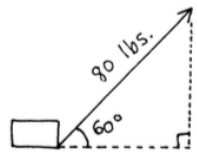
1. A force of 80 pounds is applied in the given diagram. Find the horizontal and vertical components of this force.



The y component of the force is given by

$$F \sin\theta = 80 \sin 60^\circ = 40\text{lb}$$

The x component of the force is given by

$$F \cos\theta = 80 \cos 60^\circ = 40\sqrt{3}\text{lb}$$

2. Let vector $\overrightarrow{A} = \overrightarrow{\langle 3, 4\rangle}$. Find a vector of length 4 which is

a.) parallel to $\overrightarrow{A}$

Any vector pointing in the same direction works. So,

$$2\overrightarrow{A} = \langle 6, 8\rangle$$

b.) perpendicular to $\overrightarrow{A}$

To be perpendicular we need the angle to be $\pi/2$, or the dot product equal to zero. Let $\overrightarrow{B} = \langle b_1, b_2\rangle$ then

$$\overrightarrow{A} \cdot \overrightarrow{B} = 3b_1 + 4b_2 = 0 \iff b_1 = -\frac{4}{3}b_2$$

Thus let $b_2 = 1$ and we get

$$\overrightarrow{B} = \left\langle -\frac{4}{3}, 1 \right\rangle$$

3. Find two unit vectors each of which is perpendicular to both $\overrightarrow{A} = \overrightarrow{\langle 1, 0, -2\rangle}$ and $\overrightarrow{B} = \overrightarrow{\langle 0, 3, 4\rangle}$

To be perpendicular to both of these vectors we need to be perpendicular to the plane spanned by these vectors. Thus we need to find a normal vector of this plane. The easiest way to do this is by computing the cross-product of both vectors. Thus

$$\overrightarrow{A} \times \overrightarrow{B} = \begin{bmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 0 & -2 \\ 0 & 3 & 4 \end{bmatrix} = \left\langle \left|\begin{bmatrix} 0 & -2 \\ 3 & 4 \end{bmatrix}\right|, \left|\begin{bmatrix} 1 & -2 \\ 0 & 4 \end{bmatrix}\right|, \left|\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\right| \right\rangle = \langle 6, 4, 0\rangle$$

Thus the two vectors are

$$\langle 6, 4, 0\rangle \quad \text{and} \quad \langle -6, -4, 0\rangle$$

4. Construct the projection vector $\text{proj}_{\overrightarrow{B}} \overrightarrow{A}$ for each pair of vectors.

a.) $\overrightarrow{A} = \overrightarrow{(3,4)}, \overrightarrow{B} = \overrightarrow{(-1,2)}$

The projection vector can be found by

$$\text{proj}_{\overrightarrow{B}} \overrightarrow{A} = \left(\overrightarrow{A} \cdot \hat{B}\right) \hat{B}$$

where $\hat{B}$ is the unit vector in the same direction of $\overrightarrow{B}$ given as $\hat{B} = \overrightarrow{B}/\left|\overrightarrow{B}\right|$. Then we get

$$\hat{B} = \frac{\overrightarrow{B}}{\left|\overrightarrow{B}\right|} = \frac{\langle -1,2\rangle}{|\langle(-1,2\rangle|} = \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

then we get

$$\text{proj}_{\overrightarrow{B}} \overrightarrow{A} = \left(\overrightarrow{A} \cdot \hat{B}\right) \hat{B} = \left(-\frac{3}{\sqrt{5}} + \frac{8}{\sqrt{5}}\right) \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle = \langle -1,2\rangle$$

Note, the formula that is used in class is equivalent, but I like this one because it's a little more intuitive. The projection of $\overrightarrow{A}$ onto $\overrightarrow{B}$ has magnitude of $\overrightarrow{A}$ dotted with the unit vector $\hat{B}$ and is in the direction of $\overrightarrow{B}$.

