Math 21C
Kouba
Disussion Sheet 2 - Key
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Tests

1. nth term test (Divergence test) If the a_n does not converge to 0 then the series must diverge

2. Geometric series One of our only solvable series, converges to $1/(1-r)$ for $\sum_0^n$.

3. p-series test Converges for $p < 1$.

4. Comaprison test Converges if you find a b_n such that $a_n \leq b_n \; \forall \; n$.

5. Limit comparison test Converges if you find a b_n such that $\lim a_n/b_n = c$ for finite $c > 0$ then $\sum a_n$ behaves like $\sum b_n$.

6. Ratio test For a series $\sum a_n$ we have

$$L := \lim \left| \frac{a_{n+1}}{a_n} \right|$$

Then if $L < 1 \implies$ Convergent absolutely, if $L > 1$ then divergent, if $L = 1$ then unknown.

7. Root Test For a series $\sum a_n$ define

$$L := \lim \sqrt[n]{|a_n|} = \lim |a_n|^1 /n$$

Then if $L < 1 \implies$ Convergent absolutely, if $L > 1$ then divergent, if $L = 1$ then unknown.

1.) Determine convergence or divergence of each series using the test indicated. I suggest that your read all of the assumptions and conclusions for each test as you do each problem.

a.) $\sum_{\mathbf{n=3}}^{\infty} \frac{\mathbf{2n+3}}{\mathbf{3n+2}}$ **(Use the nth term test.)**

Let $a_n := \frac{2n+3}{3n+2}$ then we obtain

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n+3}{3n+2} = \lim_{n\to\infty} \frac{\frac{2n}{n}+\frac{3}{n}}{\frac{3n}{n}+\frac{2}{n}} = \frac{2+0}{3+0} = \frac{2}{3} \neq 0$$

Therefore, since the terms in ther series $\sum_{n=3}^{\infty} a_n$ do not approach 0, we conclude this series diverges by the nth term test.

b.) $\sum_{\mathbf{n=4}}^{\infty} \mathbf{7}\frac{\mathbf{(-2)^{n+1}}}{\mathbf{3^{n-1}}}$ **(Use the geometric series test.)**

With the geometric series test we must get our series into the form of $\sum (r)^n$ for $|r| < 1$. So, for the given series we obtain

$$\sum_{n=4}^{\infty} 7\frac{(-2)^{n+1}}{3^{n-1}} = \sum_{n=4}^{\infty} 7\frac{(-2)^{n-1}(-2)^2}{3^{n-1}} = \sum_{n=4}^{\infty} 7(-2)^2 \left(\frac{-2}{3}\right)^{n-1} = 28\sum_{n=4}^{\infty} \left(-\frac{2}{3}\right)^{n-1}$$

Thus since $|-2/3| = 2/3 < 1$ we conclude that this series converges by the geometric series test.

c.) $\sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{1}}{\mathbf{n^{\sqrt{2}}}}$ **(Use the p-series test.)**

By the p-series test we have the series $\sum \frac{1}{n^p}$ converge if $p > 1$. For our series we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}} \approx \sum_{n=1}^{\infty} \frac{1}{n^{1.4\cdots}}$$

Thus, since $\sqrt{2} \approx 1.4... > 1$, we conclude that our series converges by the p-series test with $p = \sqrt{2} > 1$.

d.) $\sum_{\mathbf{n=2}}^{\infty} \frac{\mathbf{n}}{\mathbf{n^2+4}}$ **(Use the integral test.)**

The integral test states that for $a_n = f(n)$ we have

$$\int_k^{\infty} f(x)\ dx \leq \sum_{n=k}^{\infty} a_n$$

Thus define $f(x) = \frac{x}{x^2+4}$ and we obtain

$$\int_2^{\infty} f(x)\ dx = \int_2^{\infty} \frac{x}{x^2+4}\ dx = \int_2^{\infty} \frac{1}{2}\frac{1}{x^2+4}\ 2xdx = \int_4^{\infty} \frac{1}{2}\frac{1}{u}\ du = \frac{1}{2}\ \ln u|_{u=4}^{\infty} \to \infty$$

