MATH3063 Assignment 3
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4 June 2018

Consider the following system of nonlinear ordinary differential equations:

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x + \mu y(1 - x^2)\end{aligned} \tag{1}$$

1. There is one equilibrium. Where is it? For what values of μ is it an unstable focus?

   Answer:

   The equilibirum point can be found by finding the point where $\dot{x} = \dot{y} = 0$.

$$\begin{aligned}\frac{dx}{dt} &= y \\ &= 0 \\ \implies y &= 0 \\ \frac{dy}{dt} &= -x + \mu y(1 - x^2) \\ &= 0 \\ \implies x &= \mu y(1 - x^2) \\ &= \mu \cdot 0 \cdot (1 - x^2) \\ &= 0\end{aligned} \tag{2}$$

   Thus, the one equilibrium points is at $(0, 0)$.

   The values of μ which cause an unstable focus can be found by taking the Jacobian, J, and determining the values of μ for which $\det(J) > 0$, $\mathrm{tr}(J) > 0$ and $\Delta(J) < 0$.

   The Jacobian is

$$J = \begin{bmatrix} 0 & 1 \\ -1 - 2\mu yx & \mu(1 - x^2) \end{bmatrix}$$

   The determinant is

$$\begin{aligned}\det(J) &= 0 \cdot \mu(1 - x^2) - 1 \cdot (-1 - 2\mu yx) \\ &= 1 + 2\mu yx \\ \text{at } (0, 0) \implies &\det(J) > 0 \;\forall\, \mu\end{aligned} \tag{3}$$

   The trace is

$$\begin{aligned}\mathrm{tr}(J) &= 0 + \mu(1 - x^2) \\ &= \mu \\ \text{at } (0, 0) \implies &\mathrm{tr}(J) > 0 \text{ when } \mu > 0\end{aligned} \tag{4}$$

The descriminant is

$$
\begin{aligned}
\Delta(J) &= \mu^2(1-x^2)^2 - 4\cdot(1+2\mu yx) \\
&= \mu^2 - 2\mu^2x^2 + \mu^2x^4 - 4 - 8\mu yx \\
&= \mu^2 - 4 - 2\mu^2x^2 + \mu^2x^4 - 8\mu yx \\
\text{at } (0,0) &\implies \Delta(J) < 0 \text{ when } \mu^2 - 4 < 0 \\
&\implies \mu < 2 \text{ or } \mu > -2
\end{aligned} \tag{5}
$$

In summary, these conditions indicate, due to the determinant, μ can take any value, due to the trace, μ must be positive, and due to the descriminant, $-2 < \mu < 2$. Therefore, we can say for the equilibirum to be an unstable focus $0 < \mu < 2$.

2. Assume $\mu = 1/10$. The nullclines for this case are drawn on the plot provided above. Determine and draw the directions of the arrows on the nullclines in the plot provided (or your own plot as long as it's big enough to see). (The coice of $\mu = 1/10$ causes the resulting stable limit cycle to be relatively round rather than sharp and narrow.)

   Answer: The nullclines can be found by let $\dot{x} = 0$ and solving for y in terms of x, as well as doing the same for $\dot{y} = 0$.

$$
\begin{aligned}
\frac{dx}{dt} &= y \\
&= 0
\end{aligned} \tag{6}
$$

This gives a nullcline for $y = 0$, along the x-axis, for which $\dot{x} = 0$, that is, there are only direction arrows in the vertical direction.

Substituting this into the equation for $\dot{y}$, it is evident that when $x < 0$, this nullcline will have arrows pointing in the positive direction and when $x > 0$, this nullcline will have arrows pointing in the negative direction.

$$
\begin{aligned}
\frac{dy}{dt} &= -x + \frac{1}{10}y(1-x^2) \\
&= 0 \\
y &= \frac{10x}{1-x^2}
\end{aligned} \tag{7}
$$

This gives a nullcline along $y = 10x$, for which $\dot{y} = 0$, that is, there are only direction arrows in the horizontal direction. There will also be nullclines that following the curve $y = 1/(1-x^2)$ for which $\dot{y} = 0$ and this also only has direction arrows along the horizontal direction.

It is evident, directly from the nonlinear ODE for $\dot{x}$, the direction of the arrows along this nullcline, is dependent on y, that is, for $y > 0$, the nullclines will be pointing in the positive x direction and for $y < 0$, the nullclines will be pointing in the negative x direction.

This vector field illustrates the direction arrows along these nullclines and at other points in the solution and is shown in Fig. 1, while some solution curves are shown in Fig. 2.