b.) $\overrightarrow{A} = \overrightarrow{(1,2,3)}, \overrightarrow{B} = \overrightarrow{(3,2,1)}$

$$\hat{B} = \frac{\overrightarrow{B}}{\left|\overrightarrow{B}\right|} = \frac{\langle 3,2,1\rangle}{|\langle 3,2,1\rangle|} = \left\langle \frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}} \right\rangle$$

then we get

$$\text{proj}_{\overrightarrow{B}} \overrightarrow{A} = \left(\overrightarrow{A} \cdot \hat{B}\right) \hat{B} = \left(\frac{3}{\sqrt{14}} + \frac{4}{\sqrt{14}} + \frac{3}{\sqrt{14}}\right) \left\langle \frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}} \right\rangle = \frac{10}{14} \langle 3,2,1\rangle = \frac{5}{7} \langle 3,2,1\rangle$$

5. Determine the area of the triangle formed by the points

a.) (1,1),(4,2), and (-3,3)

We could go through the process of trying to compute the area with geometry, but we can also use the fact that the magnitude of the parallelogram from two vectors is the magnitude of their cross product. Thus the triangle would be half this area. Thus we get

$$\text{Area of Triangle} = \frac{1}{2}\left|\overrightarrow{A} \times \overrightarrow{B}\right|$$

where $\overrightarrow{A} = \langle 3,1\rangle$ and $\overrightarrow{B} = \langle 4,2\rangle$. Then we get

$$\text{Area of Triangle} = \frac{1}{2}\left|\overrightarrow{A} \times \overrightarrow{B}\right| = \frac{1}{2}\,|\langle 0,0,2\rangle| = \frac{1}{2}2 = 1$$

b.) (0,0,0),(3,-2,1), and (1,0,2)

$$\text{Area of Triangle} = \frac{1}{2}\left|\overrightarrow{A} \times \overrightarrow{B}\right| = \frac{1}{2}\,|\langle 3,-2,1\rangle \times \langle 1,0,2\rangle| = \frac{1}{2}\,|\langle -4,5,2\rangle| = \frac{1}{2}\sqrt{45} = \frac{3}{2}\sqrt{5}$$

6. Compute the area of the parallelogram formed by the vectors $\overrightarrow{A} = \overrightarrow{\langle 4, -1, 2\rangle}$ and $\overrightarrow{B} = \overrightarrow{\langle 2, 3, 0\rangle}$

As stated in 5 we can just compute the magnitude of the cross product, so we get

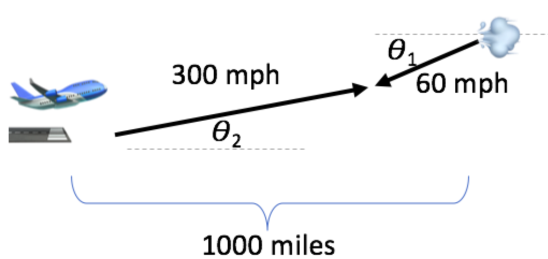
$$\left|\overrightarrow{A} \times \overrightarrow{B}\right| = |\langle 4, -1, 2\rangle \times \langle 2, 3, 0\rangle| = |\langle 6, 4, 14\rangle| = \sqrt{248} = 2\sqrt{62}$$

7. Compute the volume of the parllelpiped formed by the vectors $\overrightarrow{u} = \overrightarrow{i} + \mathbf{2}\overrightarrow{j} - \overrightarrow{k}$, $\overrightarrow{v} = \mathbf{2}\overrightarrow{i} + \overrightarrow{j} + \mathbf{3}\overrightarrow{k}$, and $\overrightarrow{w} = \overrightarrow{i} - \overrightarrow{j} + \mathbf{2}\overrightarrow{k}$

Here we can compute the magnitude of the cross product to get the area of the parallelogram spanned by two of the vectors, then multiply that by the magnitude of the third vector. I'm just going to do this in Matlab.

$$|\overrightarrow{u} \times \overrightarrow{v}|\,|\overrightarrow{w}| \approx 22.32$$

8. A jet airplane wants to fly in a straight line from airport A directly East to airport B, which is 1000 miles away. The jet is pushed by a tailwind from 30° South of West at 60 mph. If the jet flies at a constant speed of 300 mph (relative to the surrounding air space),



1. Plane travelling east with a SE headwind

a.) in what direction should the jet fly?

