1. Evaluate the following limits or determine that the limit does not exist

a.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{0},\mathbf{0})} \frac{\mathbf{x^2+y^2-4}}{\mathbf{x+y+2}}$

We can simply plug in $(x, y) = (0, 0)$ and we obtain

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2-4}{x+y+2} = \frac{-4}{2} = -2$$

b.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{1},\mathbf{1})} \frac{\mathbf{xy-y-2x+2}}{\mathbf{x-1}}$

Unlike in **a.)**, if we simply plug in (1,1) for (x, y) we get an undefined value. This is because of the $x-1$ in the denominator. Thus, we need to remove this discontinuity. To do this we need to factor the top. Notice that we have,

$$xy - y - 2x + 2 = (xy - 2x) - (y - 2) = x(y - 2) - (y - 2) = (y - 2)(x - 1)$$

Thus the limit becomes

$$\lim_{(x,y)\to(1,1)} \frac{xy-y-2x+2}{x-1} = \lim_{(x,y)\to(1,1)} \frac{(y-2)(x-1)}{x-1} = \lim_{(x,y)\to(1,1)} y - 2 = -1$$

c.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{2},\mathbf{2})} \frac{\mathbf{x+y-4}}{\sqrt{\mathbf{x+y}}-\mathbf{2}}$

Let $r = \sqrt{x+y}$, then notice that we have

$$\frac{x+y-4}{\sqrt{x+y}-2} = \frac{r^2-2^2}{r-2} = \frac{(r-2)(r+2)}{r-2} = r + 2 = \sqrt{x+y} + 2$$

Thus we can evaluate the limit as

$$\lim_{(x,y)\to(2,2)} \frac{x+y-4}{\sqrt{x+y}-2} = \lim_{(x,y)\to(2,2)} \sqrt{x+y} + 2 = \sqrt{4} + 2 = 4$$

d.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{0},\mathbf{0})} \frac{\sin(\mathbf{x^2+y^2})}{\mathbf{x^2+y^2}}$

Let $r = x^2 + y^2$, then we obtain

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0} \frac{\sin(r)}{r} = {}^{1}1$$

e.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{1},\mathbf{1})} \frac{\sin(\mathbf{x^2-y^2})}{\mathbf{x-y}}$

Here we can't use the same trick as before. However, we see that we have $x^2 - y^2 = (x - y)(x + y)$ in the numerator, which could cancel the denominator if we didn't have the sine. Note that we could use a Taylor series trick here. Recall that

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots$$

[1]Note: This limit can be easily found now that you know Taylor series. Just let $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$, then $\sin(x)/x$ as $x \to 0$ is clearly 1

so we obtain

$$\sin(x^2-y^2)=\sin((x-y)(x+y))=[(x-y)(x+y)]-\frac{[(x-y)(x+y)]^3}{3!}+\frac{[(x-y)(x+y)]^5}{5!}+\cdots$$

and so our limit is

$$\begin{aligned}\frac{\sin(x^2-y^2)}{x-y}&=\frac{(x-y)(x+y)}{x-y}-\frac{[(x-y)(x+y)]^3}{3!(x-y)}+\frac{[(x-y)(x+y)]^5}{5!(x-y)}+\cdots\\&=[(x+y)]-\frac{(x-y)^2(x+y)^3}{3!}+\frac{(x-y)^4(x+y)^5}{5!}\end{aligned}$$

and taking the limit we see that the only non-zero term is the first term, thus

$$\lim_{(x,y)\to(1,1)}\frac{\sin(x^2-y^2)}{x-y}=\lim_{(x,y)\to(1,1)}x+y=2$$

f.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{1},-\mathbf{1})}\arcsin\left(\frac{\mathbf{xy}}{\sqrt{\mathbf{x^2+y^2}}}\right)$

We can just plug into this one

$$\lim_{(x,y)\to(1,-1)}\arcsin\left(\frac{xy}{\sqrt{x^2+y^2}}\right)=\arcsin\left(\frac{-1}{\sqrt{2}}\right)=\arcsin\left(\frac{-\sqrt{2}}{2}\right)=-\frac{\pi}{4}$$

g.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{0},\mathbf{0})}\frac{\mathbf{x^3}}{\mathbf{x^4+y^3}}$

I'm guessing this doesn't exist because the function $1/x$ doesn't exist as $x\to 0$. Thus we need to approach the limit from two directions that yield different values. Consider the case when $y=0$, then from $x\to 0^+$ we have

$$\lim_{x\to 0^+}\frac{x^3}{x^4}=\lim_{x\to 0^+}\frac{1}{x}\to\infty$$

and from the left we have

$$\lim_{x\to 0^-}\frac{x^3}{x^4}=\lim_{x\to 0^-}\frac{1}{x}\to-\infty$$

thus, that is enough to show the limit does not exist.

h.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{0},\mathbf{0})}\frac{\mathbf{xy}}{\mathbf{x^2+y^2}}$

Similar to above, consider the direction when $x=y$, then we have

$$\lim_{y\to 0}\frac{y^2}{2y^2}=\frac{1}{2}$$

and when $x=0$, we have

$$\lim_{y\to 0}\frac{0}{y^2}=0$$

thus the limit does not exist.

i.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{0},\mathbf{0})}\frac{\mathbf{xy^2}}{\mathbf{x^2+y^4}}$

Same as above, but let's compare $x=y^2$ with $x=0$.

