General Relativity
Assignment 1

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4 September 2019

Question 1

In the lectures we considered a rocket undergoing a constant acceleration in the x-direction. The motion is governed by the relationships

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{u} &= \eta_{\alpha\beta} u^\alpha u^\beta = -1 \\
\mathbf{u} \cdot \mathbf{a} &= \eta_{\alpha\beta} u^\alpha a^\beta = 0 \\
\mathbf{a} \cdot \mathbf{a} &= \eta_{\alpha\beta} a^\alpha a^\beta = a^2
\end{aligned}$$

where a is a constant. The rocket begins from rest at the coordinates $(t, x) = (0, 0)$ at a time of $\tau = 0$ on the clock of the rocketeer (its *proper* time). In the following, consider spatial motion in the x-direction only.

a) Explicitly solve for the motion of the rocket, showing that the components of the position, 4-velocity and 4-acceleration are given by:

$$\begin{aligned}
x^\alpha(\tau) &= a^{-1}(\sinh(a\tau), \cosh(a\tau) - 1) \\
u^\alpha(\tau) &= (\cosh(a\tau), \sinh(a\tau)) \\
a^\alpha(\tau) &= a(\sinh(a\tau), \cosh(a\tau))
\end{aligned}$$

Answer:

We can use $\mathbf{u} \cdot \mathbf{a} = 0$ which implies $\mathbf{u} \cdot \frac{d}{d\tau}(\mathbf{u} \cdot \mathbf{a}) = 0$, $\mathbf{u} \cdot \mathbf{u} = -1$ and $\mathbf{a} \cdot \mathbf{a} = a^2$. Rearranging this gives the differential equations

$$\begin{aligned}
\mathbf{u} \cdot \frac{d}{d\tau}(\mathbf{u} \cdot \mathbf{a}) &= 0 \\
\mathbf{u} \cdot \left(\mathbf{a} \cdot \mathbf{a} + \mathbf{u} \cdot \frac{d\mathbf{a}}{d\tau} \right) &= 0 \\
a^2 \mathbf{u} + \mathbf{u} \cdot \mathbf{u} \frac{d\mathbf{a}}{d\tau} &= 0 \\
\frac{d^2 \mathbf{u}}{d\tau^2} - a^2 \mathbf{u} &= 0 \\
\implies \frac{d^2 u^x}{d\tau^2} - a^2 u^x &= 0
\end{aligned}$$

which are homogeneous and linear with a characteristic equation of $\lambda_x^2 - a^2 = 0$.

Therefore, the general solutions are

$$
\begin{aligned}
u^x &= C_1 \sinh a\tau + C_2 \cosh a\tau \\
u^x(\tau = 0) = 0 &\Longrightarrow C_2 = 0 \\
\Longrightarrow u^x &= C_1 \sinh a\tau \\
x^x &= \frac{C_1}{a} \cosh a\tau + C_3 \\
x^x(\tau = 0) = 0 &\Longrightarrow C_3 = \frac{C_1}{a} \\
\Longrightarrow x^x &= \frac{C_1}{a} \cosh a\tau - \frac{C_1}{a} \\
\Longrightarrow a^x &= aC_1 \cosh a\tau
\end{aligned}
$$

Now we can use the dot product to figure out the value of C_1 and the time components.

$$
\begin{aligned}
-(u^t)^2 + C_1^2 \sinh^2 a\tau &= -1 \\
\text{Using the identity } \cosh^2 \tau - \sinh^2 \tau &= 1 \\
\Longrightarrow u^t = C_1 \cosh a\tau,\ C_1 &= 1 \\
x^t &= \frac{1}{a} \sinh a\tau + C_4 \\
x^t(\tau = 0) = 0 &\Longrightarrow C_4 = 0 \\
\Longrightarrow x^t &= \frac{1}{a} \sinh a\tau \\
\Longrightarrow a^t &= a \sinh a\tau
\end{aligned}
$$

due to the identity again and because of $-(a^t)^2 + (a^x)^2 = a^2$. Therefore, we have found that

$$
\begin{aligned}
x^\alpha(\tau) &= a^{-1}(\sinh(a\tau), \cosh(a\tau) - 1) \\
u^\alpha(\tau) &= (\cosh(a\tau), \sinh(a\tau)) \\
a^\alpha(\tau) &= a(\sinh(a\tau), \cosh(a\tau))
\end{aligned}
$$

b) As the rocket travels, an observer at rest at the origin ($x = 0$) fires photons in the positive x-direction which are detected on the rocket. Show that the relationship between the time the photon is emitted from the origin, t_e, and the proper time on the rocket when the photon is received, τ_r, is:

$$\tau_r = -\frac{1}{a} \ln(1 - at_e)$$

Answer:

The photon is emitted at t_e, the proper time on the rocket when it is received is $\tau_r = \frac{1}{a} \sinh a\tau_r$ and its position at the proper time when it was received is $x_r = \frac{1}{a} \cosh a\tau_r - \frac{1}{a}$. Now because photons travel at 45° in space-time diagrams, we know that the difference between the emission time and the proper time at reception is equal to the spatial distance travelled by the photon

$$\begin{aligned}
\tau_r - t_e &= x(\tau_r) \\
\frac{1}{a}\sinh a\tau_r - t_e &= \frac{1}{a}\cosh a\tau_r - \frac{1}{a} \\
\frac{1}{a}\left(\frac{e^{a\tau_r} - e^{-a\tau_r}}{2} - \frac{e^{a\tau_r} + e^{-a\tau_r}}{2}\right) &= t_e - \frac{1}{a} \\
e^{-a\tau_r} &= 1 - at_e \\
\tau_r &= -\frac{1}{a}\ln(1 - at_e)
\end{aligned}$$

c) Show that the energy of an exchanged photon detected by the observer on the rocket, E_r, is compared to that emitted by an observer at rest at the origin, E_e, is given by:

$$\frac{E_r}{E_e} = e^{-a\tau_r} = 1 - at_e$$

Answer:

The energy of the photon emitted at rest is given by

$$\begin{aligned}
E_e &= -\hat{p} \cdot \hat{u}_e \\
&= \hbar k
\end{aligned}$$

where $\hat{p} = \hbar k(1,1)$ and $\hat{u}_e = (-\cosh 0, \sinh 0)$.

Then the energy when the photon was received was given by

$$\begin{aligned}
E_r &= -\hat{p} \cdot \hat{u}_r \\
&= \hbar k\left(\cosh a\tau_r - \sinh a\tau_r\right) \\
&= \hbar k e^{-a\tau_r}
\end{aligned}$$

where $\hat{p} = \hbar k(1,1)$ and $\hat{u}_e = (-\cosh a\tau_r, \sinh a\tau_r)$.

Therefore

$$\begin{aligned}
\frac{E_r}{E_e} &= e^{-a\tau_r} \\
&= e^{\ln(1-at_e)} \\
&= 1 - at_e
\end{aligned}$$

after substituting $\tau_r = -\frac{1}{a}\ln(1 - at_e)$.

d) With the use of sketches, briefly describe the view of observer at rest at the origin as seen by those on the rocket. Note that this demonstrates the existence of the Rindler Horizon.

