

1. Write each power series as an ordinary function.

a.) $\sum_{n=5}^{\infty} x^n$

Notice that we can re-write this as

$$\sum_{n=5}^{\infty} x^n = \sum_{n=0}^{\infty} x^n - \sum_{n=0}^4 x^n$$

Since we know the sum starting from 0, we get

$$\sum_{n=5}^{\infty} x^n = \frac{1}{1-x} - 1 - x - x^2 - x^3 - x^4; \quad x \in (-1, 1)$$

b.) $\sum_{n=0}^{\infty} 2^n x^n$

Notice that we can write $2^n x^n = (2x)^n$, then let $y = 2x$ and we obtain

$$\sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=2(0)}^{2(\infty)} y^n = \sum_{n=0}^{\infty} y^n = \frac{1}{1-y} = \frac{1}{1-2x}$$

Note that since $y \in (-1, 1)$ we have $2x \in (-1, 1)$ or $x \in (-1/2, 1/2)$.

c.) $\sum_{n=0}^{\infty} \frac{(-3)^{n+1} x^n}{5^{n-1}}$

We have to do some extensive algebra on this. Notice that we have

$$\frac{(-3)^{n+1} x^n}{5^{n-1}} = \frac{(-3)(-3)^n x^n}{5^{n-1}} = \frac{(5)(-3)(-3)^n x^n}{5^n} = -15 \left(\frac{(-3)^n x^n}{5^n} \right) = -15 \left(\frac{(-3x)^n}{5^n} \right) = -15 \left(\frac{-3x}{5} \right)^n$$

Thus we obtain

$$\sum_{n=0}^{\infty} \frac{(-3)^{n+1} x^n}{5^{n-1}} = \sum_{n=0}^{\infty} -15 \left(\frac{-3x}{5} \right)^n = -15 \sum_{n=0}^{\infty} \left(\frac{-3x}{5} \right)^n = -15 \left(\frac{1}{1 - \left(\frac{-3x}{5} \right)} \right) = -15 \left(\frac{1}{1 + \frac{3x}{5}} \right) = -15 \frac{5}{5 + 3x}$$

With radius of convergence $\frac{3x}{5} \in (-1, 1)$ which gives us $x \in (-5/3, 5/3)$.

d.) $\sum_{n=4}^{\infty} n x^{n-1}$

For this one we have to recognize that if this were $\sum x^{n-1}$ it would be relatively easy to fix. Notice that

$$\frac{d}{dx} x^n = n x^{n-1}$$

Moreover, we obtain

$$\frac{d}{dx} \left(\sum_{n=4}^{\infty} x^n \right) = \frac{d}{dx} (x^4 + x^5 + x^6 + \dots) = 4x^3 + 5x^4 + 6x^5 + \dots = \sum_{n=4}^{\infty} n x^{n-1}$$

Since we know the first sum above we obtain

$$\sum_{n=4}^{\infty} n x^{n-1} = \frac{d}{dx} \left(\sum_{n=4}^{\infty} x^n \right) = \frac{d}{dx} \left(\sum_{n=0}^{\infty} x^n - \sum_{n=0}^3 x^n \right) = \frac{d}{dx} \left(\frac{1}{1-x} - 1 - x - x^2 - x^3 \right) = \frac{1}{(1-x)^2} - 1 - 2x - 3x^2$$

where $x \in (-1, 1)$

e.) $\sum_{n=0}^{\infty} n^2 x^{n-1}$

Just like in d we recognize that we need one derivative to get nx^{n-1} , so a second derivative should get us close to n^2x^{n-1} . Notice that we have

$$x \frac{d}{dx} \left(\sum_{n=0}^{\infty} x^n \right) = \sum_{n=0}^{\infty} x \frac{d}{dx} x^n = \sum_{n=0}^{\infty} nx^n$$

Moreover, we have

$$\frac{d}{dx} \left(\sum_{n=0}^{\infty} nx^n \right) = \sum_{n=0}^{\infty} \frac{d}{dx} nx^n = \sum_{n=0}^{\infty} n^2 x^{n-1}$$

which gives us our target series. Thus, using geometric series, we obtain

$$\sum_{n=0}^{\infty} n^2 x^{n-1} = \frac{d}{dx} \left(\sum_{n=0}^{\infty} nx^n \right) = \frac{d}{dx} \left(x \frac{d}{dx} \left(\sum_{n=0}^{\infty} x^n \right) \right) = \frac{d}{dx} \left(x \frac{d}{dx} \left(\frac{1}{1-x} \right) \right) = \frac{d}{dx} \left(\frac{x}{(1-x)^2} \right) = \frac{(1-x)^2 - 2x(1-x)}{(1-x)^4}$$

where $x \in (-1, 1)$ because we used a geometric series.

