Old Discussion Sheet 10

1. Consider the function given by $\mathbf{f(x,y) = xy^2 - x^2y}$ and the point $\mathbf{P = (1,-1)}$. Compute

a.) the exact change of f and

b.) use a differential to estimate the exact change of f

if point P moves in a straight line to point $\mathbf{Q = (1.5, -0.7)}$.

The exact change is

$$f(x_2, y_2) - f(x_1, y_1)$$

for the differential, we notice that

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = f_x dx + f_y dy$$

since we have $f_x = y^2 - 2xy$ and $f_y = 2xy - x^2$ we get

$$df = (y^2 - 2xy)dx + (2xy - x^2)dy$$

so we can estimate the change by plugging in P for x and y and using $dx = (1.5 - 1)$ and $dy = -0.7 - (-1)$

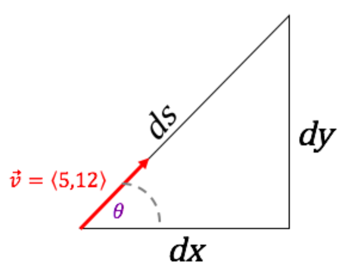
2. Consider the function given by $\mathbf{f(x,y) = \ln(3x + 4y^2)}$ and the point $\mathbf{P = (5,2)}$. Compute

a.) the exact change of f and

b.) use a differential to estimate the exact change of f

if point P moves a distance $\mathbf{ds = 1.4}$ in the direction $\mathbf{\tilde{A} = 5\tilde{i} + 12\tilde{j}}$.

This is the same as above except we don't know dx and dy explicitly. Thus we have to compute it.



Thus what we can do is use trig to get

$$dx = ds\cos\theta \quad \text{and} \quad dy = ds\sin\theta$$

However, we don't know theta, so we have to use the fact that $\tan\theta = \frac{5}{12}$. Since you have a calculator on the test, just use this, but you could also use the fact that the hypotenuse is ds in the direction of $\vec{v}$. This gives us

$$dx^2 + dy^2 = \left(ds\left|\left|\frac{\vec{v}}{||v||}\right|\right|\right)^2$$

and we could solve using Pythagorean's theorem.

New Discussion Sheet 10

1. Find and classify critical points as determining relative maximums, relative minimums, or saddle points

In general, critical points are the points of the function f where

$$\nabla f(x,y) = \langle f_x, f_y \rangle = 0$$

To classify these points as minima, maxima, or saddle points, we must consider the second derivative or Hessian, $H_f(x,y)$. In 2D, we would compare if $f''(x)$ is greater than or less than zero. Asking if the Hessian is greater than zero is not a valid statement, so we must transform $H_f(x,y)$ into a number so we can compare. We do this using the determinant. Hence we consider

$$\det(H_f(x,y)) = |H_f(x,y)| = \left| \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \right| = f_{xx}f_{yy} - f_{yx}f_{xy} = f_{xx}f_{yy} - f_{xy}^2$$

Then we compare this value at our critical points (x_0, y_0) to zero. This gives us three cases.

1. $|\mathbf{H_f(x_0, y_0)}| < \mathbf{0}$: This means that $f_{xx}(x_0,y_0)$ and $f_{yy}(x_0,y_0)$ have opposite behavior. Thus if $f_{xx}(x_0,y_0)$ is a min, then $f_{yy}(x_0,y_0)$ is a max, and vice versa. This is called a saddle point.
2. $|\mathbf{H_f(x_0, y_0)}| > \mathbf{0}$: This means that $f_{xx}(x_0,y_0)$ and $f_{yy}(x_0,y_0)$ behave the same way. However, we need to look further to see if this is a min or max. If $f_{xx}(x_0,y_0) < 0$, then $f(x_0,y_0)$ is a min in the x direction, and the Hessian being greater than zero tells us that $f_{yy}(x_0,y_0) < 0$ so $f(x_0,y_0)$ is also a min in the y direction, so this point $f(x_0,y_0)$ is a min. Similarly if $f_{xx}(x_0,y_0) < 0$ then (x_0,y_0) is a max.