Answer:

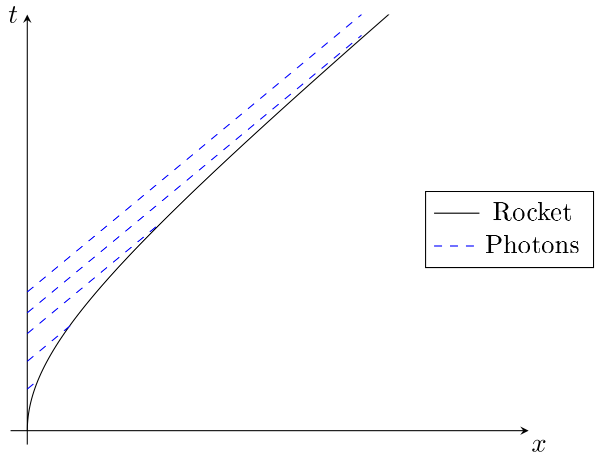

Figure 1: Space-time diagram of photons travelling to rocket

In Figure 1, we can see 5 photons, plotted as blue dashed lines, that have been sent off from the observer at rest. The four earliest photons that were emitted will all reach the rocket, at the respective intersection points. However, the final plotted photon will never reach the rocket because the world line of the rocket approaches an asymptote. This implies that as time ticks by on the rocket it will look like things are slowing down near the observer at rest because the time even though the time between emitted photons is relatively constant. The proper time at which each photon is received is growing larger and larger. This implies there is a photon that will meet them at an infinite distance after an infinite time. This is called the Rindler Horizon and everything behind it will never be received by the spaceship so long as it continues to accelerate.

Question 2

The space-time metric for a wormhole is given by

$$ds^2 = -dt^2 + dr^2 + (r^2 + b^2)(d\theta^2 + \sin^2\theta d\phi^2)$$

where b is the "size" of the wormhole. Note that when $b = 0$ this metric represents flat space-time in spherical polar coordinates. The non-zero values of the Christoffel symbols for this space-time are given in the lecture notes.

a) Using the Lagrangian approach outlined in the lectures, derive the equations of motions in each other coordinates (t, r, θ, ϕ) for objects moving in this space-time.

   Answer:

   The Lagrangian is given by

$$L\left(\frac{dx^\alpha}{d\sigma}, r, \theta\right) = \left[\left(\frac{dt}{d\sigma}\right)^2 - \left(\frac{dr}{d\sigma}\right)^2 - (b^2 + r^2)\left[\left(\frac{d\theta}{d\sigma}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\sigma}\right)^2\right]\right]^{\frac{1}{2}}$$

   and the Euler-Lagrange equation is

$$-\frac{d}{d\sigma}\left[\frac{\partial L}{\partial(dx^\alpha/d\sigma)}\right] + \frac{\partial L}{\partial x^\alpha} = 0$$

   Because the proper time between two points can be written as

$$\tau_{AB} = \int_A^B d\sigma\left(-g_{\alpha\beta}(x)\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}\right)^{\frac{1}{2}}$$

   there is a factor of L which transfers between coordinates σ and τ so that $L = d\tau/d\sigma$.

   For the time component, we have

$$\begin{aligned}
\frac{\partial L}{\partial t} &= 0 \\
\frac{\partial L}{\partial(dt/d\sigma)} &= \frac{1}{L}\frac{dt}{d\sigma} \\
&= \frac{dt}{d\tau} \\
\implies \frac{d}{d\tau}\left[\frac{dt}{d\tau}\right] &= 0 \\
\frac{d^2t}{d\tau^2} &= 0
\end{aligned}$$

Then for the radial position

$$
\begin{aligned}
\frac{\partial L}{\partial r} &= -\frac{r}{L}\left[\left(\frac{d\theta}{d\sigma}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\sigma}\right)^2\right] \\
\frac{\partial L}{\partial(dr/d\sigma)} &= -\frac{1}{L}\frac{dr}{d\sigma} \\
&= -\frac{dr}{d\tau} \\
\implies L\frac{d}{d\tau}\left[\frac{dr}{d\tau}\right] - \frac{r}{L}\left[\left(\frac{d\theta}{d\sigma}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\sigma}\right)^2\right] &= 0 \\
\frac{d^2r}{d\tau^2} &= \frac{r}{L^2}\left[\left(L\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(L\frac{d\phi}{d\tau}\right)^2\right] \\
\frac{d^2r}{d\tau^2} &= r\left[\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right]
\end{aligned}
$$

Now for the angular component, θ

$$
\begin{aligned}
\frac{\partial L}{\partial\theta} &= -\frac{b^2+r^2}{L}\left[\sin\theta\cos\theta\left(\frac{d\phi}{d\sigma}\right)^2\right] \\
\frac{\partial L}{\partial(d\theta/d\sigma)} &= -(b^2+r^2)\frac{1}{L}\frac{d\theta}{d\sigma} \\
&= -(b^2+r^2)\frac{d\theta}{d\tau} \\
\implies L\frac{d}{d\tau}\left[(b^2+r^2)\frac{d\theta}{d\tau}\right] - \frac{b^2+r^2}{L}\left[\sin\theta\cos\theta\left(\frac{d\phi}{d\sigma}\right)^2\right] &= 0 \\
b^2\frac{d^2\theta}{d\tau^2} + 2r\frac{dr}{d\tau}\frac{d\theta}{d\tau} + r^2\frac{d^2\theta}{d\tau^2} &= \frac{(b^2+r^2)}{L^2}\left[\sin\theta\cos\theta\left(\frac{d\phi}{d\sigma}\right)^2\right] \\
(b^2+r^2)\frac{d^2\theta}{d\tau^2} &= (b^2+r^2)\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 - 2r\frac{dr}{d\tau}\frac{d\theta}{d\tau} \\
\frac{d^2\theta}{d\tau^2} &= \sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 - \frac{2r}{b^2+r^2}\frac{dr}{d\tau}\frac{d\theta}{d\tau}
\end{aligned}
$$

and finally for the last angular component ϕ

$$
\begin{aligned}
\frac{\partial L}{\partial\phi} &= 0 \\
\frac{\partial L}{\partial(d\phi/d\sigma)} &= -(b^2+r^2)\sin^2\theta\frac{1}{L}\frac{d\phi}{d\sigma} \\
&= -(b^2+r^2)\sin^2\theta\frac{d\phi}{d\tau}
\end{aligned}
$$

