Math 21B
Disussion Sheet 1 - Key
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*Note: These answers are not endorsed by Dr. Gravner and may be incorrect!*

**1. Compute the following limits, in any correct way you can.**

**(a)** $\lim_{\mathbf{x}\to\infty}\left(\sqrt{\mathbf{x^2+x}}-\sqrt{\mathbf{x^2-x}}\right)$

At first glance, if we simply take $x\to\infty$ we get

$$\sqrt{x^2+x}-\sqrt{x^2-x}\Big|_{x\to\infty}=\infty-\infty$$

which is indeterminant, but not a form where L'Hôpital's rule can be used. Thus, we need to re-arrange this in order to properly take the limit. Notice that it is pretty much always the case that we should multiply by the conjugate with radical functions, so we obtain

$$\sqrt{x^2+x}-\sqrt{x^2-x}=\sqrt{x^2+x}-\sqrt{x^2-x}\left(\frac{\sqrt{x^2+x}+\sqrt{x^2-x}}{\sqrt{x^2+x}+\sqrt{x^2-x}}\right)=\frac{x^2+x-x^2+x}{\sqrt{x^2+x}+\sqrt{x^2-x}}=\frac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}}$$

Here we can divide top and bottom by the largest power of $x$, which is $x=\sqrt{x^2}$, thus

$$\begin{aligned}\lim_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)&=\lim_{x\to\infty}\frac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}}\\&=\lim_{x\to\infty}\frac{\frac{2x}{x}}{\frac{\sqrt{x^2+x}}{\sqrt{x^2}}+\frac{\sqrt{x^2-x}}{\sqrt{x^2}}}\\&=\lim_{x\to\infty}\frac{2}{\sqrt{1+1/x}+\sqrt{1-1/x}}\\&=\frac{2}{\sqrt{1+0}+\sqrt{1-0}}=\frac{2}{2}=1\end{aligned}$$

Therefore we conclude that

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)=1$$

**(b)** $\lim_{\mathbf{x}\to\mathbf{0}}\frac{\ln(\mathbf{1+x})-\mathbf{x}}{\mathbf{e^x-1-x}}$

Simply plugging in $x=0$ we obtain

$$\frac{\ln(1+x)-x}{e^x-1-x}\Big|_{x=0}=\frac{\ln(1)-0}{e^0-1-0}=\frac{0}{0}$$

Hence, we have another indeterminant form, but this time it suffices to use L'Hôpital's rule. Thus we obtain

$$\lim_{x\to0}\frac{\ln(1+x)-x}{e^x-1-x}\overset{\mathcal{L}'\mathcal{H}}{=}\lim_{x\to0}\frac{\frac{1}{1+x}-1}{e^x-1}$$

Note that we once again get a limit of the form $\frac{0}{0}$, so one more iteration of L'Hôpital's rule gives us

$$\lim_{x\to0}\frac{\ln(1+x)-x}{e^x-1-x}\overset{\mathcal{L}'\mathcal{H}}{=}\lim_{x\to0}\frac{\frac{1}{1+x}-1}{e^x-1}\overset{\mathcal{L}'\mathcal{H}}{=}\lim_{x\to0}\frac{\frac{1}{(1+x)^2}}{e^x}=\frac{\frac{1}{1}}{1}=1$$

Therefore we conclude that

$$\lim_{x\to 0} \frac{\ln(1+x)-x}{e^x-1-x} = 1$$

**(c)** $\lim_{\mathbf{h}\to\mathbf{0}} \frac{\mathbf{(2+h)^5-32}}{\mathbf{h}}$

Here we could be clever and recognize this is the limit definition of a derivative of $x^5$ at $x=2$, then we obtain

$$\lim_{h\to 0} \frac{(2+h)^5-32}{h} = \frac{d}{dx}x^5\Big|_{x=2} = 5(2)^4 = 80$$

Or we can use expand $(2+h)^5 = (1)2^5 + (5)2^4h + (10)2^3h^2 + (10)2^2h^3 + (5)2h^4 + (1)h^5$ and thus

$$\begin{aligned}\lim_{h\to 0} \frac{(2+h)^5-32}{h} &= \lim_{h\to 0} \frac{2^5 + 2^4h + 2^3h^2 + 2^2h^3 + 2h^4 + h^5 - 32}{h} \\ &= \lim_{h\to 0} (5)2^4 + (10)2^3h + (10)2^2h^2 + (5)2h^3 + (1)h^4 \\ &= 5(2)^4 = 80\end{aligned}$$

Therefore we conclude that

$$\lim_{h\to 0} \frac{(2+h)^5-32}{h} = 5(2)^4 = 80$$

**(d)** $\lim_{\mathbf{x}\to\infty} \left(\frac{\mathbf{2x-3}}{\mathbf{2x-4}}\right)^{\mathbf{3x-4}}$

The only limit I know how to take that looks similar to this is

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{ax} = e^a$$

Thus, I am going to try to get this into that form. Since I need to separate $\frac{2x-3}{2x-4}$ into 1 + something, I am going to use a substitution to make things easier. Thus, let $u = 2x-4$ and we no longer have a mixed term in the denominator. Note we lucked out that if $x\to\infty$, then $u = 2x-4$ also goes to infinity, but you need to make sure this is the case if you do a substitution. Thus using this substitution we obtain

$$\left(\frac{2x-3}{2x-4}\right)^{3x-4} = \left(\frac{u+1}{u}\right)^{3/2u+2} = \left(1+\frac{1}{u}\right)^{3/2u}\left(\frac{u+1}{u}\right)^2$$

Hence the limit becomes

$$\begin{aligned}\lim_{x\to\infty} \left(\frac{2x-3}{2x-4}\right)^{3x-4} &= \lim_{x\to\infty} \left(1+\frac{1}{u}\right)^{3/2u}\left(\frac{u+1}{u}\right)^2 \\ &= \lim_{x\to\infty} \left(1+\frac{1}{u}\right)^{3/2u} \lim_{x\to\infty} \left(\frac{u+1}{u}\right)^2 \\ &= e^{3/2}(1) = e^{3/2}\end{aligned}$$

Therefore we conclude that

$$\lim_{x\to\infty} \left(\frac{2x-3}{2x-4}\right)^{3x-4} = e^{3/2}$$