Math 21C
Kouba
Disussion Sheet 2 - Key
Answers by Doug

1.) Determine convergence or divergence of each series using the test indicated. I suggest that your read all of the assumptions and conclusions for each test as you do each problem.

a.) $\sum_{\mathbf{n=3}}^{\infty} \frac{\mathbf{2n+3}}{\mathbf{3n+2}}$ **(Use the nth term test.)**

Let $a_n := \frac{2n+3}{3n+2}$ then we obtain

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n+3}{3n+2} = \lim_{n\to\infty} \frac{\frac{2n}{n}+\frac{3}{n}}{\frac{3n}{n}+\frac{2}{n}} = \frac{2+0}{3+0} = \frac{2}{3} \neq 0$$

Therefore, since the terms in ther series $\sum_{n=3}^{\infty} a_n$ do not approach 0, we conclude this series diverges by the nth term test.

b.) $\sum_{\mathbf{n=4}}^{\infty} \mathbf{7}\frac{\mathbf{(-2)^{n+1}}}{\mathbf{3^{n-1}}}$ **(Use the geometric series test.)**

With the geometric series test we must get our series into the form of $\sum (r)^n$ for $|r| < 1$. So, for the given series we obtain

$$\sum_{n=4}^{\infty} 7\frac{(-2)^{n+1}}{3^{n-1}} = \sum_{n=4}^{\infty} 7\frac{(-2)^{n-1}(-2)^2}{3^{n-1}} = \sum_{n=4}^{\infty} 7(-2)^2\left(\frac{-2}{3}\right)^{n-1} = 28\sum_{n=4}^{\infty}\left(-\frac{2}{3}\right)^{n-1}$$

Thus since $|-2/3| = 2/3 < 1$ we conclude that this series converges by the geometric series test.

c.) $\sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{1}}{\mathbf{n^{\sqrt{2}}}}$ **(Use the p-series test.)**

By the p-series test we have the series $\sum \frac{1}{n^p}$ converge if $p > 1$. For our series we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}} \approx \sum_{n=1}^{\infty} \frac{1}{n^{1.4\cdots}}$$

Thus, since $\sqrt{2} \approx 1.4... > 1$, we conclude that our series converges by the p-series test with $p = \sqrt{2} > 1$.

d.) $\sum_{\mathbf{n=2}}^{\infty} \frac{\mathbf{n}}{\mathbf{n^2+4}}$ **(Use the integral test.)**

The integral test states that for $a_n = f(n)$ we have

$$\int_k^{\infty} f(x)\ dx \leq \sum_{n=k}^{\infty} a_n$$

Thus define $f(x) = \frac{x}{x^2+4}$ and we obtain

$$\int_2^{\infty} f(x)\ dx = \int_2^{\infty} \frac{x}{x^2+4}\ dx = \int_2^{\infty} \frac{1}{2}\frac{1}{x^2+4}\ 2xdx = \int_4^{\infty} \frac{1}{2}\frac{1}{u}\ du = \frac{1}{2}\ \ln u|_{u=4}^{\infty} \to \infty$$

Thus, by the integral test, since $\int f(x)\ dx$ diverges, we conclude that $\sum a_n$ must diverge as well.

e.) $\sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{1}}{\sqrt{\mathbf{n+1}}} - \frac{\mathbf{1}}{\sqrt{\mathbf{n+2}}}$ **(Use the sequence of partial sums test.)**

Let $a_n := \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n+2}}$ and define the nth partial sum, s_n, as $s_n := \sum_{k=1}^{n} a_k$. Then the first few partial sums are given by

$$\begin{aligned}
s_1 &= \sum_{k=1}^{1} a_k = \frac{1}{\sqrt{1+1}} - \frac{1}{\sqrt{1+2}} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} \\
s_2 &= \sum_{k=1}^{2} a_k = \frac{1}{\sqrt{1+1}} - \cancel{\frac{1}{\sqrt{1+2}}} + \cancel{\frac{1}{\sqrt{2+1}}} - \frac{1}{\sqrt{2+2}} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{4}} \\
s_3 &= \sum_{k=1}^{3} a_k = \frac{1}{\sqrt{1+1}} - \cancel{\frac{1}{\sqrt{1+2}}} + \cancel{\frac{1}{\sqrt{2+1}}} - \cancel{\frac{1}{\sqrt{2+2}}} + \cancel{\frac{1}{\sqrt{2+2}}} - \frac{1}{\sqrt{3+2}} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{5}} \\
&\vdots
\end{aligned}$$

Thus we see that the pattern for the nth partial sum is

$$\sum_{k=1}^{n} a_k = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{n+2}}$$

Thus, taking the limit as n approaches infinity we obtain,

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{n+2}} = \frac{1}{\sqrt{2}}$$

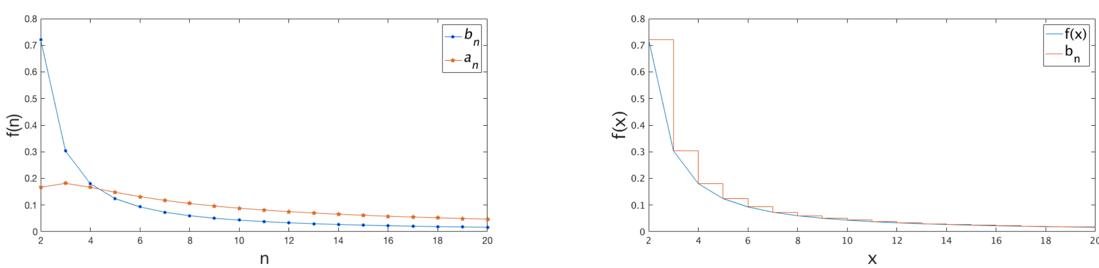
Therefore we conclude that the series converges to $1/\sqrt{2}$, so it clearly converges.