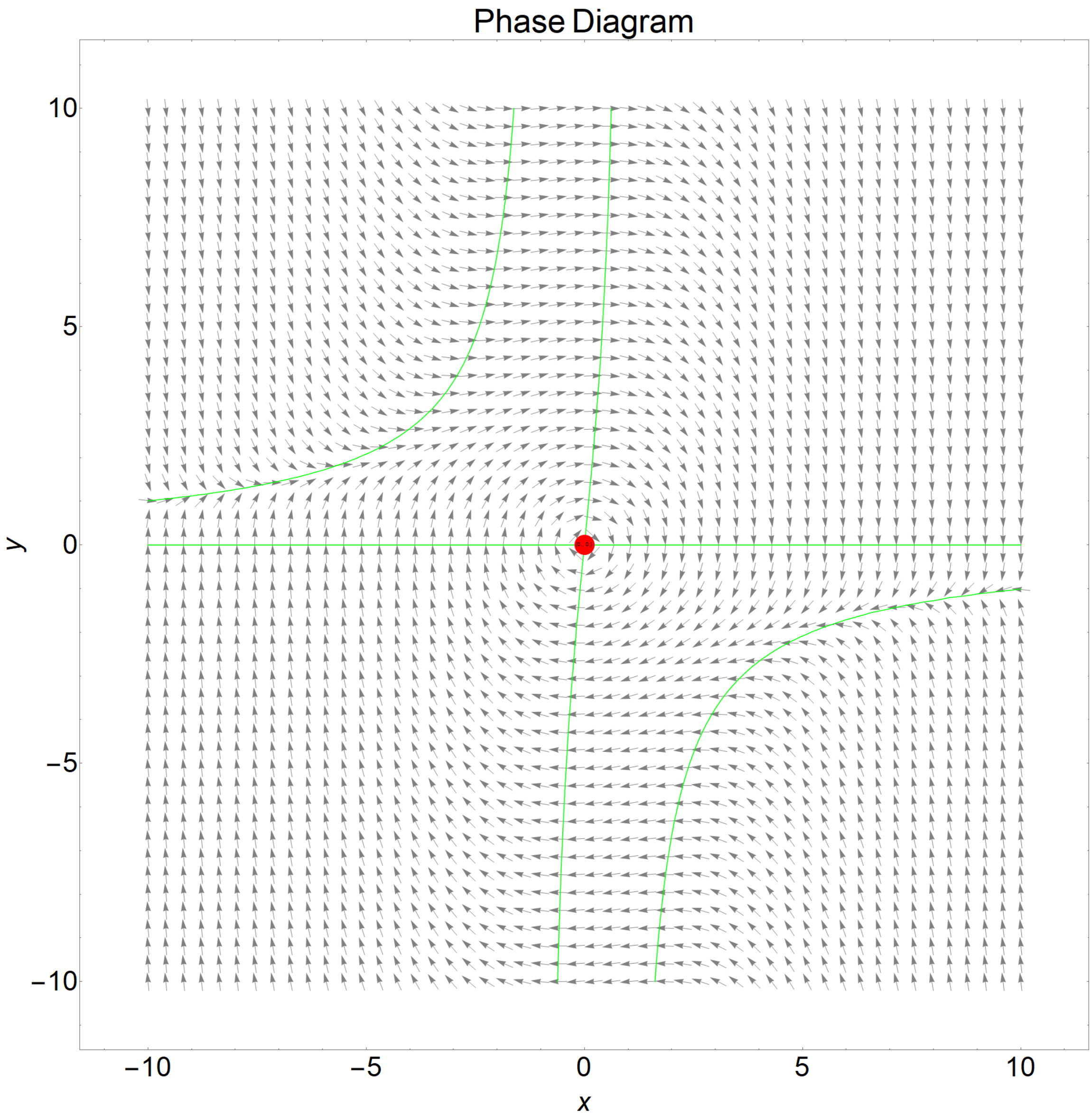

Figure 1: Vector field of the nonlinear ordinary differential equations

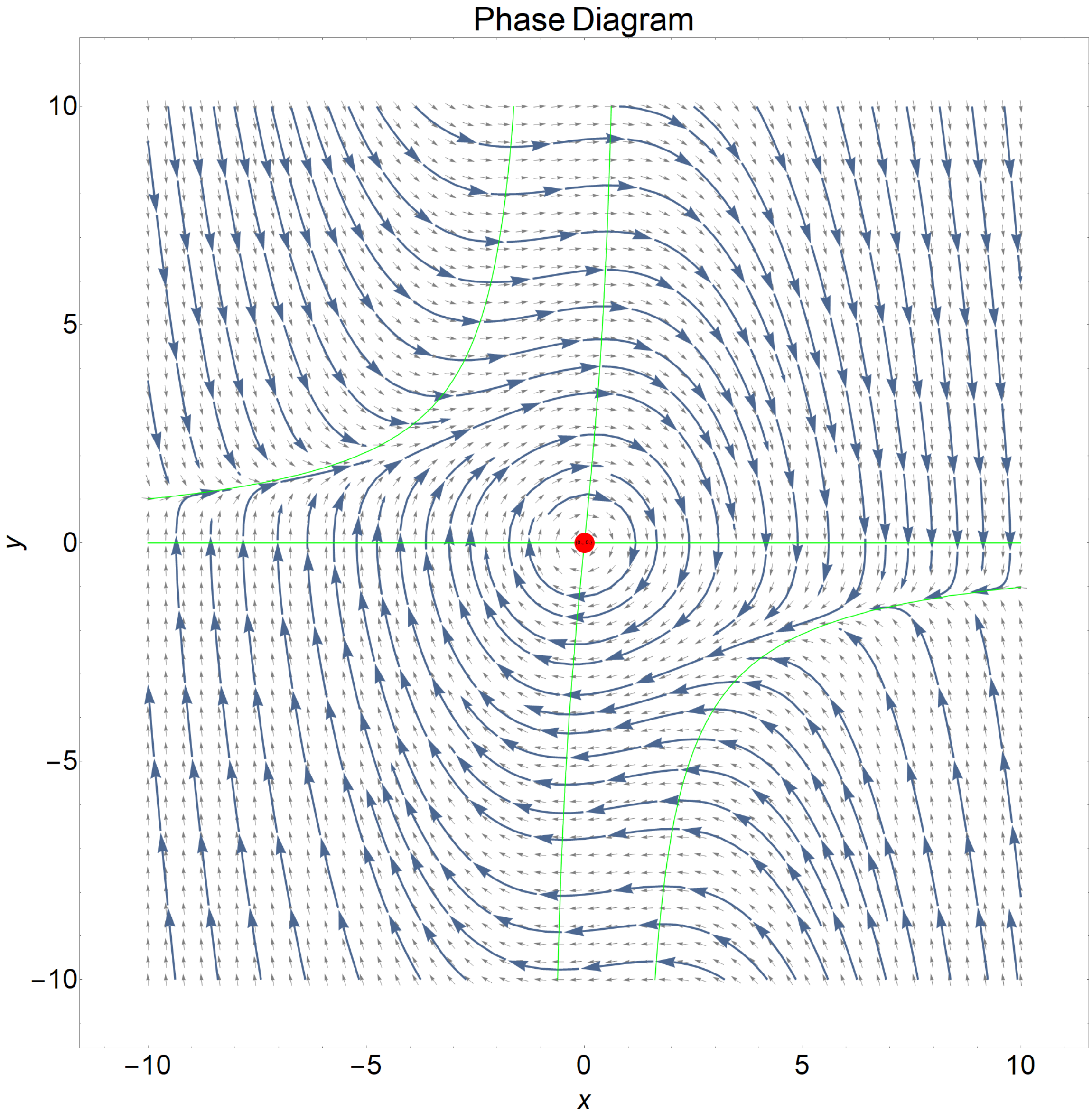

Figure 2: Solution curves and the vector field of the nonlinear ordinary differential equations

3. Find a trapping region to show that a stable limit cycle exists.

Answer:

In order to find the trapping region, we will convert to polar coordinates using $x = r\cos\theta$ and $y = r\sin\theta$ with $r^2 = x^2 + y^2$. Using $r\dot{r} = x\dot{x} + y\dot{y}$ gives

