Math 21C
Kouba
Disussion Sheet 2 - Key

1.) Determine convergence or divergence of each series using the test indicated. I suggest that your read all of the assumptions and conclusions for each test as you do each problem.

a.) $\boldsymbol{\sum_{n=3}^{\infty} \frac{2n+3}{3n+2}}$ **(Use the nth term test.)**

Let $a_n := \frac{2n+3}{3n+2}$ then we obtain

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n+3}{3n+2} = \lim_{n\to\infty} \frac{\frac{2n}{n}+\frac{3}{n}}{\frac{3n}{n}+\frac{2}{n}} = \frac{2+0}{3+0} = \frac{2}{3} \neq 0$$

Therefore, since the terms in ther series $\sum_{n=3}^{\infty} a_n$ do not approach 0, we conclude this series diverges by the nth term test.

b.) $\boldsymbol{\sum_{n=4}^{\infty} 7\frac{(-2)^{n+1}}{3^{n-1}}}$ **(Use the geometric series test.)**

With the geometric series test we must get our series into the form of $\sum(r)^n$ for $|r| < 1$. So, for the given series we obtain

$$\sum_{n=4}^{\infty} 7\frac{(-2)^{n+1}}{3^{n-1}} = \sum_{n=4}^{\infty} 7\frac{(-2)^{n-1}(-2)^2}{3^{n-1}} = \sum_{n=4}^{\infty} 7(-2)^2\left(\frac{-2}{3}\right)^{n-1} = 28\sum_{n=4}^{\infty}\left(-\frac{2}{3}\right)^{n-1}$$

Thus since $|-2/3| = 2/3 < 1$ we conclude that this series converges by the geometric series test.

c.) $\boldsymbol{\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}}$ **(Use the p-series test.)**

By the p-series test we have the series $\sum \frac{1}{n^p}$ converge if $p > 1$. For our series we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}} \approx \sum_{n=1}^{\infty} \frac{1}{n^{1.4\cdots}}$$

Thus, since $\sqrt{2} \approx 1.4... > 1$, we conclude that our series converges by the p-series test with $p = \sqrt{2} > 1$.

d.) $\boldsymbol{\sum_{n=2}^{\infty} \frac{n}{n^2+4}}$ **(Use the integral test.)**

The integral test states that for $a_n = f(n)$ we have

$$\int_k^{\infty} f(x)\ dx \leq \sum_{n=k}^{\infty} a_n$$

Thus define $f(x) = \frac{x}{x^2+4}$ and we obtain

$$\int_2^{\infty} f(x)\ dx = \int_2^{\infty} \frac{x}{x^2+4}\ dx = \int_2^{\infty} \frac{1}{2}\frac{1}{x^2+4}\ 2xdx = \int_4^{\infty} \frac{1}{2}\frac{1}{u}\ du = \frac{1}{2}\ \ln u|_{u=4}^{\infty} \to \infty$$

Thus, by the integral test, since $\int f(x)\ dx$ diverges, we conclude that $\sum a_n$ must diverge as well.

e.) $\sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{1}}{\sqrt{\mathbf{n+1}}} - \frac{\mathbf{1}}{\sqrt{\mathbf{n+2}}}$ **(Use the sequence of partial sums test.)**

Let $a_n := \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n+2}}$ and define the nth partial sum, s_n, as $s_n := \sum_{k=1}^{n} a_k$. Then the first few partial sums are given by

$$s_1 = \sum_{k=1}^{1} a_k = \frac{1}{\sqrt{1+1}} - \frac{1}{\sqrt{1+2}} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}$$

$$s_2 = \sum_{k=1}^{2} a_k = \frac{1}{\sqrt{1+1}} - \cancel{\frac{1}{\sqrt{1+2}}} + \cancel{\frac{1}{\sqrt{2+1}}} - \frac{1}{\sqrt{2+2}} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{4}}$$

$$s_3 = \sum_{k=1}^{3} a_k = \frac{1}{\sqrt{1+1}} - \cancel{\frac{1}{\sqrt{1+2}}} + \cancel{\frac{1}{\sqrt{2+1}}} - \cancel{\frac{1}{\sqrt{2+2}}} + \cancel{\frac{1}{\sqrt{2+2}}} - \frac{1}{\sqrt{3+2}} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{5}}$$

$$\vdots$$

Thus we see that the pattern for the nth partial sum is

$$\sum_{k=1}^{n} a_k = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{n+2}}$$

Thus, taking the limit as n approaches infinity we obtain,

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{n+2}} = \frac{1}{\sqrt{2}}$$

Therefore we conclude that the series converges to $1/\sqrt{2}$, so it clearly converges.

f.) $\sum_{\mathbf{n=2}}^{\infty} \frac{\mathbf{n-1}}{\mathbf{n^3+2}}$ **(Use the comparison test.)**

Let $a_n := \frac{n-1}{n^2+2}$, notice that for large n we have a_n similar to the function

$$\frac{n - \cancelto{0}{1}}{n^2 + \cancelto{0}{2}} = \frac{n}{n^2} = \frac{1}{n}$$

So we can predict that this will likely diverge because $\sum \frac{1}{n}$ diverges. One may be tempted to just use this function as a comparison, but a_n is clearly less than $1/n$ because the numerator is smaller and the denominator is larger. Since $1/n$ diverges, saying a_n is smaller tells us nothing.

(Sorry, got tied up at work, I'll continue on this tomorrow)

g.) $\sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{n^3+7n^2-3}}{\mathbf{n^4-4n+9}}$ **(Use the limit comparison test.)**

h.) $\sum_{\mathbf{n=0}}^{\infty} \frac{\mathbf{3^{n-1}}}{\mathbf{(n+1)!}}$ **(Use the ratio test.)**

i.) $\sum_{\mathbf{n=1}}^{\infty} \left(\mathbf{1.01} - \frac{\mathbf{5}}{\mathbf{n^3}}\right)^{\mathbf{n}}$ **(Use the root test.)**