f.) $\sum_{n=1}^{\infty} \frac{x^{n+3}}{n}$

Something to notice here is that

$$\frac{d}{dx} \left(\frac{x^n}{n} \right) = x^{n-1}$$

which is a geometric series that we can solve. This is backwards from what we did in e and d . Since the reverse of a derivative is an integral, we need to use an integral. Thus we obtain

$$\int \sum_{n=1}^{\infty} x^{n-1} dx = \int (1 + x + x^2 + x^3 + \dots) dx = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

Thus we can describe the series above as

$$\sum_{n=1}^{\infty} \frac{x^{n+3}}{n} = x^3 \int \sum_{n=1}^{\infty} x^{n-1} dx = x^3 \int \frac{1}{x} \left(\frac{1}{1-x} \right) dx = x^3 (\ln(|x|) - \ln(|1-x|)) = x^3 \ln \left(\left| \frac{x}{1-x} \right| \right)$$

where we used partial fractions to solve that integral. Note we used a geometric series so $x \in (-1, 1)$, moreover, $x \neq 0$ since $\ln 0$ doesn't exist.

g.) $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{2^n n!}$

Notice we obtain

$$\frac{x^n}{2^n n!} = \frac{\left(\frac{x}{2}\right)^n}{n!}$$

So what we can do is recognize that since

$$e^x = \sum_{n=1}^{\infty} \left(\frac{x^n}{n!} \right)$$

then we obtain

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{2^n n!} = \sum_{n=1}^{\infty} \left(\frac{(-x/2)^n}{n!} \right) = e^{-x/2}$$

Where $x \in \mathbb{R}$ because $\sum \frac{x^n}{n!}$ is valid everywhere.

h.) $\sum_{n=2}^{\infty} (-1)^n \left(\frac{2}{5}\right)^n \frac{x^{2n+1}}{2n+1}$

Notice that $\int x^{2n} dx = \frac{x^{2n+1}}{2n+1}$, so we will need to use that at some point in this problem. Using some algebra we obtain,

$$(-1)^n \left(\frac{2}{5}\right)^n \frac{x^{2n+1}}{2n+1} = (-1)^n \left(\frac{2}{5}\right)^n \int x^{2n} dx = \int (-1)^n \left(\frac{2}{5}\right)^n x^{2n} dx = \int (-1)^n \left(\frac{2}{5}\right)^n (x^2)^n dx = \int \left(-\frac{2}{5}x^2\right)^n dx$$

Moreover, the series $\sum \left(-\frac{2}{5}x^2\right)^n$ is relatively easy to solve using geometric series theorems. Thus we obtain

$$\begin{aligned} \sum_{n=2}^{\infty} (-1)^n \left(\frac{2}{5}\right)^n \frac{x^{2n+1}}{2n+1} &= \int \sum_{n=2}^{\infty} \left(-\frac{2}{5}x^2\right)^n dx \\ &= \int \sum_{n=0}^{\infty} \left(-\frac{2}{5}x^2\right)^n - \sum_{n=0}^1 \left(-\frac{2}{5}x^2\right)^n dx \\ &= \int \frac{1}{1 - \left(-\frac{2}{5}x^2\right)} - 1 - \left(-\frac{2}{5}x^2\right) dx \\ &= \int \frac{1}{1 + \frac{2}{5}x^2} dx - x + \frac{2}{15}x^3 \\ &= \int \frac{1}{1 + \left(\sqrt{\frac{2}{5}}x\right)^2} dx - x + \frac{2}{15}x^3 \\ &= \sqrt{\frac{5}{2}} \arctan\left(\sqrt{\frac{2}{5}}x\right) - x + \frac{2}{15}x^3 \end{aligned}$$

Notice that we used a geometric series, so we need $\frac{2}{5}x^2 \in (-1, 1)$ or $x \in \left(-\sqrt{\frac{5}{2}}, \sqrt{\frac{5}{2}}\right)$.