We need the north component of the plane's trajectory to balance the south component of the wind. Thus we get

$$300 \sin\theta_2 = 60 \sin\theta_1 = 60 \sin 30° = 30$$

solving for θ_2 gives us

$$\theta_2 = \arcsin\left(\frac{1}{10}\right)$$

b.) what is the jet's actual flying speed (relative to the ground)?

This is the plane's x-component, but we need to also subtract the wind's x component as well. Thus

$$300 \cos\theta_2 - 60 \cos\theta_1 = \sqrt{10^2 - 1} - 30\sqrt{3} \approx 268.5$$

c.) how long will the flight take?

This is just 1000 miles divided by the speed or

$$\frac{1000}{268.4962} = 3.7244 \text{ hours}$$

9. Determine parametric equations for the line L passing through the points (1,-1,2) and (3,0,-4).

The line passing through this equation must solve the system of equations given by

$$\begin{cases} x - y + 2z = 0 \\ 3x - 4z = 0 \end{cases}$$

Solving this we get $x = \frac{4}{3}z$, $y = \frac{10}{3}z$, and let $z = t$. Then we have

$$L : \begin{cases} x = \frac{4}{3}t \\ y = \frac{10}{3}t \\ z = y \end{cases}$$

10. Determine parametric equation for the line L passing through the point (2,1,-3) and which is parallel to the line M given by.

$$M : \begin{cases} x & = & 1 + t \\ y & = & 2 - t \\ z & = & 3t \end{cases}$$

11. Determine if the following lines intersect. If they do, find the point of intersection and the angle between the lines.

$$L : \begin{cases} x & = & 3 - t \\ y & = & 2 + t \\ z & = & t \end{cases}$$

$$M : \begin{cases} x & = & 3 + s \\ y & = & 3 - s \\ z & = & 2s + 7 \end{cases}$$

For the points to intersect we need

$$3 - t = 3 + s \iff t = -s$$

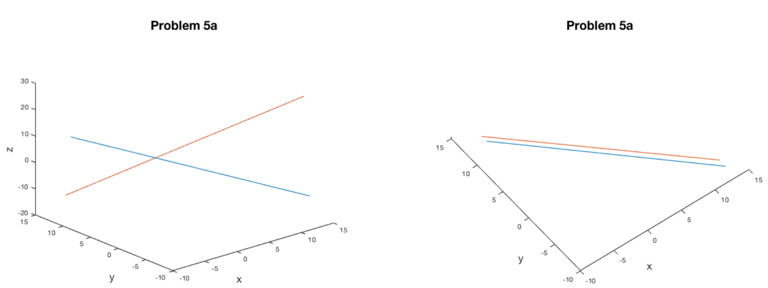
and

$$2 + t = 3 - s \iff t = 1 - s$$

and

$$t = 2s + 7$$

and since each of these equations has different solutions for t and s, we can't find a value for t or s that satisfies each of them. Thus they don't intersect.



2. Plotting two different angles of the Lines M and L

12. Determine parametric equations for the line L representing the intersection of the planes $\mathbf{x+y-2z=10}$ and $\mathbf{3x+2y+z=5}$.

13. Determine the angle between the intersecting planes $\mathbf{x+y-2z=10}$ and $\mathbf{3x+2y+z=5}$

14. Consider the plane given by $\mathbf{2x-4y+5z=20}$.

a.) Find 3 points lying on this plane.

We can just plug in anything for x, y, and z that satisfies this equation. Chooosing points with as many zeros as possible is always my go to. Thus

$$2(0)-4(0)+5z=20 \iff z=4$$

thus $(0,0,4)$ is on the plane. The next two points can be found similarly as $(10,0,0)$ and $(0,-5,0)$. Thus we have

$$(0,0,4),\ (10,0,0),\ \text{and } (0,-5,0)$$

b.) Find 2 vectors perpendicular (normal) to the plane.

