

1. Use what you know about converging geometric series to write each power series as an ordinary function

a.)  $\sum_{n=2}^{\infty} \frac{x^n}{3^n}$

Notice that, for  $m = n - 2$ , we have

$$\sum_{n=2}^{\infty} \frac{x^n}{3^n} = \sum_{m=0}^{\infty} \frac{x^{m+2}}{3^{m+2}} = \sum_{m=0}^{\infty} \frac{x^2}{3^2} \left(\frac{x}{3}\right)^m = \left(\frac{x}{3}\right)^2 \left(\frac{1}{1 - \frac{x}{3}}\right) = \frac{x^2}{3(3-x)}$$

so long as  $|x/3| < 1$ . Thus we obtain,

$$\sum_{n=2}^{\infty} \frac{x^n}{3^n} = \frac{x^2}{3(3-x)}$$

Note, one can show that  $x \in (-3, 3)$ .

b.)  $\sum_{n=0}^{\infty} \frac{x^n}{3^n}$

This one is easier since the series starts at  $n = 0$ , thus we have

$$\sum_{n=0}^{\infty} \frac{x^n}{3^n} = \frac{1}{1 - \frac{x}{3}} = \frac{3}{3-x}$$

for  $|x/3| < 1 \iff x \in (-3, 3)$ .

c.)  $x^2 - x^{5/2} + x^3 - x^{7/2} + x^4 - x^{9/2} + \dots$

The pattern in this expression is given by

$$x^2 - x^{5/2} + x^3 - x^{7/2} + x^4 - x^{9/2} + \dots = \sqrt{x^4} - \sqrt{x^5} + \sqrt{x^6} - \sqrt{x^7} + \sqrt{x^8} - \sqrt{x^9} + \dots = \sum_{n=4}^{\infty} -(-\sqrt{x})^n = \sum_{n=0}^{\infty} -x^4 (-\sqrt{x})^n$$

This is a geometric series with  $r = -\sqrt{x}$ , thus we need  $|-\sqrt{x}| < 1 \iff x \in (0, 1)$ . Thus we obtain

$$x^2 - x^{5/2} + x^3 - x^{7/2} + x^4 - x^{9/2} + \dots = -x^4 \sum_{n=0}^{\infty} (\sqrt{x})^n = -x^4 \frac{1}{1 - (-\sqrt{x})} = -\frac{x^4}{1 + \sqrt{x}}$$

d.)  $\sum_{n=0}^{\infty} (n+1)x^n$

For this series, notice that we have

$$\sum_{n=0}^{\infty} (n+1)x^n = \sum_{n=0}^{\infty} nx^n + \sum_{n=0}^{\infty} x^n$$

For  $\sum x^n$ , we have  $\sum x^n = \frac{1}{1-x}$  if  $x \in (-1, 1)$ . For the other term, let  $g(x) := \sum x^n$ , then notice that

$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x} \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \frac{\partial}{\partial x} x^n = \sum_{n=0}^{\infty} n x^{n-1} = \frac{1}{x} \sum_{n=0}^{\infty} x(n x^{n-1}) = \frac{1}{x} \sum_{n=0}^{\infty} n x^n$$

Thus we have  $\sum n x^n = x \frac{\partial}{\partial x} g(x)$ . Moreover,  $g(x) = \sum x^n = \frac{1}{1-x}$  for  $x \in (-1, 1)$ , so we obtain

$$\sum_{n=0}^{\infty} n x^n = x \frac{\partial}{\partial x} g = x \frac{\partial}{\partial x} \frac{1}{1-x} = \frac{-x}{(1-x)^2}$$

Therefore the functional form of the series  $\sum (n+1)x^n$  is given by

$$\sum_{n=0}^{\infty} (n+1)x^n = \frac{-x}{(1-x)^2} + \frac{1}{1-x} = \frac{(1-x) - x}{(1-x)^2} = \frac{1-2x}{(1-x)^2}$$

## 2. Recall that if $y = f(x)$ is a function and

$$a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 + \cdots = \sum_{n=0}^{\infty} a_n(x-a)^n$$

is the Taylor Series (or Maclaurin for series if  $a = 0$ ) centered at  $x = a$  for  $y = f(x)$ , then  $a_n = \frac{f^{(n)}(a)}{n!}$ . Use this formula to compute the first four nonzero terms and the general formula for the Taylor series expansion for each function about the given value of  $a$ .

This is just a bunch of gross computation, so I'm probably not going to do this. However, I have written the answers to the ones centered at 0 below.

a.)  $f(x) = e^x$  centered at  $x = 0$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

b.)  $f(x) = e^x$  centered at  $x = \ln 2$

c.)  $f(x) = \frac{1}{1-x}$  centered at  $x = 0$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

d.)  $f(x) = \sin x$  centered at  $x = 0$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

e.)  $f(x) = \frac{1}{x}$  centered at  $x = 1$

f.)  $f(x) = \sqrt{x+5}$  centered at  $x = -1$

3. Use the suggested method to find the first four nonzero terms of the Maclaurin series for each function

a.)  $f(x) = \frac{1}{1+x^2}$  (Substitute  $-x^2$  into the Maclaurin series for  $\frac{1}{1-x}$ .)