2. Use any method to find the exact value of each of the following convergent series.

a.) $\sum_{n=0}^{\infty} 3 \left(\frac{-2}{3}\right)^n$

Let $f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, the target series is given by

$$\sum_{n=0}^{\infty} 3 \left(\frac{-2}{3}\right)^n = 3f\left(\frac{-2}{3}\right) = 3\left(\frac{1}{1 - \left(\frac{-2}{3}\right)}\right) = 3\left(\frac{3}{5}\right) = \frac{9}{5}$$

Note, in order to do this we needed $x \in (-1, 1)$.

b.) $\sum_{n=4}^{\infty} \frac{(-1)^{n+2} n^{-3}}{2}$

With this we need to get the same exponent on both terms. So notice that

$$\frac{(-1)^{n+2} n^{-3}}{2} = \frac{(-1)^n (-1)^2}{2^{n-3}} = \frac{(-1)^n (-1)^2 2^3}{2^n} = 2^3 \left(\frac{-1}{2}\right)^n$$

Thus we can solve this as a geometric series as

$$\sum_{n=4}^{\infty} \frac{(-1)^{n+2} n^{-3}}{2} = \sum_{n=4}^{\infty} 2^3 \left(\frac{-1}{2}\right)^n = 2^3 \left(\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n - \sum_{n=0}^3 \left(\frac{-1}{2}\right)^n \right) = 2^3 \left(\left(\frac{1}{1 - (-1/2)}\right) - 1 + \frac{1}{2} - \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 \right)$$

Thus the final answer is

$$\sum_{n=4}^{\infty} \frac{(-1)^{n+2} n^{-3}}{2} = \frac{2}{3} - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} = \frac{1}{24}$$

c.) $\sum_{n=1}^{\infty} n^2 \left(\frac{1}{2}\right)^n$

Note that suppose $f(x) = \sum_{n=1}^{\infty} n^2 x^n$ then $f(1/2)$ gives us the series above. From before we recognize that we need to take two derivatives of x^n to get $n^2 x^n$, thus we obtain

$$x \frac{d}{dx} \left(x \frac{d}{dx} x^n \right) = x \frac{d}{dx} (n x^n) = n^2 x^n$$

Thus we obtain

$$\sum_{n=1}^{\infty} n^2 x^n = x \frac{d}{dx} \left(x \frac{d}{dx} \sum_{n=1}^{\infty} x^n \right) = x \frac{d}{dx} \left(x \frac{d}{dx} \frac{1}{1-x} - 1 \right) = x \frac{d}{dx} \left(\frac{x}{(1-x)^2} \right) = \frac{x(1-3x)}{(1-x)^3}$$

since our series is $f(1/2)$ we obtain

$$\sum_{n=1}^{\infty} n^2 \left(\frac{1}{2}\right)^n = f(1/2) = \frac{\frac{1}{2}(1-3(\frac{1}{2}))}{(1-(\frac{1}{2}))^3} = -2$$

This looks wrong to me

d.) $\sum_{n=0}^{\infty} n(n-1) \left(\frac{3}{4}\right)^{n+1}$

Notice that if $f(x) = \sum n(n-1)x^{n+1}$, then the series above is $f(3/4)$. Then, similar to problem c), we have

$$\frac{d^2}{dx^2} x^n = n(n-1)x^{n-1}$$

thus what we need is

$$x^2 \frac{d^2}{dx^2} x^n = x^2 n(n-1)x^{n-1} = n(n-1)x^{n+1}$$

so, putting it all together, we obtain

$$f(x) = \sum_{n=0}^{\infty} n(n-1)x^{n+1} = x^2 \frac{d^2}{dx^2} \left(\sum_{n=0}^{\infty} x^n \right) = x^2 \frac{d^2}{dx^2} \left(\frac{1}{1-x} \right) = \frac{4x^2}{(1-x)^5}$$