To summarize, extrema are categorized as

1. **Max:** $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$, $f_{xx}(x_0,y_0) > 0$, and $|H_f(x_0,y_0)| > 0$.
2. **Min:** $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$, $f_{xx}(x_0,y_0) < 0$, and $|H_f(x_0,y_0)| > 0$.
3. **Saddle:** $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$ and $|H_f(x_0,y_0)| < 0$.

a.) $\mathbf{z = 3x^2 - 6xy + y^2 + 12x - 16y + 1}$

Critical points are when the derivative is zero or undefined, so

$$\nabla z = \langle 6x - 6y + 12, 2y - 6x - 16 \rangle = \langle 0, 0 \rangle$$

Thus, this happens when $y = x + 2$ and when $(x+2) - 3x = -2x + 2 = 8 \iff x = -3$. Thus our critical point is $p = (-3, -1)$. To check if this is a min,max, or saddle point we need to consider the second derivative, or Hessian matrix as this point. Thus we obtain

$$H_z = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ -6 & 2 \end{bmatrix}$$

Note, that clearly the hessian is the same at any point, so it's the same at our ciritical point. In single variable calculus, we would check if the 2nd derivative is positive or negative, so the equivalent in this case is if the Hessian is positive semi-definite or negative semi-definite. An easy way to do this is look at the determiniant of the Hessian. Thus we have

$$|H_z| := \det(H_z) = 12 - 36 = -24 < 0$$

This shows that we have a saddle point at $(-3, -1)$.

b.) $\mathbf{z = x^2y - x^2 - 2y^2}$

$$\nabla f(x,y) = 0 \iff \langle 2xy - 2x, x^2 - 4y \rangle = \langle 0,0 \rangle \iff 2xy - 2x = 0 \text{ and } x^2 - 4y = 0$$

Thus we have the system of equations

$$\begin{cases} 2xy - 2x &= 0 \\ x^2 - 4y &= 0 \end{cases} \iff \begin{cases} 2xy &= 2x \\ x^2 &= 4y \end{cases} \iff \begin{cases} xy &= x \\ x &= \pm 2\sqrt{y} \end{cases}$$

For equation one, we want to divide by x and get $y = 1$, however, we can't say for sure that $x \neq 0$. Thus we need to consider 2 cases.

Case 1: $x = 0$
This satisfies equation 1, and from equation 2 we get $y = 0$. Thus $(0,0)$ is a critical point.

Case 2: $x \neq 0$
Now we can divide by x and equation 1 becomes $y = 1$. Plugging into equation 2 we get $x = 2$ or $x = -2$. Thus $(1,2)$ and $(1,-2)$ are critical points.

Therefore our critical points are $(0,0)$,$(1,2)$, and $(1,-2)$. In general, the determinant of the Hessian is

$$|H_f(x,y)| = \left| \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \right| = \left| \begin{bmatrix} 2y-2 & 2x \\ 2x & -4 \end{bmatrix} \right| = -4(2y-2) - 4x^2 = 8 - 8y - 4x^2$$

Thus the critical points are

(a) $(\mathbf{0},\mathbf{0})$: $|H_f(0,0) = 8 > 0|$, $f_{xx}(0,0) = -2 < 0$, thus $(0,0)$ is a max.
(b) $(\mathbf{1},\mathbf{2})$: $|H_f(1,2) = -12 < 0|$, thus $(1,2)$ is a saddle point.
(c) $(\mathbf{1},-\mathbf{2})$: $|H_f(1,-2) = 4 > 0|$, $f_{xx}(1,-2) = -6 < 0$, thus $(1,-2)$ is a max.

c.) $\mathbf{z = x^2 - 8\ln(xy) + y^2}$

$$\nabla f(x,y) = 0 \iff \left\langle 2x - \frac{8}{x}, 2y - \frac{8}{y} \right\rangle = \langle 0,0 \rangle \iff 2x - \frac{8}{x} = 0 \text{ and } 2y - \frac{8}{y} = 0$$