Note that  $\sum x^n = \frac{1}{1-x}$ , so we obtain

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

b.)  $f(x) = x^3 e^{-3x}$  (Substitute  $-3x$  into the Maclaurin series for  $e^x$  and then multiply by  $x^3$ )

The Maclaurin series for  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ , thus we obtain

$$x^3 e^{-3x} = x^3 \left( \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!} \right) = x^3 \left( \sum_{n=0}^{\infty} (-1)^n \frac{3^n x^n}{n!} \right) = \sum_{n=0}^{\infty} (-1)^n \frac{3^n x^{n+3}}{n!}$$

c.)  $f(x) = e^x \frac{1}{1-x}$  (Multiply the Maclaurin series for  $e^x$  and  $\frac{1}{1-x}$  term by term and then group like powers of  $x$ )

Note that  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$  and  $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ , thus we obtain

$$\begin{aligned} e^x \frac{1}{1-x} &= \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} \right) \left( \sum_{n=0}^{\infty} x^n \right) \\ &= \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right) (1 + x + x^2 + x^3 + \cdots) \\ &= (1 + x + x^2 + x^3 + \cdots) + x(1 + x + x^2 + x^3 + \cdots) + \frac{x^2}{2!} (1 + x + x^2 + x^3 + \cdots) + \frac{x^3}{3!} (1 + x + x^2 + x^3 + \cdots) + \cdots \\ &= (1 + x + x^2 + x^3 + \cdots) + (x + x^2 + x^3 + x^4 + \cdots) + \left( \frac{x^2}{2!} + \frac{x^3}{2!} + \frac{x^4}{2!} + \frac{x^5}{2!} + \cdots \right) + \left( \frac{x^3}{3!} + \frac{x^4}{3!} + \frac{x^5}{3!} + \frac{x^6}{3!} + \cdots \right) + \cdots \\ &= 1 + (1+1)x + \left( 1+1+\frac{1}{2!} \right) x^2 + \left( 1+1+\frac{1}{2!}+\frac{1}{3!} \right) x^3 + \left( 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!} \right) x^4 + \cdots \\ &= \sum_{k=0}^0 \frac{1}{k!} x^0 + \sum_{k=0}^1 \frac{1}{k!} x^1 + \sum_{k=0}^2 \frac{1}{k!} x^2 + \sum_{k=0}^3 \frac{1}{k!} x^3 + \cdots \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{1}{k!} x^n \end{aligned}$$

d.)  $f(x) = \frac{e^x}{1-x}$  (Use polynomial division. Divide the Maclaurin series for  $e^x$  by  $1-x$ )

e.)  $f(x) = 3x^2 \cos(x^3)$  (Substitute  $x^3$  into the Maclaurin series for  $\sin x$  then differentiate term by term)

f.)  $f(x) = \arctan x$  (Integrate the Maclaurin series for  $\frac{1}{1+t^2}$  from 0 to  $x$ )

4. The Maclaurin series for  $f(x) = \frac{1}{1+x}$  is  $1 - x + x^2 - x^3 + x^4 - x^5 + \dots$
- a.) Show that  $f(x) = \frac{1}{1+x}$  and  $1 - x + x^2 - x^3 + x^4 - x^5 + \dots$  have the same value at  $x = 0$ .
- b.) Show that  $f(x) = \frac{1}{1+x}$  and  $1 - x + x^2 - x^3 + x^4 - x^5 + \dots$  have the same value at  $x = 1/2$ .
- c.) Show that  $f(x) = \frac{1}{1+x}$  and  $1 - x + x^2 - x^3 + x^4 - x^5 + \dots$  have the same value at  $x = 1$ .
- d.) For what  $x$ -values is  $f(x) = \frac{1}{1+x}$  defined?
- d.) For what  $x$ -values is  $1 - x + x^2 - x^3 + x^4 - x^5 + \dots$  defined?

**Note:** It can be shown that  $f(x) = \frac{1}{1+x}$  and its Maclaurin series  $1 - x + x^2 - x^3 + x^4 - x^5 + \dots$  are equal on the interval  $(-1, 1)$ .

5. Determine (Use shortcuts.) the third-degree Taylor polynomial,  $P_3(x; 0)$ , for the function  $f(x) = \frac{x}{1+x}$ . Use  $\int_0^1 P_3(x; 0) dx$  to estimate the value of  $\int_0^1 \frac{x}{1+x} dx$ . Now evaluate  $\int_0^1 \frac{x}{1+x} dx$  directly to see how good the estimate is.