Thus our target series is $f(3/4)$ which is

$$f(3/4) = \frac{4 \left(\frac{3}{4}\right)^2}{\left(1 - \left(\frac{3}{4}\right)\right)^5} = 3^2 4^4$$

e.) $\sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!}$

Note that $e^x = \sum \frac{x^n}{n!}$, then we obtain

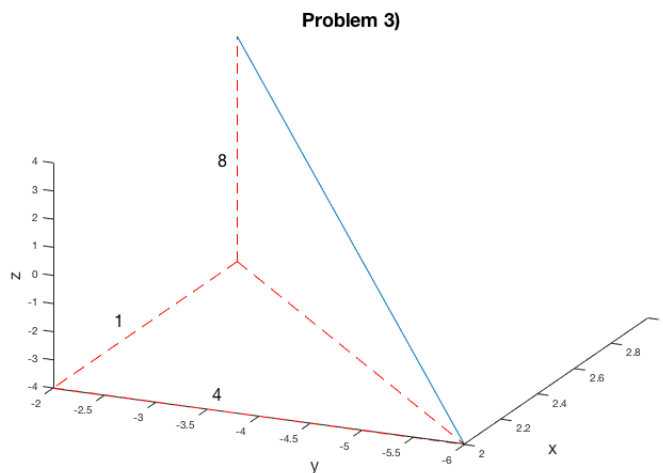
$$\sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!} = e^{\ln 2} = 2$$

f.) $\sum_{n=2}^{\infty} (-1)^n \frac{9^n}{(2n)!}$

For this one, recognize that $\cos(x) = \sum (-1)^n \frac{x^{2n}}{(2n)!}$, which is the only series we know with $(2n)!$ in the denominator and an alternator. However, we don't have the x^{2n} term in the numerator. So we have to recognize that we can re-write our series as

$$\sum_{n=2}^{\infty} (-1)^n \frac{9^n}{(2n)!} = \sum_{n=2}^{\infty} (-1)^n \frac{(3^2)^n}{(2n)!} = \sum_{n=2}^{\infty} (-1)^n \frac{(3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{(3)^{2n}}{(2n)!} - \sum_{n=0}^1 (-1)^n \frac{(3)^{2n}}{(2n)!} = \cos(3) - 1 + \frac{9}{2}$$

3. Find the distance between the points (3,-2,4) and (2,-6,-4).



1. Plot of line segment between both points with annotations for computing distance

Notice with this image above we see that there is a triangle from the target line segment to the xy plane. However, we don't know the length of the side in the plane. To compute this we use pythagorean's theorem on the x and y sides. Thus let r be the length of the diagonal in the xy plane, then we get

$$r^2 = \delta x^2 + \delta y^2 = (3 - 2)^2 + (-6 - (-2))^2 = 1 + 4^2 = 17$$

Thus $r = \sqrt{17}$. Then we use this to compute the length of the segment, d , as

$$d^2 = r^2 + \delta z^2 = (\sqrt{17})^2 + (4 - (-4))^2 = 17 + 8^2 = 81$$

Thus $d = \sqrt{81} = 9$. Note that we can cheat this and do

$$d^2 = r^2 + \delta z^2 = (\delta x^2 + \delta y^2) + \delta z^2 = \delta x^2 + \delta y^2 + \delta z^2$$

Thus we have determined Pythagorean's Theorem in 3D.

4. Find an equation of the sphere whose diameter has endpoints (2, 4, -5) and (0, -2, 4)

For defining a basic sphere, we have

$$x^2 + y^2 + z^2 = r^2$$

then when we have a centered sphere at $p = (x_1, y_1, z_1)$ we get

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2$$

Thus, we need to find the center of the sphere and the radius to define a sphere. Since we know the diameter, the center is just the midpoint of the diameter, or

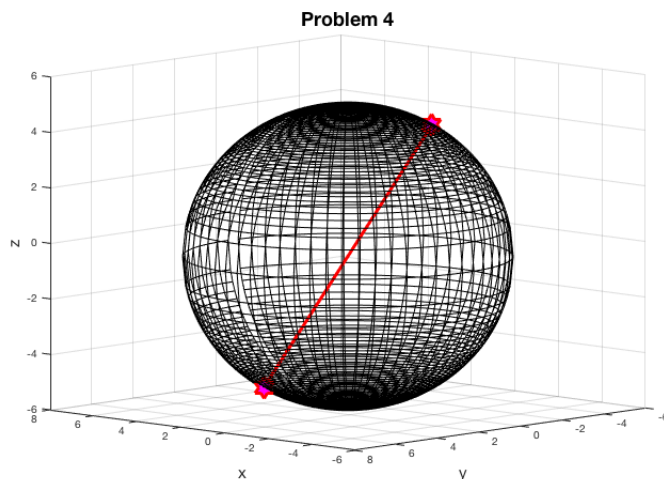
$$\text{center} = \left(\frac{0+2}{2}, \frac{-2+4}{2}, \frac{4+(-5)}{2} \right) = \left(1, 1, -\frac{1}{2} \right)$$

