1. Write each power series as an ordinary function.

a.) $\sum_{\mathbf{n=5}}^{\infty} \mathbf{x^n}$

Notice that we can re-write this as

$$\sum_{n=5}^{\infty} x^n = \sum_{n=0}^{\infty} x^n - \sum_{n=0}^{4} x^n$$

Since we know the sum starting from 0, we get

$$\sum_{n=5}^{\infty} x^n = \frac{1}{1-x} - \sum_{n=0}^{4} x^n = \frac{1}{1-x} - 1 - x - x^2 - x^3 - x^4; \quad x \in (-1,1)$$

b.) $\sum_{\mathbf{n=0}}^{\infty} \mathbf{2^n x^n}$

Notice that we can write $2^n x^n = (2x)^n$, then let $y = 2x$ and we obtain

$$\sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=2(0)}^{2(\infty)} y^n = \sum_{n=0}^{\infty} y^n = \frac{1}{1-y} = \frac{1}{1-2x}$$

Note that since $y \in (-1,1)$ we have $2x \in (-1,1)$ or $x \in (-1/2, 1/2)$.

c.) $\sum_{\mathbf{n=0}}^{\infty} \frac{\mathbf{(-3)^{n+1} x^n}}{\mathbf{5^{n-1}}}$

We have to do some extensive alebra on this. Notice that we have

$$\frac{(-3)^{n+1}x^n}{5^{n-1}} = \frac{(-3)(-3)^n x^n}{5^{n-1}} = \frac{(5)(-3)(-3)^n x^n}{5^n} = -15\left(\frac{(-3)^n x^n}{5^n}\right) = -15\left(\frac{(-3x)^n}{5^n}\right) = -15\left(\frac{-3x}{5}\right)^n$$

Thus we obtain

$$\sum_{n=0}^{\infty} \frac{(-3)^{n+1}x^n}{5^{n-1}} = \sum_{n=0}^{\infty} -15\left(\frac{-3x}{5}\right)^n = -15\sum_{n=0}^{\infty}\left(\frac{-3x}{5}\right)^n = -15\left(\frac{1}{1-\left(\frac{-3x}{5}\right)}\right) = -15\left(\frac{1}{1+\frac{3x}{5}}\right) = -15\frac{5}{5+3x}$$

With radius of convergence $\frac{3x}{5} \in (-1,1)$ which gives us $x \in (-5/3, 5/3)$.

d.) $\sum_{\mathbf{n=4}}^{\infty} \mathbf{n x^{n-1}}$

For this one we have to recognize that if this were $\sum x^{n-1}$ it would be relatively easy to fix. Notice that

$$\frac{d}{dx} x^n = n x^{n-1}$$

Moreover, we obtain

$$\frac{d}{dx}\left(\sum_{n=4}^{\infty} x^n\right) = \frac{d}{dx}\left(x^4 + x^5 + x^6 + \cdots\right) = 4x^3 + 5x^4 + 6x^5 + \cdots = \sum_{n=4}^{\infty} n x^{n-1}$$

Since we know the first sum above we obtain

$$\sum_{n=4}^{\infty} n x^{n-1} = \frac{d}{dx}\left(\sum_{n=4}^{\infty} x^n\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n - \sum_{n=0}^{3} x^n\right) = \frac{d}{dx}\left(\frac{1}{1-x} - 1 - x - x^2 - x^3\right) = \frac{1}{(1-x)^2} - 1 - 2x - 3x^2$$

where $x \in (-1,1)$

e.) $\boldsymbol{\sum_{n=0}^{\infty} n^2 x^{n-1}}$

Just like in d we recognize that we need one derivative to get nx^{n-1}, so a second derivative should get us close to n^2x^{n-1}. Notice that we have

$$x\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} x\frac{d}{dx}x^n = \sum_{n=0}^{\infty} nx^n$$

Moreover, we have

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} nx^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}nx^n = \sum_{n=0}^{\infty} n^2x^{n-1}$$

which gives us our target series. Thus, using geometric series, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} n^2x^{n-1} &= \frac{d}{dx}\left(\sum_{n=0}^{\infty} nx^n\right) \\
&= \frac{d}{dx}\left(x\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right)\right) \\
&= \frac{d}{dx}\left(x\frac{d}{dx}\left(\frac{1}{1-x}\right)\right) \\
&= \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) \\
&= \frac{(1-x)^2 - 2x(1-x)}{(1-x)^4} \\
&= \frac{1-3x}{(1-x)^3}
\end{aligned}$$

where $x \in (-1, 1)$ because we used a geometric series.

f.) $\boldsymbol{\sum_{n=1}^{\infty} \frac{x^{n+3}}{n}}$

Something to notice here is that

$$\frac{d}{dx}\left(\frac{x^n}{n}\right) = x^{n-1}$$

which is a geometric series that we can solve. This is backwards from what we did in e and d. Since the reverse of a derivative is an integral, we need to use an integral. Thus we obtain

$$\int \sum_{n=1}^{\infty} x^n \; dx = \int (x + x^2 + x^3 + \cdots) \; dx = \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=2}^{\infty} \frac{x^n}{n}$$

Thus we can describe the series above as

$$\sum_{n=1}^{\infty} \frac{x^{n+3}}{n} = x^3 \int \sum_{n=1}^{\infty} x^n \; dx = x^3 \int \frac{1}{1-x} \; dx = x^3 \left(-\ln(|1-x|)\right) = x^3 \ln\left(\left|\frac{1}{1-x}\right|\right)$$

where we used partial fractions to solve that integral. Note we used a geometric series so $x \in (-1, 1)$.

g.) $\sum_{\mathbf{n=1}}^{\infty}(-\mathbf{1})^{\mathbf{n}}\frac{\mathbf{x^n}}{\mathbf{2^n n!}}$

Notice we obtain

$$\frac{x^n}{2^n n!} = \frac{\left(\frac{x}{2}\right)^n}{n!}$$

So what we can do is recognize that since

$$e^x = \sum_{n=1}^{\infty}\left(\frac{x^n}{n!}\right)$$

then we obtain

$$\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{2^n n!} = \sum_{n=1}^{\infty}\left(\frac{(-x/2)^n}{n!}\right) = e^{-x/2}$$

Where $x \in \mathbb{R}$ because $\sum \frac{x^n}{n!}$ is valid everywere.

h.) $\sum_{\mathbf{n=2}}^{\infty}(-\mathbf{1})^{\mathbf{n}}\left(\frac{\mathbf{2}}{\mathbf{5}}\right)^{\mathbf{n}}\frac{\mathbf{x^{2n+1}}}{\mathbf{2n+1}}$

Notice that $\int x^{2n}\ dx = \frac{x^{2n+1}}{2n+1}$, so we will need to use that at some point in this problem. Using some algebra we obtain,

$$(-1)^n\left(\frac{2}{5}\right)^n\frac{x^{2n+1}}{2n+1} = (-1)^n\left(\frac{2}{5}\right)^n\int x^{2n}\ dx = \int(-1)^n\left(\frac{2}{5}\right)^n x^{2n}\ dx = \int(-1)^n\left(\frac{2}{5}\right)^n (x^2)^n\ dx = \int\left(-\frac{2}{5}x^2\right)^n\ dx$$