$$\implies \frac{d}{d\tau}\left[(b^2+r^2)\sin^2\theta\frac{d\phi}{d\tau}\right] = 0$$
$$b^2\frac{d}{d\tau}\left[\sin^2\theta\frac{d\phi}{d\tau}\right] + \frac{d}{d\tau}\left[r^2\sin^2\theta\frac{d\phi}{d\tau}\right] = 0$$
$$2b^2\sin\theta\cos\theta\frac{d\theta}{d\tau}\frac{d\phi}{d\tau} + b^2\sin^2\theta\frac{d^2\phi}{d\tau^2} + \frac{d\phi}{d\tau}\left[2r\frac{dr}{d\tau}\sin^2\theta + 2r^2\sin\theta\cos\theta\right] + r^2\sin^2\theta\frac{d^2\phi}{d\tau^2} = 0$$

$$\begin{aligned}
\implies (b^2+r^2)\frac{d^2\phi}{d\tau^2} &= -\frac{2}{\tan\theta}\frac{d\theta}{d\tau}\frac{d\phi}{d\tau}\left(b^2+r^2\right) - 2r\frac{d\phi}{d\tau}\frac{dr}{d\tau} \\
\frac{d^2\phi}{d\tau^2} &= -2\left[\cot\theta\frac{d\theta}{d\tau}\frac{d\phi}{d\tau} + \frac{r}{b^2+r^2}\frac{d\phi}{d\tau}\frac{dr}{d\tau}\right]
\end{aligned}$$

b) Using the results derived in (a), determine the non-zero Christoffel symbols for the wormhole space-time.

The Christoffel symbols can also be determined directly from the metric through:

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}(g_{\beta\delta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta})$$

where the comma refers to a partial derivative.

Answer:

The relevant non-zero Christoffel symbols can be read out of the equations above, taking careful note of the factors of 2 for when two indices are not repeated and the Christoffel symbols are defined to be positive on the other side of the equation shown in part a. Therefore, they are

$$\begin{aligned}
\Gamma^{r}_{\theta\theta} &= -r & \Gamma^{r}_{\phi\phi} &= -r\sin^2\theta \\
\Gamma^{\theta}_{r\theta} &= \frac{r}{b^2+r^2} & \Gamma^{\theta}_{\phi\phi} &= -\sin\theta\cos\theta \\
\Gamma^{\phi}_{r\phi} &= \frac{r}{b^2+r^2} & \Gamma^{\phi}_{\phi\theta} &= -\cot\theta
\end{aligned}$$

c) Noting that the wormhole metric is diagonal, explicitly determine the non-zero value of the Christoffel symbols using the above expression.

Answer:

The Lagrangian can be written as $L\left(\frac{dx^\alpha}{d\sigma}, x^\alpha\right) = \left[-g_{\alpha\beta}dx^\alpha dx^\beta\right]^{1/2}$ so the metric can be written as

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (b^2+r^2) & 0 \\ 0 & 0 & 0 & (b^2+r^2)\sin^2\theta \end{pmatrix}$$

Now we can calculate the same non zero terms found in part b. This was first done by determining α, β and γ from the terms found in part b. Then the $g^{\alpha\delta}$ term was moved to the other side to become $g_{\alpha\delta}$ and because the metric is symmetric $g_{\alpha\delta} = g_{\alpha\alpha}$. Then terms were substituted in from the metric above, differentiated where necessary and then simplified.

$$
\begin{aligned}
\Gamma^{\alpha}_{\beta\gamma} &= \frac{1}{2} g^{\alpha\delta}(g_{\beta\delta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta}) \\
\text{For } r:\ g_{rr}\Gamma^{r}_{\theta\theta} &= \frac{1}{2}(g_{\theta r,\theta} + g_{r\theta,\theta} - g_{\theta\theta,r}),\ g_{rr}\Gamma^{r}_{\phi\phi} = \frac{1}{2}(g_{\phi r,\phi} + g_{r\phi,\phi} - g_{\phi\phi,r}) \\
\Longrightarrow \Gamma^{r}_{\theta\theta} &= -r,\ \Gamma^{r}_{\phi\phi} = -r\sin^2\theta \\
\text{For } \theta:\ g_{\theta\theta}\Gamma^{\theta}_{r\theta} &= \frac{1}{2}(g_{r\theta,\theta} + g_{\theta\theta,r} - g_{r\theta,\theta}),\ g_{\theta\theta}\Gamma^{\theta}_{\phi\phi} = \frac{1}{2}(g_{\phi\theta,\phi} + g_{\theta\phi,\phi} - g_{\phi\phi,\theta}) \\
\Longrightarrow \Gamma^{r}_{r\theta} &= \frac{r}{b^2 + r^2},\ \Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta \\
\text{For } \phi:\ g_{\phi\phi}\Gamma^{\phi}_{r\phi} &= \frac{1}{2}(g_{r\phi,\phi} + g_{\phi\phi,r} - g_{r\phi,\phi}),\ g_{\phi\phi}\Gamma^{\phi}_{\phi\theta} = \frac{1}{2}(g_{\phi\phi,\theta} + g_{\phi\theta,\phi} - g_{\phi\theta,\theta}) \\
\Longrightarrow \Gamma^{\phi}_{r\phi} &= \frac{r}{b^2 + r^2},\ \Gamma^{\phi}_{\phi\theta} = \cot\theta
\end{aligned}
$$

where a lot of the terms have dropped to zero and only $g_{rr} = 1$, $g_{\theta\theta,r} = 2r$, $g_{\phi\phi,r} = 2r\sin^2\theta$, $g_{\theta\theta} = (b^2 + r^2)$, $g_{\phi\phi,\theta} = 2\sin\theta\cos\theta$, $g_{\phi\phi} = (b^2 + r^2)\sin^2\theta$ are non zero.

Question 3

Now consider motion in the wormhole metric described in Question 2. Using the results for the equations of motion derived above (and given in the lecture notes and textbook), and considering motion only in the equatorial plane, so $\theta = \pi/2$ and $u^\theta = 0$, show that:

a) For a massive object at an initial location of

$$x_o^\alpha(\tau = 0) = (t_o, r_o, \theta_o, \phi_o) = \left(0, R, \frac{\pi}{2}, 0\right)$$

with an initial spatial motion only in the ϕ-direction, with $u^\phi(\tau = 0) = C$. Show that the initial 4-velocity is given by:

$$u^\alpha(\tau = 0) = \left(\sqrt{1 + C^2(R^2 + b^2)}, 0, 0, C\right)$$

Answer:

We are given $u^r(\tau) = 0$, $u^\theta(\tau) = 0$ and $u^\phi(\tau = 0) = C$, so we can use the metric to calculate the other components

$$\begin{aligned} u^\alpha(\tau = 0) &= \left(u^t, 0, C\right) \\ u^\alpha \cdot u^\alpha &= -(u^t)^2 + (b^2 + r^2)\sin\pi/2C^2 \\ &= -1 \\ u^t &= \sqrt{1 + C^2(b^2 + r^2)} \\ \implies u^t(\tau) &= \sqrt{1 + C^2(b^2 + R^2)} \end{aligned}$$

b) Repeat a) for a photon (massless particle) and show that initial 4-velocity is given by

$$u^\alpha(\tau = 0) = \left(\sqrt{C^2(R^2 + b^2)}, 0, 0, C\right)$$

Answer:

For a photon, the normalization constant is $u^\alpha \cdot u^\alpha = 0$ so

$$\begin{aligned} u^\alpha(\tau = 0) &= \left(u^t, 0, C\right) \\ u^\alpha \cdot u^\alpha &= -(u^t)^2 + (b^2 + r^2)\sin\pi/2C^2 \\ &= 0 \\ u^t &= \sqrt{C^2(b^2 + r^2)} \\ \implies u^t(\tau) &= \sqrt{C^2(b^2 + R^2)} \end{aligned}$$

In the following, you will be required to numerically integrate the equations of motion in the wormhole metric. You may use any numerical approach (matlab, python, Mathematica, bespoke integrator) but your code must be included as part of your solution.

