Math 21C
Kouba
Disussion Sheet 2 - Key
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Alternating Series Test $\sum(-1)^n a_n$ converges if and only if $|a_n|$ is a decreasing sequence and $\lim_{n\to\infty} a_n = 0$.

Absolute Convergence Test If the series $\sum |a_n|$ converges, then $\sum a_n$ converges as well. In this case we say that $\sum a_n$ converges absolutely.

1. Determine convergence or divergence of each series using the test indicated

a.) $\boldsymbol{\sum_{n=1}^{\infty}(-1)^{n+1}\frac{7}{n^2+3}}$ **(Use the alternating series test)**

a.) $\boldsymbol{1+\frac{1}{3}-\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}-\frac{1}{3^5}+\frac{1}{3^6}+\frac{1}{3^7}-\frac{1}{3^8}}$ **(Absolute convergence test)**

2. Determine if the following series, which contain both positive and negative terms, converge conditionally, converge absolutely, or diverge. Remember what the absolute convernce test says.

a.) $\boldsymbol{\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n+2}}$

Let $f(n) = \frac{1}{n+2}$ and $a_n := \frac{1}{n+2}$. Note that

$$\frac{\partial f}{\partial n} = -\frac{1}{(n+2)^2}$$

which is strictly negative, thus f is decreasing for all $n > 0$. Moreover

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n+2} = 0$$

Thus $\sum(-1)^{n+1}a_n$ converges by the alternating series test. Note that $\sum \frac{1}{n+2}$ diverges by the limit comparison test when compared with $\sum \frac{1}{n}$. Thus this series is conditionally convergent.

b.) $\boldsymbol{\sum_{n=4}^{\infty}(-1)^{n+1}\frac{1}{n^2+4}}$

This series converges by alternating series test, and the series $\sum \frac{1}{n^2+4}$ converges by comparison test; compared with $\sum \frac{1}{n^2}$. Thus this is absolutely convergent.

c.) $\sum_{\mathbf{n=1}}^{\infty}(-\mathbf{1})^{\mathbf{n+1}}\frac{\mathbf{n+1}}{\mathbf{n^2+99}}$ **(Use a derivative to show that $\mathbf{f(x) = (x+1)/(x^2+99)}$ decreases after some point)**

Let $f(x) := \frac{x+1}{x^2+99}$, then notice that

$$\frac{\partial f}{\partial x} = \frac{x^2+99-2x^2-2x}{(x^2+99)^2} = \frac{-x^2-2x+99}{(x^2+99)^2}$$

Note that this function has zeros for x where $-x^2-2x+99=0$, and the zeros of this quadratic function are $(9-x)(11+x)=0$ or $x=9$ and $x=-11$. Since this is a negative quadratic, we know that for $x>9$, $f'(x)<0$ and thus $f(x)$ is decreasing. Moreover, $\lim_{x\to\infty} f(x) = 0$ so this series converges by the alternating series test. Note, the absolute value of this series does not converge by limit comparison with $1/n$. Therefore the series converges conditionally.

d.) $\sum_{\mathbf{n=1}}^{\infty}(-\mathbf{1})^{\mathbf{n+1}}\frac{\mathbf{n^2+50}}{\mathbf{n^5+50}}$

Converges absolutely by limit comparison with $1/n^3$.

e.) $\frac{\mathbf{5}}{\mathbf{1^4}}+\frac{\mathbf{5}}{\mathbf{2^4}}-\frac{\mathbf{5}}{\mathbf{3^4}}-\frac{\mathbf{5}}{\mathbf{4^4}}+\frac{\mathbf{5}}{\mathbf{5^4}}+\frac{\mathbf{5}}{\mathbf{6^4}}-\frac{\mathbf{5}}{\mathbf{7^4}}-\frac{\mathbf{5}}{\mathbf{8^4}}+\cdots$

