1. Compute the derivative of $\mathbf{f(x,y) = x^2 + xy}$ at the point $\mathbf{P = (1,-1)}$ in the direction of vector $\tilde{\mathbf{A}} = \tilde{\mathbf{i}} - 2\tilde{\mathbf{j}}$.

The direction derivative is how much of the tangent plane at P is in the direction $\vec{A}$. First, since we are only concerned with the direction of $\vec{A}$, we need to compute

$$\widehat{A} = \frac{\vec{A}}{\left|\vec{A}\right|} = \frac{\langle 1,-2\rangle}{|\langle 1,-2\rangle|} = \left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle$$

Then, just as we had done with projections, we need the projections of the tangent plane $\nabla f(P)$ in the direction $\widehat{A}$. Thus we use a dot product. Note that

$$\nabla f(P) = \langle 2x+y, x\rangle|_{P=(1,-1)} = \langle 1,1\rangle$$

Putting everything together we obtain

$$D_{\widehat{A}} f(P) = \langle 1,1\rangle \cdot \left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle = \frac{1}{\sqrt{5}} - \frac{2}{\sqrt{5}} = \frac{1}{\sqrt{5}}$$

2. Compute the derivative of $\mathbf{f(x,y,z) = x - y^2 + z^3}$ at the point $\mathbf{P = (2,0,-1)}$ in the direction of vector $\tilde{\mathbf{A}} = \tilde{\mathbf{i}} - \tilde{\mathbf{j}} + \tilde{\mathbf{k}}$

Exactly the same as before, but now we're talking about a 4 variables function (f,x,y,z), so

$$\begin{aligned} D_{\widehat{A}} f(P) &= \nabla f(P) \cdot \frac{\vec{A}}{\left|\vec{A}\right|} \\ &= \left\langle 1, -2y, 3z^2 \right\rangle|_{P=(2,0,-1)} \cdot \frac{\langle 1,-1,1\rangle}{\sqrt{3}} \\ &= \langle 1,0,3\rangle \cdot \left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle \\ &= \frac{1}{\sqrt{3}} + \frac{3}{\sqrt{3}} = \frac{4}{\sqrt{3}} \end{aligned}$$

3. Consider the function $\mathbf{f(x,y) = xy^3}$ and the point $\mathbf{P = (2,1)}$. Determine all unit vectors $\tilde{\mathbf{u}}$ so that $\mathbf{D_{\tilde{u}} f(2,1)}$ is

Just doing some preliminary work first.

$$\nabla f(P) = \left\langle y^3, 3xy^2 \right\rangle|_{P=(2,1)} = \langle 1,6\rangle$$

a.) as large as possible.

Here we recognize that the gradient direction is the largest increase we can obtain, so we let $\vec{u} = \nabla f(P)$, thus the derivative in the largest direction is

$$\nabla f(P) \cdot \frac{\nabla f(P)}{|\nabla f(P)|} = \frac{|\nabla f(P)|^2}{|\nabla f(P)|} = |\nabla f(P)| = \sqrt{1+6^2} = \sqrt{37}$$

b.) as small as possible.

Smallest is the negative direction, so the only change from *a.*) is that $\vec{u} = -\nabla f(P)$. Thus we get

$$\nabla f(P) \cdot -\frac{\nabla f(P)}{|\nabla f(P)|} = -\frac{|\nabla f(P)|^2}{|\nabla f(P)|} = -|\nabla f(P)| = -\sqrt{1+6^2} = -\sqrt{37}$$

c.) equal to zero.

We want to find $\widehat{u}$ such that

$$\nabla f(P) \cdot \widehat{u} = 0$$

thus we obtain

$$\langle 1, 6 \rangle \cdot \langle u_1, u_2 \rangle = u_1 + 6u_2 = 0$$

However, this a single equation with two variables, so we need a 2nd equation. We get this since $|\widehat{u}| = 1$ or $u_1^2 + u_2^2 = 1$. Thus our system of equations is

$$\begin{cases} u_1 + 6u_2 & = 0 \\ u_1^2 + u_2^2 & = 1 \end{cases}$$

which gives us $u_1 = -6u_2$ and $36u_2^2 + u_2^2 = 1 \iff u_2 = 1/\sqrt{37}$. Thus our direction is

$$\widehat{u} = \left\langle \frac{-6}{\sqrt{37}}, \frac{1}{\sqrt{37}} \right\rangle$$

d.) equal to 1.

