1. Write each power series as an ordinary function.

a.) $\sum_{\mathbf{n=5}}^{\infty} \mathbf{x^n}$
Notice that we can re-write this as

$$\sum_{n=5}^{\infty} x^n = \sum_{n=0}^{\infty} x^n - \sum_{n=0}^{4} x^n$$

Since we know the sum starting from 0, we get

$$\sum_{n=5}^{\infty} x^n = \frac{1}{1-x} - 1 - x - x^2 - x^3 - x^4; \quad x \in (-1,1)$$

b.) $\sum_{\mathbf{n=0}}^{\infty} \mathbf{2^n x^n}$

Notice that we can write $2^n x^n = (2x)^n$, then let $y = 2x$ and we obtain

$$\sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} (2x)^n = \sum_{n=2(0)}^{2(\infty)} y^n = \sum_{n=0}^{\infty} y^n = \frac{1}{1-y} = \frac{1}{1-2x}$$

Note that since $y \in (-1,1)$ we have $2x \in (-1,1)$ or $x \in (-1/2, 1/2)$.

c.) $\sum_{\mathbf{n=0}}^{\infty} \frac{\mathbf{(-3)^{n+1} x^n}}{\mathbf{5^{n-1}}}$

We have to do some extensive alebra on this. Notice that we have

$$\frac{(-3)^{n+1}x^n}{5^{n-1}} = \frac{(-3)(-3)^n x^n}{5^{n-1}} = \frac{(5)(-3)(-3)^n x^n}{5^n} = -15\left(\frac{(-3)^n x^n}{5^n}\right) = -15\left(\frac{(-3x)^n}{5^n}\right) = -15\left(\frac{-3x}{5}\right)^n$$

Thus we obtain

$$\sum_{n=0}^{\infty} \frac{(-3)^{n+1}x^n}{5^{n-1}} = \sum_{n=0}^{\infty} -15\left(\frac{-3x}{5}\right)^n = -15\sum_{n=0}^{\infty}\left(\frac{-3x}{5}\right)^n = -15\left(\frac{1}{1-\left(\frac{-3x}{5}\right)}\right) = -15\left(\frac{1}{1+\frac{3x}{5}}\right) = -15\frac{5}{5+3x}$$

With radius of convergence $\frac{3x}{5} \in (-1,1)$ which gives us $x \in (-5/3, 5/3)$.

d.) $\sum_{\mathbf{n=4}}^{\infty} \mathbf{n x^{n-1}}$

For this one we have to recognize that if this were $\sum x^{n-1}$ it would be relatively easy to fix. Notice that

$$\frac{d}{dx} x^n = n x^{n-1}$$

Moreover, we obtain

$$\frac{d}{dx}\left(\sum_{n=4}^{\infty} x^n\right) = \frac{d}{dx}\left(x^4 + x^5 + x^6 + \cdots\right) = 4x^3 + 5x^4 + 6x^5 + \cdots = \sum_{n=4}^{\infty} n x^{n-1}$$

Since we know the first sum above we obtain

$$\sum_{n=4}^{\infty} n x^{n-1} = \frac{d}{dx}\left(\sum_{n=4}^{\infty} x^n\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n - \sum_{n=0}^{3} x^n\right) = \frac{d}{dx}\left(\frac{1}{1-x} - 1 - x - x^2 - x^3\right) = \frac{1}{(1-x)^2} - 1 - 2x - 3x^2$$

where $x \in (-1,1)$

e.) $\sum_{\mathbf{n=0}}^{\infty} \mathbf{n^2 x^{n-1}}$

Just like in d we recognize that we need one derivative to get nx^{n-1}, so a second derivative should get us close to n^2x^{n-1}. Notice that we have

$$x\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty} x\frac{d}{dx}x^n = \sum_{n=0}^{\infty} nx^n$$

Moreover, we have

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} nx^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}nx^n = \sum_{n=0}^{\infty} n^2x^{n-1}$$

which gives us our target series. Thus, using geometric series, we obtain

$$\sum_{n=0}^{\infty} n^2x^{n-1} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} nx^n\right) = \frac{d}{dx}\left(x\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right)\right) = \frac{d}{dx}\left(x\frac{d}{dx}\left(\frac{1}{1-x}\right)\right) = \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) = \frac{(1-x)^2 - 2x(1-x}{(1-x)^4}$$

where $x \in (-1, 1)$ because we used a geometric series.

f.) $\sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{x^{n+3}}}{\mathbf{n}}$

Something to notice here is that

$$\frac{d}{dx}\left(\frac{x^n}{n}\right) = x^{n-1}$$

which is a geometric series that we can solve. This is backwards from what we did in e and d. Since the reverse of a derivative is an integral, we need to use an integral. Thus we obtain

$$\int \sum_{n=1}^{\infty} x^{n-1}\ dx = \int (1 + x + x^2 + x^3 + \cdots)\ dx = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