And our radius is the distance from one of the diameter points to the radius or using our equation in 3 we get

$$r^2 = (0 - (1))^2 + (-2 - (1))^2 + \left(4 - \left(-\frac{1}{2} \right) \right)^2 = 1 + 9 + \frac{49}{4} = \frac{11}{2}$$

Thus, putting it all together, we obtain an equation for the sphere as

$$(x - 1)^2 + (y - 1)^2 + \left(z + \frac{1}{2} \right)^2 = \left(\frac{11}{2} \right)^2$$



2. Sphere with diameter defined by $(2, 4, -5)$ and $(0, -2, 4)$.

5. Find the center and radius of the following sphere: $x^2 + y^2 + z^2 = 2x - 4y + 6z - 5$

If this equation were in the form

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

we could trivially determine the center and radius. Thus, we have to do some algebra to get our equation into this form. Notice that

$$\begin{aligned} x^2 + y^2 + z^2 &= 2x - 4y + 6z - 5 \\ (x^2 - 2x) + (y^2 + 4y) + (z^2 - 6z) &= -5 \\ (x^2 - 2x + 1) + (y^2 + 4y + 4) + (z^2 - 6z + 9) &= -5 + 1 - 4 + 9 \\ (x - 1)^2 + (y + 2)^2 + (z - 3)^2 &= 1 \end{aligned}$$

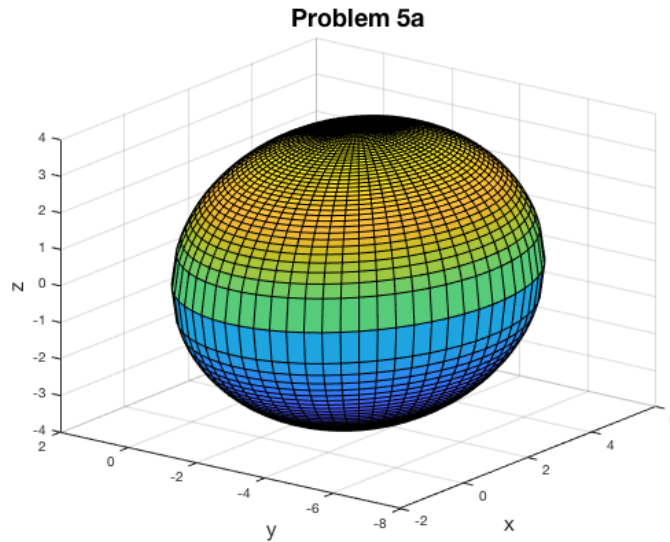
Notice that we needed to add some number to both sides to make perfect squares in each variable. Thus we can clearly see the center is at $p = (1, -2, 3)$ and the radius is $\sqrt{1} = 1$.

6. Determine a formula (and sketch the surface) for the set of all points (x,y,z) in three-dimensional space which are

a.) 4 units from the point $(2,-3,0)$

Notice that if we want to be equidistant from the point $(2,-3,0)$, then we are discussing a sphere with center $(2,-3,0)$ and radius 4. Note, many people will assume a cube, but note that the corners of the cube are actually $2\sqrt{2}$ if a side is length 4. Thus the equation for this sphere is