$$\begin{aligned}
r\dot{r} &= x\dot{x} + y\dot{y} \\
&= xy - xy + \frac{1}{10}y^2(1-x^2) \\
&= \frac{1}{10}y^2(1-x^2) \\
&= \frac{1}{10}(y^2 - y^2x^2) \\
&= \frac{1}{10}(y^2 - (r^2-x^2)(r^2-y^2)) \\
&= \frac{1}{10}(y^2 - r^4 + r^2y^2 + r^2x^2 - x^2y^2) \\
&= \frac{1}{10}(y^2(1+r^2-x^2) - r^2(r^2-x^2)) \\
&= \frac{1}{10}(y^2(1+y^2) - r^4 + r^2x^2) \\
&= \frac{1}{10}(r^2(1-r^2) - x^2(1-r^2) + y^2(r^2-x^2)) \\
&= \frac{1}{10}(r^2(1-r^2) - x^2(1-r^2) + y^2(r^2-x^2)) \\
&= \frac{1}{10}(r^2(1-r^2) - x^2 + x^2r^2 + y^4) \\
&= \frac{1}{10}(r^2(1-r^2) - r^2\cos^2\theta + r^4\cos^2\theta + r^4\sin^4\theta) \\
&= \frac{1}{10}r^2((1-r^2) - \cos^2\theta + r^2\cos^2\theta + r^2\sin^4\theta) \\
&= \frac{1}{10}r^2\left((1-r^2) - \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) + r^2\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) + r^2\left(\frac{3}{8} - \frac{\cos 2\theta}{2} + \frac{\cos 4\theta}{8}\right)\right) \\
&= \frac{1}{10}r^2\left(\left(1 - \frac{1}{2}\right) + (-r^2 + \frac{1}{2}r^2 + \frac{3}{8}r^2 + \frac{1}{2}r^2\cos 2\theta - \frac{1}{2}r^2\cos 2\theta + \frac{1}{8}r^2\cos 4\theta)\right) \\
&= \frac{1}{10}r^2\left(\frac{1}{2} + r^2\left(-\frac{1}{8} + \frac{1}{8}\cos 4\theta\right)\right) \\
&= \frac{1}{10}r^2\left(\frac{1}{2} - r^2\sin^2\theta\cos^2\theta\right)
\end{aligned}$$

$$\dot{r} = \frac{1}{10}r\left(\frac{1}{2} - r^2\sin^2\theta\cos^2\theta\right) \tag{8}$$

As is evident in the equation for $\dot{r}$, the radius will increase when $r^2\sin^2\theta\cos^2\theta < 1/2$ and the radius will decrease when $r^2\sin^2\theta\cos^2\theta > 1/2$ as $\sin^2\theta$ and $\cos^2\theta$ have positive values for all θ.

We can also determine the maximum radius such that all trajectories have a radially inwards trajectory, that is, the maximum radius such that $\dot{r} < 0$ for all θ. The maximum value of $\sin^2\theta\cos^2\theta$ for all θ is 1/4. Therefore, we can solve the equation for $\dot{r}$.

$$\dot{r} = \frac{1}{10}r\left(\frac{1}{2} - \frac{1}{4}r^2\right) \tag{9}$$

This gives intercepts at $r = \sqrt{2}$ with a maximum at at $r = \sqrt{2/3}$, where we have only taken the meaninful solutions for a radius. Therefore, it is clear that there are certain values of r where $\dot{r}$ will be positive and negative. This indicates that there is a trapping region as $\dot{r}$ is positive below a certain radius and negative greater than a certain radius so, there must be region in between where both meet.

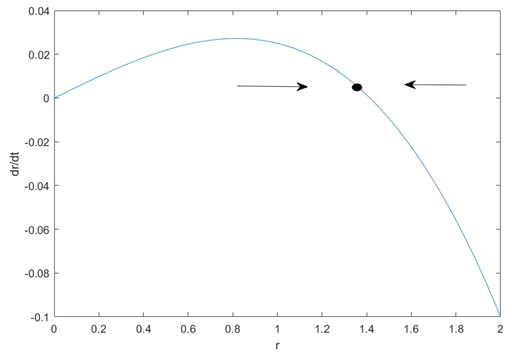

Figure 3: Change of radius at maximum value of $\sin^2\theta\cos^2\theta$

4. Use the Hopf Bifurcation Theorem to show that a stable limit cycle exists for some $\mu > 0$.

Answer:

$$\begin{aligned}\det(J) &= \begin{vmatrix} -\lambda & 1 \\ -1-2\mu yx & \mu(1-x^2)-\lambda \end{vmatrix} \\ &= \lambda\mu(1-x^2) + \lambda^2 + 1 + 2\mu yx \\ &= \lambda^2 + \lambda\mu(1-x^2) + (1+2\mu yx)\end{aligned}$$

Therefore the eigenvalues are

$$\begin{aligned}\lambda &= \frac{-\mu(1-x^2) \pm \sqrt{\mu^2(1-x^2)^2 - 4(1+2\mu yx)}}{2} \\ &= \frac{-\mu + \mu x^2 \pm \sqrt{\mu^2 - 2\mu^2x^2 + \mu^2x^4 - 4 - 8\mu yx)}}{2} \\ &= \frac{-\mu + \mu x^2 \pm \sqrt{\mu^2 - 2\mu^2x^2 + \mu^2x^4 - 4 - 8\mu yx)}}{2}\end{aligned}$$

For $\mu = 0$, the eigenvalues are purely complex, that is, $\lambda = \pm i$. So there is a point where it transitions from begin stable to unstable, therefore, there exists some μ with a stable limit cycle.