Thus, by the integral test, since $\int f(x)\ dx$ diverges, we conclude that $\sum a_n$ must diverge as well.

e.) $\sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{1}}{\sqrt{\mathbf{n+1}}} - \frac{\mathbf{1}}{\sqrt{\mathbf{n+2}}}$ **(Use the sequence of partial sums test.)**

Let $a_n := \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n+2}}$ and define the nth partial sum, s_n, as $s_n := \sum_{k=1}^{n} a_k$. Then the first few partial sums are given by

$$s_1 = \sum_{k=1}^{1} a_k = \frac{1}{\sqrt{1+1}} - \frac{1}{\sqrt{1+2}} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}$$
$$s_2 = \sum_{k=1}^{2} a_k = \frac{1}{\sqrt{1+1}} - \cancel{\frac{1}{\sqrt{1+2}}} + \cancel{\frac{1}{\sqrt{2+1}}} - \frac{1}{\sqrt{2+2}} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{4}}$$
$$s_3 = \sum_{k=1}^{3} a_k = \frac{1}{\sqrt{1+1}} - \cancel{\frac{1}{\sqrt{1+2}}} + \cancel{\frac{1}{\sqrt{2+1}}} - \cancel{\frac{1}{\sqrt{2+2}}} + \cancel{\frac{1}{\sqrt{2+2}}} - \frac{1}{\sqrt{3+2}} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{5}}$$
$$\vdots$$

Thus we see that the pattern for the nth partial sum is

$$\sum_{k=1}^{n} a_k = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{n+2}}$$

Thus, taking the limit as n approaches infinity we obtain,

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{n+2}} = \frac{1}{\sqrt{2}}$$

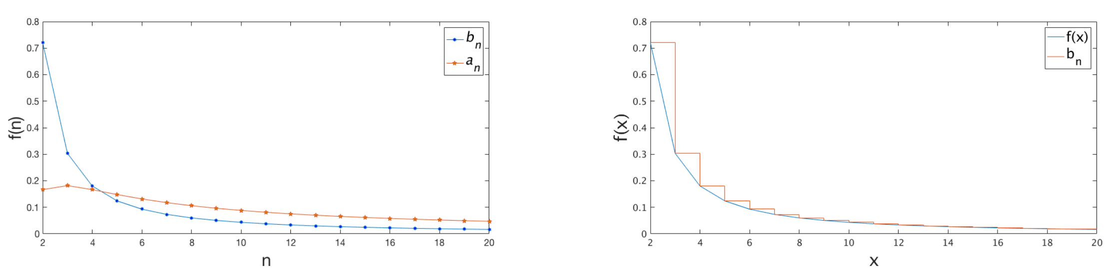
Therefore we conclude that the series converges to $1/\sqrt{2}$, so it clearly converges.

f.) $\sum_{\mathbf{n=2}}^{\infty} \frac{\mathbf{n-1}}{\mathbf{n^3+2}}$ **(Use the comparison test.)**

Let $a_n := \frac{n-1}{n^2+2}$, notice that for large n we have a_n similar to the function

$$\frac{n - \cancelto{0}{1}}{n^2 + \cancelto{0}{2}} \sim \frac{n}{n^2} = \frac{1}{n}$$

So we can predict that this will likely diverge because $\sum \frac{1}{n}$ diverges. One may be tempted to just use this function as a comparison, but a_n is clearly less than $1/n$ because the numerator is smaller and the denominator is larger. Since $1/n$ diverges, saying a_n is smaller tells us nothing. So instead, let $b_n := \frac{1}{n \ln n}$ and then we notice that for $n > 4$ we have $a_n > b_n$. Moreover, we have for $f(x) = \frac{1}{x \ln x}$ that $\int_1^{\infty} f(x)\, dx \leq \sum_{n=1}^{\infty} b_n$. This is more clear in the images below, thus we say that $a_n > b_n$ for $n > 4$, and $\sum b_n$ diverges by the integral test, so $\sum a_n$ diverges by the comparison test.



