Math 21B
Disussion Sheet 5 - Key
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Note: These answers are not endorsed by Dr. Gravner and may be incorrect!

1. Compute

a) $\int \mathbf{x^3 e^{x^2}}\ \mathbf{dx}$
Let $u = x^2$, then $du = 2x\ dx$, so we obtain

$$\int x^3 e^{x^2}\ dx = \int \frac{1}{2} u e^u\ du$$

Then by integration by parts, let $w = \frac{1}{2}u$, and $dv = e^u$, then we get

$$\int \frac{1}{2} u e^u\ du = \frac{1}{2} u e^u - \int \frac{1}{2} e^u\ du = \frac{1}{2}(x^2 - 1)e^{x^2}$$

b) $\int \arcsin \mathbf{x}\ \mathbf{dx}$

By integration by parts, let $u = \arcsin x$ and $dv = dx$, then

$$\int \arcsin x\ dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}}\ dx = x \arcsin x + \frac{1}{2} \ln \left|1 - x^2\right| + C$$

c) $\int \arccos \mathbf{x}\ \mathbf{dx}$

We would solve this the same way, so this would be

$$x \arccos x + \frac{1}{2} \ln \left|1 + x^2\right| + C$$

2. Compute

a) $\int \sin \mathbf{3x} \cos \mathbf{4x}\ \mathbf{dx}$

Recall that

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \iff \sin(a)\cos(b) = \sin(a+b) - \cos(a)\sin(b)$$

so let $a = 3x$ and $b = 4x$ then we obtain

$$\int \sin 3x \cos 4x\ dx = \int \sin(3x+4x) - \sin 4x \cos 3x\ dx = -\frac{1}{7}\cos(7x) - \int \sin 4x \cos 3x\ dx$$

b) $\int \sin^{\mathbf{4}} \mathbf{x} \cos^{\mathbf{4}} \mathbf{x}\ \mathbf{dx}$

Recall that we have

$$\sin^2 x = \frac{1}{2}(1 + \cos 2x) \quad \text{and} \quad \cos^2 = \frac{1}{2}(1 - \cos 2x)$$

Then we obtain

$$
\begin{aligned}
\int \sin^4 x \cos^4 x \, dx &= \int \left(\sin^2 x\right)^2 \left(\cos^2 x\right)^2 \, dx \\
&= \int \left(\frac{1}{2}(1+\cos 2x)\right)^2 \left(\frac{1}{2}(1-\cos 2x)\right)^2 \, dx \\
&= \int \left(\frac{1}{2}\right)^2 \left(1+2\cos 2x+\cos^2 2x\right) \left(\frac{1}{2}\right)^2 \left(1-2\cos 2x+\cos^2 2x\right) \, dx \\
&= \frac{1}{16} \int 1-2\cos^2 2x+\cos^4 2x \, dx \\
&= \frac{1}{16} \int 1-2\left(\frac{1}{2}(1-\cos 4x)\right)+\left(\frac{1}{2}(1-\cos 4x)\right)^2 \, dx \\
&= \frac{1}{16} \int \frac{1}{2}\cos 4x+\frac{1}{2}\left(1-2\cos 4x+\cos^2 4x\right) \, dx \\
&= \frac{1}{16} \int \frac{1}{2}\cos 4x+\frac{1}{2}-\cos 4x+\frac{1}{2}\left(\frac{1}{2}(1-\cos 8x)\right) \, dx \\
&= \frac{1}{16} \int \frac{1}{4}-\frac{1}{2}\cos 4x-\frac{1}{4}\cos 8x \, dx \\
&= \frac{1}{16}\left(\frac{1}{4}x-\frac{1}{2}\frac{\sin 4x}{4}-\frac{1}{4}\frac{\sin 8x}{8}\right)+C \\
&= \frac{24x-8\sin 4x+\sin 8x}{1024}+C
\end{aligned}
$$

c) $\int (\sin \mathbf{2x})^{\mathbf{2}} \sin^{\mathbf{2}} \mathbf{x} \ \mathbf{dx}$

Recall that the half-angle formula for $\cos 2x$ gives us

$$
\sin^2 x = \frac{1}{2}(\cos 2x + 1)
$$

Then we obtain

$$
\begin{aligned}
\int (\sin 2x)^2 \sin^2 x \, dx &= \int (\sin 2x)^2 \frac{1}{2}(\cos 2x+1) \, dx \\
&= \frac{1}{2}\int (\sin 2x)^2 \cos 2x \, dx + \frac{1}{2}\int \sin^2 2x \, dx \\
&= \frac{1}{2}\int (\sin 2x)^2 \cos 2x \, dx + \frac{1}{2}\int \frac{1}{2}(\cos 2x+1) \, dx \\
&= \frac{1}{12}\sin^3 2x + \frac{1}{8}\sin 2x + \frac{1}{4}x + C
\end{aligned}
$$

d) $\int \frac{\tan \mathbf{x}}{\cos^{\mathbf{2}} \mathbf{x}} \ \mathbf{dx}$

Note that $\tan x = \sin x / \cos x$, then let $u = \cos x$ and $du = -\sin x dx$, thus

$$
\int \frac{\tan x}{\cos^2 x} \, dx = \int \frac{\sin x}{\cos^3 x} \, dx = \int \frac{1}{u^3} \, dx = \frac{1}{2}\frac{1}{\cos^2 x} + C
$$

3. Compute

a) $\int_{\pi}^{\pi} (\sin \mathbf{2x})^{\mathbf{3}} \sin^{\mathbf{2}} \mathbf{x}\ \mathbf{dx}$

b) $\int_{\pi}^{\mathbf{17}\pi} (\sin \mathbf{2x})^{\mathbf{3}} \sin^{\mathbf{2}} \mathbf{x}\ \mathbf{dx}$

c) $\int_{\pi/\mathbf{2}}^{\pi/\mathbf{2}} \cos^{\mathbf{2}}(\mathbf{2x})\ \mathbf{dx}$

4. Compute $\int \frac{\mathbf{x^2}}{\sqrt{\mathbf{x^2+9}}}\ \mathbf{dx}$

Note that with some algebra we get

$$\frac{x^2}{x^2+9} = \frac{1}{9}\frac{x^2}{\left(\frac{x}{3}\right)^2+1}$$

Thus, let $\tan(\theta) = \frac{x}{3}$, then $dx = 3\sec^2\theta d\theta$

$$\begin{aligned}
\int \frac{x^2}{x^2+9}\ dx &= \frac{1}{9}\int \frac{9\tan^2\theta}{\tan^2\theta+1}\ 3\sec^2\theta d\theta \\
&= \frac{9}{9}\int 3\frac{\tan^2\theta}{\sec^2\theta}\sec^2\theta\ d\theta \\
&= \int 3\tan^2\theta\ d\theta \\
&= 3\int \sec^2\theta - 1\ d\theta \\
&= 3\tan\theta - 3\theta + C \\
&= 3\frac{x}{3} - 3\arctan\left(\frac{x}{3}\right) + C \\
&= x - 3\arctan\left(\frac{x}{3}\right) + C
\end{aligned}$$