Math 21C
Kouba
Disussion Sheet 4 - Key
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1. Use what you know about converging geometric series to write each power series as an ordinary function

a.) $\sum_{\mathbf{n=2}}^{\infty} \frac{\mathbf{x^n}}{\mathbf{3^n}}$

Notice that, for $m = n - 2$, we have

$$\sum_{n=2}^{\infty} \frac{x^n}{3^n} = \sum_{m=0}^{\infty} \frac{x^{m+2}}{3^{m+2}} = \sum_{m=0}^{\infty} \frac{x^2}{3^2} \left(\frac{x}{3}\right)^m = \left(\frac{x}{3}\right)^2 \left(\frac{1}{1 - \frac{x}{3}}\right) = \frac{x^2}{3(3-x)}$$

so long as $|x/3| < 1$. Thus we obtain,

$$\sum_{n=2}^{\infty} \frac{x^n}{3^n} = \frac{x^2}{3(3-x)}$$

Note, one can show that $x \in (-3, 3)$.

b.) $\sum_{\mathbf{n=0}}^{\infty} \frac{\mathbf{x^n}}{\mathbf{3^n}}$

This one is easier since the series starts at $n = 0$, thus we have

$$\sum_{n=0}^{\infty} \frac{x^n}{3^n} = \frac{1}{1 - \frac{x}{3}} = \frac{3}{3 - x}$$

for $|x/3| < 1 \iff x \in (-3, 3)$.

c.) $\mathbf{x^2 - x^{5/2} + x^3 - x^{7/2} + x^4 - x^{9/2} + \cdots}$

The pattern in this expression is given by

$$x^2 - x^{5/2} + x^3 - x^{7/2} + x^4 - x^{9/2} + \cdots = \sqrt{x}^4 - \sqrt{x}^5 + \sqrt{x}^6 - \sqrt{x}^7 + \sqrt{x}^8 - \sqrt{x}^9 + \cdots = \sum_{n=4}^{\infty} -\left(-\sqrt{x}\right)^n = \sum_{n=0}^{\infty} -x^4 \left(-\sqrt{x}\right)^n$$

This is a geometric series with $r = -\sqrt{x}$, thus we need $|-\sqrt{x}| < 1 \iff x \in (0, 1)$. Thus we obtain

$$x^2 - x^{5/2} + x^3 - x^{7/2} + x^4 - x^{9/2} + \cdots = -x^4 \sum_{n=0}^{\infty} \left(\sqrt{x}\right)^n = -x^4 \frac{1}{1 - (-\sqrt{x})} = -\frac{x^4}{1 + \sqrt{x}}$$

d.) $\sum_{\mathbf{n=0}}^{\infty} \mathbf{(n+1)x^n}$

For this series, notice that we have

$$\sum_{n=0}^{\infty} (n+1)x^n = \sum_{n=0}^{\infty} nx^n + \sum_{n=0}^{\infty} x^n$$

For $\sum x^n$, we have $\sum x^n = \frac{1}{1-x}$ if $x \in (-1,1)$. For the other term, let $g(x) := \sum x^n$, then notice that

$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}\frac{\partial}{\partial x}x^n = \sum_{n=0}^{\infty} nx^{n-1} = \frac{1}{x}\sum_{n=0}^{\infty} x(nx^{n-1}) = \frac{1}{x}\sum_{n=0}^{\infty} nx^n$$

Thus we have $\sum nx^n = x\frac{\partial}{\partial x}g(x)$. Moreover, $g(x) = \sum x^n = \frac{1}{1-x}$ for $x \in (-1,1)$, so we obtain

$$\sum_{n=0}^{\infty} nx^n = x\frac{\partial}{\partial x}g = x\frac{\partial}{\partial x}\frac{1}{1-x} = \frac{-x}{(1-x)^2}$$

Therefore the functional form of the series $\sum(n+1)x^n$ is given by

$$\sum_{n=0}^{\infty}(n+1)x^n = \frac{-x}{(1-x)^2} + \frac{1}{1-x} = \frac{(1-x)-x}{(1-x)^2} = \frac{1-2x}{(1-x)^2}$$

2. Recall that if $\mathbf{y = f(x)}$ is a function and

$$\mathbf{a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 + \cdots = \sum_{n=0}^{\infty} a_n(x-a)^n}$$

is the Taylor Series (or Maclaurin for series if $\mathbf{a = 0}$) centered at $\mathbf{x = a}$ for $\mathbf{y = f(x)}$, then $\mathbf{a_n = \frac{f^{(n)}(a)}{n!}}$. Use this formula to compue the first four nonzero terms and the general formula for the Taylor series expansion for each function about the given value of a.

