}$$

where $q$ is the conserved quantity in the $x$ direction.

Then we can use the normalisation of the 4-velocity because there is only motion in the $x$-direction.

$$\begin{aligned} -(u^t)^2 + a^2(t)(u^x)^2 &= -1 \\ (u^t)^2 &= 1 + \frac{q^2}{a^2(t)} \\ u^t &= \sqrt{1 + \frac{q^2}{a^2(t)}} \end{aligned}$$

b) Derive an expression for the three velocity, $\frac{dx}{dt}$, and demonstrate that this implies that the particle will come to rest with regards to the spatial coordinates. Express the asymptotic coordinates where the particle comes to rest, $x_f$, as an integral over $a(t)$. Briefly comment on what this means for conservation of energy in an expanding universe.

**Answer:**

The three velocity is found by

$$\begin{aligned}\frac{dx}{dt} &= \frac{\frac{q}{a^t(t)}}{\left(1+\frac{q^2}{a^2(t)}\right)^{1/2}} \\ &= \frac{1}{\frac{a^2(t)}{q}\left(1+\frac{q^2}{a^2(t)}\right)^{1/2}} \\ &= \frac{1}{\left(\frac{a^4(t)}{q^2}+a^2(t)\right)^{1/2}}\end{aligned}$$

The particle will therefore come to rest with respect to the spatial coordinates when $a(t) \to \infty$.

$$x_f = \int_{a(t_*)}^{\infty} \frac{1}{\left(\frac{a^4(t)}{q^2}+a^2(t)\right)^{1/2}} da$$

This means in an expanding universe the particle will appear to accelerate away from the observer and gain energy such that it will reach the same velocity with respect to the spatial coordinated expansion at infinity. This means that there is no conservation of energy over time in an expanding universe.

By identifying an appropriate orthonormal frame at the origin

c) Show that the spatial component of the 4-velocity of the particle seen in the orthonormal frame is

$$u^{\hat{x}} = \frac{q}{a(t_*)}$$

**Answer:**

The coordinate basis vector in the orthonormal frame is $(\mathbf{e}_x)^{\hat{x}} = a(t)$ and then we see that

$$\begin{aligned}u^{\hat{x}} &= u^x(\mathbf{e}_x)^{\hat{x}} \\ &= \frac{q}{a(t_*)}\end{aligned}$$

d) Show that the conserved quantity is related to the initial speed by

$$q = a(t_*) \frac{V_*}{\sqrt{1 - V_*^2}}$$

**Answer:**

Firstly we need to convert to the orthonormal frame and we already know $u^{\hat{x}} = \dfrac{q}{a(t_*)}$

$$u^{\hat{t}} = u^t = \sqrt{1 + \frac{q^2}{a^2(t)}}$$

Now we know the velocity in the orthonormal frame is constant and equal to $V_*$.

$$\begin{aligned}
\frac{dx}{dt} &= \frac{\frac{q}{a(t_*)}}{\sqrt{1 + \frac{q^2}{a^2(t_*)}}} \\
\frac{a^2(t_*)V_*^2}{q^2} &= \frac{1}{1 + \frac{q^2}{a^2(t_*)}} \\
\frac{q^2}{a^2(t_*)V_*^2} &= 1 + \frac{q^2}{a^2(t_*)} \\
\frac{q^2}{a^2(t_*)} \left( \frac{1}{V_*^2} - 1 \right) &= 1 \\
q &= \frac{a(t_*)}{\sqrt{\frac{1}{V_*^2} - 1}} \\
&= a(t_*) \frac{V_*}{\sqrt{1 - V_*^2}}
\end{aligned}$$

# Appendix

## Listings



```
import numpy as np
from scipy.integrate import quad
import matplotlib.pyplot as plt
%matplotlib inline
def integrand(x,e):
    return (2*e+2/x)**(-0.5)
M=1
energy = np.arange(-0.5,0,0.001)
propert = []
for i in np.arange(len(energy)):
    e = energy[i]
    propert.append(quad(integrand,2,0,args=(e))[0])

fig, ax = plt.subplots(1,1,figsize=(10,5))
ax.plot(energy,propert)
ax.set_xlabel('Energy',fontsize=15)
ax.set_ylabel('Experienced Proper time',fontsize=15)
ax.set_title('Proper time as a function of energy',fontsize=15)
ax.tick_params(axis='both', which='major', labelsize=15)
plt.savefig('q2c',dpi=300)
```

Listing 1: Code to integrate radial path and produce energy and proper time plot 2$c$