1. Visual representation of the results of problem f

g.) $\sum_{n=1}^{\infty} \frac{n^3+7n^2-3}{n^4-4n+9}$ **(Use the limit comparison test.)**

When looking at this sequence, we are really only concerned the largest power of n in both the numerator and the denomination. So we notice that

$$\frac{n^3+\cancel{7n^2}-\cancel{3}}{n^4-\cancel{4n}+\cancel{9}} \sim \frac{n^3}{n^4} = \frac{1}{n}$$

So this means we will likely want to use $b_n := 1/n$ for our comparison sequence. This is different than in f) because in the limit comparison test we are simply saying that one function is like the other, so we need not be concerned if one is greater or less than the other. So, let $a_n := \frac{n^3+7n^2-3}{n^4-4n+9}$, and by the Limit Comparison Test we obtain

$$\left|\frac{a_n}{b_n}\right| = \left|\frac{\frac{n^3+7n^2-3}{n^4-4n+9}}{\frac{1}{n}}\right| = \left|\left(\frac{n^3+7n^2-3}{n^4-4n+9}\right)\cdot n\right| = \left|\frac{n^4+7n^3-3n}{n^4-4n+9}\right|$$

Thus

$$\lim_{n\to\infty}\left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty}\left|\frac{n^4+7n^3-3n}{n^4-4n+9}\right| = \lim_{n\to\infty}\left|\frac{\frac{n^4}{n^4}+7\cancelto{0}{\frac{n^3}{n^4}}-3\cancelto{0}{\frac{n}{n^4}}}{\frac{n^4}{n^4}-4\cancelto{0}{\frac{n}{n^4}}+9\cancelto{0}{\frac{1}{n^4}}}\right| = 1$$

Therefore $\sum a_n$ has the same convergence behavior as $\sum b_n$. Since $\sum b_n$ is the harmonic series and diverges, by p-series test ($p \leq 1$), we conclude that $\sum a_n$ must diverge as well.

h.) $\sum_{n=0}^{\infty} \frac{3^{n-1}}{(n+1)!}$ **(Use the ratio test.)**

Let $a_n := \frac{3^{n-1}}{(n+1)!}$, then the ratio of a_{n+1} and a_n is given by

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{3^n}{(n+2)!}}{\frac{3^{n-1}}{(n+1)!}}\right| = \left|\left(\frac{3^n}{(n+2)!}\right)\left(\frac{(n+1)!}{3^{n-1}}\right)\right| = \left|\left(\frac{3\cdot\cancel{3^{n-1}}}{(n+2)\cancel{(n+1)!}}\right)\left(\frac{\cancel{(n+1)!}}{\cancel{3^{n-1}}}\right)\right| = \left|\frac{3}{n+2}\right|$$

Taking the limit as n goes to infinity gives us,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3}{n+2}\right| = 0$$

Thus, by the Ratio Test, we conclude that $\sum a_n$ converges.

i.) $\sum_{n=1}^{\infty}\left(1.01-\frac{5}{n^3}\right)^n$ **(Use the root test.)**

Let $a_n := \left(1.01-\frac{5}{n^3}\right)^n$ then we obtain

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left|\left(1.01-\frac{5}{n^3}\right)^n\right|} = \sqrt[n]{\left|\left(1.01-\frac{5}{n^3}\right)\right|^n} = \left|1.01-\frac{5}{n^3}\right|$$

Thus, taking the limit as n goes to infinity, we obtain

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left|1.01-\frac{5}{n^3}\right| = 1.01 > 1$$

Therefore, by the Root Test, the series $\sum a_n$ diverges.