This is just a bunch of gross computation, so I'm probably not going to do this. However, I have written the answers to the ones centered at 0 below.

a.) $\mathbf{f(x) = e^x}$ centered at $\mathbf{x = 0}$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

b.) $\mathbf{f(x) = e^x}$ centered at $\mathbf{x = \ln 2}$

c.) $\mathbf{f(x) = \frac{1}{1-x}}$ centered at $\mathbf{x = 0}$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

d.) $\mathbf{f(x) = \sin x}$ centered at $\mathbf{x = 0}$

$$\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

e.) $\mathbf{f(x) = \frac{1}{x}}$ centered at $\mathbf{x = 1}$

f.) $\mathbf{f(x) = \sqrt{x+5}}$ centered at $\mathbf{x = -1}$

3. Use the sugessted method to find the first four nonzero terms of the Maclaurin series for each function

a.) $\mathbf{f(x) = \frac{1}{1+x^2}}$ **(Substitute** $\mathbf{-x^2}$ **into the Maclaurin series for** $\mathbf{\frac{1}{1-x}}$**.)**

Note that $\sum x^n = \frac{1}{1-x}$, so we obtain

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

b.) $\mathbf{f(x) = x^3 e^{-3x}}$ **(Substitute** $\mathbf{-3x}$ **into the Maclaurin series for** $\mathbf{e^x}$ **and then multiply by** $\mathbf{x^3}$**)**

The Maclaurin series for $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, thus we obtain

$$x^3 e^{-3x} = x^3 \left(\sum_{n=0}^{\infty} \frac{(-3x)^n}{n!}\right) = x^3 \left(\sum_{n=0}^{\infty}(-1)^n \frac{3^n x^n}{n!}\right) = \sum_{n=0}^{\infty}(-1)^n \frac{3^n x^{n+3}}{n!}$$

c.) $\mathbf{f(x) = e^x \frac{1}{1-x}}$ **(Multiply the Maclaurin series for** $\mathbf{e^x}$ **and** $\mathbf{\frac{1}{1-x}}$ **term by term and then group like powers of x)**

Note that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, thus we obtain

$$\begin{aligned}
e^x \frac{1}{1-x} &= \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} x^n\right) \\
&= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(1 + x + x^2 + x^3 + \cdots\right) \\
&= \left(1 + x + x^2 + x^3 + \cdots\right) + x\left(1 + x + x^2 + x^3 + \cdots\right) + \frac{x^2}{2!}\left(1 + x + x^2 + x^3 + \cdots\right) + \frac{x^3}{3!}\left(1 + x + x^2 + x^3 + \cdot\right. \\
&= \left(1 + x + x^2 + x^3 + \cdots\right) + \left(x + x^2 + x^3 + x^4 + \cdots\right) + \left(\frac{x^2}{2!} + \frac{x^3}{2!} + \frac{x^4}{2!} + \frac{x^5}{2!} + \cdots\right) + \left(\frac{x^3}{3!} + \frac{x^4}{3!} + \frac{x^5}{3!} + \frac{x^6}{3!}\right. \\
&= 1 + (1+1)x + \left(1 + 1 + \frac{1}{2!}\right)x^2 + \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!}\right)x^3 + \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}\right)x^4 + \cdots \\
&= \sum_{k=0}^{0} \frac{1}{k!}x^0 + \sum_{k=0}^{1} \frac{1}{k!}x^1 + \sum_{k=0}^{2} \frac{1}{k!}x^2 + \sum_{k=0}^{3} \frac{1}{k!}x^3 + \cdots \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{k!}x^n
\end{aligned}$$

d.) $\mathbf{f(x) = \frac{e^x}{1-x}}$ **(Use polynomial division. Divide the Maclaurin series for** $\mathbf{e^x}$ **by** $\mathbf{1-x}$**)**

e.) $\mathbf{f(x) = 3x^2 \cos(x^3)}$ **(Substitute** $\mathbf{x^3}$ **into the Maclaurin series for** $\sin \mathbf{x}$ **then differentiate term by term)**

f.) $\mathbf{f(x) = \arctan x}$ **(Integrate the Maclaurin series for** $\mathbf{\frac{1}{1+t^2}}$ **from 0 to x)**