f.) $\sum_{\mathbf{n=2}}^{\infty} \frac{\mathbf{n-1}}{\mathbf{n^3+2}}$ **(Use the comparison test.)**

Let $a_n := \frac{n-1}{n^2+2}$, notice that for large n we have a_n similar to the function

$$\frac{n - \cancelto{0}{1}}{n^2 + \cancelto{0}{2}} \sim \frac{n}{n^2} = \frac{1}{n}$$

So we can predict that this will likely diverge because $\sum \frac{1}{n}$ diverges. One may be tempted to just use this function as a comparison, but a_n is clearly less than $1/n$ because the numerator is smaller and the denominator is larger. Since $1/n$ diverges, saying a_n is smaller tells us nothing. So instead, let $b_n := \frac{1}{n \ln n}$ and then we notice that for $n > 4$ we have $a_n > b_n$. Moreover, we have for $f(x) = \frac{1}{x \ln x}$ that $\int_1^{\infty} f(x)\, dx \leq \sum_{n=1}^{\infty} b_n$. This is more clear in the images below, thus we say that $a_n > b_n$ for $n > 4$, and $\sum b_n$ diverges by the integral test, so $\sum a_n$ diverges by the comparison test.



1. Visual representation of the results of problem f

g.) $\sum_{n=1}^{\infty} \frac{n^3+7n^2-3}{n^4-4n+9}$ (Use the limit comparison test.)

When looking at this sequence, we are really only concerned the largest power of n in both the numerator and the denomination. So we notice that

$$\frac{n^3+\cancel{7n^2}-\cancel{3}}{n^4-\cancel{4n}+\cancel{9}} \sim \frac{n^3}{n^4} = \frac{1}{n}$$

So this means we will likely want to use $b_n := 1/n$ for our comparison sequence. This is different than in f) because in the limit comparison test we are simply saying that one function is like the other, so we need not be concerned if one is greater or less than the other. So, let $a_n := \frac{n^3+7n^2-3}{n^4-4n+9}$, and by the Limit Comparison Test we obtain

$$\left|\frac{a_n}{b_n}\right| = \left|\frac{\frac{n^3+7n^2-3}{n^4-4n+9}}{\frac{1}{n}}\right| = \left|\left(\frac{n^3+7n^2-3}{n^4-4n+9}\right)\cdot n\right| = \left|\frac{n^4+7n^3-3n}{n^4-4n+9}\right|$$

Thus

$$\lim_{n\to\infty}\left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty}\left|\frac{n^4+7n^3-3n}{n^4-4n+9}\right| = \lim_{n\to\infty}\left|\frac{\frac{n^4}{n^4}+7\cancelto{0}{\frac{n^3}{n^4}}-3\cancelto{0}{\frac{n}{n^4}}}{\frac{n^4}{n^4}-4\cancelto{0}{\frac{n}{n^4}}+9\cancelto{0}{\frac{1}{n^4}}}\right| = 1$$

Therefore $\sum a_n$ has the same convergence behavior as $\sum b_n$. Since $\sum b_n$ is the harmonic series and diverges, by p-series test ($p \leq 1$), we conclude that $\sum a_n$ must diverge as well.

h.) $\sum_{n=0}^{\infty} \frac{3^{n-1}}{(n+1)!}$ (Use the ratio test.)

Let $a_n := \frac{3^{n-1}}{(n+1)!}$, then the ratio of a_{n+1} and a_n is given by

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{3^n}{(n+2)!}}{\frac{3^{n-1}}{(n+1)!}}\right| = \left|\left(\frac{3^n}{(n+2)!}\right)\left(\frac{(n+1)!}{3^{n-1}}\right)\right| = \left|\left(\frac{3\cdot\cancel{3^{n-1}}}{(n+2)\cancel{(n+1)!}}\right)\left(\frac{\cancel{(n+1)!}}{\cancel{3^{n-1}}}\right)\right| = \left|\frac{3}{n+2}\right|$$

Taking the limit as n goes to infinity gives us,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3}{n+2}\right| = 0$$

Thus, by the Ratio Test, we conclude that $\sum a_n$ converges.

i.) $\sum_{n=1}^{\infty}\left(1.01-\frac{5}{n^3}\right)^n$ (Use the root test.)

Let $a_n := \left(1.01-\frac{5}{n^3}\right)^n$ then we obtain

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left|\left(1.01-\frac{5}{n^3}\right)^n\right|} = \sqrt[n]{\left|\left(1.01-\frac{5}{n^3}\right)\right|^n} = \left|1.01-\frac{5}{n^3}\right|$$

Thus, taking the limit as n goes to infinity, we obtain

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left|1.01-\frac{5}{n^3}\right| = 1.01 > 1$$

Therefore, by the Root Test, the series $\sum a_n$ diverges.