This series can be expressed as

$$\sum_{n=1}^{\infty} C_n \frac{5}{n^4}$$

where C_n is some function that goes like $\{1,1,-1,-1,1,1,-1,-1,\cdots\}$. In order to determine C_n we need to employ sin and cos. So to get this pattern we use

$$C_n := \cos(n\pi/2) + \sin(n\pi/2)$$

Notice that this works because

$$\begin{aligned}
\cos(n\pi/2) &= \{0,-1,0,1,0,-1,0,1,\cdots,\} \\
\sin(n\pi/2) &= \{1,0,-1,0,1,0,-1,0,\cdots,\} \\
C_n &= \{1,-1,-1,1,1,-1,-1,1,\cdots,\}
\end{aligned}$$

We're slightly off by one, so instead we will use $C_n = \cos((n-1)\pi/2) + \sin((n-1)\pi/2)$. Now that we've gone through all that work, it doesn't matter because we are just going to look at the series

$$\sum_{n=1}^{\infty}\left|C_n\frac{5}{n^4}\right| = \sum_{n=1}^{\infty}\frac{5}{n^4}$$

which converges by p-series with $p=4>1$. Thus this series is absolutely convergent.

f.) $\sum_{\mathbf{n=1}}^{\infty}(-\mathbf{1})^{\mathbf{n}}\frac{(\ln \mathbf{n})^{\mathbf{3}}}{\mathbf{n}}$ **(Use a derivative to show that** $\mathbf{f(x)} = (\ln \mathbf{x})^{\mathbf{3}}/\mathbf{x}$ **decreases after some point)**

Let $f(x) = \frac{(\ln n)^3}{n}$, then the derivative of f is given by

$$\frac{\partial f}{\partial x} = \frac{(\ln x)^2(3 - \ln(x))}{n^2}$$

Note that $\ln(x)^2$ and n^2 are always positive, so this function is negative when $\ln(x) > 3 \iff x > e^3$. Thus at some point it is strictly decreasing and, by 3 iterations of L'Hospitale's, $\lim_{x\to\infty} f(x) = 0$. Thus this series converges by the alternating series test. However, $\sum \ln(n)^3/n > \sum \frac{1}{n}$ for all n, which diverges, thus the absolute series diverges by the comparison test. Therefore this series is conditionally convergent.

3. Use the absolute ratio test to determine the interval of convergence for each series.

a.) $\sum_{\mathbf{n=1}}^{\infty}\frac{\mathbf{x^n}}{\mathbf{n+1}}$ $x \in [-1, 1)$

Let $a_n := \frac{x^n}{n+1}$, then by the ratio test we obtain

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{n+2}}{\frac{x^n}{n+1}}\right| = \lim_{n\to\infty}\left|\left(\frac{x^{n+1}}{n+2}\right)\left(\frac{n+1}{x^n}\right)\right| = |x|$$

Thus, for this series to converge, we need $|x| < 1$. However, this test is inconclusive when the limit is 1, so we must test the series for $x = 1$ and $x = -1$ seperately. For $x = 1$ we get $\sum \frac{1}{n+1}$ which diverges (do a sub $m = n + 1$ then p-series). For $x = -1$ we get the series $\sum(-1)^n\frac{1}{n+1}$ which converges by alternating series test. Therefore the interval of convergence is $x \in [-1, 1)$.

b.) $\sum_{\mathbf{n=1}}^{\infty}\frac{\mathbf{x^n}}{\mathbf{n^2+1}}$ $x \in [-1, 1]$

Let $a_n := \frac{x^n}{n^2+1}$, then by the ratio test we obtain

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{n^2+2n+2}}{\frac{x^n}{n^2+1}}\right| = \lim_{n\to\infty}\left|\left(\frac{x^{n+1}}{n^2+2n+2}\right)\left(\frac{n^2+1}{x^n}\right)\right| = |x|$$

