Definition

The formal definition of a multivariable limit given by

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

is given as follows.

Let $\epsilon > 0$, then for $\delta > 0$, if $\sqrt{(x-a)^2+(y-b)^2} < \delta$ then $|f(x,y)-L| < \epsilon$.

Thus, what we need to do is find δ as some function ϵ that makes this true. The way to do this is start from $|f(x,y)-L|$ and use algebra to get $\sqrt{(x-a)^2+(y-b)^2}$. For example,

$$|f(x,y)-L| < ... < C\sqrt{(x-a)^2+(y-b)^2}$$

thus if say that $\sqrt{(x-a)^2+(y-b)^2} < \delta = \epsilon/C$ then the above equation shows that $|f(x,y)-L| < \epsilon$. Thus the challenge is going from $|f(x,y)-L|$ to $C\sqrt{(x-a)^2+(y-b)^2}$. Below are some tricks for this.

Examples

1. Prove that

$$\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{0},\mathbf{0})} \frac{\mathbf{x^3y-y^3x}}{\mathbf{390}} + \mathbf{10} = \mathbf{10}$$

Let $\sqrt{(x-0)^2+(y-0)^2} = \sqrt{x^2+y^2} < \delta$. Notice that since $\sqrt{x^2+y^2}$ is the hypotenuse of a triangle with sides x and y, we have $|x| \leq \sqrt{x^2+y^2}$ and $|y| \leq \sqrt{x^2+y^2}$. Thus we have $|x| < \delta$ and $|y| < \delta$. Thus we obtain,

$$\begin{aligned}
|f(x,y)-L| &= \left|\frac{x^3y-y^3x}{390} + 10 - 10\right| \\
&= \left|\frac{x^3y-y^3x}{390}\right| \\
&= \frac{1}{390}\left|x^3y-y^3x\right| && \text{Because } 1/390 > 0 \\
&\leq \left|x^3y-y^3x\right| && \text{Get rid of constant because it's gross} \\
&\leq \left|x^3y\right| + \left|-y^3x\right| && \text{By the triangle inequality} \\
&= \left|x^3y\right| + \left|y^3x\right| && \text{By absolute value} \\
&= |x|^3|y| + |y|^3|x| && \text{Again, by absolute value} \\
&< |\delta|^3|\delta| + |\delta|^3|\delta| && \text{Since } |x| < \delta \text{ and } |y| < \delta \\
&= 2\delta^4 && \text{since } 0 < \delta
\end{aligned}$$

Thus what we need is $2\delta^4 < \epsilon \iff \delta < \sqrt[4]{\epsilon/2}$. Therefore we conclude that

$$\lim_{(x,y)\to(0,0)} \frac{x^3y-y^3x}{390} + 10 = 10$$

So, in this problem we were able to "cheat" by saying that $|x| < \delta$ and $|y| < \delta$, so we just substituded the seperate x's and y's with δ's and got a function of only 1 variable.

2. Prove that

$$\lim_{\mathbf{(x,y)\to(1,1)}} \frac{\mathbf{xy}}{\mathbf{x+y}} = \frac{\mathbf{1}}{\mathbf{2}}$$

Here, we can't use the trick that $|x|$ and $|y|$ are less than δ. However, just as above, we can say $\sqrt{(x-1)^2+(y-1)^2} < \delta \iff |x-1| < \delta$ and $|y-1| < \delta$.

$$\begin{aligned}
|f(x,y) - L| &= \left| \frac{xy}{x+y} - \frac{1}{2} \right| \\
&= \left| \frac{xy - x - y}{2x+2y} \right| \\
&\leq \left| \frac{xy + xy - x - y}{2x+2y} \right| \\
&= \left| \frac{x(y-1) + y(x-1)}{2x+2y} \right| \\
&\leq \left| \frac{x(y-1)}{2(x+y)} \right| + \left| \frac{y(x-1)}{2(x+y)} \right| && \text{By Triangle Inequality} \\
&= \frac{1}{2} \left| \frac{x}{x+y} \right| |y-1| + \frac{1}{2} \left| \frac{y}{x+y} \right| |x-1|
\end{aligned}$$

Since we are consider values of x and y where $|x-1| < \delta$ or $|y-1| < \delta$, ensure that $\delta < 1$ then x and y must be positive. Thus $|x/(x+y)|$ or $|y/(x+y)|$ must be less than 1. Thus we obtain

$$\frac{1}{2} \left| \frac{x}{x+y} \right| |y-1| + \frac{1}{2} \left| \frac{y}{x+y} \right| |x-1| \leq \frac{1}{2} |y-1| + \frac{1}{2} |x-1| < \frac{1}{2}\delta + \frac{1}{2}\delta = \delta$$

Thus if we set $\delta = \min\{1, \epsilon\}$, then we conclude that $f(x,y) \to 1/2$ as $(x,y) \to (1,1)$.