Moreover, the series $\sum\left(-\frac{2}{5}x^2\right)^n$ is relatively easy to solve using geometric series theorems. Thus we obtain

$$\begin{aligned}
\sum_{n=2}^{\infty}(-1)^n\left(\frac{2}{5}\right)^n\frac{x^{2n+1}}{2n+1} &= \int\sum_{n=2}^{\infty}\left(-\frac{2}{5}x^2\right)^n\ dx \\
&= \int\sum_{n=0}^{\infty}\left(-\frac{2}{5}x^2\right)^n - \sum_{n=0}^{1}\left(-\frac{2}{5}x^2\right)^n\ dx \\
&= \int\frac{1}{1-(-\frac{2}{5}x^2)} - 1 - \left(-\frac{2}{5}x^2\right)\ dx \\
&= \int\frac{1}{1+\frac{2}{5}x^2}\ dx - x + \frac{2}{15}x^3 \\
&= \int\frac{1}{1+\left(\sqrt{\frac{2}{5}}x\right)^2}\ dx - x + \frac{2}{15}x^3 \\
&= \sqrt{\frac{5}{2}}\arctan\left(\sqrt{\frac{2}{5}}x\right) - x + \frac{2}{15}x^3
\end{aligned}$$

Notice that we used a geometric series, so we need $\frac{2}{5}x^2 \in (-1,1)$ or $x \in \left(-\sqrt{\frac{5}{2}}, \sqrt{\frac{5}{2}}\right)$.

2. Use any method to find the exact value of each of the following convergent series.

a.) $\sum_{\mathbf{n=0}}^{\infty} \mathbf{3}\left(\frac{\mathbf{-2}}{\mathbf{3}}\right)^{\mathbf{n}}$

Let $f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, the the target series is given by

$$\sum_{n=0}^{\infty} 3\left(\frac{-2}{3}\right)^n = 3f\left(\frac{-2}{3}\right) = 3\left(\frac{1}{1-\left(\frac{-2}{3}\right)}\right) = 3\left(\frac{3}{5}\right) = \frac{9}{5}$$

Note, in order to do this we needed $x \in (-1,1)$.

b.) $\sum_{\mathbf{n=4}}^{\infty} \frac{(\mathbf{-1})^{\mathbf{n+2}}}{\mathbf{2}^{\mathbf{n-3}}}$

WIth this we need to get the same exponent on both terms. So notice that

$$\frac{(-1)^{n+2}}{2^{n-3}} = \frac{(-1)^n(-1)^2}{2^{n-3}} = \frac{(-1)^n(-1)^2 2^3}{2^n} = 2^3\left(\frac{-1}{2}\right)^n$$

Thus we can solve this as a geometric series as

$$\sum_{n=4}^{\infty} \frac{(-1)^{n+2}}{2^{n-3}} = \sum_{n=4}^{\infty} 2^3\left(\frac{-1}{2}\right)^n = 2^3\left(\sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n - \sum_{n=0}^{3}\left(\frac{-1}{2}\right)^n\right) = 2^3\left(\left(\frac{1}{1-(-1/2)}\right) - 1 + \frac{1}{2} - \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3\right.$$

Thus the final answer is

$$\sum_{n=4}^{\infty} \frac{(-1)^{n+2}}{2^{n-3}} = \frac{2}{3} - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} = \frac{1}{24}$$

c.) $\sum_{\mathbf{n=1}}^{\infty} \mathbf{n^2}\left(\frac{\mathbf{1}}{\mathbf{2}}\right)^{\mathbf{n}}$

Note that suppose $f(x) = \sum_{n=1}^{\infty} n^2 x^n$ then $f(1/2)$ gives us the series above. From before we recognize that we need to take two derivatives of x^n to get $n^2 x^n$, thus we obtain

$$x\frac{d}{dx}\left(x\frac{d}{dx}x^n\right) = x\frac{d}{dx}(nx^n) = n^2 x^n$$

Thus we obtain

$$\sum_{n=1}^{\infty} n^2 x^n = x\frac{d}{dx}\left(x\frac{d}{dx}\sum_{n=1}^{\infty} x^n\right) = x\frac{d}{dx}\left(x\frac{d}{dx}\frac{1}{1-x} - 1\right) = x\frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) = \frac{x(1-3x)}{(1-x)^3}$$

since our series is $f(1/2)$ we obtain

$$\sum_{n=1}^{\infty} n^2\left(\frac{1}{2}\right)^n = f(1/2) = \frac{\frac{1}{2}(1-3\left(\frac{1}{2}\right))}{(1-\left(\frac{1}{2}\right))^3} = -2$$

This looks wrong to me

d.) $\sum_{\mathbf{n=0}}^{\infty} \mathbf{n(n-1)} \left(\frac{\mathbf{3}}{\mathbf{4}}\right)^{\mathbf{n+1}}$

Notice that if $f(x) = \sum n(n-1)x^{n+1}$, then the series above is $f(3/4)$. Then, similar to problem c), we have

$$\frac{d^2}{dx^2}x^n = n(n-1)x^{n-1}$$

thus what we need is

$$x^2\frac{d^2}{dx^2}x^n = x^2 n(n-1)x^{n-1} = n(n-1)x^{n+1}$$

so, putting it all together, we obtain

$$f(x) = \sum_{n=0}^{\infty} n(n-1)x^{n+1} = x^2\frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty} x^n\right) = x^2\frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) = \frac{4x^2}{(1-x)^5}$$

Thus our target series is $f(3/4)$ which is

$$f(3/4) = \frac{4\left(\frac{3}{4}\right)^2}{(1-\left(\frac{3}{4}\right))^5} = 3^2 4^4$$

e.) $\sum_{\mathbf{n=0}}^{\infty} \frac{(\ln \mathbf{2})^{\mathbf{n}}}{\mathbf{n}!}$

Note that $e^x = \sum \frac{x^n}{n!}$, then we obtain

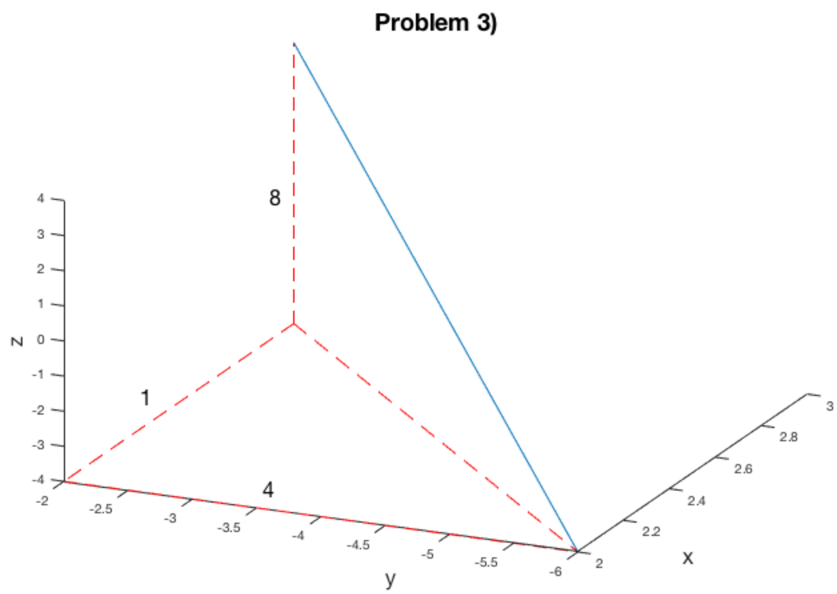
$$\sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!} = e^{\ln 2} = 2$$

f.) $\sum_{\mathbf{n=2}}^{\infty} (-\mathbf{1})^{\mathbf{n}} \frac{\mathbf{9^n}}{(\mathbf{2n})!}$