Thus we have the system of equations

$$\begin{cases} 2x - \frac{8}{x} &= 0 \\ 2y - \frac{8}{y} &= 0 \end{cases} \iff \begin{cases} x &= \frac{4}{x} \\ y &= \frac{4}{y} \end{cases} \iff \begin{cases} x &= \pm 2 \\ y &= \pm 2 \end{cases}$$

Notice that we were able to solve both equations without using the other. Thus we need to consider all such points that make these true. This gives us critical points of $(2,2),(2,-2),(-2,2)$, and $(-2,-2)$. In general, the determinant of the Hessian is

$$|H_f(x,y)| = \left| \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \right| = \left| \begin{bmatrix} 8/x^2 & 0 \\ 0 & 8/y^2 \end{bmatrix} \right| = \left(\frac{8}{xy} \right)^2$$

Thus the critical points are

(a) $(\mathbf{2},\mathbf{2})$: $|H_f(2,2) = 4 > 0|$, $f_{xx}(2,2) = 2 > 0$, so this is a min.
(b) $(\mathbf{2},-\mathbf{2})$: $|H_f(2,-2) = 4 > 0|$, $f_{xx}(2,-2) = -2 < 0$, so this is a max.
(c) $(-\mathbf{2},\mathbf{2})$: $|H_f(-2,2) = 4 > 0|$, $f_{xx}(-2,2) = -2 < 0$, so this is a max.
(d) $(-\mathbf{2},-\mathbf{2})$: $|H_f(-2,-2) = 4 > 0|$, $f_{xx}(-2,-2) = 2 > 0$, so this is a min.

d.) $\mathbf{z = 3x^2y - 6x^2 + y^3 - 6y^2}$

$$\nabla f(x,y) = 0 \iff \left\langle 6xy - 12x, 3x^2 + 3y^2 - 12y \right\rangle = \langle 0,0\rangle \iff 6xy - 12x = 0 \text{ and } 3x^2 + 3y^2 - 12y = 0$$

Thus we have the system of equations

$$\begin{cases} 6xy - 12x & = 0 \\ 3x^2 + 3y^2 - 12y & = 0 \end{cases} \iff \begin{cases} xy & = 2x \\ x^2 & = y(4-y) \end{cases}$$

For equation one, we want to divide by x and get $y = 2$, however, we can't say for sure that $x \neq 0$. Thus we need to consider 2 cases.

Case 1: $x = 0$
This satisfies equation 1, and from equation 2 we get $y(4-y) = 0$. Thus $(0,0)$ and $(0,4)$ are critical points.

Case 2: $x \neq 0$
Now we can divide by x and equation 1 becomes $y = 2$. Plugging into equation 2 we get $x = 2$ or $x = -2$. Thus $(2,2)$ and $(2,-2)$ are critical points.

Therefore our critical points are $(0,0)$,$(0,4)$,$(2,2)$, and $(2,-2)$. In general, the determinant of the Hessian is

$$|H_f(x,y)| = \left| \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \right| = \left| \begin{bmatrix} 6y - 12 & 6x \\ 6x & 6y - 12 \end{bmatrix} \right| = 6^2(y-2)^2 - 6^2x^2 = 6^2\left((y-2)^2 - x^2\right)$$

Thus the critical points are

(a) $(\mathbf{0},\mathbf{0})$: $\left|H_f(0,0) = 6^2 4^2 > 0\right|$, $f_{xx}(0,0) = -12 < 0$, thus $(0,0)$ is a max.
(b) $(\mathbf{0},\mathbf{4})$: $\left|H_f(0,4) = 6^2 4^2 > 0\right|$, $f_{xx}(0,4) = 12 > 0$, thus $(0,4)$ is a min.
(c) $(\mathbf{2},\mathbf{2})$: $|H_f(2,2) = -104 < 0|$, thus $(2,2)$ is a saddle point.
(d) $(\mathbf{2},\mathbf{-2})$: $|H_f(2,-2) = (14)(36) > 0|$, $f_{xx}(2,-2) = -24 < 0$, thus $(2,-2)$ is a max.