Consider a wormhole with $b = 1$, and an initial radius of $R = 1.2$.

c) Integrate the path of a massive object from $\tau = 0$, to $\tau = 10$ for $C = 0, 0.5$ and 1.0. Plot the resultant t, R and ϕ, as a function of the proper time. Demonstrate that the normalization of the 4-velocity holds along the world-line of the massive object.

Answer:

First are the values for $C = 0$ in Fig. 2, which involve no change in variables because initial velocity is zero, thus acceleration is zero and therefore, there is no change in the normalization of the four velocity, as is shown in Fig. 3.

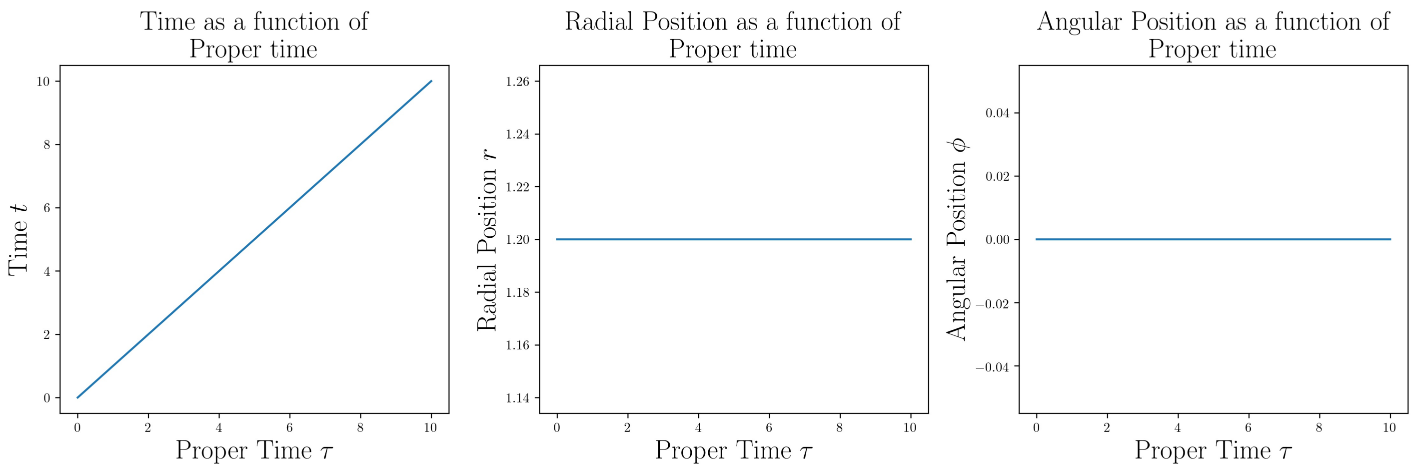

Figure 2: Time (t), radial component (r) and angular component (ϕ) of orbit with $C = 0$

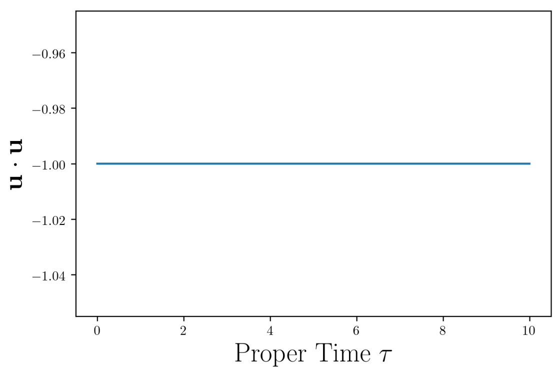

Figure 3: Change in normalization of 4-velocity for massive particle with $C = 0$

Secondly we have $C = 0.5$ in Fig. 4, where there is some changes noticed, for example, there is some time dilation with a linear relationship and the radial position and angular position both increase. Here we can also see, in Fig. 5 that the normalization of the four velocity fluctuates very short distances from $\mathbf{u} \cdot \mathbf{u} = -1$.

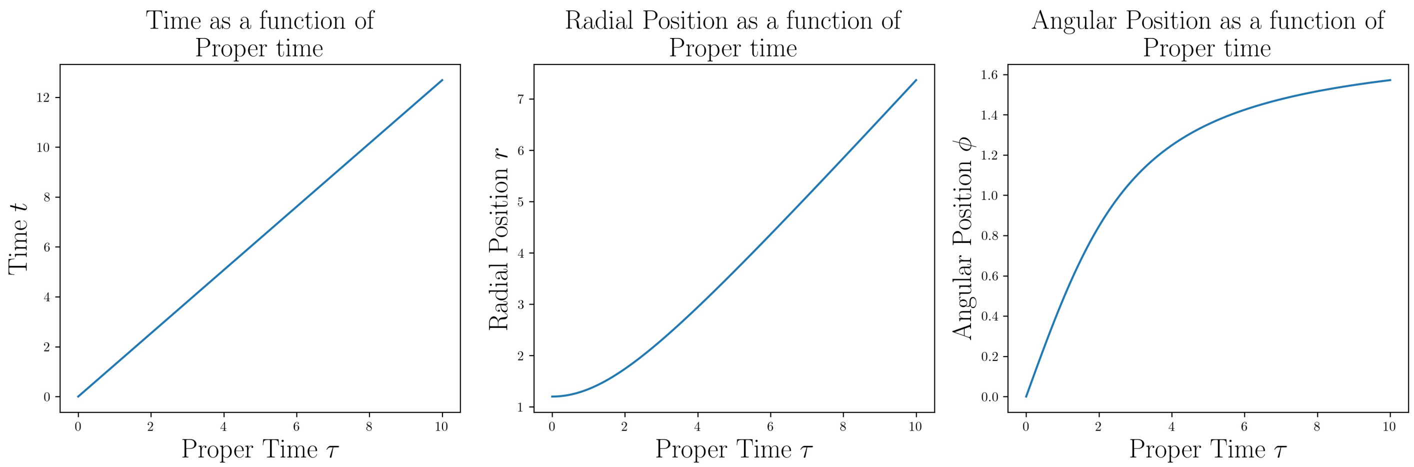

Figure 4: Time (t), radial component (r) and angular component (ϕ) of orbit for massive particle with $C = 0.5$