The easiest way to find two vectors normal to the plane is to cheat and recognize that any plane given by $ax+by+cz=d$ has normal vector $\langle a,b,c\rangle$. Thus the two vectors are

$$\langle 2,-4,5\rangle \text{ and } \langle -2,4,-5\rangle$$

Note, I think the point of this problem was to construct two vectors from the 3 points in **a)** and then compute the cross product of those vectors. Should give you the same answer (scaled by some value).

c.) Find a vector parallel to the plane's surface.

Can just use two of the 3 points we computed. Thus a vector is

$$\langle 10-0, 0-0, 0-4\rangle = \langle 10,0,-4\rangle$$

15. Determine an equation of the plane passing through the points (0,0,0),(1,0,-2), and (0,3,4).

Can just produce a system of equation by plugging each point into a general plane $ax+by+cz=d$. Thus we have

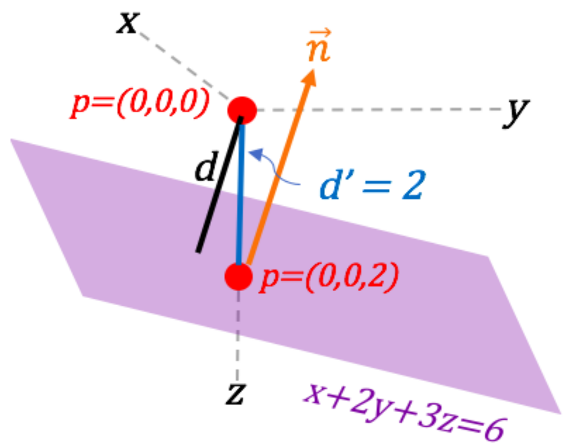
$$\begin{cases} 0=d \\ a-2c=d \\ 3b+4c=d \end{cases}$$

Thus we have $d=0$, $a=2c$, and $b=\frac{4}{3}c$. Since 3 equations can't solve for 4 variables, we have to settle for a parametrized solution. Let $c=1$, then the equation for this plane is

$$2x+\frac{4}{3}y+z=0$$

16. Compute the distance from the origin to the plane given by $\mathbf{x + 2y + 3z = 6}$.

We can easily pick a random point in the plane by setting $x = y = 0$ then $z = 2$ solves the plane equation. Then we can compute the normal vector to the plane as $\overrightarrow{n} = \langle 1, 2, 3 \rangle$. What we need to do is find a way to use these pieces of information to determine the distance between the point and plane. Look at the diagram for this problem here.



3. Diagram of computing distance from a plane to the origin

Notice in this diagram I flipped the axes so the view was better (remember the "right-hand rule"). For here, we can easily compute the distance our point $(0, 0, 2)$ to the origin as 2. However, this isn't the "closest" part of the plane to our target point. The closest point will be directly from the plane to the origin following an orthogonal path. This is where the normal vector comes into play. If we project the distance we just calculated onto the normal vector, we will get the part of that distance that is orthogonal to the plane; or the closest distance to the origin. Thus we are looking for

$$d = \left|\text{proj}_{\overrightarrow{n}} \overrightarrow{d'}\right| = \left|\frac{\overrightarrow{d'} \cdot \overrightarrow{n}}{|\overrightarrow{n}|^2} \overrightarrow{n}\right| = \left|\frac{\langle 0, 0, 2 \rangle \cdot \langle 1, 2, 3 \rangle}{|\langle 1, 2, 3 \rangle|^2} \langle 1, 2, 3 \rangle\right| = \left|\frac{6}{13} \langle 1, 2, 3 \rangle\right| = \frac{6}{\sqrt{13}} \approx 1.664$$

Note, if what I just computed was larger than 2 I would have been nervous the answer is wrong. Why? Because when we project a vector, $\overrightarrow{A}$, onto another vector, $\overrightarrow{B}$, we are only looking at the fraction that $\overrightarrow{A}$ points in $\overrightarrow{B}$. Thus the amount of $\overrightarrow{A}$ pointing in $\overrightarrow{B}$ can not be more than $\overrightarrow{A}$ in total.