2. Use any test to determine convergence or divergence of each series

a.) $\sum_{\mathbf{n=1}}^{\infty} \cos\left(\frac{\mathbf{1}}{\mathbf{n^2}}\right)$

Diverges by the nth term test because

$$\lim_{n\to\infty} \cos\left(\frac{1}{n^2}\right) = \cos\left(\lim_{n\to\infty}\frac{1}{n^2}\right) = \cos(0) = 1 \neq 0$$

b.) $\sum_{\mathbf{n=1}}^{\infty} \sin\left(\frac{\mathbf{1}}{\mathbf{n^2}}\right)$

Note that nth term test is inconclusive because $\sin(0) = 0$. Instead, let's use the comparison test. Note that $1/n^2 \in [0,1]$ when $n \in [1,\infty)$, and $\sin(x) \in [0,1]$ when $x \in [0,1]$, so we are going to compare against $\sum 1/n^2$. Notice that

$$0 \leq \sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right) \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Since $\sum 1/n^2$ converges by the p-series test ($p = 2 > 1$) we conclude that $\sum \sin(1/n^2)$ must also converge.

c.) $\sum_{\mathbf{n=2}}^{\infty} \frac{\mathbf{1}}{\mathbf{n(\ln n)^2}}$

The only way I know how to handle logarithms in series is with the integral test, so that's what we will use. Notice that $1/n$ is positive decreasing and $1/(\ln n)^2$ is also positive decreasing, so $1/n(\ln n)^2$ must also be positive decreasing. Hence

$$\int_2^{\infty} \frac{1}{x\ln(x)^2}\, dx \leq \sum_{n=2}^{\infty} \frac{1}{n\ln(n)^2}$$

Moreover, let $u = \ln x$, then $du = \frac{1}{x}dx$, and we obtain

$$\int_2^{\infty} \frac{1}{x\ln(x)^2} = \int_{\ln 2}^{\ln\infty} \frac{1}{u^2}\, du = -\frac{1}{u}\bigg|_{u=\ln 2}^{\infty} = \frac{1}{\ln 2}$$

So we have shown that $\sum \frac{1}{n\ln(n)^2} > \frac{1}{\ln 2}$, which gives us nothing. I'm stuck now so let me think about this some more.

d.) $\sum_{\mathbf{n=1}}^{\infty} \mathbf{3}\left(\mathbf{2^{-n}}\right)$

This sort of looks geometric, but it is not quite in the right form. So, using some algebra, we obtain

$$3(2^{-n}) = 3\left(\frac{1}{2^n}\right) = 3\left(\frac{1}{2}\right)^n$$

Thus this converges by the Geometric Series Test since $|r| = \left|\frac{1}{2}\right| < 1$.

e.) $\sum_{\mathbf{n=1}}^{\infty} \mathbf{2}\left(\mathbf{n^{-3}}\right)$

One may be tempted to say that this diverges by p-series because $p = -3 \not> 1$. However, this series is in the wrong form, notice that

$$\sum 2\left(n^-3\right) = \sum \frac{2}{n^3}$$

which converges by the p-series test $p = 3 > 1$.

f.) $\sum_{\mathbf{n=0}}^{\infty} \sqrt{\frac{\mathbf{n+3}}{\mathbf{n^3+8}}}$

Notice that for large n we have

$$\sqrt{\frac{n+\cancel{3}}{n^3+\cancel{8}}} \sim \sqrt{\frac{n}{n^3}} = \frac{1}{n}$$

Thus, a limit comparison test with $b_n := 1/n$ will likely work. Let $a_n := \sqrt{\frac{n+3}{n^3+8}}$ then we have

$$\lim_{n\to\infty}\left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{\frac{n+3}{n^3+8}}}{\frac{1}{n}}\right| = \lim_{n\to\infty}\left|\sqrt{\left(\frac{n+3}{n^3+8}\right)}\left(\frac{n}{1}\right)\right| = \lim_{n\to\infty}\left|\sqrt{\frac{n^3+3n^3}{n^3+8}}\right| = 1$$