Thus, for this series to converge, we need $|x| < 1$. However, this test is inconclusive when the limit is 1, so we must test the series for $x = 1$ and $x = -1$ seperately. For $x = 1$ our series becomes $\sum \frac{1}{n^2+1} \leq \sum \frac{1}{n^2}$, thus this converges by the comparison and p-series tests. For $x = -1$ we have $\sum(-1)^n\frac{1}{n^2+1}$ which converges by the alternating series test. Therefore the interval of convergence is $x \in [-1, 1]$.

c.) $\sum_{\mathbf{n=1}}^{\infty}(-\mathbf{1})^{\mathbf{n}}\frac{(\mathbf{x-1})^{\mathbf{n+1}}}{\mathbf{n2^n}}$ $x \in (-1,3]$

Let $a_n := (-1)^n \frac{(x-1)^{n+1}}{n2^n}$, then by the ratio test we obtain

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(x-1)^{n+1}}{(n+1)2^{n+1}}}{\frac{(-1)^n(x-1)^n}{n2^n}}\right| = \lim_{n\to\infty}\left|\left(\frac{(x-1)^{n+1}}{(n+1)2^{n+1}}\right)\left(\frac{n2^n}{(x-1)^n}\right)\right| = \lim_{n\to\infty}\left|\left(\frac{n}{n+1}\right)\left(\frac{x-1}{2}\right)\right| = \left|\frac{x-1}{2}\right|$$

Thus, for this series to converge, we need $|(x-1)/2| < 1 \iff x \in (-1,3)$. However, this test is inconclusive when the limit is 1, so we must test the series for $x = 3$ and $x = -1$ seperately. For $x = 3$ we obtain

$$\sum_{n=1}^{\infty}(-1)^n\frac{2^{n+1}}{n2^n} = \sum_{n=1}^{\infty}(-1)^n\frac{2}{n}$$

which converges by the alternating series test (basically alternating harmonic series). For $x = -1$ we obtain

$$\sum_{n=1}^{\infty}(-1)^n\frac{(-2)^{n+1}}{n2^n} = \sum_{n=1}^{\infty}(-1)^n(-1)^{n+1}\frac{2^{n+1}}{n2^n} = \sum_{n=1}^{\infty}(-1)(-1)^{2n}\frac{2}{n} = \sum_{n=1}^{\infty}-\frac{2}{n}$$

Which diverges (basically regular harmonic series). Therefore the interval of convergence is $x \in (-1,3]$.

d.) $\sum_{\mathbf{n=1}}^{\infty}\frac{\mathbf{x^{n-1}}}{\mathbf{(n+1)!}}$ $x \in (-\infty,\infty)$

Let $a_n := \frac{x^{n-1}}{(n+1)!}$, then by the ratio test we obtain

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^n}{(n+2)!}}{\frac{(n+1)!}{x^{n-1}}}\right| = \lim_{n\to\infty}\left|\left(\frac{x^n}{(n+2)!}\right)\left(\frac{(n+1)!}{x^{n-1}}\right)\right| = lim_{n\to\infty}\left|\frac{x}{n+2}\right| = |0|$$

Since this limit is 0 regardless of the value of x, then x can be any real number. Therefore the interval of convergence is $x \in \mathbb{R}$ or $x \in (-\infty,\infty)$. Note that the denominator is a factorial; for any power series of this form the interval will always be $\mathbb{R}$.

e.) $\sum_{\mathbf{n=1}}^{\infty}(\mathbf{n!})^{\mathbf{2}}\mathbf{x^n}$ $x = 0$

Let $a_n := (n!)^2x^n$, then by the ratio test we obtain

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{((n+1)!)^2x^{n+1}}{(n!)^2x^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(n!)(n+1)(n!)x}{n!n!}\right| = \lim_{n\to\infty}|(n+1)(n+1)x| \to \infty$$

For this series, the limit diverges, so our only choices for convergence is the trivial case of $x = 0$ and the series becomes $\sum 0 = 0$. So the interval of convergence is only $x = 0$ or $x = [0,0]$.