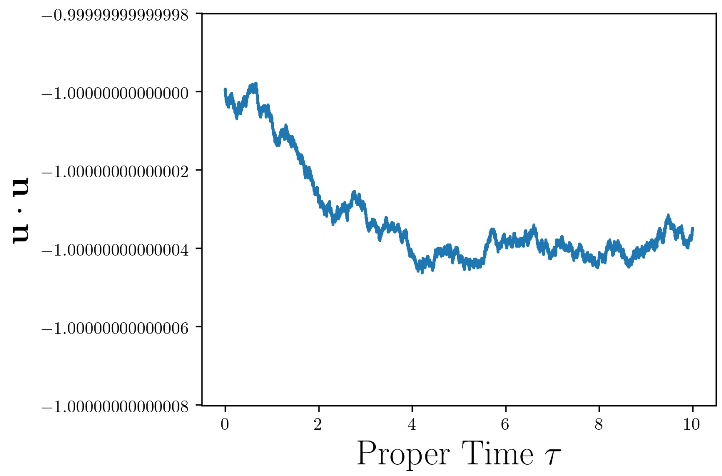

Figure 5: Change in normalization of 4-velocity for massive particle with $C = 0.5$

Finally we have the case for $C = 1$ in Fig. 6, where there is even more time dilation and further changes in distance because of the larger initial velocity. The radial position has approached a straight line relationship with proper time. We can also see the fluctuation in the normalization in Fig. 7.

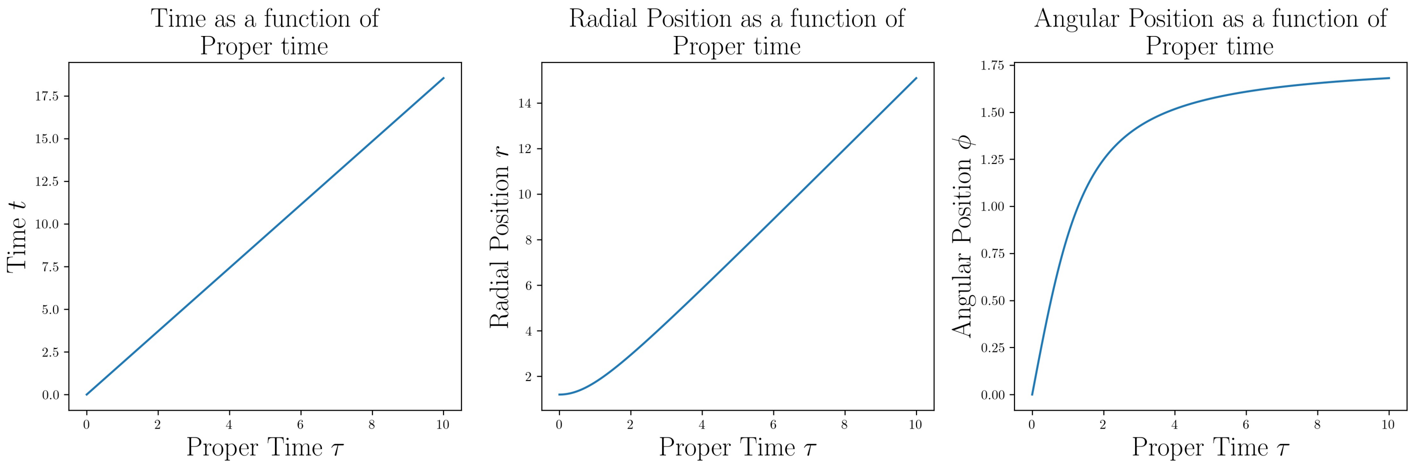

Figure 6: Time (t), radial component (r) and angular component (ϕ) of orbit for massive particle with $C = 1$

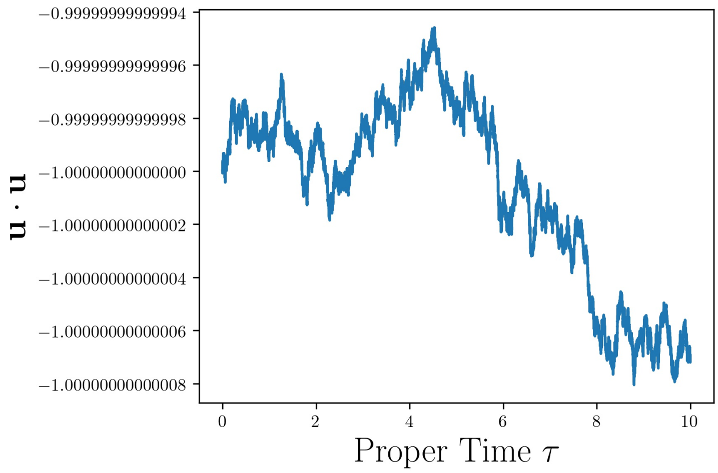

Figure 7: Change in normalization of 4-velocity for massive particle with $C = 1$

d) Repeat (c) for a massless particle, but for the affine parameter from $\lambda = 0$ to $\lambda = 10$, and $C = 1, 5$ and 10. Briefly comment on the relationship between the initial components of the 4-velocity and the affine parameter.

Answer:

Firstly, we can see the relationship for the massless particle for $C = 1$ in Fig. 8, where it appears the only difference between this and the massive particle with $C = 1$ is the time dilation, the radial and angular positions are very similar, however the one is with respect to proper time and the other is with respect to an affine parameter. Again normalisation oscillates around the expected value of 0 with very high precision in Fig. 9.

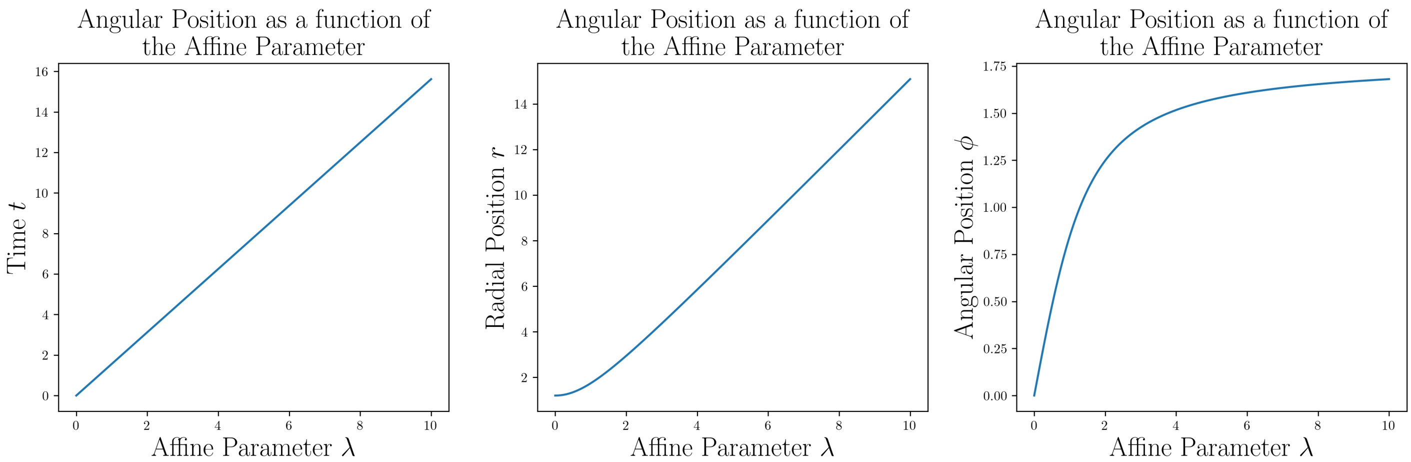

Figure 8: Time (t), radial component (r) and angular component (ϕ) of orbit for massless photon with $C = 1$