For this one, recognize that $\cos(x) = \sum(-1)^n \frac{x^{2n}}{(2n)!}$, which is the only series we know with $(2n)!$ in the denominator and an alternator. However, we don't have the x^{2n} term in the numerator. So we have to recognize that we can re-write our series as

$$\sum_{n=2}^{\infty}(-1)^n\frac{9^n}{(2n)!} = \sum_{n=2}^{\infty}(-1)^n\frac{\left(3^2\right)^n}{(2n)!} = \sum_{n=2}^{\infty}(-1)^n\frac{(3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{(3)^{2n}}{(2n)!} - \sum_{n=0}^{\infty}(-1)^1\frac{(3)^{2n}}{(2n)!} = \cos(3) - 1 + \frac{9}{2}$$

3. Find the distance between the points (3,-2,4) and (2,-6,-4).



1. Plot of line segment between both points with annotations for computing distance

Notice with this image above we see that there is a triangle from the target line segment to the xy plane. However, we don't know the length of the side in the plane. To compute this we use pythagorean's theorem on the x and y sides. Thus let r be the length of the diagonal in the xy plane, then we get

$$r^2 = \Delta x^2 + \Delta y^2 = (3-2)^2 + (-6-(-2))^2 = 1 + 4^2 = 17$$

Thus $r = \sqrt{17}$. Then we use this to compute the length of the segment, d, as

$$d^2 = r^2 + \Delta z^2 = \left(\sqrt{17}\right)^2 + (4-(-4))^2 = 17 + 8^2 = 81$$

Thus $d = \sqrt{81} = 9$. Note that we can cheat this and do

$$d^2 = r^2 + \Delta z^2 = \left(\Delta x^2 + \Delta y^2\right) + \Delta z^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$$

Thus we have determined Pythagorean's Theorem in 3D.

4. Find an equation of the sphere whose diameter has endpoints $(2, 4, -5)$ and $(0, -2, 4)$

For defining a basic sphere, we have

$$x^2 + y^2 + z^2 = r^2$$

then when we have a centered sphere at $p = (x_1, y_1, z_1)$ we get

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2$$

Thus, we need to find the center of the sphere and the radius to define a sphere. Since we know the diameter, the center is just the midpoint of the diameter, or

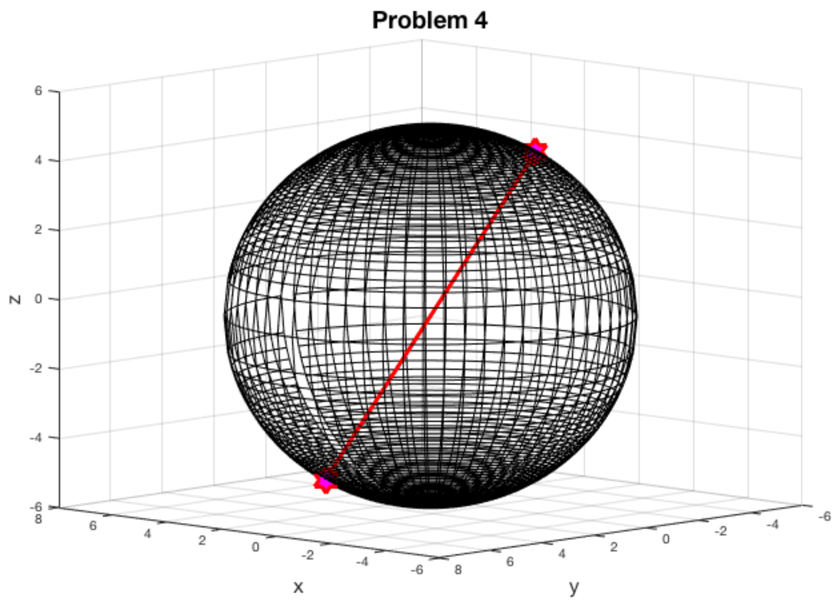
$$\text{center} = \left(\frac{0+2}{2}, \frac{-2+4}{2}, \frac{4+(-5)}{2}\right) = \left(1, 1, -\frac{1}{2}\right)$$

And our radius is the distance from one of the diameter points to the radius or using our equation in 3 we get

$$r^2 = (0-(1))^2 + (-2-(1))^2 + \left(4 - \frac{1}{2}\right)^2 = 1 + 9 + \frac{49}{4} = \frac{11}{2}$$

Thus, putting it all together, we obtain an equation for the sphere as

$$(x-1)^2 + (y-1)^2 + \left(z + \frac{1}{2}\right)^2 = \left(\frac{11}{2}\right)^2$$



2. Sphere with diamater defined by $(\mathbf{2}, \mathbf{4}, -\mathbf{5})$ and $(\mathbf{0}, -\mathbf{2}, \mathbf{4})$.

5. Find the center and radius of the following sphere: $\mathbf{x^2 + y^2 + z^2 = 2x - 4y + 6z - 5}$

If this equation were in the form

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

we could trivially determine the center and radius. Thus, we have to do some algebra to get our equation into this form. Notice that

$$\begin{aligned} x^2 + y^2 + z^2 &= 2x - 4y + 6z - 5 \\ (x^2 - 2x) + (y^2 + 4y) + (z^2 - 6z) &= -5 \\ (x^2 - 2x + 1) + (y^2 + 4y + 4) + (z^2 - 6z + 9) &= -5 + 1 - 4 + 9 \\ (x-1)^2 + (y+2)^2 + (z-3)^2 &= 1 \end{aligned}$$

