Derivation of 4th Order Runge-Kutta Method

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Consider the initial value problem given by

$$\begin{cases} y'(t) & = f(t, y(t)) \\ y(t_0) & = y_0 \end{cases}$$

for a function Lipschitz continuous function f. To approximate the solution $y(t)$, the Runge-Kutta method uses an intermediate y value to enhance the approximation of each successive approximation $y_k \approx y(t_k)$ as

$$\begin{aligned} \widetilde{y}_{k+\alpha} &= y_k + \alpha h f(t_k, y_k) \\ \widetilde{y}_{k+\delta} &= y_k + \delta h f(t_k, y_k) \\ y_{k+1} &= y_k + \beta h f(t_k, y_k) + \gamma h f(t_k + \alpha h, \widetilde{y}_{k+\alpha}) + \eta h f(t_k + \delta h, \widetilde{y}_{k+\delta}) \end{aligned}$$

for some $\alpha, \beta, \gamma, \delta$, and η that must be consistent with the Taylor expandion of $y' = f$. Define f_y and f_t as the partial derivatives of f with respect to y and t respectively. Moreover, since f is continuously differentiable, the mixed partials of f are equivalent. Then the chain rule and Taylor's theorem gives us

$$\begin{aligned} f(t_k + \alpha h, \widetilde{y}_{k+\alpha}) &= f(t_k + \alpha h, y_k + \alpha h f(t_k, y_k)) \\ &= f(t_k, y_k) + \alpha h \left[f_t + f f_y\right] + \frac{(\alpha h)^2}{2}\left[f_{tt} + f f_{ty} + f^2 f_{yy}\right] + \mathcal{O}(h^3) \end{aligned}$$

as α is arbitrary this also holds for $f(t_k + \delta h, \widetilde{y}_{k+\delta})$. Thus the iterative step for y_{k+1} is given by

$$\begin{aligned} y_{k+1} &= y_k + \beta h f(t_k, y_k) + \gamma h f(t_k + \alpha h, \widetilde{y}_{k+\alpha}) + \eta h f(t_k \\ &= \beta h f(t_k, y_k) + \gamma h \Bigg(f(t_k, y_k) + \alpha h \left[f_t + f f_y\right] + \frac{(\alpha \end{aligned}$$

$$+\eta h \left(f(t_k, y_k) + \delta h \left[f_t + f f_y\right] + \frac{(\delta h)^2}{2}\left[f_{tt} + f f_{ty} + f^2 f_{yy}\right] \right) + \mathcal{O}(h^3)$$

Moreover, the Taylor expansion of the actual solution $y(t_{k+1})$ is given by

$$\begin{aligned} y(t_{k+1}) &= y(t_k) + h y'(t_k) + \frac{h^2}{2} y''(t_k) + \frac{h^3}{6} y'''(t_k) + \frac{h^4}{24} y^{(4)}(t_k) + \frac{h^5}{120} y^{(5)}(t_k) + \mathcal{O}(h^6) \\ &= y(t_k) + h f + \frac{h^2}{2}\left[f_t + f f_y\right] + \frac{h^3}{6}\left[f_{tt} + f f_{ty} + f^2 f_{yy}\right] + \mathcal{O}(h^4) \end{aligned}$$

Hence, we must choose $\alpha, \beta, \gamma, \delta$, and η so that $y(t_{k+1}) = y_{k+1}$. Since, the seperate terms are independent the corresponding coefficients of h must be equal. Hene we obtain a system of equations given by

$$\begin{aligned}\gamma + \eta &= 1\\ \gamma\alpha &= \frac{1}{2}\\ \eta\delta &= \frac{1}{2}\end{aligned}$$

Thus we have a parametrized solution for the 5 variables.