Thus since $1 \in (0,\infty)$ we concldue that $\sum a_n$ converges by the Limit Comparison Test.

g.) $\sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{2^n+3^n}}{\mathbf{5^n-2^n}}$

Let $a_n := \frac{2^n+3^n}{5^n-2^n}$ then we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{2^{n+1}+3^{n+1}}{5^{n+1}-2^{n+1}}}{\frac{2^n+3^n}{5^n-2^n}}\right| = \left|\left(\frac{2^{n+1}+3^{n+1}}{5^{n+1}-2^{n+1}}\right)\left(\frac{5^n-2^n}{2^n+3^n}\right)\right|$$

So taking the limit as $n \to \infty$ we obtain

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\left(\frac{2^{n+1}+3^{n+1}}{5^{n+1}-2^{n+1}}\right)\left(\frac{5^n-2^n}{2^n+3^n}\right)\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}5^n+5^n3^{n+1}-2^n2^{n+1}-3^{n+1}2^n}{2^n5^{n+1}-2^n2^{n+1}+3^n5^{n+1}-3^n2^{n+1}}\right| = \frac{3}{5}$$

Note, the limit is $3/5$ because the largest terms on the top and bottom are $3^{n+1}5^n$ and 3^n5^{n+1} respectively. Thus, by the ratio test, this series converges.

h.) $\sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{1}}{\mathbf{1+2+3+\cdots+n}}$

For this series, let's use the fact that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Then we obtain,

$$\sum_{n=1}^{\infty}\frac{1}{1+2+3+\cdots+n} = \sum_{n=1}^{\infty}\frac{1}{\sum_{i=1}^{n} i} = \sum_{n=1}^{\infty}\frac{1}{\frac{n(n+1)}{2}} = \sum_{n=1}^{\infty}\frac{2}{n(n+1)} = 2\sum_{n=1}^{\infty}\frac{1}{n^2+n}$$

Moreover, since $n^2+n \geq n^2$ for any $n > 0$, this series converges by the comparison test against $\sum \frac{1}{n^2}$ where $\sum \frac{1}{n^2}$ converges by the p-series test ($p = 2 > 1$).

i.) $\sum_{\mathbf{n=3}}^{\infty} \frac{\mathbf{1}}{\ln \mathbf{n}}$

Notice that $\ln n < n$ for any $n \geq 3$. So we use the comparison test as

$$\sum_{n=3}^{\infty}\frac{1}{n} \leq \sum_{n=3}^{\infty}\frac{1}{\ln n}$$

Thus, since $\sum 1/n$ diverges, our series must diverge by the Comparison Test.

j.) $\sum_{\mathbf{n=2}}^{\infty} \frac{\ln \mathbf{n}}{\mathbf{n^3}}$

Note that for $n > 0$, $\ln n < n$ so we obtain

$$\sum_{n=2}^{\infty} \frac{\ln n}{n^3} \leq \sum_{n=2}^{\infty} \frac{n}{n^3} = \sum_{n=2}^{\infty} \frac{1}{n^2}$$

Since $\sum 1/n^2$ converges by the p-series test ($p = 2 > 1$) we conclude that our series converges by the Comparison Test.

k.) $\sum_{\mathbf{n=3}}^{\infty} \frac{\mathbf{1}}{\mathbf{(n+1/n)^n}}$

Let $a_n := 1/(n + 1/n)^n$, then we obtain

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{(n+1/n)^n}\right|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{1}{n+1/n}\right)^n\right|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{1}{n+1/n}\right)\right|^n} = \lim_{n\to\infty} \left|\left(\frac{1}{n+1/n}\right)\right| = 0$$

Thus, by the Root Test, since $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$ we conclude that our series converges.