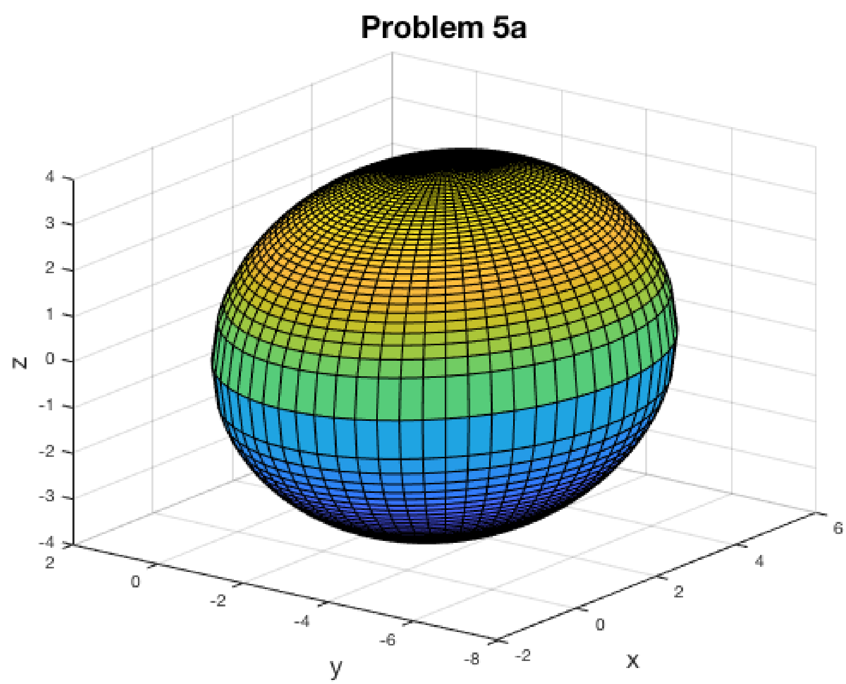
Notice that we needed to add some number to both sides to make perfect squares in each variable. Thus we can clearly see the center is at $p = (1, -2, 3)$ and the radius is $\sqrt{1} = 1$.

6. Determine a formula (and sketch the surface) for the set of all points $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ in three-dimensional space which are

a.) 4 units from the point (2,-3,0)

Notice that if we want to be equidistant from the point (2,-3,0), then we are discussing a sphere with center (2,-3,0) and radius 4. Note, many people will assume a cube, but note that the corners of the cube are actually $2\sqrt{2}$ if a side is length 4. Thus the equation for this sphere is

$$(x-2)^2 + (y+3)^2 + z^2 = 4^2$$

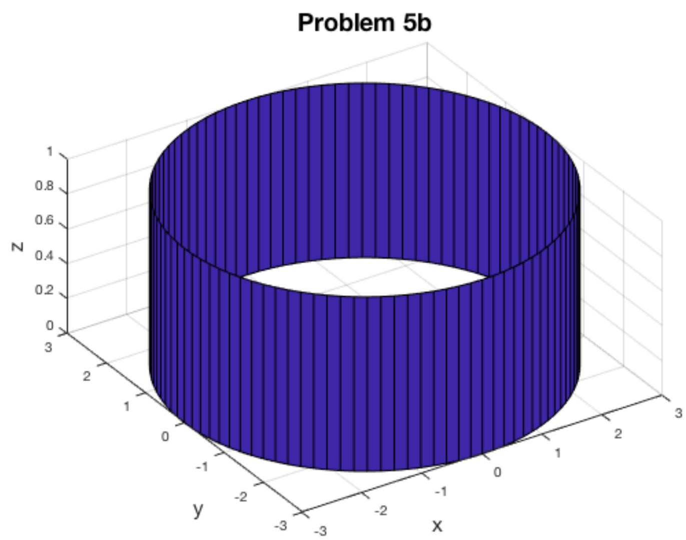


b.) 3 units from the point z-axis

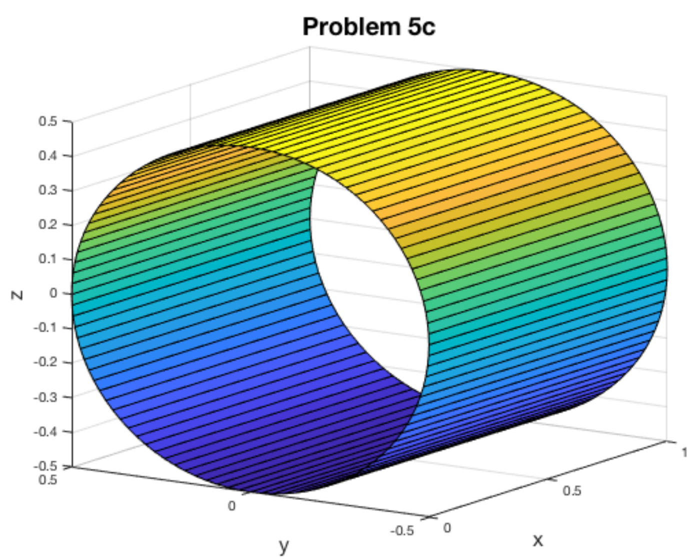
Since we are now 3 units from the z axis, this is similar to the sphere above. Consider a slice of the 3-d space where $z = 0$, then we need a circle around the z-axis. However, if $z = 1$ it would also be a circle. So this is a circle around every point of the z-axis; or a cylinder. For a cylinder, z can be anything, so we need only define what x and y do. Thus we obtain

$$x^2 + y^2 = 3^2$$

Note that a cylinder in 3D is the same equation as a circle in 2D.

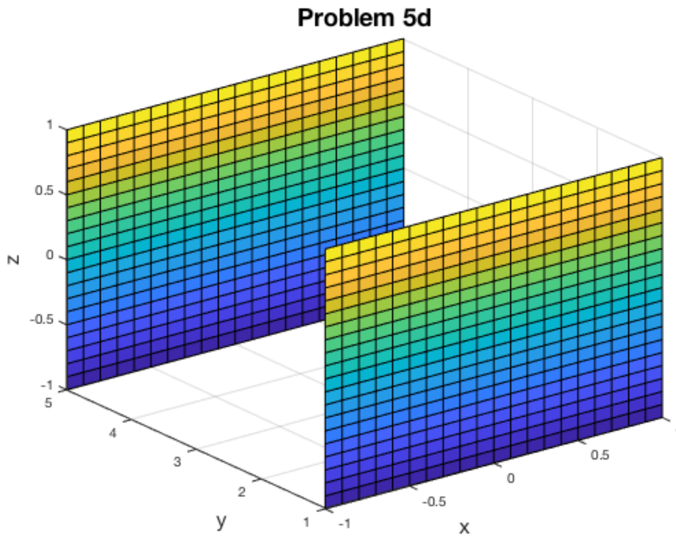


c.) 1/2 unit from the x-axis
Same thing as in problem b. A Cyclinder around the x axis



d.) 2 units from the plane y = 3

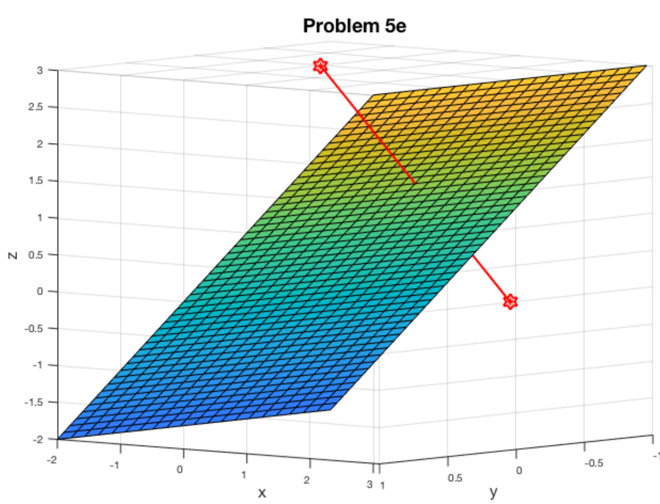
If we are 2 units from the plane, then we have to be 2 units everywhere. Thus we are looking at the planes $y = 3 + 2 = 5$ and $y = 3 - 2 = 1$.



e.) equidistant from the points $(3, 0, 0)$ and $(0, 0, 3)$.