4. The Maclaurin series for $\mathbf{f(x) = \frac{1}{1+x}}$ is $\mathbf{1 - x + x^2 - x^3 + x^4 - x^5 + \cdots}$

a.) Show that $\mathbf{f(x) = \frac{1}{1+x}}$ and $\mathbf{1 - x + x^2 - x^3 + x^4 - x^5 + \cdots}$ have the same value at $\mathbf{x = 0}$.

b.) Show that $\mathbf{f(x) = \frac{1}{1+x}}$ and $\mathbf{1 - x + x^2 - x^3 + x^4 - x^5 + \cdots}$ have the same value at $\mathbf{x = 1/2}$.

c.) Show that $\mathbf{f(x) = \frac{1}{1+x}}$ and $\mathbf{1 - x + x^2 - x^3 + x^4 - x^5 + \cdots}$ have the same value at $\mathbf{x = 1}$.

d.) For what x-values is $\mathbf{f(x) = \frac{1}{1+x}}$ defined?

d.) For what x-values is $\mathbf{1 - x + x^2 - x^3 + x^4 - x^5 + \cdots}$ defined?

Note: It can be shown that $\mathbf{f(x) = \frac{1}{1+x}}$ and its Maclaurin series $\mathbf{1 - x + x^2 - x^3 + x^4 - x^5 + \cdots}$ are equal on the interval $\mathbf{(-1, 1)}$.

5. Determine (Use shortcuts.) the third-degree Taylor polynomial, $\mathbf{P_3(x; 0)}$, for the function $\mathbf{f(x) = \frac{x}{1+x}}$. Use $\int_0^1 \mathbf{P_3(x; 0)}\,\mathbf{dx}$ to estimate the value of $\int_0^1 \frac{\mathbf{x}}{\mathbf{1+x}}\,\mathbf{dx}$. Now evaluate $\int_0^1 \frac{\mathbf{x}}{\mathbf{1+x}}\,\mathbf{dx}$ directly to see how good the estimate is.

6. The following definite integral cannot be evaluated using the Fundamental Theorem of Calculus. Use the Maclaurin series for $\cos \mathbf{x}$ and the absolute error $|\mathbf{R_n}|$ for an alternating series to estimate the value of this integral with error at most 0.0001:$\int_0^1 \cos(\mathbf{x^2})\,\mathbf{dx}$.

7. Write each Maclaurin series as an ordinary function.

a.) $\mathbf{(3x) - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!} + \frac{(3x)^9}{9!} - \cdots}$ (HINT: Use $\sin \mathbf{x}$)

Note that we have

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Then the series in question is given by

$$\sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1} x^{2n+1}}{(2n+1)!} = \sin(3x)$$

b.) $\mathbf{x^2 - x^3 + x^4 - x^5 + x^6 - \cdots}$ (HINT: Factor)

Factoring out x^2 gives us,

$$x^2 - x^3 + x^4 - x^5 + x^6 - \cdots = x^2 \left[1 - x + x^2 - x^3 + x^4 - \cdots\right] = x^2 \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-x^2)^n = \frac{1}{1 - (-x^2)} = \frac{1}{1 + x^2}$$

c.) $\mathbf{\frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{6!} + \frac{x^3}{5!} + \frac{x^4}{6!} + \cdots}$ (HINT: Use $\mathbf{e^x}$)

Note that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

But the provided series has x^n divided by $(n+2)!$. Therefore we have

$$\frac{1}{2!}+\frac{x}{3!}+\frac{x^2}{6!}+\frac{x^3}{5!}+\frac{x^4}{6!}+\cdots = \sum_{n=0}^{\infty} \frac{x^n}{(n+2)!} = \sum_{n=0}^{\infty} \frac{1}{x^2}\frac{x^(n+2)}{(n+2)!} = \frac{1}{x^2}\sum_{n=2}^{\infty}\frac{x^n}{n!} = \frac{1}{x^2}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!} - \sum_{n=0}^{1}\frac{x^n}{n!}\right) = \frac{1}{x^2}\left(e^x - 1 - x\right)$$

d.) $\mathbf{x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \cdots}$ **(Challenging)**

Challenging probably means using the differentiation rule we used 1d); just a hunch since its of the form $\sum nx^n$. From 1d), we know that

$$\sum_{n=0}^{\infty} nx^n = \frac{-x}{(1-x)^2}$$

Thus this is the answer.