l.) $\sum_{\mathbf{n=2}}^{\infty} \frac{\mathbf{(2n)!}}{\mathbf{n^2+100}}$

This series has factorials, so Ratio Test is probably the best bet. Thus let $a_n := \frac{(2n)!}{n^2+100}$ and we obtain

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left| \frac{\frac{(2(n+1))!}{(n+1)^2+100}}{\frac{(2n)!}{n^2+100}} \right| \\
&= \lim_{n\to\infty} \left| \left(\frac{(2n+2)!}{(n+1)^2+100}\right)\left(\frac{n^2+100}{(2n)!}\right) \right| \\
&= \lim_{n\to\infty} \left| \left(\frac{((2n+2)(2n+1)\cancel{(2n)!}}{n^2+2n+101}\right)\left(\frac{n^2+100}{\cancel{(2n)!}}\right) \right| \\
&= \lim_{n\to\infty} \left| \frac{(4n^2+6n+2)(n^2+100)}{(n^2+2n+101)} \right| \to \infty
\end{aligned}$$

Thus, we conclude that this series diverges by the Ratio Test. Note, the nth term test would also work since $n! >> n^2$ and the limit would be ∞.

3. Consider the series $\sum_{\mathbf{n=3}}^{\infty} \frac{\mathbf{1}}{\mathbf{4n^2-1}}$

a.) Use the limit comparison test to show that the series converges.

b.) **Use partial fractions then the sequence of partial sums to find the exact value of the series**

Using partial fraction decomposition we obtain,

$$\frac{1}{4n^2-1} = \frac{1}{(2n-1)(2n+1)} = \frac{1/2}{2n-1} - \frac{1/2}{2n+1}$$

Then the sequence of partial sums is given by

$$\begin{aligned}
S_3 &= \sum_{n=3}^{3} \frac{1/2}{2n-1} - \frac{1/2}{2n+1} = \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) \\
S_4 &= \sum_{n=3}^{4} \frac{1/2}{2n-1} - \frac{1/2}{2n+1} = \frac{1}{2}\left(\frac{1}{5} - \cancel{\frac{1}{7}} + \cancel{\frac{1}{7}} - \frac{1}{9}\right) \\
S_5 &= \sum_{n=3}^{5} \frac{1/2}{2n-1} - \frac{1/2}{2n+1} = \frac{1}{2}\left(\frac{1}{5} - \cancel{\frac{1}{7}} + \cancel{\frac{1}{7}} - \cancel{\frac{1}{9}} + \cancel{\frac{1}{9}} - \frac{1}{11}\right) \\
&\vdots \\
S_n &= \sum_{k=3}^{k} \frac{1/2}{2n-1} - \frac{1/2}{2n+1} = \frac{1}{2}\left(\frac{1}{5} - \frac{1}{2n+1}\right)
\end{aligned}$$

Thus, $S = \lim_{n\to\infty} S_n = \frac{1}{10}$.

4. Find the exact value of the following convergent series:

$$\mathbf{\frac{3^{-3}}{10^1} - \frac{3^{-1}}{10^2} + \frac{3^1}{10^3} - \frac{3^3}{10^4} + \frac{3^5}{10^5} - \frac{3^7}{10^6}} + \cdots$$

This series can be written as

$$\sum_{n=1}^{\infty} (-1)^n \frac{3^{2n-5}}{10^n}$$

Since the only series we really know how to solve are geometric and telescoping, we need to get this into one of those forms. Generally, telescoping series would arise from a partial fractions problem, so this is likely a geometric series. Notice the following transformation of our series,

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^{2n-5}}{10^n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^{2n} \cdot 3^{-5}}{10^n} = \sum_{n=1}^{\infty} 3^{-5}(-1)(-1)^n \frac{(3^2)^n}{10^n} = -3^{-5} \sum_{n=1}^{\infty} \left(-\frac{9}{10}\right)^n$$