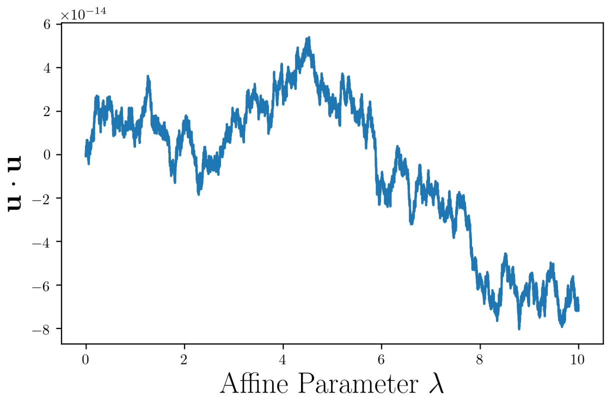

Figure 9: Change in normalization of 4-velocity for massless photon with $C = 1$

Secondly, for $C = 5$, we also see, in Fig. 10, that time has increased, radial position has increased however angular position has not and the positions have a sharper gradient. Fig. 11 also shows the expected fluctuations of the normalization around 0.

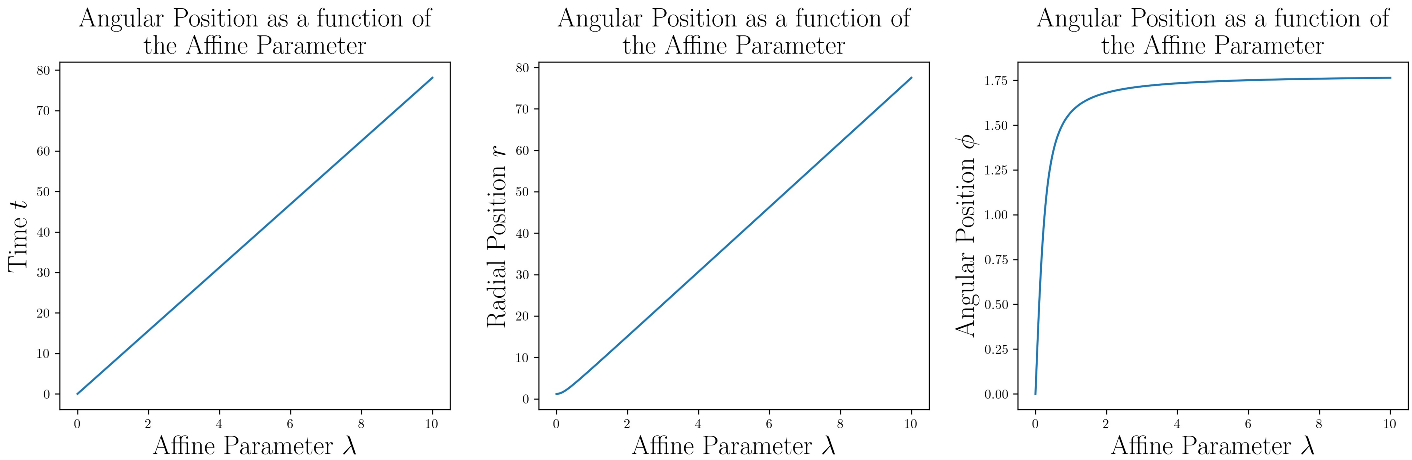

Figure 10: Time (t), radial component (r) and angular component (ϕ) of orbit for massless photon with $C = 5$

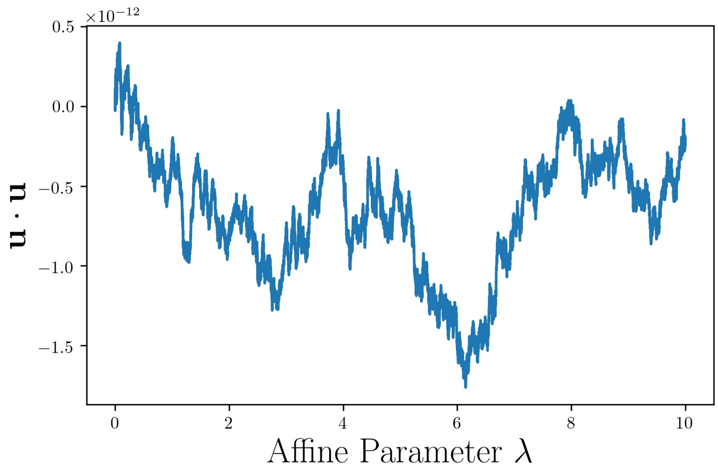

Figure 11: Change in normalization of 4-velocity for massless photon with $C = 5$

Finally, for $C = 10$, in Fig. 12, the time component has doubled, the radial position has also doubled and it looks closer to a straight line now and again, the angular position doesn't seem to increase and only looks to reach the same value much faster. Lastly, we see the fluctuations of the normalization of in Fig. 13.

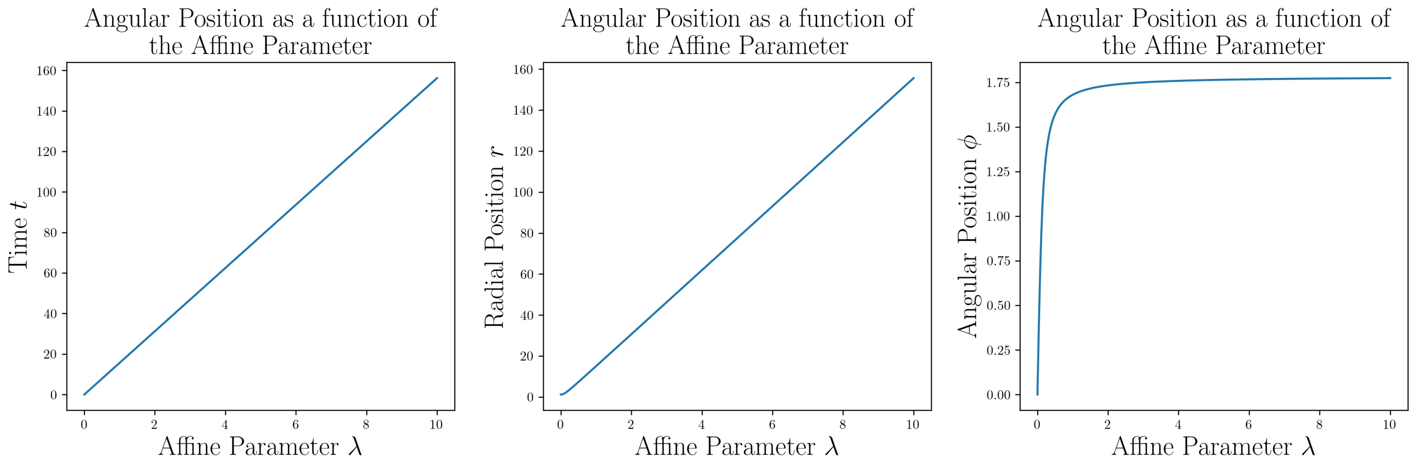

Figure 12: Time (t), radial component (r) and angular component (ϕ) of orbit for massless photon with $C = 10$

