

1. Evaluate the following limits or determine that the limit does not exist

a.) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2+y^2-4}{x+y+2}$

We can simply plug in $(x, y) = (0, 0)$ and we obtain

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2 - 4}{x + y + 2} = \frac{-4}{2} = -2$$

b.) $\lim_{(x,y) \rightarrow (1,1)} \frac{xy-y-2x+2}{x-1}$

Unlike in a.), if we simply plug in $(1,1)$ for (x, y) we get an undefined value. This is because of the $x - 1$ in the denominator. Thus, we need to remove this discontinuity. To do this we need to factor the top. Notice that we have,

$$xy - y - 2x + 2 = (xy - 2x) - (y - 2) = x(y - 2) - (y - 2) = (y - 2)(x - 1)$$

Thus the limit becomes

$$\lim_{(x,y) \rightarrow (1,1)} \frac{xy - y - 2x + 2}{x - 1} = \lim_{(x,y) \rightarrow (1,1)} \frac{(y - 2)(x - 1)}{x - 1} = \lim_{(x,y) \rightarrow (1,1)} y - 2 = -1$$

c.) $\lim_{(x,y) \rightarrow (2,2)} \frac{x+y-4}{\sqrt{x+y}-2}$

Let $r = \sqrt{x + y}$, then notice that we have

$$\frac{x + y - 4}{\sqrt{x + y} - 2} = \frac{r^2 - 2^2}{r - 2} = \frac{(r - 2)(r + 2)}{r - 2} = r + 2 = \sqrt{x + y} + 2$$

Thus we can evaluate the limit as

$$\lim_{(x,y) \rightarrow (2,2)} \frac{x + y - 4}{\sqrt{x + y} - 2} = \lim_{(x,y) \rightarrow (2,2)} \sqrt{x + y} + 2 = \sqrt{4} + 2 = 4$$

d.) $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

Let $r = x^2 + y^2$, then we obtain

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = \lim_{r \rightarrow 0} \frac{\sin(r)}{r} = {}^1 1$$

e.) $\lim_{(x,y) \rightarrow (1,1)} \frac{\sin(x^2-y^2)}{x-y}$

Here we can't use the same trick as before. However, we see that we have $x^2 - y^2 = (x - y)(x + y)$ in the numerator, which could cancel the denominator if we didn't have the sine. Note that we could use a Taylor series trick here. Recall that

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \dots$$

¹Note: This limit can be easily found now that you know Taylor series. Just let $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$, then $\sin(x)/x$ as $x \rightarrow 0$ is clearly 1

so we obtain

$$\sin(x^2 - y^2) = \sin((x - y)(x + y)) = [(x - y)(x + y)] - \frac{[(x - y)(x + y)]^3}{3!} + \frac{[(x - y)(x + y)]^5}{5!} + \dots$$

and so our limit is

$$\begin{aligned} \frac{\sin(x^2 - y^2)}{x - y} &= \frac{(x - y)(x + y)}{x - y} - \frac{[(x - y)(x + y)]^3}{3!(x - y)} + \frac{[(x - y)(x + y)]^5}{5!(x - y)} + \dots \\ &= [(x + y)] - \frac{(x - y)^2(x + y)^3}{3!} + \frac{(x - y)^4(x + y)^5}{5!} \end{aligned}$$

and taking the limit we see that the only non-zero term is the first term, thus

$$\lim_{(x,y) \rightarrow (1,1)} \frac{\sin(x^2 - y^2)}{x - y} = \lim_{(x,y) \rightarrow (1,1)} x + y = 2$$

f.) $\lim_{(\mathbf{x}, \mathbf{y}) \rightarrow (1, -1)} \arcsin\left(\frac{\mathbf{xy}}{\sqrt{\mathbf{x}^2 + \mathbf{y}^2}}\right)$

We can just plug into this one

$$\lim_{(x,y) \rightarrow (1,-1)} \arcsin\left(\frac{xy}{\sqrt{x^2 + y^2}}\right) = \arcsin\left(\frac{-1}{\sqrt{2}}\right) = \arcsin\left(\frac{-\sqrt{2}}{2}\right) = -\frac{\pi}{4}$$

g.) $\lim_{(\mathbf{x}, \mathbf{y}) \rightarrow (0,0)} \frac{\mathbf{x}^3}{\mathbf{x}^4 + \mathbf{y}^3}$