Similar to above except that

$$\nabla f(P) \cdot \widehat{u} = 1$$

thus we obtain

$$\begin{cases} u_1 + 6u_2 & = 1 \\ u_1^2 + u_2^2 & = 1 \end{cases}$$

which gives us $u_1 = 1 - 6u_2$ and $(1-6u_2)^2 + u_2^2 = 1 \iff 37u_2^2 - 12u_2 + 1 = 1 \iff u_2(37u_2 - 12) = 0$. This gives us two solutions of

$$\widehat{u} = \langle 1, 0 \rangle \quad \text{and} \quad \widehat{u} = \left\langle -\frac{35}{37}, \frac{12}{37} \right\rangle$$

Check my arithmetic here; did most of it in my head

4. Consider the surface given by $\mathbf{x^2 + 2y^2 + 3z^2 = 3}$ and the point $\mathbf{P = (1, -1, 0)}$ on the surface. Find equations for

a.) the plane tangent to the surface at point P

A plane can be entirely determined by a normal vector and a point. This is because the plane

$$ax + by + cz = d$$

has normal vector defined by $\langle a, b, c \rangle$ and is equal to d at the point $(0, 0, 0)$. Thus we are looking for the plane with normal vector $\nabla f(P)$ and the point is defined by P. Notice that

$$\nabla f(P) = \langle 2x, 4y, 4z \rangle|_{P=(1,-1,0)} = \langle 2, -4, 0 \rangle$$

so our plane is

$$2x - 4y = d$$

and we need the plane at P to equal the surface at P so let's just slide the plane's origin to P and then force d to equal 3. Thus

$$2(x - 1) - 4(y + 1) = 3$$

is our tangent plane. Note that another good answer is

$$2x - 4y = 9$$

b.) the line normal to the surface at the point P.

A simple way to give the equation of a line is to take a point p and then travel in the direction of some vector $\vec{A}$ from that point. As a parametric equation this is given by

$$L := P + t\vec{A}$$

thus, our point is already given to us, and our vector we want to travel in is the normal vector to the surface at P; or $\nabla f(P)$. Thus we have

$$L := P + t\nabla f(P) = (1, -1, 0) + t(2, -4, 0)$$

which gives us

$$L := \begin{cases} 1 + 2t \\ -1 - 4t \\ 0 \end{cases}$$

5. Consider the surface (hyperbolic paraboloid) given by $\mathbf{f(x, y) = 3x^2 - 2y^2 + 5}$ and the point $\mathbf{P = (2, 3, -1)}$ on the surface. Find equations for

Same as above, so I might not do this

a.) the plane tangent to the surface at point P

b.) the line normal to the surface at the point P.

6. Consider the function $\mathbf{f(x,y) = xe^{xy}}$ and the point $\mathbf{P = (0,1)}$. Use a differential to estimate the change in the values f if

a.) point P moves a distance of $\mathbf{ds = 0.15}$ in the direction of vector $\vec{\mathbf{A}} = \mathbf{3}\vec{\mathbf{i}} - \mathbf{4}\vec{\mathbf{j}}$.

The differential df is given by

$$df = f_x dx + f_y dy$$

thus we have

$$df = (1+xy)e^{xy}dx + (1+xy)e^{xy}dy$$

thus we can find df by determining dx and dy. Notice that if ds moves in the direction $\vec{A} = \langle 3, -4\rangle$ then trigonemtry gives us $dx = (3/4)ds$ and $dy = (-4/5)ds$. Thus we have

$$df = (1+xy)e^{xy}\left(ds(3/5) + ds(-4/5)\right) = 0.15((3/5) - (4/5)) = -\frac{0.15}{5} = -0.3$$

b.) point P moves in a straight line to point $\mathbf{Q = (1,0)}$

Similar to above we have $dx = \frac{1}{\sqrt{2}}ds$ and $dy = -\frac{1}{\sqrt{2}}ds$, thus $df = 0$.