f.) $\sum_{\mathbf{n=1}}^{\infty}\frac{\mathbf{(n!)x^n}}{\mathbf{n^{2n}}}$ $x \in (-\infty,\infty)$

Let $a_n := \frac{(n!)x^n}{n^{2n}}$, then by the ratio test we obtain

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)!x^{n+1}}{(n+1)^{2(n+1)}}}{\frac{n!x^n}{n^{2n}}}\right| = \lim_{n\to\infty}\left|\left(\frac{(n+1)!x^{n+1}}{(n+1)^{2(n+1)}}\right)\left(\frac{n^{2n}}{n!x^n}\right)\right| = \lim_{n\to\infty}\left|(n+1)x\frac{(n^n)^2}{(n+1)^2((n+1)^n)^2}\right| = 0$$

Since this limit is 0 regardless of the value of x, then x can be any real number. Therefore the interval of convergence is $x \in \mathbb{R}$ or $x \in (-\infty,\infty)$. Note that the denominator is a factorial; for any power series of this form the interval will always be $\mathbb{R}$.

g.) $\sum_{\mathbf{n=0}}^{\infty} \left(\frac{\mathbf{n+1}}{\mathbf{n+3}}\right)^{\mathbf{n}} \mathbf{x^n}$ **(Hint: Use the absolute root test.)** $x \in (-1, 1)$

Let $a_n := \left(\frac{n+1}{n+3}\right)^n x^n$, then by the root test we obtain

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n+1}{n+3}\right)^n x^n\right|} = \lim_{n\to\infty} \left|\sqrt[n]{\left(\frac{n+1}{n+3}\right)^n} \sqrt[n]{x^n}\right| = |x|$$

Thus, for this series to converge, we need $|x| < 1$. However, this test is inconclusive when the limit is 1, so we must test the series for $x = 1$ and $x = -1$ seperately. For $x = 1$ our series becomes $\sum \left(\frac{n+1}{n+3}\right)^n$, which diverges by the nth term test. For $x = -1$ we have $\sum (-1)^n \left(\frac{n+1}{n+3}\right)^n$ which also diverges. Therefore the interval of convergence is $x \in (-1, 1)$.

4. a.) Consider the series $\sum_{\mathbf{n=1}}^{\infty} (\cos(\mathbf{2n}\pi) + \cos((\mathbf{2n+1})\pi))$

i.) Write out the first 4 terms of the series.

The first 4 terms are given by

$$(1 + -1) + (1 + -1) + (1 + -1) + (1 + -1)$$

ii.) Use a particular test to determine if the series converges or diverges.

Diverges by nth term test

b.) Consider the series $\sum_{\mathbf{n=1}}^{\infty} (-\mathbf{1})^{\mathbf{n+1}}$

i.) Write out the first 8 terms of the series.
The first 8 terms are given by

$$1 + -1 + 1 + -1 + 1 + -1 + 1 + -1$$

ii.) Use a particular test to determine if the series converges or diverges.

Divergent by nth term test.

c.) The series in a.) and b.) appear to be the same, but the asnwers in a.) ii.) and b.) ii.); show otherwise

Although both of these series appear to be the same, we can use the indentity $\cos(a+b) = \cos(a)\cos(b) + \sin(a)\sin(b)$ to change the sum into

$$\sum_{n=1}^{\infty} (\cos(2n\pi) + \cos((2n+1)\pi)) = \sum_{n=1}^{\infty} (\cos(2n\pi) + \cos(2n\pi)\cos(n\pi) + \sin(2n\pi)\sin(n\pi))$$

Note that $\sin(2n\pi) = 0$ for any n, so this series becomes

$$\sum_{n=1}^{\infty} \cos(2n\pi)(1 + \cos(n\pi))$$

and the first 4 terms are now

$$1 \cdot (1 + -1) + 1 \cdot (1 + 1) + 1 \cdot (1 + -1) + 1 \cdot (1 + 1) + 1 \cdot (1 + -1) = 0 + 1 + 0 + 1$$

Thus the two series are not the same.