$x = y^2$:

$$\lim_{y \to 0} \frac{y^4}{2y^4} = \frac{1}{2}$$

$x = 0$:

$$\lim_{y \to 0} \frac{0}{y^4} = 0$$

thus the limit does not exist.

j.) $\lim_{(\mathbf{x},\mathbf{y}) \to (\mathbf{2},-\mathbf{2})} \frac{\mathbf{4-xy}}{\mathbf{4+xy}}$

$x = y$:

$$\lim_{y \to -2} \frac{4 - y^2}{4 + y^2} = 0$$

$x = 0$:

$$\lim_{y \to -2} \frac{4}{4} = 1$$

thus the limit does not exist.

k.) $\lim_{(\mathbf{x},\mathbf{y}) \to (\mathbf{0},\mathbf{0})} \left(\mathbf{1 + 3xy^2}\right)^{\frac{\mathbf{2}}{\mathbf{xy^2}}}$

This looks like the goofy e^x limits of the past. Let $u = \frac{2}{xy^2}$, then we have

$$\left(1 + 3xy^2\right)^{2/xy^2} = \left(1 + \frac{3}{u/2}\right)^u = \left(1 + \frac{6}{u}\right)^u$$

Moreover, notice that

$$\lim_{(x,y) \to (0,0)} \frac{xy^2}{2} = \lim_{u \to \infty} \frac{2}{u}$$

thus we have

$$\lim_{(x,y) \to (0,0)} \left(1 + 3xy^2\right)^{\frac{2}{xy^2}} = \lim_{u \to \infty} \left(1 + \frac{6}{u}\right)^u = e^6$$

l.) $\lim_{(\mathbf{x},\mathbf{y}) \to (\mathbf{0},\mathbf{0})} \frac{\mathbf{xy^2}}{\mathbf{x^2+y^4}}$

Same as above, but let's compare $x = y^2$ with $x = 0$.

$x = y^2$:

$$\lim_{y \to 0} \frac{y^4}{2y^4} = \frac{1}{2}$$

$x = 0$:

$$\lim_{y\to 0} \frac{0}{y^4} = 0$$

thus the limit does not exist.

m.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{1},-\mathbf{2})} \frac{(\mathbf{x}-\mathbf{1})^2+\mathbf{3}(\mathbf{y}+\mathbf{2})^2}{\mathbf{x}-\mathbf{1}+(\mathbf{y}+\mathbf{2})^2}$

Let $y = -2$:

$$\lim_{x\to 1} \frac{(x-1)^2}{(x-1)} = \lim_{x\to 1} x - 1 = 0$$

Let $x = 1$:

$$\lim_{y\to -2} \frac{3(y+2)^2}{(y+2)^2} = \lim_{y\to -2} 3 = 3$$

Thus the limit does not exist.

n.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{1},\mathbf{2})} \frac{\mathbf{xy}+\mathbf{2x}-\mathbf{y}-\mathbf{2}}{\mathbf{xy}-\mathbf{y}+\mathbf{3x}-\mathbf{3}}$

2.) Compute $\mathbf{z_x}$ and $\mathbf{z_y}$ for each of the following functions

a.) $\mathbf{z} = \mathbf{xy^2} + \ln\mathbf{x} + \mathbf{e^y} + \mathbf{5}$

$$z_x = \frac{\partial z}{\partial x} = y^2 + \frac{1}{x}$$
$$z_y = \frac{\partial z}{\partial y} = 2xy + e^y$$

b.) $\mathbf{z} = \mathbf{xe^{2y}}\arctan\mathbf{x}$

$$z_x = \frac{\partial z}{\partial x} = e^2 y \arctan(x) + xe^{2y}\frac{1}{1+x^2}$$
$$z_y = \frac{\partial z}{\partial y} = 2x \arctan xe^{2y}$$

c.) $\mathbf{z} = \sqrt{\mathbf{x}-\mathbf{y^2}}$

$$z_x = \frac{\partial z}{\partial x} = \frac{1}{\sqrt{x-y^2}}$$
$$z_y = \frac{\partial z}{\partial y} = \frac{-2y}{\sqrt{x-y^2}}$$

d.) $\mathbf{z = \frac{x^3}{y^2} + \sin(xy)}$

$$z_x = \frac{\partial z}{\partial x} = \frac{3x^2}{y^2} + y\cos(xy)$$
$$z_y = \frac{\partial z}{\partial y} = -\left(\frac{x}{y}\right)^3 + x\cos(xy)$$