Thus, we have a geometric series multiplied by some value. Following the rule for geometric series we see that the sum is

$$-3^{-5} \sum_{n=1}^{\infty} \left(-\frac{9}{10}\right)^n = -3^{-5} \frac{-9/10}{1-(-9/10)} = 3^{-5}\left(\frac{90}{19}\right) = 0.0195$$

Bonus: Why must the we explicitly state that this is a convergent series?
Find out in the intro here: Divergent Series

5. Use a geometric series to convert the decimal number 0.25252525... to a fraction

Notice that we we can express this fraction as

$$0.25 + 0.0025 + 0.000025 + \cdots$$

As a series, this is given by

$$\sum_{n=1}^{\infty} \frac{25}{10^{2n}} = 25 \sum_{n=1}^{\infty} \left(\frac{1}{100}\right)^n = 25\left(\frac{1/100}{1-\frac{1}{100}}\right) = \frac{25}{100}\,\frac{100}{99} = 0.2525252...$$

Thus we see this series is equivalent to the provided decimal.

6. The series $\sum \frac{1}{n}$ diverges.

a.) Use equation (*) to determine between which two numbers the partial sum S_{50} lies.

The equation (∗) is given by

$$\int_1^{n+1} f(x)\ dx < S_n < f(1) + \int_1^n f(x)\ dx$$

So for $n = 50$ we obtain,

$$\int_1^5 1\frac{1}{x}\ dx = \ln(51) < S_{50} < \frac{1}{1} + \int_1^5 0\frac{1}{x}\ dx = \ln(50)$$

Thus $S_{50} \in (\ln 51, 1 + \ln 50) \approx (3.932, 4.912)$.

b.) What should n be in order that the partial sum S_n be at least 20?

Since, by (∗), the lower bound is $\int_1^{n+1} f(x)\ dx$, so we need to find n such that $\ln(n+1) > 20$. Thus we obtain

$$\ln(n+1) > 20 \iff n+1 > e^{20} \iff n > e^{20} - 1$$

c.) What is the largest value of n for which the partial sum n for which the partial sum S_n does not exceed 50?

Using the other side of (∗) we want to find n such that

$$\frac{1}{1} + \ln n < 50 \iff n < e^{49}$$

Thus n must be less than 49 so that $S_n < 50$.

7. The series $\sum \frac{1}{n^3}$

a.) Compute the partial sum S_5. Use (*)(*) to estimate the error

Notice that the partioal sum S_5 is given as

$$S_5 = \sum_{n=1}^{5} \frac{1}{n^3} = \frac{1}{1} + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} \approx 1.187$$

We have equation (∗)(∗) defined as,

$$\int_{n+1}^{\infty} f(x)\ dx < \sum_{k=n+1}^{\infty} f(k) < \int_n^{\infty} f(x)\ dx$$

So, the resulting error of S_5 is given by $S - S_5 = \sum_{n=6}^{\infty} \frac{1}{n^3}$, thus for $f(x) = \frac{1}{x^3}$ we obtain

$$\int_6^{\infty} f(x)\ dx = \frac{1}{2}\left(\frac{1}{6^2}\right) \approx 0.0139 < S - S_5 < \int_5^{\infty} f(x)\ dx = \frac{1}{2}\frac{1}{5^2} = 0.02$$

Thus we conclude that $\sum \frac{1}{n^3} \approx 1.187 \pm 0.02$.

b.) What should n be for the partial sum S_n to estimate the series with error at most 0.0001?

Following (*)(*) we have

$$S - S_n < \int_n^{\infty} f(x)\ dx$$

So we must find n such that

$$\int_n^{\infty} f(x)\ dx = \int_n^{\infty} \frac{1}{n^3}\ dx = \frac{1}{2}\left(\frac{1}{n^2}\right) < 0.0001$$

Solving for n we obtain,

$$n > \sqrt{\frac{1}{0.0001}} = \sqrt{10000} = 100$$

Thus n must be more than 100 for the error to be 0.0001.