To be equidistant from these two points, we need to consider a plane that goes through their midpoint $m = (3/2, 0, 3/2)$ and points towards one of the points. Thus the normal vector of the plane is $n = (3, 0, 0) - (3/2, 0, 3/2) = (3/2, 0, -3/2)$. Thus we obtain an equation of the plane

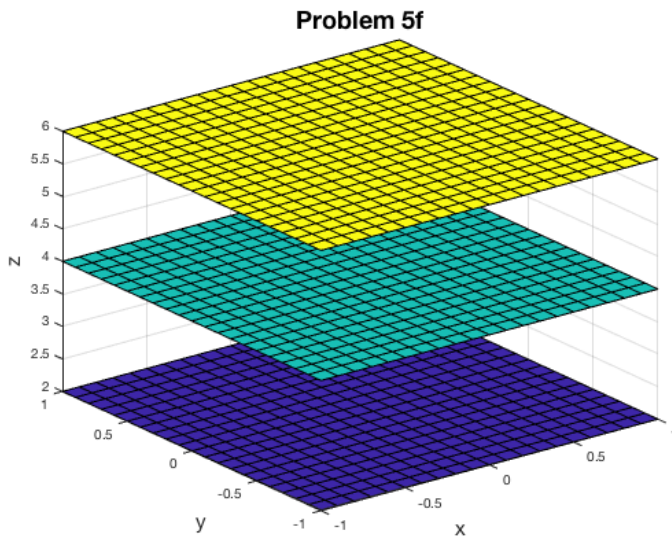
$$Ax + By + Cz = D \iff \frac{3}{2}x - \frac{3}{2}y = 0$$



f.) equidistant from the planes z = 2 and z = 6

In order to be equidistant between two planes, we must specify another plane in the middle. Thus the equation is just

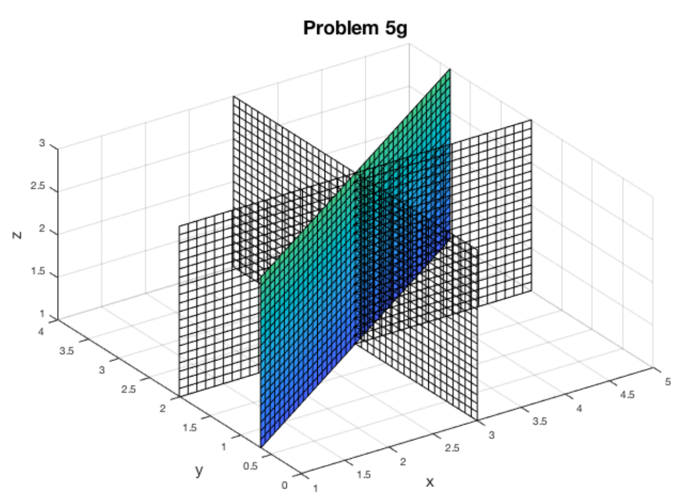
$$z = \frac{2+6}{2} = 4$$



g.) equidistant from the planes x = 3 and y = 2

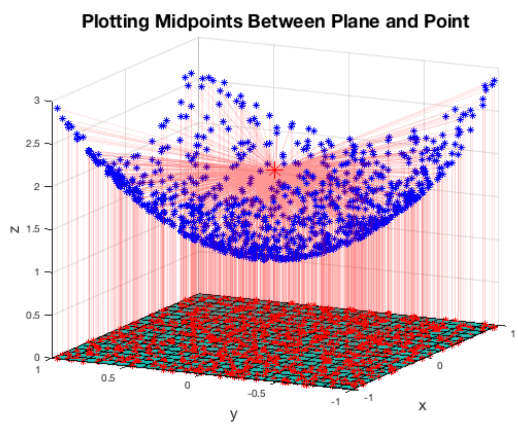
This one is pretty weird. If this were in 2d, the problem could be "find something equidistant from the x and y axes". In this case, we clearly have the function $y = x$ satisfy this. So in 3D we need to come up with a plane that goes through these planes' diagonals. Thus we obtain

$$y = \frac{2}{3}x$$



h.) equidistant from the point $(0, 0, 2)$ and the xy-plane

So to be equidistant from a plane we need another plane, and to be equidistant from a point we need a sphere. Let's take a look at this point and plane and see if this helps us figure something out. In this next image we plot a few line segments from the point into space and those same segments down onto the plane. The blue dots are where the line segments need to be in order for both segments to be equal.



Here we see that the surfacr curves around the point at a steeper angle than a sphere. This shape can be recognized as a parabola in 3D, or a paraboloid. In 2D, the equation for a parbols is

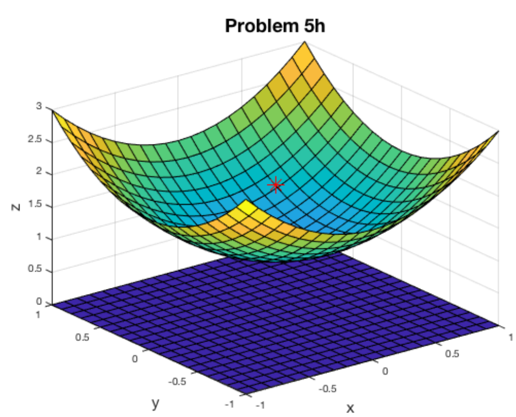
$$y = x^2$$

Which abstracts to 3D as

$$z = x^2 + y^2$$

However, we want the bottom of our parabaloid to be at $z = 1$, so we get

$$z = x^2 + y^2 + 1$$



7. a.) If vector $\overrightarrow{A} = \overrightarrow{\langle 1, 0, -2 \rangle}$, then what is the unit vector in the same direction as $\overrightarrow{A}$?

What we are looking for is a some scalar multiple of $\overrightarrow{A}$ that has magnitude 1. Thus what we are looking for is

$$||cA|| = 1$$

which, we can solve for c as

$$||cA|| = c\left|\left|\overrightarrow{A}\right|\right| = 1 \iff c = \frac{1}{\left|\left|\overrightarrow{A}\right|\right|}$$

Thus, c must equal 1 over the magnitude of $\overrightarrow{A}$. So our unit vector is

$$c\overrightarrow{A} = \frac{\overrightarrow{A}}{\left|\left|\overrightarrow{A}\right|\right|} = \frac{\langle 1, 0, -2 \rangle}{||\langle 1, 0, -2 \rangle||} = \frac{\langle 1, 0, -2 \rangle}{\sqrt{1^2 + 0^2 + (-2)^2}} = \left\langle \frac{1}{\sqrt{5}}, 0, -\frac{2}{\sqrt{5}} \right\rangle$$

b.) If vector $\overrightarrow{A} = \overrightarrow{\langle a, b, c \rangle}$, and a, b, and c are not all zero, then what is the unit vector in the same direction as $\overrightarrow{A}$?