5. Draw the phase plane that includes a stable limit cycle.

   Answer:

   As is evident in Fig. 4, a solution spirals out when the initial radius is below a certain value (see red line) and all solutions spiral in from a radius greater than a certain value (see black line). Fig. 5 may be easier on the eyes to witness these trends. Therefore, there is clearly a stable limit cycle at some radius between.

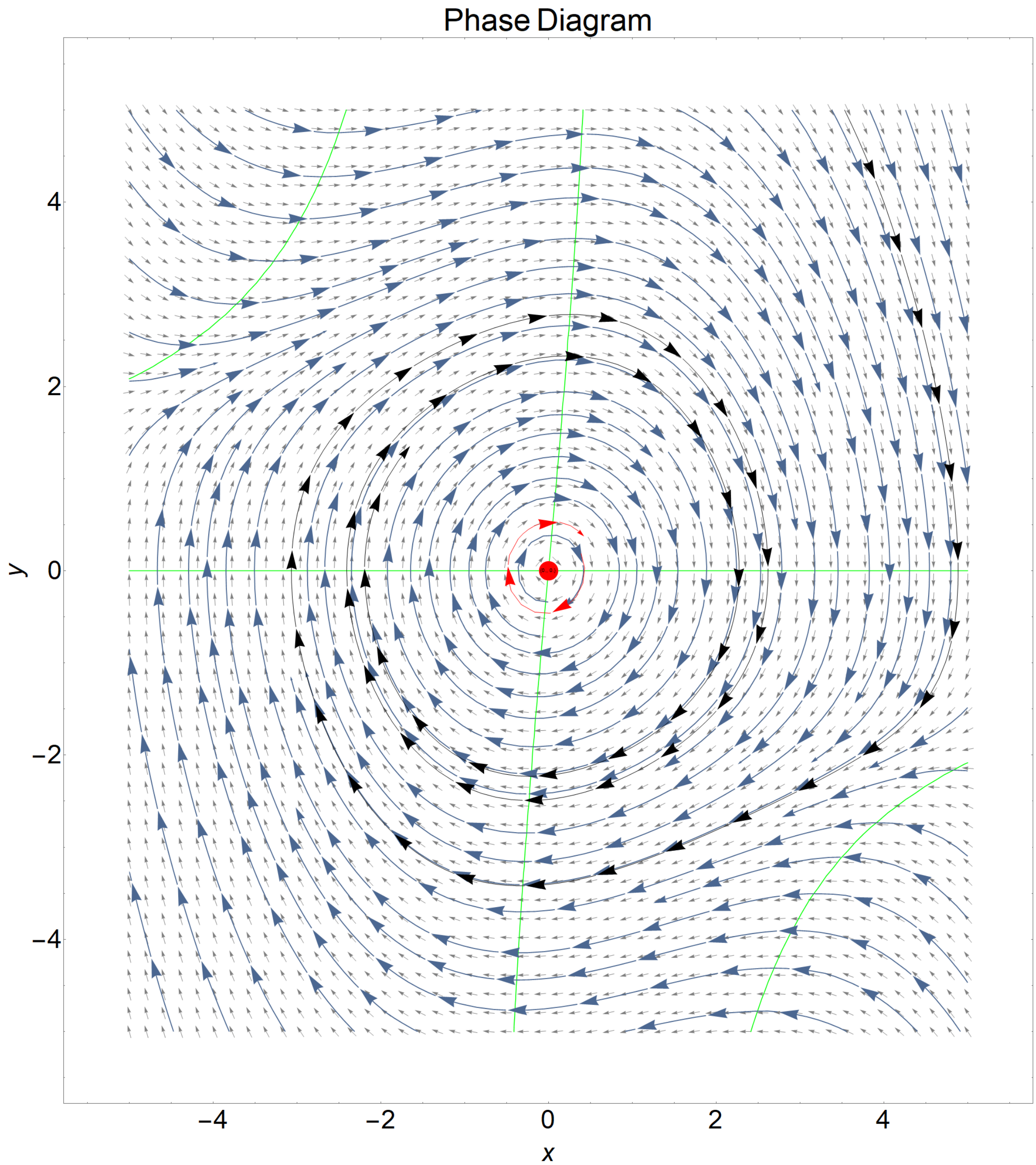

Figure 4: Solution curves and the vector field of the nonlinear ordinary differential equations with 2 solutions trending towards a stable limit cycle