I'm guessing this doesn't exist because the function $1/x$ doesn't exist as $x \rightarrow 0$. Thus we need to approach the limit from two directions that yield different values. Consider the case when $y = 0$, then from $x \rightarrow 0^+$ we have

$$\lim_{x \rightarrow 0^+} \frac{x^3}{x^4} = \lim_{x \rightarrow 0^+} \frac{1}{x} \rightarrow \infty$$

and from the left we have

$$\lim_{x \rightarrow 0^-} \frac{x^3}{x^4} = \lim_{x \rightarrow 0^-} \frac{1}{x} \rightarrow -\infty$$

thus, that is enough to show the limit does not exist.

h.) $\lim_{(\mathbf{x}, \mathbf{y}) \rightarrow (0,0)} \frac{\mathbf{xy}}{\mathbf{x}^2 + \mathbf{y}^2}$

Similar to above, consider the direction when $x = y$, then we have

$$\lim_{y \rightarrow 0} \frac{y^2}{2y^2} = \frac{1}{2}$$

and when $x = 0$, we have

$$\lim_{y \rightarrow 0} \frac{0}{y^2} = 0$$

thus the limit does not exist.

i.) $\lim_{(\mathbf{x}, \mathbf{y}) \rightarrow (0,0)} \frac{\mathbf{xy}^2}{\mathbf{x}^2 + \mathbf{y}^4}$

Same as above, but let's compare $x = y^2$ with $x = 0$.

$$x = y^2:$$

$$\lim_{y \rightarrow 0} \frac{y^4}{2y^4} = \frac{1}{2}$$

$$x = 0:$$

$$\lim_{y \rightarrow 0} \frac{0}{y^4} = 0$$

thus the limit does not exist.

$$\textbf{j.) } \lim_{(\mathbf{x}, \mathbf{y}) \rightarrow (2, -2)} \frac{4 - \mathbf{xy}}{4 + \mathbf{xy}}$$

$$x = y:$$

$$\lim_{y \rightarrow -2} \frac{4 - y^2}{4 + y^2} = 0$$

$$x = 0:$$

$$\lim_{y \rightarrow -2} \frac{4}{4} = 1$$

thus the limit does not exist.

$$\textbf{k.) } \lim_{(\mathbf{x}, \mathbf{y}) \rightarrow (0, 0)} (1 + 3\mathbf{xy}^2)^{\frac{2}{\mathbf{xy}^2}}$$

This looks like the goofy e^x limits of the past. Let $u = \frac{2}{xy^2}$, then we have

$$(1 + 3xy^2)^{2/xy^2} = \left(1 + \frac{3}{u/2}\right)^u = \left(1 + \frac{6}{u}\right)^u$$

Moreover, notice that

$$\lim_{(x, y) \rightarrow (0, 0)} \frac{xy^2}{2} = \lim_{u \rightarrow \infty} \frac{2}{u}$$

thus we have

$$\lim_{(x, y) \rightarrow (0, 0)} (1 + 3xy^2)^{\frac{2}{xy^2}} = \lim_{u \rightarrow \infty} \left(1 + \frac{6}{u}\right)^u = e^6$$

$$\textbf{l.) } \lim_{(\mathbf{x}, \mathbf{y}) \rightarrow (0, 0)} \frac{\mathbf{xy}^2}{\mathbf{x}^2 + \mathbf{y}^4}$$

Same as above, but let's compare $x = y^2$ with $x = 0$.

$$x = y^2:$$

$$\lim_{y \rightarrow 0} \frac{y^4}{2y^4} = \frac{1}{2}$$

$x = 0$:

$$\lim_{y \rightarrow 0} \frac{0}{y^4} = 0$$

thus the limit does not exist.

m.) $\lim_{(\mathbf{x}, \mathbf{y}) \rightarrow (1, -2)} \frac{(\mathbf{x}-1)^2 + 3(\mathbf{y}+2)^2}{\mathbf{x}-1 + (\mathbf{y}+2)^2}$

Let $y = -2$:

$$\lim_{x \rightarrow 1} \frac{(x-1)^2}{(x-1)} = \lim_{x \rightarrow 1} x - 1 = 0$$

Let $x = 1$:

$$\lim_{y \rightarrow -2} \frac{3(y+2)^2}{(y+2)^2} = \lim_{y \rightarrow -2} 3 = 3$$

Thus the limit does not exist.

n.) $\lim_{(\mathbf{x}, \mathbf{y}) \rightarrow (1, 2)} \frac{\mathbf{x}\mathbf{y} + 2\mathbf{x} - \mathbf{y} - 2}{\mathbf{x}\mathbf{y} - \mathbf{y} + 3\mathbf{x} - 3}$