5. a.) Consider the series $\sum_{\mathbf{n=1}}^{\infty}(-\mathbf{1})^{\mathbf{n+1}}\frac{\mathbf{n}}{\mathbf{n+1}}$

i.) Write out the first 8 terms of the series.

The first 8 terms of this series are

$$\frac{1}{2}-\frac{2}{3}+\frac{3}{4}-\frac{4}{5}+\frac{5}{6}-\frac{6}{7}+\frac{7}{8}-\frac{8}{9}$$

ii.) Use a particular test to determine if the series converges or diverges.
Diverges because $\lim a_n = 1$.

b.) Consider the series $\sum_{\mathbf{n=1}}^{\infty}\left(\frac{\mathbf{2n-1}}{\mathbf{2n}}-\frac{\mathbf{2n}}{\mathbf{2n+1}}\right)$

i.) Write out the first 4 terms of the series.

The first 4 terms are given by

$$\left(\frac{1}{2}-\frac{2}{3}\right)+\left(\frac{3}{4}-\frac{4}{5}\right)+\left(\frac{5}{6}-\frac{6}{7}\right)+\left(\frac{7}{8}-\frac{8}{9}\right)$$

ii.) Use a particular test to determine if the series converges or diverges.

Combining the two fractions gives us

$$\sum_{n=1}^{\infty}\left(\frac{2n-1}{2n}-\frac{2n}{2n+1}\right)=\sum_{n=1}^{\infty}\left(\frac{1}{2n(2n+1)}\right)$$

Note that

$$\frac{1}{4n^2+2n}<\frac{1}{n^2}$$

for any $n>0$, so by the comparison test, we conclude that this series converges.

c.) The series in a.) and b.) appear to be the same, but the asnwers in a.) ii.) and b.) ii.); show otherwise

One converges and the other diverges, so they are not equal; wow.

6. a.) Consider the series $\sum_{\mathbf{n=0}}^{\infty}(-\mathbf{1/2})^{\mathbf{n}}$

i.) Write out the first 8 terms of the series.

The first 8 terms are

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}-\frac{1}{128}$$

ii.) Use a particular test to determine if the series converges or diverges.
Converges absolutely by geometric series test.

b.) Consider the series $\sum_{\mathbf{n=0}}^{\infty}\left(\frac{\mathbf{1}}{\mathbf{2^{2n}}}-\frac{\mathbf{1}}{\mathbf{2^{2n+1}}}\right)$

i.) Write out the first 4 terms of the series.

The first 4 terms are

$$\left(1-\frac{1}{2}\right)+\left(\frac{1}{4}-\frac{1}{8}\right)+\left(\frac{1}{16}-\frac{1}{32}\right)+\left(\frac{1}{64}-\frac{1}{128}\right)$$

ii.) Use a particular test to determine if the series converges or diverges.
Some algebra gives us

$$\sum_{n=0}^{\infty}\left(\frac{1}{2^{2n}}-\frac{1}{2^{2n+1}}\right)=\sum_{n=0}^{\infty}\frac{2^{2n+1}-2^{2n}}{2^{2n}2^{2n+1}}=\sum_{n=0}^{\infty}\frac{2^{2n}(2-1)}{2^{2n}2^{2n+1}}=\sum_{n=0}^{\infty}\frac{1}{2^{2n+1}}$$

Which converges by geometric series test

c.) The series in a.) and b.) appear to be the same, but the asnwers in a.) ii.) and b.) ii.); bear out that they are. What charactertistic in 6.) have that those in 4.) and 5.) do not? The series (All are the same.) in 6.) is an *absolutely convergent* series, which converges to the same number no matter how the terms in the series are grouped or in what order they are added. This is not always true of *conditionally convergent* or *divergent* series.

Both series converge so these are the same. In Divergent series, terms can be arbitrarily grouped and thus they can look similar without being the same.