Math 21B
Disussion Sheet 1 - Key
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Note: These answers are not endorsed by Dr. Gravner and may be incorrect!

1. Compute the following limits, in any correct way you can.

(a) $\lim_{\mathbf{x}\to\infty}\left(\sqrt{\mathbf{x^2+x}}-\sqrt{\mathbf{x^2-x}}\right)$

At first glance, if we simply take $x\to\infty$ we get

$$\sqrt{x^2+x}-\sqrt{x^2-x}\Big|_{x\to\infty}=\infty-\infty$$

which is indeterminant, but not a form where L'Hôpital's rule can be used. Thus, we need to re-arrange this in order to properly take the limit. Notice that it is pretty much always the case that we should multiply by the conjugate with radical functions, so we obtain

$$\sqrt{x^2+x}-\sqrt{x^2-x}=\sqrt{x^2+x}-\sqrt{x^2-x}\left(\frac{\sqrt{x^2+x}+\sqrt{x^2-x}}{\sqrt{x^2+x}+\sqrt{x^2-x}}\right)=\frac{x^2+x-x^2+x}{\sqrt{x^2+x}+\sqrt{x^2-x}}=\frac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}}$$

Here we can divide top and bottom by the largest power of x, which is $x=\sqrt{x^2}$, thus

$$\begin{aligned}\lim_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)&=\lim_{x\to\infty}\frac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}}\\&=\lim_{x\to\infty}\frac{\frac{2x}{x}}{\frac{\sqrt{x^2+x}}{\sqrt{x^2}}+\frac{\sqrt{x^2-x}}{\sqrt{x^2}}}\\&=\lim_{x\to\infty}\frac{2}{\sqrt{1+1/x}+\sqrt{1-1/x}}\\&=\frac{2}{\sqrt{1+0}+\sqrt{1-0}}=\frac{2}{2}=1\end{aligned}$$

Therefore we conclude that

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)=1$$

(b) $\lim_{\mathbf{x}\to\mathbf{0}}\frac{\ln(\mathbf{1+x})-\mathbf{x}}{\mathbf{e^x-1-x}}$

Simply plugging in $x=0$ we obtain

$$\frac{\ln(1+x)-x}{e^x-1-x}\bigg|_{x=0}=\frac{\ln(1)-0}{e^0-1-0}=\frac{0}{0}$$

Hence, we have another indeterminant form, but this time it suffices to use L'Hôpital's rule. Thus we obtain

$$\lim_{x\to0}\frac{\ln(1+x)-x}{e^x-1-x}\overset{\mathcal{L}'\mathcal{H}}{=}\lim_{x\to0}\frac{\frac{1}{1+x}-1}{e^x-1}$$

Note that we once again get a limit of the form $\frac{0}{0}$, so one more iteration of L'Hôpital's rule gives us

$$\lim_{x\to0}\frac{\ln(1+x)-x}{e^x-1-x}\overset{\mathcal{L}'\mathcal{H}}{=}\lim_{x\to0}\frac{\frac{1}{1+x}-1}{e^x-1}\overset{\mathcal{L}'\mathcal{H}}{=}\lim_{x\to0}\frac{\frac{1}{(1+x)^2}}{e^x}=\frac{\frac{1}{1}}{1}=1$$

Therefore we conclude that

$$\lim_{x\to 0}\frac{\ln(1+x)-x}{e^x-1-x}=1$$

(c) $\lim_{\mathbf{h}\to\mathbf{0}}\frac{\mathbf{(2+h)^5-32}}{\mathbf{h}}$

Here we could be clever and recognize this is the limit definition of a derivative of x^5 at $x=2$, then we obtain

$$\lim_{h\to 0}\frac{(2+h)^5-32}{h}=\left.\frac{d}{dx}x^5\right|_{x=2}=5(2)^4=80$$

Or we can use expand $(2+h)^5=(1)2^5+(5)2^4h+(10)2^3h^2+(10)2^2h^3+(5)2h^4+(1)h^5$ and thus

$$\begin{aligned}\lim_{h\to 0}\frac{(2+h)^5-32}{h}&=\lim_{h\to 0}\frac{2^5+2^4h+2^3h^2+2^2h^3+2h^4+h^5-32}{h}\\&=\lim_{h\to 0}(5)2^4+(10)2^3h+(10)2^2h^2+(5)2h^3+(1)h^4\\&=5(2)^4=80\end{aligned}$$

Therefore we conclude that

$$\lim_{h\to 0}\frac{(2+h)^5-32}{h}=5(2)^4=80$$

(d) $\lim_{\mathbf{x}\to\infty}\left(\frac{\mathbf{2x-3}}{\mathbf{2x-4}}\right)^{\mathbf{3x-4}}$