$$(x-2)^2 + (y+3)^2 + z^2 = 4^2$$



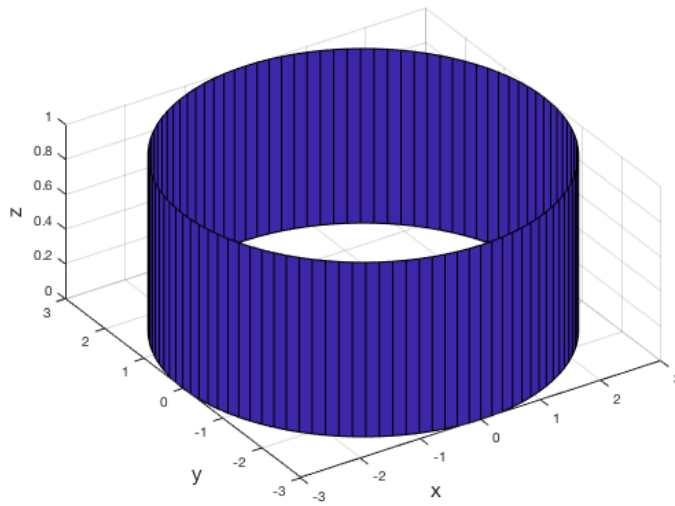
b.) 3 units from the point z -axis

Since we are now 3 units from the z axis, this is similar to the sphere above. Consider a slice of the 3-d space where $z = 0$, then we need a circle around the z -axis. However, if $z = 1$ it would also be a circle. So this is a circle around every point of the z -axis; or a cylinder. For a cylinder, z can be anything, so we need only define what x and y do. Thus we obtain

$$x^2 + y^2 = 3^2$$

Note that a cylinder in 3D is the same equation as a circle in 2D.

Problem 5b



- c.) $1/2$ unit from the x -axis
- d.) 2 units from the plane $y = 3$
- e.) equidistant from the points $(3, 0, 0)$ and $(0, 0, 3)$.
- f.) equidistant from the planes $z = 2$ and $z = 6$
- g.) equidistant from the planes $x = 3$ and $y = 2$
- h.) equidistant from the point $(0, 0, 2)$ and the xy -plane

Code

% Problem 3) – How to plot two points with connections

```
figure;
plot3([3 2]',[-2 -6]',[4 -4]'); hold on;
plot3([2 2]',[-2 -6]',[-4 -4]','r—');
plot3([3 2]',[-2 -2]',[-4 -4]','r—');
plot3([3 2]',[-2 -6]',[-4 -4]','r—');
plot3([3 3]',[-2 -2]',[-4 4]','r—');
xlabel('x','FontSize',16);
ylabel('y','FontSize',16);
zlabel('z','FontSize',16);
title('Problem_3','FontSize',18);
```

% Problem 4 – Plot a sphere with given diameter

```
figure;
d1 = [2 4 -5]; d2 = [0 -2 4];
a = (d1(1)+d2(1))/2; b=(d1(2)+d2(2))/2; c=(d1(3)+d2(3))/2;
r = sqrt((d1(1)-a)^2 + (d1(2)-b)^2 + (d1(3)-c)^2);
% Since taking the sqrt has two answers
Zp = @(x,y) sqrt(-(x-a).^2 - (y-b).^2 + r.^2) + c;
Zn = @(x,y) -sqrt(-(x-a).^2 - (y-b).^2 + r.^2) + c;
% Create a circle (using polar coordinates) then
% generate many points inside the circle with
% meshgrid. Convert these points to cartesian for
% plotting our surface. I want you to think about
% why using polar makes this easier to do.
[R,T] = meshgrid(0:.1:r,0:.1:2*pi);
X = R.*cos(T)+a; Y = R.*sin(T)+b;
% Plot top half, then bottom
surf(X,Y,real(Zp(X,Y))); hold on;
surf(X,Y,real(Zn(X,Y)));
% Plot diameter
plot3([d1(1) d2(1)], [d1(2) d2(2)], [d1(3) d2(3)], '*-r');
% Prettify the plot
title('Problem_4','FontSize',18);
xlabel('x','FontSize',15);
ylabel('y','FontSize',15);
zlabel('z','FontSize',15);
```

% Prob 5 – Plot objects

% a) Sphere about (2,-3,0)

```
figure;
a = 2; b=-3; c=0; r = 4;
Zp = @(x,y) sqrt(-(x-a).^2 - (y-b).^2 + r.^2) + c;
Zn = @(x,y) -sqrt(-(x-a).^2 - (y-b).^2 + r.^2) + c;
[R,T] = meshgrid(0:.1:r,0:.1:2*pi);
X = R.*cos(T)+a; Y = R.*sin(T)+b;
% Plot top half, then bottom
surf(X,Y,real(Zp(X,Y))); hold on; surf(X,Y,real(Zn(X,Y)));
title('Problem_5a','FontSize',18); xlabel('x','FontSize',15);
```

```

ylabel('y','FontSize',15);zlabel('z','FontSize',15);

% a) Cylinder 3 units about z-axis
figure;
r = 3;
[X,Y,Z] = cylinder(r,100); surf(X,Y,Z);
title('Problem_5b','FontSize',18);xlabel('x','FontSize',15);
ylabel('y','FontSize',15);zlabel('z','FontSize',15);

```