Following the same logic as part a), a unit vector in the same direction as $\overrightarrow{A}$ is given by

$$c\overrightarrow{A} = \frac{\overrightarrow{A}}{\left|\left|\overrightarrow{A}\right|\right|} = \frac{\langle a, b, c \rangle}{||\langle a, b, c \rangle||} = \frac{\langle a, b, c \rangle}{\sqrt{a^2 + b^2 + c^2}} = \left\langle \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}} \right\rangle$$

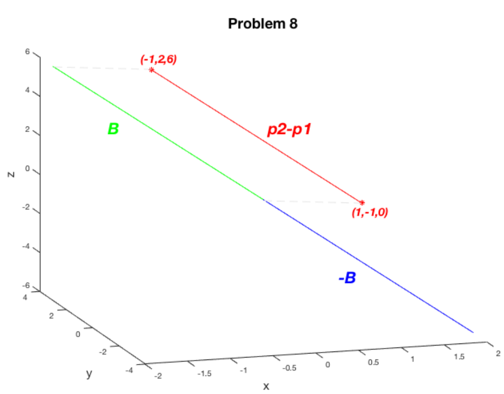
8. Determine the vector $\overrightarrow{B}$, which starts at point (1,-1,0) and ends at point (-1,2,6). Find a vector of length 2 point in the opposite direction of $\overrightarrow{B}$.

A vector that starts at one point and ends at another is simply the difference of those two points. This vector will point in the direction of the first point and start from the point you subtract from it. Thus we have

$$\overrightarrow{B} = \langle -1 - 1, 2 - (-1), 6 - 0 \rangle = \langle -2, 3, 6 \rangle$$

A vector pointing in the opposite direction is simply $(-1)\overrightarrow{B}$, given by

$$(-1)\overrightarrow{B} = \langle 2, -3, -6 \rangle$$



3. Plotting the vectors $\overrightarrow{B}$ and $-\overrightarrow{B}$

9. Find two vectors of length 3 which are both perpendicular to

a) vector $\overrightarrow{W} = \mathbf{3}\overrightarrow{i} + \mathbf{4}\overrightarrow{j}$

To be perpendicular to $\overrightarrow{W}$, our vector must have the angle between it and $\overrightarrow{W}$ be equal to $\pi/2$. Note, that by the definition of the dot product we have

$$\overrightarrow{A} \cdot \overrightarrow{B} = |A||B| \cos\theta$$

where θ is the angle between the vectors. If that angle is $\pi/2$, we have $\cos\theta = 0$, and thus $\overrightarrow{A} \cdot \overrightarrow{B} = 0$. So to find a vector perpendicular, we need the dot product to be zero. Thus let $\overrightarrow{A} = \langle a_1, a_2 \rangle$ and we obtain

$$0 = \overrightarrow{A} \cdot \overrightarrow{W} = 3a_1 + 4a_2 = 0$$

Note, we now have one equation with two variables. In order to solve this we need another equation (equal number of equations per variables). To get the second we notice that the length must be 3.get Thus our system of equations is

$$\begin{aligned} 3a_1 + 4a_2 &= 0 \\ a_1^2 + a_2^2 &= 3^2 \end{aligned}$$

Solving the first equation for a_2 we get $a_2 = -\frac{3}{4}a_1$, and plugging into equation 2 we obtain

$$a_1^2 + \left(\frac{-3a_1}{4}\right)^2 = 3^2 \iff a_1^2\left(1 + \frac{9}{4}\right) = 3^2 \iff a_1 = \pm\sqrt{\frac{4}{13}3^2} = \pm\frac{6}{\sqrt{13}}$$

Since we got two different values for a_1, we have

$$\overrightarrow{A} = \left\langle \frac{6}{\sqrt{13}}, \sqrt{6} \right\rangle \quad \text{or} \quad \overrightarrow{A} = \left\langle -\frac{6}{\sqrt{13}}, \sqrt{6} \right\rangle$$

b) vector $\overrightarrow{W} = \overrightarrow{i} - \mathbf{2}\overrightarrow{j} + \mathbf{2}\overrightarrow{k}$

Following the steps in a), let $\overrightarrow{A} = \langle a_1, a_2, a_3 \rangle$ we get

$$\begin{aligned} 0 = \overrightarrow{A} \cdot \overrightarrow{W} = a_1 - 2a_2 + 2a_3 &= 0 \\ a_1^2 + a_2^2 + a_3^2 &= 3^2 \end{aligned}$$

Note in this problem we now don't have enough equations to solve exactly, so we obtain

$$y = \frac{1}{4}\left(x - 3\sqrt{8 - x^2}\right) \quad \text{and} \quad z = \frac{1}{4}\left(-3\sqrt{8 - x^2} - x\right)$$

or

$$y = \frac{1}{4}\left(3\sqrt{8 - x^2} + x\right) \quad \text{and} \quad z = \frac{1}{4}\left(3\sqrt{8 - x^2} - x\right)$$

So, we can just set x equal to anything. Setting $x = 0$, we have

$$\overrightarrow{A} = \left\langle 0, -\frac{3}{2}\sqrt{2}, -\frac{3}{2}\sqrt{2} \right\rangle \quad \text{or} \quad \overrightarrow{A} = \left\langle 0, \frac{3}{2}\sqrt{2}, \frac{3}{2}\sqrt{2} \right\rangle$$

10. A sailboat starts at the origin (0,0), then sails

a.) 3 `km` `NW`, then turns and sails

This vector is

$$\langle -3\cos 45°, 3\sin 45° \rangle = \left\langle -\frac{3}{2}\sqrt{2}, \frac{3}{2}\sqrt{2} \right\rangle$$

b.) 2 `km` 60° North of East, then turns and sails

This vector is

$$\langle 2\cos 60°, 2\sin 60° \rangle = \left\langle 1, \sqrt{3} \right\rangle$$

c.) 4 `km` `SE`, then turns and sails

This vector is

$$\langle 4\cos 45°, -4\sin 45° \rangle = \left\langle 2\sqrt{2}, -2\sqrt{2} \right\rangle$$

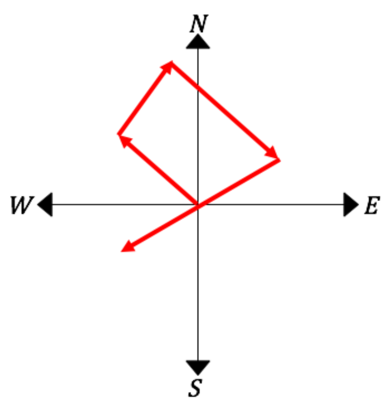
d.) 10 `km` 30° South of West, and stops. What are the coordinates now?