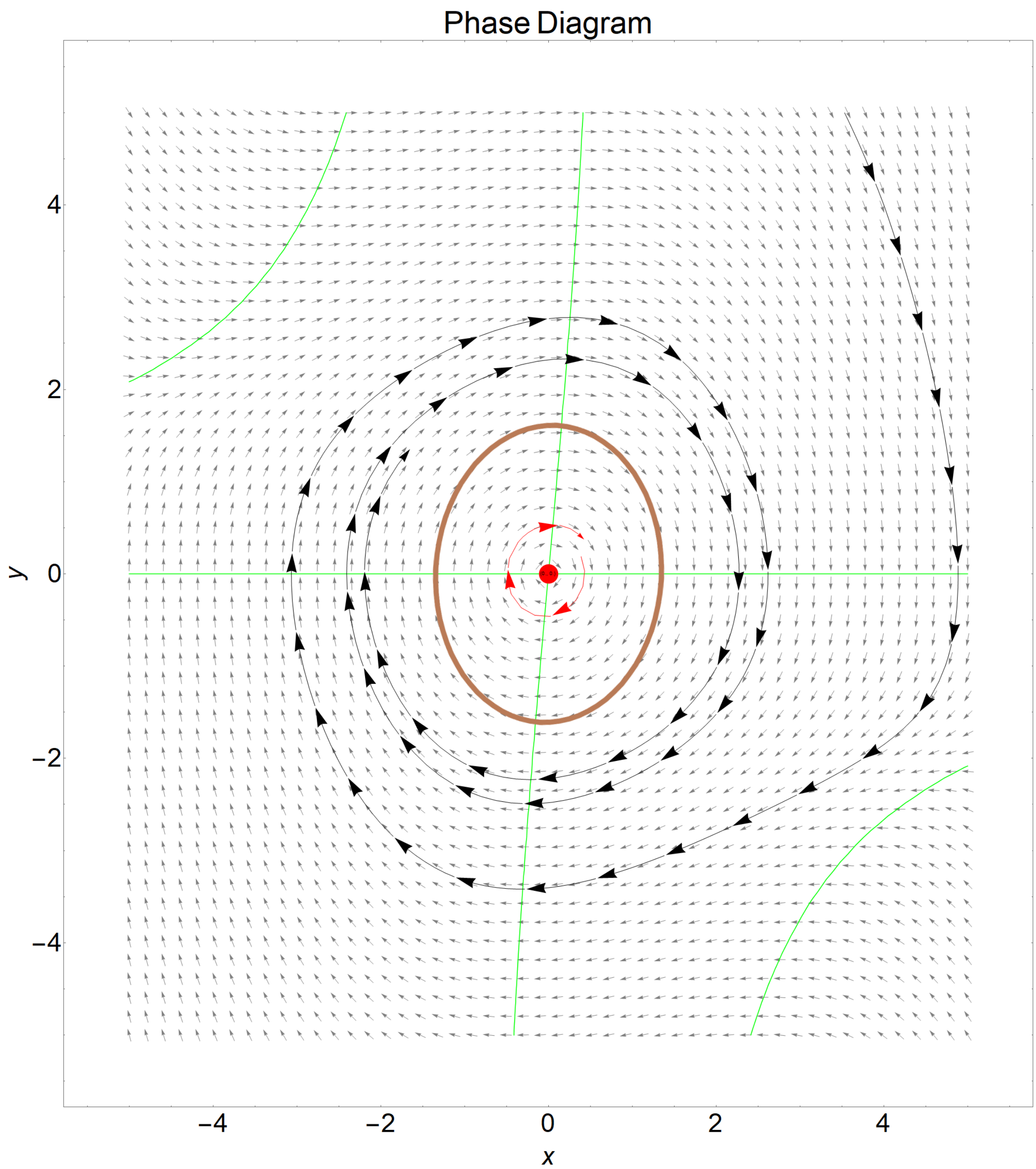

Figure 5: The vector field of the nonlinear ordinary differential equations with 2 solutions trending towards a stable limit cycle