e.) $\mathbf{z = \frac{x+4}{x^2+y^2}}$

$$z_x = \frac{\partial z}{\partial x} = \frac{x^2+y^2-2x^2-8x}{(x^2+y^2)^2}$$
$$z_y = \frac{\partial z}{\partial y} = -\frac{2y(x+4)}{(x^2+y^2)^2}$$

f.) $\mathbf{z = \left[e^{x^2y} + \tan(3y+4x)\right]^5}$

$$z_x = \frac{\partial z}{\partial x} = 5\left[e^{x^2y} + \tan(3y+4x)\right]^4\left[2xye^{x^2y} + 4\sec(3y+4x)\right]$$
$$z_y = \frac{\partial z}{\partial y} = 5\left[e^{x^2y} + \tan(3y+4x)\right]^4\left[x^2e^{x^2y} + 3\sec(3y+4x)\right]$$

g.) $\mathbf{z = y^{1+x^3}}$

$$z_x = \frac{\partial z}{\partial x} = 3x^2y^{1+x^3}\ln y$$
$$z_y = \frac{\partial z}{\partial y} = (1+x^3)y^{x^3}$$

3.) Show that $\mathbf{z = \ln(1+x^2+y^2)}$ satisfies the equation $\mathbf{z_{xy} + z_xz_y = 0}$

We are going to show this by showing that $z_{xy} = -z_xz_y$. Let $z = \ln(1+x^2+y^2)$, then

$$z_{xy} = \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{2y}{1+x^2+y^2}\right) = \frac{-4xy}{(1+x^2+y^2)^2}$$
$$-z_xz_y = -\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) = -\left(\frac{2x}{1+x^2+y^2}\right)\left(\frac{2y}{1+x^2+y^2}\right) = \frac{-4xy}{(1+x^2+y^2)^2}$$

Thus we conclude that $z_{xy} + z_xz_y = 0$.

4. Verify that $\mathbf{w_{xy} = w_{yx}}$ for $\mathbf{w = y + \frac{x}{y}}$

Since this function is a combination of continuous functions it is also continuous. Thus this must be true. To verify, notice that

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(1 - \frac{x}{y^2}\right) = -\frac{1}{y^2}$$
$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{1}{y}\right) = -\frac{1}{y^2}$$

Thus $w_{xy} = w_{yx}$

5. Determine functions z whose partial derivatives are given, or state that this is impossible

a.) $\mathbf{z_x = 2x}$ and $\mathbf{z_y = 3y^2 + 1}$

$$\int z_x \ dx = x^2 + g(y) \int z_y \ dy \qquad = y^3 + y + f(x)$$

Thus we can define a function of the form

$$z = x^2 + y + y^3 + c$$

b.) $\mathbf{z_x = xy^2 - y}$ and $\mathbf{z_y = x^2y - x}$

$$\int z_x \ dx = \frac{1}{2}x^2y^2 - xy + g(y) \int z_y \ dy \qquad = \frac{1}{2}x^2y^2 - xy + f(x)$$

Thus

$$z = \frac{1}{2}(xy)^2 - xy + c$$

c.) $\mathbf{z_x = e^y - 1}$ and $\mathbf{z_x = e^x - x}$

$$\int z_x \ dx = xe^y - x + g(y) \int z_y \ dy \qquad = ye^x - xy + g(x)$$

This can't be solved because the the difference between $\int z_x \ dx$ and $\int z_y \ dy$ are not functions of purely x or y.

d.) $\mathbf{ye^x \cos(xy) + e^x \sin(xy) - 2}$ and $\mathbf{z_y = xe^x \cos(xy) + 1}$

$$\int z_x \ dx = \int z_y \ dy \qquad =$$

6. Consider the function

$$\mathbf{f(x,y)} = \begin{cases} \frac{\sin(x^3+y)}{x^2+y^2}, & \text{if}(x,y) \neq (0,0) \\ 0, & \text{if}(x,y) = (0,0) \end{cases}$$

a.) Determine $\mathbf{f_x(x,y)}$ when $\mathbf{(x,y) \neq (0,0)}$.

$$f_x(x,y) = \frac{3x^2\cos(x^3+y) - 2x\sin(x^3+y)}{(x^2+y^2)^2}$$

b.) Determine $\mathbf{f_x(0,0)}$ (Use limit definition of partial derivatives).

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{\sin(h^3)}{h^2}}{h} = \lim_{h\to 0} \frac{\sin(h^3)}{h^3} = 1$$

c.) Determine $\mathbf{f_y(0,0)}$ (Use limit definition of partial derivatives).

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{\sin(h)}{h^2}}{h} = \lim_{h\to 0} \frac{\sin(h)}{h}\frac{1}{h^2} = \text{DNE}$$