This vector is

$$-\langle 10\cos 30°, -10\sin 30° \rangle = \left\langle -5\sqrt{3}, -5 \right\rangle$$

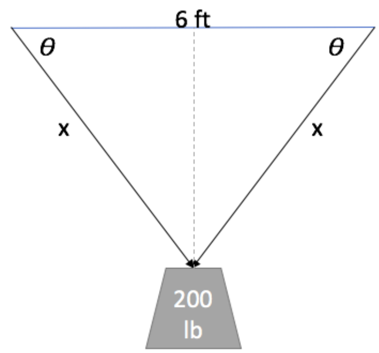
To find the final coordinate we just add all the vectors together

$$\left\langle -\frac{3}{2}\sqrt{2}, \frac{3}{2}\sqrt{2} \right\rangle + \left\langle 1, \sqrt{3} \right\rangle + \left\langle 2\sqrt{2}, -2\sqrt{2} \right\rangle + \left\langle -5\sqrt{3}, -5 \right\rangle = \left\langle \frac{1}{2}\sqrt{2} + 1 - 5\sqrt{3}, -\frac{1}{2}\sqrt{2} + \sqrt{3} - 5 \right\rangle$$



4. Trajectory of the boat

11. Two strong wires of equal length are hung from two supports which are at the same height and 6 feet apart. Each wire is attached to the same point of a 200 pound weight. What is the force of tension (in pounds) on each wire if the wires are each



5. Weight hanging from two strong wires

For this we can just solve it for an arbitrary length of the wires. Since if we knew θ, the y component of the force would be $F \sin\theta$. We can use trigonometry in order to find θ since $\cos\theta = 3/x \iff \theta = \arccos\left(\frac{3}{x}\right)$. Thus we have the y component of the force on one wire would be

$$\frac{F}{2}\sin\theta = \frac{F}{2}\sin\arccos\left(\frac{3}{x}\right) = \frac{F}{2}\sqrt{x^2 - 3^2} = \frac{F}{2}\sqrt{x^2 - 9}$$

a.) 5 feet long?

$$\frac{F}{2}\sqrt{x^2 - 9} = 100\sqrt{25 - 9} = 400\text{lb}$$

b.) 20 feet long?

$$\frac{F}{2}\sqrt{x^2 - 9} = 100\sqrt{20^2 - 9} \approx 2008\text{lb}$$

c.) 3.1 feet long?

$$\frac{F}{2}\sqrt{x^2 - 9} = 100\sqrt{3.1^2 - 9} \approx 78\text{lb}$$

d.) 3.01 feet long?

$$\frac{F}{2}\sqrt{x^2 - 9} = 100\sqrt{3.01^2 - 9} \approx 25\text{lb}$$

12. You can swim at a constant speed of 5 mph. You wish to swim across a river 1 mile wide and land at a point directly across the river from where you start swimming. If the river flows at the constant rate of 3 mph, in what direction should you swim to accomplish this? How long will it take you to swim across the river?

6. Diagram of swimmer crossing the river

In the diagram we see that we need the x component of our swim to match the flow of the river. Thus the x-component of our swim $\overrightarrow{S}$ is given by

$$\overrightarrow{S}\cos\theta = \overrightarrow{F}$$

So the angle is given by

$$\theta = \arccos\left(\frac{\left|\overrightarrow{F}\right|}{\left|\overrightarrow{S}\right|}\right) = \arccos\left(\frac{3}{5}\right) \approx 0.93 \approx 53^\circ$$

The time it takes is the time it takes the y-component of $\overrightarrow{S}$ to equal d. Thus we need

$$t\left|\overrightarrow{S}\right|\sin\theta = d \iff t = \frac{d}{|S|\sin\theta} = \frac{1}{(5)\frac{4}{5}} = 0.25 \text{ Hours} = 15 \text{ Mins}$$

13. A circus is witnessed by 120 people who have paid a total of $120. The women paid $5 each, the men paid $2 each, and the childern paid 10 cents each. How many women and children went to the circus?

Here we just set up a system of equations. Let w be the number of women, m be the number of men, and c be the number of children. Then we represent this is a system of equations given by

$$\begin{cases} w + m + c & = & 120 \\ 5w + 2m + \frac{1}{10}c & = & 120 \end{cases}$$

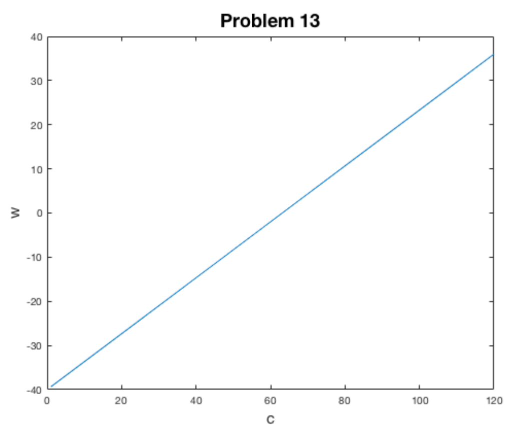
Here we need to recognize that our system has 3 variables but only 2 equations, thus it can not be solved explicitly. However, we only need to know c and w. Using the first equation we get

$$m = 160 - \frac{49}{30}c$$

and plugging into the second equation get the relation between w and c as

$$w = \frac{19}{30}c - 40$$

Thus we have an infinite set of solutions that looks like the graph here.



7. Women as a function of children

Here we see we need at least 64 children to give us a positive number of women. However, this gives us about 0.5 women; which is clearly not possible. Thus we need the children in batches of 30 to have whole number women, thus we need 90 children, which gives us 17 women, and 13 men.