The only limit I know how to take that looks similar to this is

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{ax}=e^a$$

Thus, I am going to try to get this into that form. Since I need to separate $\frac{2x-3}{2x-4}$ into 1 + something, I am going to use a substitution to make things easier. Thus, let $u=2x-4$ and we no longer have a mixed term in the denominator. Note we lucked out that if $x\to\infty$, then $u=2x-4$ also goes to infinity, but you need to make sure this is the case if you do a substitution. Thus using this substitution we obtain

$$\left(\frac{2x-3}{2x-4}\right)^{3x-4}=\left(\frac{u+1}{u}\right)^{3/2u+2}=\left(1+\frac{1}{u}\right)^{3/2u}\left(\frac{u+1}{u}\right)^2$$

Hence the limit becomes

$$\begin{aligned}\lim_{x\to\infty}\left(\frac{2x-3}{2x-4}\right)^{3x-4}&=\lim_{x\to\infty}\left(1+\frac{1}{u}\right)^{3/2u}\left(\frac{u+1}{u}\right)^2\\&=\lim_{x\to\infty}\left(1+\frac{1}{u}\right)^{3/2u}\lim_{x\to\infty}\left(\frac{u+1}{u}\right)^2\\&=e^{3/2}(1)=e^{3/2}\end{aligned}$$

Therefore we conclude that

$$\lim_{x\to\infty}\left(\frac{2x-3}{2x-4}\right)^{3x-4}=e^{3/2}$$

2. Compute $\mathbf{f'(x)}$. Do not simplify

(a) $\mathbf{f(x) = \arctan\sqrt{\sin x}}$

Let $g(x) = \arctan(x)$, $h(x) = \sqrt{x}$, and $s(x) = \sin(x)$, then by the Chain Rule we obtain

$$\frac{d}{dx}f(x) = \frac{dg}{dh}\frac{dh}{ds}\frac{ds}{dx} = \left(\frac{1}{\sqrt{1-(\sqrt{\sin x})^2}}\right)\left(\frac{1/2}{\sqrt{\sin(x)}}\right)\cos x = \frac{\cos x}{2\sqrt{\sin(x)(1-\sin(x))}}$$

(b) $\mathbf{f(x) = x \arcsin \ln x}$

Using the product and chain rule we have

$$\frac{d}{dx}f(x) = \frac{d}{dx}(x)\arcsin\ln x + x\frac{d}{dx}\arcsin\ln x = \arcsin\ln x + x\left(\frac{1}{\sqrt{1-\ln^2 x}}\right)\frac{1}{x} = \arcsin\ln x + \frac{1}{\sqrt{1-\ln^2 x}}$$

(c) Suppose that $\mathbf{f(x)}$ satisfies the equation $\mathbf{xf(x)^3 + x^3 + xe^{f(x)-1} = 12}$ and that $\mathbf{f(2) = 1}$. Compute $\mathbf{f'(2)}$.

With implicit differentiation we obtain

$$0 = x3f(x)^2\frac{df}{dx} + 3x^2 + e^{f(x)-1} + xe^{f(x)-1}\frac{df}{dx} = \frac{df}{dx}\left(3xf(x)^2 + xe^{f(x)-1}\right) + 3x^2 + e^{f(x)-1}$$

Solving for $f'(x) = \frac{df}{dx}$ we have

$$\frac{df}{dx} = \frac{3x^2 + e^{f(x)-1}}{3xf(x)^2 + xe^{f(x)-1}}$$

Then at $x = 2$, since $f(2) = 1$, we obtain

$$f'(2) = \frac{3(2)^2 + e^{f(2)-1}}{3(2)f(2)^2 + (2)e^{f(2)-1}} = \frac{12+e^0}{6(2)^2+2e^0} = \frac{13}{26} = \frac{1}{2}$$

3. Roughly graph $\mathbf{y = \frac{1}{2}x^{3/2}}$. Then find the point on the graph which is closest to (2,0). (Don't forget that it is enough to maximize the *square* of the distance between (2,0) and a point $\mathbf{(x, y)}$ on the graph!)