17. Compute the distance between the parllel planes given by $\mathbf{x + 2y + 3z = 6}$ and $\mathbf{x + 2y + 3z = 0}$

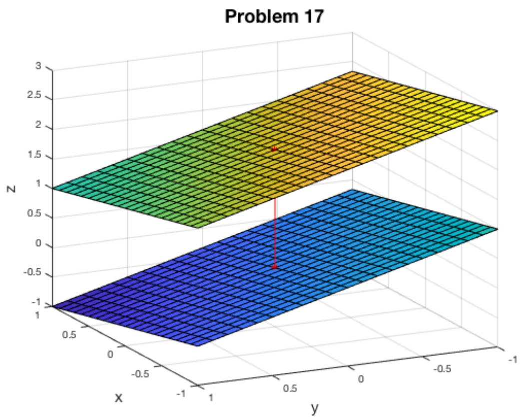
We can compute the distance using the equation

$$d = \left|\frac{d_1 - d_2}{|\overrightarrow{n}|}\right|$$

where $\overrightarrow{n}$ is the normal vector from either plane. This comes from choosing a random point on one plane and then computing the distance from the other plane to that point (like in 16). Thus we have

$$d = \left|\frac{d_1 - d_2}{|\overrightarrow{n}|}\right| = \left|\frac{6}{\sqrt{13}}\right| = \frac{6}{\sqrt{13}} \approx 1.664$$

Note this can be easily determined because the point (0,0,0) is on the second plane, and we already computed the distance to the origin in problem 16.



4. Distance between two parallel planes

18. Find the point of intersection of the plane given by $\mathbf{x + 2y + 3z = 6}$ and the line given by

$$L : \begin{cases} x &= 3 - t \\ y &= 2 + t \\ z &= t \end{cases}$$

19. Consider the vectors $\overrightarrow{A} = \overrightarrow{(a_1, a_2, a_3)}$, $\overrightarrow{B} = \overrightarrow{(b_1, b_2, b_3)}$, and $\overrightarrow{C} = \overrightarrow{(c_1, c_2, c_3)}$. Prove that $\overrightarrow{A} \cdot (\overrightarrow{B} + \overrightarrow{C}) = \overrightarrow{A} \cdot \overrightarrow{B} + \overrightarrow{A} \cdot \overrightarrow{C}$

Let $\overrightarrow{A} = \overrightarrow{(a_1, a_2, a_3)}, \overrightarrow{B} = \overrightarrow{(b_1, b_2, b_3)}$, and $\overrightarrow{C} = \overrightarrow{(c_1, c_2, c_3)}$. The we obtain

$$\begin{aligned}
\overrightarrow{A} \cdot (\overrightarrow{B} + \overrightarrow{C}) &= vec(a_1, a_2, a_3) \cdot \left(\overrightarrow{(b_1, b_2, b_3)} + \overrightarrow{(c_1, c_2, c_3)}\right) \\
&= \overrightarrow{(a_1, a_2, a_3)} \cdot \left(\overrightarrow{(b_1 + c_1, b_2 + c_2, b_3 + c_3)}\right) \\
&= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\
&= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3b_3 \\
&= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3b_3) \\
&= \overrightarrow{A} \cdot \overrightarrow{B} + \overrightarrow{A} \cdot \overrightarrow{C}
\end{aligned}$$

20. Consider the vectors $\overrightarrow{A} = \overrightarrow{(a_1, a_2, a_3)}$, $\overrightarrow{B} = \overrightarrow{(b_1, b_2, b_3)}$, and $\overrightarrow{C} = \overrightarrow{(c_1, c_2, c_3)}$. Prove that $\overrightarrow{A} \times (\overrightarrow{B} + \overrightarrow{C}) = \overrightarrow{A} \times \overrightarrow{B} + \overrightarrow{A} \times \overrightarrow{C}$

Just follow the definition of cross product for this. It's really just a lot of algebra, so I'm probably not going to do this.

