1. Consider the function given by $\mathbf{f(x,y) = xy^2 - x^2y}$ and the point $\mathbf{P = (1,-1)}$. Compute

a.) the exact change of f and

b.) use a differential to estimate the exact change of f

if point P moves in a straight line to point $\mathbf{Q = (1.5,-0.7)}$.

The exact change is

$$f(x_2,y_2) - f(x_1,y_1)$$

for the differential, we notice that

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = f_x dx + f_y dy$$

since we have $f_x = y^2 - 2xy$ and $f_y = 2xy - x^2$ we get

$$df = (y^2 - 2xy)dx + (2xy - x^2)dy$$

so we can estimate the change by plugging in P for x and y and using $dx = (1.5-1)$ and $dy = -0.7-(-1)$

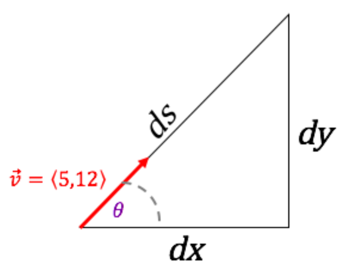
2. Consider the function given by $\mathbf{f(x,y) = \ln(3x+4y^2)}$ and the point $\mathbf{P = (5,2)}$. Compute

a.) the exact change of f and

b.) use a differential to estimate the exact change of f

if point P moves a distance $\mathbf{ds = 1.4}$ in the direction $\mathbf{\tilde{A} = 5\tilde{i} + 12\tilde{j}}$.

This is the same as above except we don't know dx and dy explicitly. Thus we have to compute it.



Thus what we can do is use trig to get

$$dx = ds\cos\theta \quad \text{and} \quad dy = ds\sin\theta$$

However, we don't know theta, so we have to use the fact that $\tan\theta = \frac{5}{12}$. Since you have a calculator on the test, just use this, but you could also use the fact that the hypotenuse is ds in the direction of $\vec{v}$. This gives us

$$dx^2 + dy^2 = \left(ds\left|\left|\frac{\vec{v}}{||v||}\right|\right|\right)^2$$

and we could solve using Pythagorean's theorem.

3. Find the point on the plane $\mathbf{x + 2y + 3z = 6}$ nearest the origin.

Way 1

Nearest the origin means that some circle around the origin is minimized. Thus we have a minimization problem of

$$\min x^2 + y^2 + z^2 \quad s.t \quad x + 2y + 3z = 6$$

and we can solve using Lagrange multipliers (if you learned this yet).

Way 2

Another way to solve this is the shortest distance between a point and a plane, which you've done in the past. Thus we can do this by picking any point on the plane, and projecting the vector from the origin to that point onto the normal vector of the plane. Notice that $(0, 0, 2)$ is on the plane, so let $\vec{v} = \langle 0-0, 0-0, 2-0\rangle = \langle 0, 0, 2\rangle$. Also, the normal vector $\vec{n} = \langle 1, 2, 3\rangle$ then the distance between the point and plane is

$$d = \left|\frac{\vec{v}\cdot\vec{n}}{||\vec{n}||\,||\vec{n}||}\vec{n}\right| = \left|\frac{6}{14}\langle 1, 2, 3\rangle\right| = \frac{6}{\sqrt{14}}$$

Then we have a 3D triangle with sides some multiple of $1, 2, 3$ respectively, with hypotenuse the distance we just computed. Thus we have

$$(1c)^2 + (2c)^2 + (3c^2) = \left(\frac{6}{\sqrt{14}}\right)^2$$

This gives us

$$14c^2 = \frac{36}{14} \iff c = \sqrt{\frac{6^2}{14^2}} = \frac{6}{14} = \frac{3}{7}$$

Thus we conclude that the point is at

$$(1c, 2c, 3c) = \left(\frac{3}{7}, \frac{6}{7}, \frac{9}{7}\right)$$

Way 3

Another way, the normal vector of the plane from the origin must hit the plane at some point and is the closest direction from the plane to the origin. So we can simply do

$$|c\vec{n}| = |\langle c, 2c, 3c\rangle| = \sqrt{c^2 + 4^2 + 9^2 = 14c^2} = 6$$

which clearly gives us the same answer as before.

4. Determine the dimensions and minimum surface area of a closed rectangular box with volume 8ft^3.

A rectangular box has volume given by xyz where x, y,and z are the sides of the box. The surface area is given by $2xy + 4zy$, so we get an optimization problem of

$$\max 2xy + 4zy \quad s.t. \quad xyz = 8$$

So we could solve this by say

$$\nabla(2xy + 4zy) = \langle 2y, 2x + 4z, 4y\rangle = 0$$

since each part of the gradient must be zero, we can use the f_y part to get $x = z$. So we can conclude that $x = y = z$ and since $xyz = 8$, we have $x^3 = y^3 = z^3 = 8$ or $x = y = z = 2$.

5. determine the dimensions and minimum surface area of the closed right circular cylinder with volume 16 πft^3

6. Material for the top and bottom of a rectangular box costs \$4/ft^2 and that for the sides costs \$2/ft^2. Determine the dimensions of the least expensive box of volume \$4/ft^2.

7. Among all open (no top) rectangular boxes with surface area $300in^2$, determine the dimensions of the box of maximum volume.