Code

```
%-------------------------------------------------------------
% Problem 3) - How to plot two points with connections
%-------------------------------------------------------------
figure;
plot3([3 2]',[-2 -6]',[4 -4]'); hold on;
plot3([2 2]',[-2 -6]',[-4 -4]','r--');
plot3([3 2]',[-2 -2]',[-4 -4]','r--');
plot3([3 2]',[-2 -6]',[-4 -4]','r--');
plot3([3 3]',[-2 -2]',[-4 4]','r--');
xlabel('x','Fontsize',16);
ylabel('y','Fontsize',16);
zlabel('z','Fontsize',16);
title('Problem 3)','Fontsize',18);


%-------------------------------------------------------------
% Problem 4 - Plot a sphere with given diameter
%-------------------------------------------------------------
figure;
d1 = [2 4 -5]; d2 = [0 -2 4];
a = (d1(1)+d2(1))/2; b=(d1(2)+d2(2))/2; c=(d1(3)+d2(3))/2;
r = sqrt( (d1(1)-a)^2 + (d1(2)-b)^2 + (d1(3)-c)^2);
% Since taking the sqrt has two answers
Zp = @(x,y)  sqrt(-(x-a).^2 - (y-b).^2 + r.^2) + c;
Zn = @(x,y) -sqrt(-(x-a).^2 - (y-b).^2 + r.^2) + c;
% Create a circle (using polar coordinates) then
% generate many points inside the circle with
% meshgrid. Convert these points to cartesian for
% plotting our surface. I want you to think about
% why using polar makes this easier to do.
[R,T] = meshgrid(0:.1:r,0:.1:2*pi);
X = R.*cos(T)+a;    Y = R.*sin(T)+b;
% Plot top half, then bottom
surf(X,Y,real(Zp(X,Y)));hold on;
surf(X,Y,real(Zn(X,Y)));
% Plot diameter
plot3([d1(1) d2(1)],[d1(2) d2(2)], [d1(3) d2(3)],'*-r');
% Prettify the plot
title('Problem 4','Fontsize',18);
xlabel('x','Fontsize',15);
ylabel('y','Fontsize',15);
zlabel('z','Fontsize',15);


%-------------------------------------------------------------
% Prob 5 - Plot objects
%-------------------------------------------------------------
% a) Sphere about (2,-3,0)
figure;
a = 2; b=-3; c=0; r = 4;
Zp = @(x,y)  sqrt(-(x-a).^2 - (y-b).^2 + r.^2) + c;
Zn = @(x,y) -sqrt(-(x-a).^2 - (y-b).^2 + r.^2) + c;
[R,T] = meshgrid(0:.1:r,0:.1:2*pi);
X = R.*cos(T)+a;    Y = R.*sin(T)+b;
% Plot top half, then bottom
surf(X,Y,real(Zp(X,Y)));hold on; surf(X,Y,real(Zn(X,Y)));
title('Problem 5a','Fontsize',18);xlabel('x','Fontsize',15);
ylabel('y','Fontsize',15);zlabel('z','Fontsize',15);

% b) Cylinder 3 units about z-axis
figure;
r = 3;
[X,Y,Z] = cylinder(r,100); surf(X,Y,Z);
```

```
title('Problem 5b','Fontsize',18);xlabel('x','Fontsize',15);
ylabel('y','Fontsize',15);zlabel('z','Fontsize',15);

% c) Cylinder 1/2 units about x-axis
figure;
r = 1/2;
[x,y] = meshgrid(-0.5:.05:0.5);
zp =  sqrt(r.^2-y.^2); % independent of x
zn = -sqrt(r.^2-y.^2); % independent of x
surf(x,y,zp); hold on; surf(x,y,zn);
title('Problem 5c','Fontsize',18);xlabel('x','Fontsize',15);
ylabel('y','Fontsize',15);zlabel('z','Fontsize',15);
% Another way to do this is using the cylinder function
figure;
r = 1/2;
[X,Y,Z] = cylinder(r,50); surf(Z,Y,X);
title('Problem 5c (alternate)','Fontsize',18);xlabel('x','Fontsize',15);
ylabel('y','Fontsize',15);zlabel('z','Fontsize',15);

% d) Planes at y=1 and y=5
figure;
[x,z] = meshgrid(-1:0.1:1);
y = zeros(size(x, 1));
surf(x, y+1, z); hold on;
surf(x, y+5, z);
title('Problem 5d','Fontsize',18);xlabel('x','Fontsize',15);
ylabel('y','Fontsize',15);zlabel('z','Fontsize',15);

% e) Planes at y=1 and y=5
figure;
p1 = [3 0 0]; p2 = [0 0 3];
[x,y] = meshgrid(-4:0.1:4); % Generate x and y data
A=3/2; B=0; C=-3/2; D=0;
z = -1/C*(A*x + B*y + D); % Solve for z data
surf(x, y, z); hold on;
plot3([p1(1) p2(1)], [p1(2) p2(2)], [p1(3) p2(3)],'*-r');
title('Problem 5e','Fontsize',18);xlabel('x','Fontsize',15);
ylabel('y','Fontsize',15);zlabel('z','Fontsize',15);

% f) Planes at z=4
figure;
[x,y] = meshgrid(-1:0.1:1);
z = zeros(size(x, 1));
surf(x, y, z+4); hold on;
surf(x, y, z+6);
surf(x, y, z+2);
title('Problem 5f','Fontsize',18);xlabel('x','Fontsize',15);
ylabel('y','Fontsize',15);zlabel('z','Fontsize',15);

% g) Planes at y=2, x=3, and y=3/2 x
figure;
[x,y] = meshgrid(1:0.1:5,0:.1:4);
z = zeros(size(x, 1));
surf(x, z+2, y); hold on;
surf(z+3, y, x);
surf(x,x*2/3,y);
title('Problem 5f','Fontsize',18);xlabel('x','Fontsize',15);
ylabel('y','Fontsize',15);zlabel('z','Fontsize',15);

% h) xy plane and point (0,0,2)
figure;
p = [0 0 2];
[x,y] = meshgrid(-1:.1:1);
z = zeros(size(x, 1));
% Plot point and plane
```

```matlab
surf(x, y, z); hold on;
plot3(p(1),p(2),p(3),'r*','Markersize',15);
% plot some lines and their midpoints
r = 1;
zp = @(x,y)  x.^2 + y.^2 + 1;
for i=1:1000
    xr = rand(1)*(-1)^randi([0 1],1);
    yr = rand(1)*(-1)^randi([0 1],1);
    zr = zp(xr,yr);
    p2 = [xr yr 0]; % random point in xy
    plt1=plot3([p(1) xr], [p(2) yr], [p(3) zr],'*-r');
    plt2=plot3([xr p2(1)], [yr p2(2)], [zr p2(3)],'*-r');
    plt1.Color(4) = 0.1;
    plt2.Color(4) = 0.1;
    plot3(xr, yr, zr,'*b');
    %plot3((p(1)+p2(1))/2, (p(2) +p2(2))/2, (p(3) +p2(3))/2,'*b'); %mid
end
title('Plotting Midpoints Between Plane and Point','Fontsize',18);xlabel('x','Fontsize',15);
ylabel('y','Fontsize',15);zlabel('z','Fontsize',15);

figure
p = [0 0 2];
[x,y] = meshgrid(-1:.1:1);
z = zeros(size(x, 1));
% Plot point and plane
surf(x, y, z); hold on;
plot3(p(1),p(2),p(3),'r*','Markersize',15);
surf(x,y,zp(x,y))
title('Problem 5h','Fontsize',18);xlabel('x','Fontsize',15);
ylabel('y','Fontsize',15);zlabel('z','Fontsize',15);

% ------------------------------------------------------------
% Problem 8 - Plotting a Vector and It's Reverse
% ------------------------------------------------------------
p1 = [1,-1,0]; p2 = [-1,2,6];
B = p2-p1;
figure;
plot3([p1(1) p2(1)],[p1(2) p2(2)],[p1(3) p2(3)],'*-r'); hold on;
plot3([0 B(1)], [0 B(2)], [0 B(3)],'g');
plot3([0 -B(1)],[0 -B(2)],[0 -B(3)],'b');
title('Problem 8','Fontsize',18);xlabel('x','Fontsize',15);
ylabel('y','Fontsize',15);zlabel('z','Fontsize',15);

% ------------------------------------------------------------
% Problem 13 - Women as f(Children)
% ------------------------------------------------------------
w = @(c) (19/30)*c-40;
c = 1:120;
figure;
plot(c,w(c)); hold on;
title('Problem 13','Fontsize',18);
xlabel('c','Fontsize',15);ylabel('w','Fontsize',15);
```