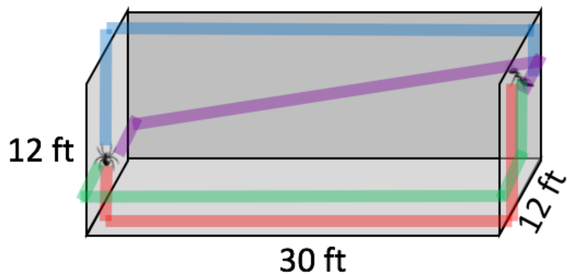
21. Consider the vectors $\overrightarrow{A} = \overrightarrow{(a_1, a_2, a_3)}$ and $\overrightarrow{B} = \overrightarrow{(b_1, b_2, b_3)}$. Prove that $\left|\overrightarrow{A} \times \overrightarrow{B}\right|^{\mathbf{2}} = \left|\overrightarrow{A}\right|^{\mathbf{2}} \left|\overrightarrow{B}\right|^{\mathbf{2}} - (\overrightarrow{A} \cdot \overrightarrow{B})^{\mathbf{2}}$

Just follow the definition of cross product for this. It's really just a lot of algebra, so I'm probably not going to do this.

22. Consider the vectors $\overrightarrow{A} = \overrightarrow{(a_1, a_2, a_3)}$ and $\overrightarrow{B} = \overrightarrow{(b_1, b_2, b_3)}$. Prove that $\overrightarrow{A} \perp \overrightarrow{A} \times \overrightarrow{B}$

23. Consider the vector $\overrightarrow{A} = \overrightarrow{(a_1, a_2, a_3)}$. Prove that $\overrightarrow{A} \times \overrightarrow{A} = \overrightarrow{O}$

24. A 12ft. by 30 ft. room has a 12 ft. ceiling. In the middle of one end wall, one foot above the floor, is a spider. The spider wants to capture a fly in the middle of the opposite end wall, one foot below the ceiling. What is the length of the shortest path the spider can walk (no spider webs allowed) in order to reach the fly?



5. Diagram of different paths spider can take

We have a few different paths the spider can take. The length of each path is

Red :	1 + 30 + (12-1)	= 42 `ft`
Blue :	(12-1) + 30 + 1	= 42 `ft`
Purple :	$6 + \sqrt{30^2 + (12-2)^2} + 6 \approx 12 + 31.6$	= 43.6 `ft`
Green :	6 + 30 + 6 + 10	= 52 `ft`

Thus 42 `ft` is the shortest path I can find (without webs).

Code

```
% ------------------------------------------------------------
% Problem 6 - Compute area of parallelpiped
% ------------------------------------------------------------
u = [1,2,-1];
v = [2,1,3];
w = [1,-1,2];
area = norm(cross(u,v))*norm(w);
sprintf('Area of Parallelpiped is %f',area)

% ------------------------------------------------------------
% Problem 11 - Plot lines L and M
% ------------------------------------------------------------
t = -10:.1:10;
s = -10:.1:10;
L = [3-t;2+t;t];
M = [3+s;3-s;2*s+7];
figure;
plot3(L(1,:),L(2,:),L(3,:)); hold on;
plot3(M(1,:),M(2,:),M(3,:));
title('Problem 5a','Fontsize',18);xlabel('x','Fontsize',15);
ylabel('y','Fontsize',15);zlabel('z','Fontsize',15);

% ------------------------------------------------------------
% Problem 17 - Distance between 2 parallel planes
% ------------------------------------------------------------
[x,y] = meshgrid(-1:.1:1);
z1 = (1/3)*(6 - 2*y-x);
z2 = (1/3)*(0 - 2*y-x);
figure;
surf(x,y,z1); hold on;
surf(x,y,z2);
title('Problem 17','Fontsize',18);xlabel('x','Fontsize',15);
ylabel('y','Fontsize',15);zlabel('z','Fontsize',15);

% Plot normal vector of plane pointing
% down, since (0,0,0) is below plane
n  = -[1,2,3];
p  = [0,0,2]; % point right under origin
pn = [0,0,0];
plot3([p(1) pn(1)], [p(2) pn(2)], [p(3) pn(3)],'*-r');
```