2. Find the point on the plane $\mathbf{x + 2y + 3z = 6}$ nearest the origin.

Way 1
Nearest the origin means that some circle around the origin is minimized. Thus we have a minimization problem of

$$\min x^2 + y^2 + z^2 \quad s.t \quad x + 2y + 3z = 6$$

and we can solve using Lagrange multipliers (if you learned this yet).

Way 2

Another way to solve this is the shortest distance between a point and a plane, which you've done in the past. Thus we can do this by picking any point on the plane, and projecting the vector from the origin to that point onto the normal vector of the plane. Notice that $(0,0,2)$ is on the plane, so let $\vec{v} = \langle 0-0, 0-0, 2-0\rangle = \langle 0,0,2\rangle$. Also, the normal vector $\vec{n} = \langle 1,2,3\rangle$ then the distance between the point and plane is

$$d = \left| \frac{\vec{v}\cdot\vec{n}}{||\vec{n}||\,||\vec{n}||}\vec{n} \right| = \left| \frac{6}{14}\langle 1,2,3\rangle \right| = \frac{6}{\sqrt{14}}$$

Then we have a 3D triangle with sides some multiple of $1, 2, 3$ respectively, with hypotenuse the distance we just computed. Thus we have

$$(1c)^2 + (2c)^2 + (3c^2) = \left(\frac{6}{\sqrt{14}}\right)^2$$

This gives us

$$14c^2 = \frac{36}{14} \iff c = \sqrt{\frac{6^2}{14^2}} = \frac{6}{14} = \frac{3}{7}$$

Thus we conclude that the point is at

$$(1c, 2c, 3c) = \left(\frac{3}{7}, \frac{6}{7}, \frac{9}{7}\right)$$

Way 3

Another way, the normal vector of the plane from the origin must hit the plane at some point and is the closest direction from the plane to the origin. So we can simply do

$$|c\vec{n}| = |\langle c, 2c, 3c\rangle| = \sqrt{c^2 + 4^2 + 9^2 = 14c^2} = 6$$

which clearly gives us the same answer as before.

3. Determine the dimensions and minimum surface area of a closed rectangular box with volume 8ft^3.

A rectangular box has volume given by xyz where x, y,and z are the sides of the box. The surface area is given by $2xy + 4zy$, so we get an optimization problem of

$$\max 2xy + 4zy \quad s.t. \quad xyz = 8$$

So we could solve this by say

$$\nabla(2xy + 4zy) = \langle 2y, 2x + 4z, 4y\rangle = 0$$

since each part of the gradient must be zero, we can use the f_y part to get $x = z$. So we can conclude that $x = y = z$ and since $xyz = 8$, we have $x^3 = y^3 = z^3 = 8$ or $x = y = z = 2$.

4. determine the dimensions and minimum surface area of the closed right circular cylinder with volume 16 πft^3

5. Material for the top and bottom of a rectangular box costs \$4/ft^2 and that for the sides costs \$2/ft^2. Determine the dimensions of the least expensive box of volume \$4/ft^2.

6. Among all open (no top) rectangular boxes with surface area $300in^2$, determine the dimensions of the box of maximum volume.

7. Determine the absolute extrema for each function on the indicated region.

a.) $f(x, y) = 2x + 4y + 12$ **on**

i) the triangle with vertices (0,0), (0,3), and (3,0) and its interior.

i) the circle $x^2 + y^2 = 4$ **and its interior.**

b.) $f(x, y) = xy - x - 3y$ **on the triangle with vertices (0,0), (0,3), and (5,0) and its interior.**

c.) $f(x, y) = x^2 - 3y^2 - 2x + 6y$ **on the square with vertices (0,0), (0,2), (2,0) and (2,2) and its interior.**