So in general, the Euclidean distance between two points, (x_1, y_1) and (x_2, y_2), is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Thus we are interested in the distance between (2,0) and some (x, y) on $y = \frac{1}{2}x^{3/2}$. Thus we can write the distance as

$$d = \sqrt{(x-2)^2 + y^2}$$

Moreover, since our point (x, y) is on the curve, we have $y = \frac{1}{2}x^{3/2}$ we get

$$d = \sqrt{(x-2)^2 + \left(\frac{1}{2}x^{3/2}\right)^2} = \sqrt{\frac{1}{4}x^3 + x^2 - 4x + 4}$$

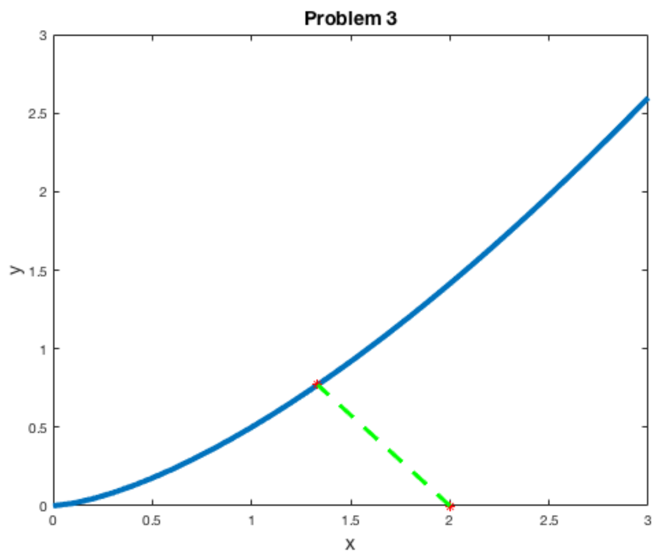
then our goal is to minimize this distance. To simplify this, we can recognize that any value that minimizes d must also minimize d^2. Thus we have the minimization problem

$$\min_x \frac{1}{4}x^3 + x^2 - 4x + 4$$

To solve this problem we need to set the derivative of this function equal to zero. Thus

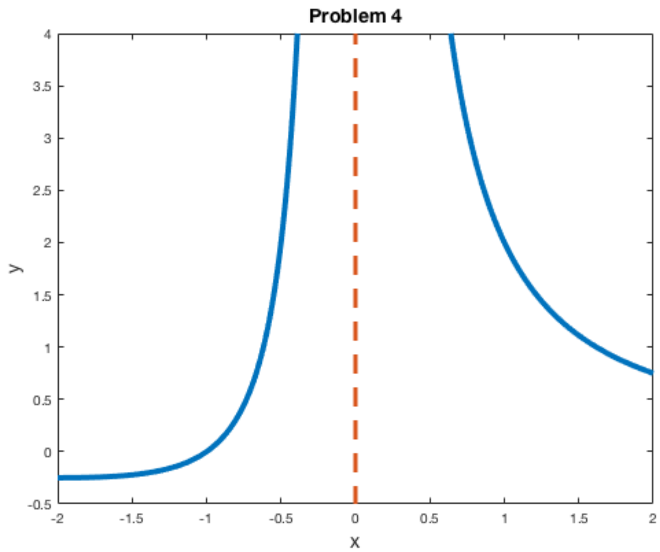
$$\frac{3}{4}x^2 + 2x - 4 = 0$$

which has zeros of $x = -4$ and $x = 4/3$. So which is the correct solution? Note that the domain of $y = \frac{1}{2}x^{3/2}$ is $x \geq 0$, so we throw out the solution $x = -4$ since it's negative. Thus the point closes to $(2, 0)$ is $\left(\frac{4}{3}, \frac{1}{2}\left(\frac{4}{3}\right)^{3/2}\right) = \left(\frac{4}{3}, \frac{4}{3\sqrt{3}}\right)$.



1. Finding the minimum distance between $y = \frac{1}{2}x^{3/2}$ and $(2, 0)$.

4. Let $\mathbf{f(x) = \frac{x+1}{x^2} = \frac{1}{x} + \frac{1}{x^2}}$. Sketch the graph of $\mathbf{y = f(x)}$ using the first and second derivative. Be sure to label clearly all important points on the graph.



1. **What is the range of this function?**

   Note that we have
   $$\lim_{x\to 0}\frac{1}{x^2}\to\infty$$
   So at least any positive value of y is in the range. Moreover, we can find the min by setting $f'(x)=0$ and we get
   $$f'(x)=-\frac{1}{x^2}-\frac{2}{x^3}=0 \iff \frac{1}{x^2}=-\frac{2}{x^3} \iff x=-2$$
   Thus the minimum of this function is when $x=-2$, and $f(-2)=\frac{1}{-2}+\frac{1}{(-2)^2}=-\frac{1}{4}$. Thus the range is $\{y|y\in[-1/4,\infty)\}$.

2. **Is this function one-to-one on the interval $(\mathbf{0},\infty)$?**

   One-to-one means that for any y value there is only a single x value. However, clearly we see that $f(1)=f(-0.5)=2$, thus this is not one-to-one.