6. The following definite integral cannot be evaluated using the Fundamental Theorem of Calculus. Use the Maclaurin series for  $\cos x$  and the absolute error  $|R_n|$  for an alternating series to estimate the value of this integral with error at most 0.0001:  $\int_0^1 \cos(x^2) dx$ .

7. Write each Maclaurin series as an ordinary function.

- a.)  $(3x) - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!} + \frac{(3x)^9}{9!} - \dots$  (HINT: Use  $\sin x$ )  
Note that we have

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Then the series in question is given by

$$\sum_{n=0}^{\infty} (-1)^n \frac{(3x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1} x^{2n+1}}{(2n+1)!} = \sin(3x)$$

- b.)  $x^2 - x^3 + x^4 - x^5 + x^6 - \dots$  (HINT: Factor )

Factoring out  $x^2$  gives us,

$$x^2 - x^3 + x^4 - x^5 + x^6 - \dots = x^2 [1 - x + x^2 - x^3 + x^4 - \dots] = x^2 \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-x^2)^n = \frac{1}{1 - (-x^2)} = \frac{1}{1 + x^2}$$

- c.)  $\frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{6!} + \frac{x^3}{5!} + \frac{x^4}{6!} + \dots$  (HINT: Use  $e^x$ )

Note that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

But the provided series has  $x^n$  divided by  $(n+2)!$ . Therefore we have

$$\frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{6!} + \frac{x^3}{5!} + \frac{x^4}{6!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{(n+2)!} = \sum_{n=0}^{\infty} \frac{1}{x^2} \frac{x^{(n+2)}}{(n+2)!} = \frac{1}{x^2} \sum_{n=2}^{\infty} \frac{x^n}{n!} = \frac{1}{x^2} \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^1 \frac{x^n}{n!} \right) = \frac{1}{x^2} (e^x - 1 - x)$$

d.)  $x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \dots$  (Challenging)

Challenging probably means using the differentiation rule we used 1d); just a hunch since its of the form  $\sum nx^n$ . From 1d), we know that

$$\sum_{n=0}^{\infty} nx^n = \frac{-x}{(1-x)^2}$$

Thus this is the answer.

**8. Use any method to find the given Taylor polynomial for each function. Then estimate the Absolute Taylor Error on the indicated interval.**

Taylor's remainder theorem states that a function with a Taylor series approximation has error, denoted  $R_n$ , given by

$$\left| \frac{f^{n+1}(\xi)}{(n+1)!} \right|$$

where  $\xi$  is some value in your desired interval,  $I$ . We can choose  $\xi$  such that  $f(\xi) = \max\{f(x) : x \in I\}$  is the maximum value  $f$  can take in that interval.

a.)  $f(x) = e^{-2x}$ ,  $P_3(x; 0)$ , for  $[-1/2, 1/3]$

We have the third degree polynomial of  $e^{-2x}$  given by

$$\sum_{n=0}^3 \frac{(-2x)^n}{n!}$$

Then by Taylor's theorem we have the error given by

$$R_n = \left| \frac{f^{(4)}(\xi)}{4!} \right|; \quad \xi \in [-1/2, 1/3]$$

Since  $f^{(4)}(x) = (-2)^4 e^{-2x} = 16e^{-2x}$ , then  $f^{(4)}(x)$  is strictly decreasing. Thus we have

$$f^{(4)}(\xi) \leq f^{(4)}(-1/2)$$

so we can estimate the absolute error as

$$R_n \approx \frac{f^{(4)}(-1/2)}{4!} = \frac{16e^{-2(-1/2)}}{4!} = \frac{16}{24}e$$

b.)  $f(x) = \sin 2x$ ,  $P_5(x; 0)$ , for  $[0, 3/4]$

c.)  $f(x) = \frac{x}{1-x}$ ,  $P_4(x; 0)$ , for  $[-1/3, 0]$

9. What should  $n$  be so that the  $n$ th-degree Taylor Polynomial  $P_n(x; a)$  estimates the value of the given function on the indicated interval with Absolute Taylor Error at most 0.00001?

a.)  $f(x) = e^{-x}$  for  $a = 1$  and  $[0, 1]$

a.)  $f(x) = \frac{x+3}{x+1}$  for  $a = 0$  and  $[0, 1/2]$

10. (Challenging) Use shortcuts to find the first three nonzero terms in the Taylor Series centered at  $x = -1$  for  $f(x) = \frac{x}{3-x}$ .

Using some algebra we can write  $f(x)$  in a form we are familiar with, such as

$$f(x) = \frac{x}{3-x} = \frac{1}{1 - \frac{x}{3}}$$

Since we know  $\sum x^n = \frac{1}{1-x}$  we just sub  $x/3$  into this equation and we obtain

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$$

Then use Taylor's remainder theorem from here.