Thus we can describe the series above as

$$\sum_{n=1}^{\infty} \frac{x^{n+3}}{n} = x^3 \int \sum_{n=1}^{\infty} x^{n-1}\ dx = x^3 \int \frac{1}{x}\left(\frac{1}{1-x}\right)\ dx = x^3\left(\ln(|x|) - \ln(|1-x|)\right) = x^3 \ln\left(\left|\frac{x}{1-x}\right|\right)$$

where we used partial fractions to solve that integral. Note we used a geometric series so $x \in (-1, 1)$, moreover, $x \neq 0$ since $\ln 0$ doesn't exist.

g.) $\sum_{\mathbf{n=1}}^{\infty} (\mathbf{-1})^{\mathbf{n}} \frac{\mathbf{x^n}}{\mathbf{2^n n!}}$

Notice we obtain

$$\frac{x^n}{2^n n!} = \frac{\left(\frac{x}{2}\right)^n}{n!}$$

So what we can do is recognize that since

$$e^x = \sum_{n=1}^{\infty} (-1)^n \left(\frac{x^n}{n!}\right)$$

then we obtain

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{2^n n!} = \sum_{n=1}^{\infty} (-1)^n \left(\frac{(x/2)^n}{n!}\right) = e^{x/2}$$

Where $x \in\ \mathbb{R}$ because e^x is valid everywere.

h.) $\sum_{\mathbf{n=2}}^{\infty}(-\mathbf{1})^{\mathbf{n}}\left(\frac{\mathbf{2}}{\mathbf{5}}\right)^{\mathbf{n}}\frac{\mathbf{x^{2n+1}}}{\mathbf{2n+1}}$

Notice that $\int x^{2n}\,dx = \frac{x^{2n+1}}{2n+1}$, so we will need to use that at some point in this problem. Using some algebra we obtain,

$$(-1)^n\left(\frac{2}{5}\right)^n\frac{x^{2n+1}}{2n+1} = (-1)^n\left(\frac{2}{5}\right)^n\int x^{2n}\,dx = \int(-1)^n\left(\frac{2}{5}\right)^n x^{2n}\,dx = \int(-1)^n\left(\frac{2}{5}\right)^n(x^2)^n\,dx = \int\left(-\frac{2}{5}x^2\right)^n\,dx$$

Moreover, the series $\sum\left(-\frac{2}{5}x^2\right)^n$ is relatively easy to solve using geometric series theorems. Thus we obtain

$$\begin{aligned}
\sum_{n=2}^{\infty}(-1)^n\left(\frac{2}{5}\right)^n\frac{x^{2n+1}}{2n+1} &= \int\sum_{n=2}^{\infty}\left(-\frac{2}{5}x^2\right)^n\,dx\\
&= \int\sum_{n=0}^{\infty}\left(-\frac{2}{5}x^2\right)^n - \sum_{n=0}^{1}\left(-\frac{2}{5}x^2\right)^n\,dx\\
&= \int\frac{1}{1-(-\frac{2}{5}x^2)} - 1 - \left(-\frac{2}{5}x^2\right)\,dx\\
&= \int\frac{1}{1+\frac{2}{5}x^2}\,dx - x + \frac{2}{15}x^3\\
&= \int\frac{1}{1+\left(\sqrt{\frac{2}{5}}x\right)^2}\,dx - x + \frac{2}{15}x^3\\
&= \sqrt{\frac{5}{2}}\arctan\left(\sqrt{\frac{2}{5}}x\right) - x + \frac{2}{15}x^3
\end{aligned}$$

Notice that we used a geometric series, so we need $\frac{2}{5}x^2 \in (-1,1)$ or $x \in \left(-\sqrt{\frac{5}{2}},\sqrt{\frac{5}{2}}\right)$.

2. Use any method to find the exact value of each of the following convergent series.

a.) $\sum_{\mathbf{n=0}}^{\infty}\mathbf{3}\left(\frac{\mathbf{-2}}{\mathbf{3}}\right)^{\mathbf{n}}$

Let $f(x) = \sum_{n=0}^{\infty}x^n = \frac{1}{1-x}$, the the target series is given by

$$\sum_{n=0}^{\infty}3\left(\frac{-2}{3}\right)^n = 3f\left(\frac{-2}{3}\right) = 3\left(\frac{1}{1-\left(\frac{-2}{3}\right)}\right) = 3\left(\frac{3}{5}\right) = \frac{9}{5}$$

b.) $\sum_{\mathbf{n=4}}^{\infty}\frac{(-\mathbf{1})^{\mathbf{n+2}}}{\mathbf{2}}^{\mathbf{n-3}}$

c.) $\sum_{\mathbf{n=1}}^{\infty}\mathbf{n^2}\left(\frac{\mathbf{1}}{\mathbf{2}}\right)^{\mathbf{n}}$

d.) $\sum_{\mathbf{n=0}}^{\infty}\mathbf{n(n-1)}\left(\frac{\mathbf{3}}{\mathbf{4}}\right)^{\mathbf{n+1}}$

e.) $\sum_{\mathbf{n=0}}^{\infty}\frac{(\ln\mathbf{2})^{\mathbf{n}}}{\mathbf{n!}}$

f.) $\sum_{\mathbf{n=2}}^{\infty}(-\mathbf{1})^{\mathbf{n}}\frac{\mathbf{9^n}}{\mathbf{(2n)!}}$