8. Use any method to find the given Taylor polynomial for each function. Then estimate the Absolute Taylor Error on the indicated interval.

Taylor's remainder theorem states that a function with a Taylor series approximation has error, denoted R_n, given by

$$\left|\frac{f^{n+1}(\xi)}{(n+1)!}\right|$$

where ξ is some value in your desired interval, I. We can choose ξ such that $f(\xi) = \max\{f(x) : x \in I\}$ is the maximum value f can take in that interval.

a.) $\mathbf{f(x) = e^{-2x},\ P_3(x;0),}$ **for** $\mathbf{[-1/2, 1/3]}$

We have the third degree polynomial of e^{-2x} given by

$$\sum_{n=0}^{3} \frac{(-2x)^n}{n!}$$

Then by taylor's theorem we have the error given by

$$R_n = \left|\frac{f^{(4)}(\xi)}{4!}\right|; \quad \xi \in [-1/2, 1/3]$$

Since $f^{(4)}(x) = (-2)^4 e^{-2x} = 16e^{-2x}$, then $f^{(4)}(x)$ is strictly decreasing. Thus we have

$$f^{(4)}(\xi) \le f^{(4)}(-1/2)$$

so we can estimate the absolute error as

$$R_n \approx \frac{f^{(4)}(-1/2)}{4!} = \frac{16e^{-2(-1/2)}}{4!} = \frac{16}{24}e$$

b.) $\mathbf{f(x) = \sin 2x,\ P_5(x;0),}$ **for** $\mathbf{[0, 3/4]}$

c.) $\mathbf{f(x) = \frac{x}{1-x},\ P_4(x;0),}$ **for** $\mathbf{[-1/3, 0]}$

9. What should n be so that the nth-degree Taylor Polynomial $\mathbf{P_n(x;a)}$ estimates the value of the given function on the indicated interval with Absolute Taylor Error at most 0.00001?

a.) $\mathbf{f(x) = e^{-x}}$ for $\mathbf{a = 1}$ and $\mathbf{[0, 1]}$

a.) $\mathbf{f(x) = \frac{x+3}{x+1}}$ for $\mathbf{a = 0}$ and $\mathbf{[0, 1/2]}$

10. (Challenging) Use shortcuts to find the first three nonzero terms in the Taylor Series centered at $\mathbf{x = -1}$ for $\mathbf{f(x) = \frac{x}{3-x}}$.

Using some algebra we can write $f(x)$ in a form we are familiar with, such as

$$f(x) = \frac{x}{3-x} = \frac{\frac{x}{3}}{1 - \frac{x}{3}}$$

Since we know the Taylor series of $\frac{1}{1-x}$ pretty easily and we can get $f(x)$ as

$$f(x) = \frac{x}{3}\left(\frac{1}{1-\frac{x}{3}}\right) = \frac{x}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n$$

However, this is centered at $x = 0$ and we want this at $x = -1$. We can't just substitute x for $x + 1$, because the Taylor series centered at $x = 0$ is given by

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}(x)^n$$

and centered at $x = -1$ it is

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(-1)}{n!}(x+1)^n$$

So we need to adjust the derivative term. Note that the first few derivatives of $\frac{1}{1-x/3} = 3\frac{1}{3-x}$ are given by

n	$f^{(n)}(x)$	$x = 0$	$x = -1$
0	$3\frac{1}{3-x}$	1	$3[\frac{1}{4}]$
1	$3\frac{1}{(3-x)^2}$	1	$3[\frac{1}{4^2}]$
2	$3\frac{2}{(3-x)^3}$	2	$3[2\frac{1}{4^3}]$
3	$3\frac{2(3)}{(3-x)^4}$	$2(3)$	$3[2(3)\frac{1}{4^4}]$
4	$3\frac{2(3)(4)}{(3-x)^4}$	$2(3)(4)$	$3[2(3)(4)\frac{1}{4^5}]$
$\vdots$			
n	$\frac{n!}{(1-x)^{n+1}}$	$n!$	$3n!\frac{1}{4^{n+1}}$

Thus, we need to adjust the $f^{(n)}$ term by $3\frac{1}{4^{n+1}}$. Therefore we have

$$f(x) = x\sum_{n=0}^{\infty}3\frac{1}{4^{n+1}}(x+1)^n = \sum_{n=0}^{\infty}3\left(\frac{x+1}{4}\right)^{n+1}$$