7. Consider the function $\mathbf{f(x,y,z) = xy^2 + yz - x^3z}$ and the point $\mathbf{P = (1,-1,2)}$. Use a differential to estimate the change in the values of f if the point P moves a distance of $\mathbf{ds = 0.2}$ in the direction of vectors $\vec{\mathbf{A}} = -\vec{\mathbf{i}} - \mathbf{2}\vec{\mathbf{j}} + \mathbf{2}\vec{\mathbf{k}}$.

Similar to above. The differential is given by

$$df = f_x dx = f_y dy + f_z dz$$

where $f_x = y^2 - 3x^2z$, $f_y = 2xy + z$, and $f_z = y - x^3$. Moreover, by pythagorean's theorem we have $dx = (-1/3)ds, dy = (-2/3)ds$, and $dz = (2/3)ds$. Thus we obtain

$$df = ds\left(-\frac{1}{3}f_x - \frac{2}{3}f_y + \frac{2}{3}f_z\right) = 0.2\left(-\frac{1}{3}(-5) - \frac{2}{3}(0) + \frac{2}{3}(-9)\right) = 0.2(-23/3) = -\frac{23}{15}$$

8. Find and classify critical points as determining relative maximums, relative minimums, or saddle points.

a.) $\mathbf{z = 3x^2 - 6xy + y^2 + 12x - 16y + 1}$

Critical points are when the derivative is zero or undefined, so

$$\nabla z = \langle 6x - 6y + 12, 2y - 6x - 16 \rangle = \langle 0, 0 \rangle$$

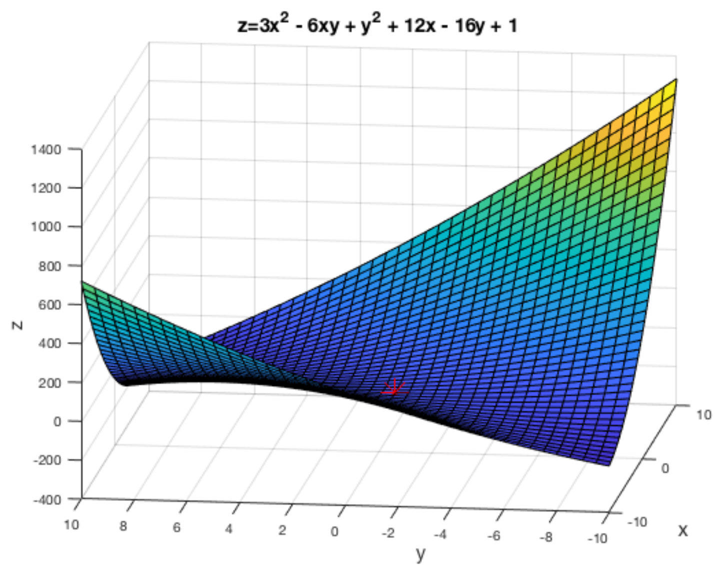
Thus, this happens when $y = x + 2$ and when $(x + 2) - 3x = -2x + 2 = 8 \iff x = -3$. Thus our critical point is $p = (-3, -1)$. To check if this is a min,max, or saddle point we need to consider the second derivative, or Hessian matrix as this point. Thus we obtain

$$H_z = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ -6 & 2 \end{bmatrix}$$

Note, that clearly the hessian is the same at any point, so it's the same at our ciritical point. In single variable calculus, we would check if the 2nd derivative is positive or negative, so the equivalent in this case is if the Hessian is positive semi-definite or negative semi-definite. An easy way to do this is look at the determiniant of the Hessian. Thus we have

$$|H_z| := \det(H_z) = 12 - 36 = -24 < 0$$

This shows that this is a saddle point.



1. Saddle point at $(-3, -1)$

b.) $\mathbf{z = x^2y - x^2 - 2y^2}$

See discussion sheet 10

c.) $\mathbf{z = x^2 - 8\ln(xy) + y^2}$

d.) $\mathbf{z = 3x^2y - 6x^2 + y^3 - 6y^2}$