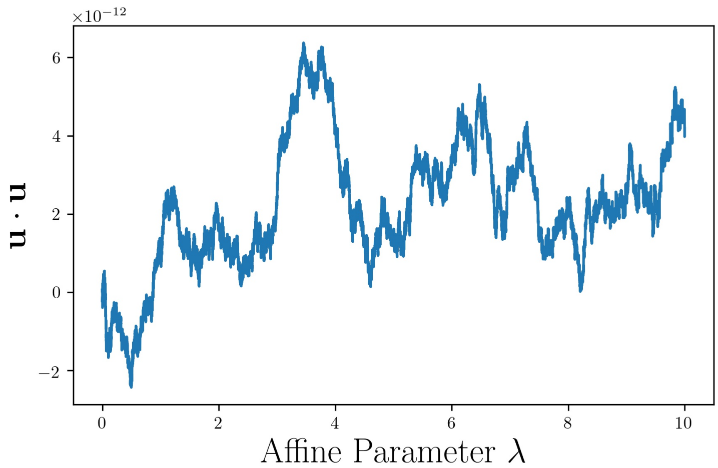

Figure 13: Change in normalization of 4-velocity for massless photon with $C = 10$

We can see the radial and angular relationship of the initial velocity and the affine parameter in Fig. 14 and a close up of the origin in Fig. 15. When comparing the initial components of the four velocity and the affine parameter, it is clear that the larger the initial velocity, the faster in initial change in positions are (gradient) and thus, the more radial distance can be covered in the same amount of “affine parameter”. It appears that the final value of the radial position does depend on initial velocity and will reach a larger final radial position and it appears that the final value of angular position does not depend on initial velocities and looks to approach the same value, however will approach this value faster with faster initial velocities and thus be closer in the same amount of “affine parameter”.

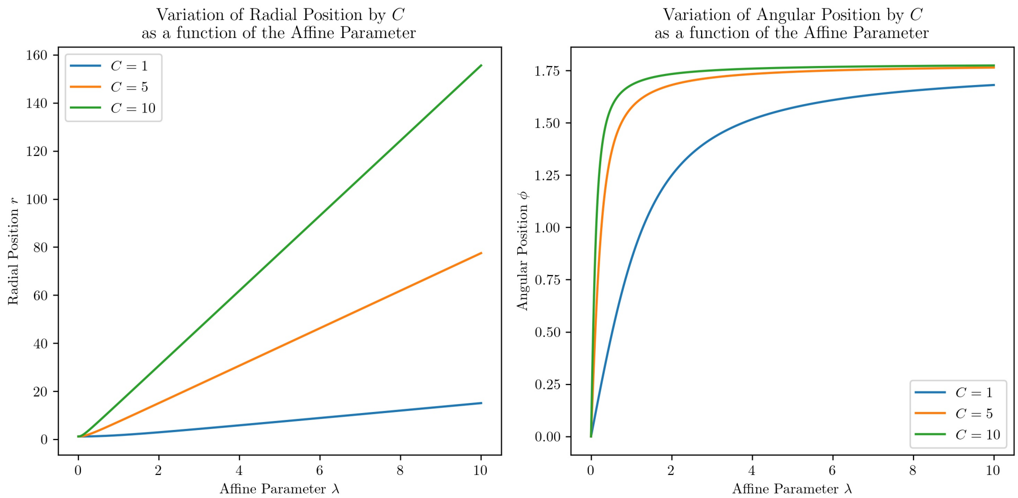

Figure 14: Varying initial velocity over the affine parameter

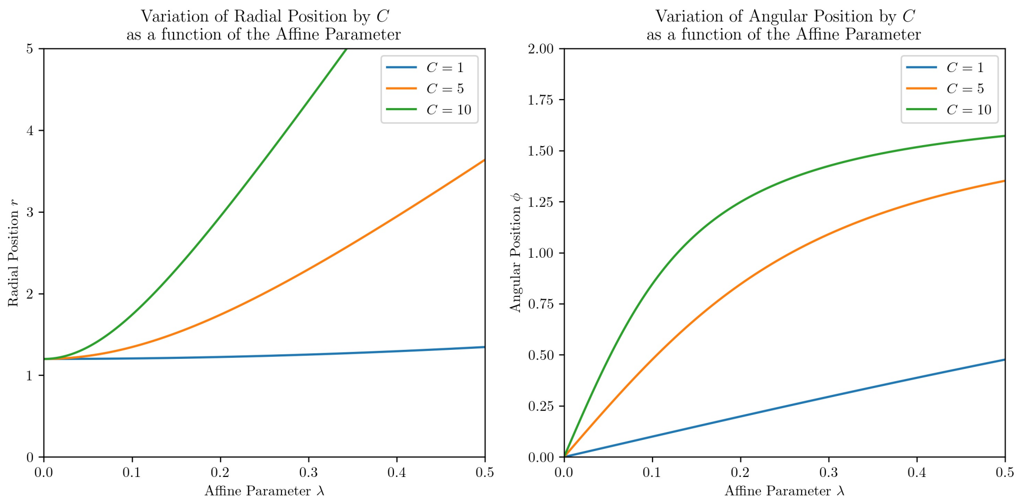

Figure 15: Varying initial velocity over the affine parameter - close up of origin

Appendix

Listings


```
import numpy as np
import matplotlib.pyplot as plt
import time
%matplotlib inline

def IVP(t,x0,u0,b):

    def rhs(x,u,dt):
        dx[0],dx[1],dx[2],dx[3] = u[0],u[1],u[2],u[3]

        du[0] = 0
        du[1] = x[1]*(u[2]**2+np.sin(x[2])**2*u[3]**2)
        du[2] = np.sin(x[2])*np.cos(x[2])*u[3]**2-(2*x[1]*u[1]*u[2])/(b**2+x[1]**2)
        du[3] = -2*((u[2]*u[3])/(np.tan(x[2]))+(x[1]*u[1]*u[3])/(b**2+x[1]**2))

        return dt*dx, dt*du

    xstep = np.array([x0,u0])
    x = np.zeros((len(t),4))
    u = np.zeros((len(t),4))
    udotu = np.zeros(len(t))
    udota = np.zeros(len(t))
    adota = np.zeros(len(t))
    x[0] = x0
    u[0] = u0
    udotu[0] = -u0[0]**2+ u0[1]**2+(b**2+x0[1]**2)*u0[2]**2+(b**2+x0[1]**2)*np.sin(x0
        [2])**2*u0[3]**2

    f1 = np.zeros((1,4))
    f2 = np.zeros((1,4))
    f3 = np.zeros((1,4))
    f4 = np.zeros((1,4))
    dx = np.zeros(4)
    du = np.zeros(4)
    diff = np.diff(t)
    for n in np.arange(1,len(t)):
        dt = diff[n-1]

        f1 = rhs(xstep[0],xstep[1],dt)

        f2 = rhs(xstep[0]+0.5*f1[0],xstep[1] + 0.5*f1[1],dt)

        f3 = rhs(xstep[0]+0.5*f2[0],xstep[1] + 0.5*f2[1],dt)

        f4 = rhs(xstep[0]+f3[0],xstep[1] + f3[1],dt)

        xstep[0] = xstep[0] + (f1[0]+2*f2[0]+2*f3[0]+f4[0])/6

        xstep[1] = xstep[1] + (f1[1]+2*f2[1]+2*f3[1]+f4[1])/6

        x[n] = xstep[0]
        u[n] = xstep[1]
        udotu[n] = -u[n,0]**2+ u[n,1]**2+(b**2+x[n,1]**2)*u[n,2]*2+(b**2+x[n,1]**2)*
            np.sin(x[n,2])**2*u[n,3]**2
```

```
    return x, u, udotu

C=10
b=1
R=1.2
nsteps = 20000
t = np.arange(0,10,1/nsteps)
t1 = time.time()
x0 = np.array([0,R,np.pi/2,0])
u0 = np.array([(C**2*(R**2+b**2))**(1/2),0,0,C])
x_10,u_10,udotu = IVP(t,x0,u0,b)
print(time.time() -t1)
tau=t
t = x[:,0]
pos = x[:,1]
theta = x[:,2]
phi = x[:,3]
print(udotu)

fig = plt.subplots()
plt.rc('text', usetex=True)
plt.rc('font', family='serif')
plt.plot(tau,udotu)

plt.ticklabel_format(useOffset=False)

ax.set_xlabel(r'Affine Parameter $\lambda$',fontsize=20)
plt.xlabel(r'Proper Time $\tau$',fontsize=20)
plt.ylabel(r'$\mathbf{u}\cdot\mathbf{u}$',fontsize=20);
plt.tight_layout()
plt.ylim(-1.00000000000008, -0.99999999999998) for massive c=0.5
plt.savefig('tau_udotu_c00.jpg',dpi=300)

fig , ax = plt.subplots(1,3,figsize=(15,5))
plt.rc('text', usetex=True)
plt.rc('font', family='serif')

ax[0].plot(tau,t)
ax[0].set_xlabel(r'Affine Parameter $\lambda$',fontsize=20)
ax[0].set_xlabel(r'Proper Time $\tau$',fontsize=20)
ax[0].set_ylabel(r'Time $t$',fontsize=20)
ax[0].set_title('Time as a function of\n Proper time',fontsize=20)
ax[0].set_title('Angular Position as a function of\n the Affine Parameter',fontsize
    =20)

ax[1].plot(tau,pos)
ax[1].set_xlabel(r'Affine Parameter $\lambda$',fontsize=20)
ax[1].set_xlabel(r'Proper Time $\tau$',fontsize=20)
ax[1].set_ylabel(r'Radial Position $r$',fontsize=20)
ax[1].set_title('Radial Position as a function of\n Proper time',fontsize=20)
ax[1].set_title('Angular Position as a function of\n the Affine Parameter',fontsize
    =20)

ax[2].plot(tau,phi)
ax[2].set_xlabel(r'Affine Parameter $\lambda$',fontsize=20)
ax[2].set_xlabel(r'Proper Time $\tau$',fontsize=20)
ax[2].set_ylabel(r'Angular Position $\phi$',fontsize=20)
ax[2].set_title('Angular Position as a function of\n Proper time',fontsize=20)
ax[2].set_title('Angular Position as a function of\n the Affine Parameter',fontsize
    =20)
```

```python
plt.tight_layout()
plt.savefig('massive_plots_c00.jpg',dpi=300)

fig, ax = plt.subplots(1,2,figsize=(10,5))
plt.rc('text', usetex=True)
plt.rc('font', family='serif')
ax[0].plot(tau,x_1[:,1])
ax[0].plot(tau,x_5[:,1])
ax[0].plot(tau,x_10[:,1])
ax[0].set_xlabel(r'Affine Parameter $\lambda$')
ax[0].set_ylabel(r'Radial Position $r$')
ax[0].set_title('Variation of Radial Position by $C$\n as a function of the Affine Parameter')
ax[0].legend([r'$C=1$',r'$C=5$',r'$C=10$'])

ax[1].plot(tau,x_1[:,3])
ax[1].plot(tau,x_5[:,3])
ax[1].plot(tau,x_10[:,3])
ax[1].set_xlabel(r'Affine Parameter $\lambda$')
ax[1].set_ylabel(r'Angular Position $\phi$')
ax[1].set_title('Variation of Angular Position by $C$\n as a function of the Affine Parameter')
ax[1].legend([r'$C=1$',r'$C=5$',r'$C=10$'])
plt.tight_layout()
plt.savefig('affine.jpg',dpi=300)

fig, ax = plt.subplots(1,2,figsize=(10,5))
plt.rc('text', usetex=True)
plt.rc('font', family='serif')
ax[0].plot(tau,x_1[:,1])
ax[0].plot(tau,x_5[:,1])
ax[0].plot(tau,x_10[:,1])
ax[0].set_xlim([0,0.5])
ax[0].set_ylim([0,5])
ax[0].set_xlabel(r'Affine Parameter $\lambda$')
ax[0].set_ylabel(r'Radial Position $r$')
ax[0].set_title('Variation of Radial Position by $C$\n as a function of the Affine Parameter')
ax[0].legend([r'$C=1$',r'$C=5$',r'$C=10$'])

ax[1].plot(tau,x_1[:,3])
ax[1].plot(tau,x_5[:,3])
ax[1].plot(tau,x_10[:,3])
ax[1].set_xlim([0,0.5])
ax[1].set_ylim([0,2])
ax[1].set_xlabel(r'Affine Parameter $\lambda$')
ax[1].set_ylabel(r'Angular Position $\phi$')
ax[1].set_title('Variation of Angular Position by $C$\n as a function of the Affine Parameter')
ax[1].legend([r'$C=1$',r'$C=5$',r'$C=10$'])
plt.tight_layout()
plt.savefig('affine_zoom.jpg',dpi=300)
```

Listing 1: Code to integrate path and produce plots in $4c, d$