MAT 125B - Homework # 7

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January 27th, 2017

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1. Evaluate the following limits or determine that the limit does not exist

a.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{0},\mathbf{0})} \frac{\mathbf{x^2+y^2-4}}{\mathbf{x+y+2}}$

We can simply plug in $(x, y) = (0, 0)$ and we obtain

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2-4}{x+y+2} = \frac{-4}{2} = -2$$

b.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{1},\mathbf{1})} \frac{\mathbf{xy-y-2x+2}}{\mathbf{x-1}}$

Unlike in **a.)**, if we simply plug in (1,1) for (x, y) we get an undefined value. This is because of the $x - 1$ in the denominator. Thus, we need to remove this discontinuity. To do this we need to factor the top. Notice that we have,

$$xy - y - 2x + 2 = (xy - 2x) - (y - 2) = x(y - 2) - (y - 2) = (y - 2)(x - 1)$$

Thus the limit becomes

$$\lim_{(x,y)\to(1,1)} \frac{xy-y-2x+2}{x-1} = \lim_{(x,y)\to(1,1)} \frac{(y-2)(x-1)}{x-1} = \lim_{(x,y)\to(1,1)} y - 2 = -1$$

c.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{2},\mathbf{2})} \frac{\mathbf{x+y-4}}{\sqrt{\mathbf{x+y}}-\mathbf{2}}$

Let $r = \sqrt{x+y}$, then notice that we have

$$\frac{x+y-4}{\sqrt{x+y}-2} = \frac{r^2-2^2}{r-2} = \frac{(r-2)(r+2)}{r-2} = r + 2 = \sqrt{x+y} + 2$$

Thus we can evaluate the limit as

$$\lim_{(x,y)\to(2,2)} \frac{x+y-4}{\sqrt{x+y}-2} = \lim_{(x,y)\to(2,2)} \sqrt{x+y} + 2 = \sqrt{4} + 2 = 4$$

d.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{0},\mathbf{0})} \frac{\sin(\mathbf{x^2+y^2})}{\mathbf{x^2+y^2}}$

Let $r = x^2 + y^2$, then we obtain

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0} \frac{\sin(r)}{r} = {}^{1}1$$

e.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{1},\mathbf{1})} \frac{\sin(\mathbf{x^2-y^2})}{\mathbf{x-y}}$

Here we can't use the same trick as before. However, we see that we have $x^2 - y^2 = (x-y)(x+y)$ in the numerator, which could cancel the denominator if we didn't have the sine. Note that we could use a Taylor series trick here. Recall that

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots$$

so we obtain

$$\sin(x^2-y^2) = \sin((x-y)(x+y)) = [(x-y)(x+y)] - \frac{[(x-y)(x+y)]^3}{3!} + \frac{[(x-y)(x+y)]^5}{5!} + \cdots$$

and so our limit is

$$\begin{aligned}\frac{\sin(x^2-y^2)}{x-y} &= \frac{(x-y)(x+y)}{x-y} - \frac{[(x-y)(x+y)]^3}{3!(x-y)} + \frac{[(x-y)(x+y)]^5}{5!(x-y)} + \cdots \\ &= [(x+y)] - \frac{(x-y)^2(x+y)^3}{3!} + \frac{(x-y)^4(x+y)^5}{5!}\end{aligned}$$

and taking the limit we see that the only non-zero term is the first term, thus

$$\lim_{(x,y)\to(1,1)} \frac{\sin(x^2-y^2)}{x-y} = \lim_{(x,y)\to(1,1)} x+y = 2$$

f.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{1},\mathbf{-1})} \arcsin\left(\frac{\mathbf{xy}}{\sqrt{\mathbf{x^2+y^2}}}\right)$

We can just plug into this one

$$\lim_{(x,y)\to(1,-1)} \arcsin\left(\frac{xy}{\sqrt{x^2+y^2}}\right) = \arcsin\left(\frac{-1}{\sqrt{2}}\right) = \arcsin\left(\frac{-\sqrt{2}}{2}\right) = -\frac{\pi}{4}$$

g.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{0},\mathbf{0})} \frac{\mathbf{x^3}}{\mathbf{x^4+y^3}}$

I'm guessing this doesn't exist because the function $1/x$ doesn't exist as $x\to 0$. Thus we need to approach the limit from two directions that yield different values. Consider the case when $y=0$, then from $x\to 0^+$ we have

$$\lim_{x\to 0^+} \frac{x^3}{x^4} = \lim_{x\to 0^+} \frac{1}{x} \to \infty$$

and from the left we have

$$\lim_{x\to 0^-} \frac{x^3}{x^4} = \lim_{x\to 0^-} \frac{1}{x} \to -\infty$$

thus, that is enough to show the limit does not exist.

[1] Note: This limit can be easily found now that you know Taylor series. Just let $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$, then $\sin(x)/x$ as $x\to 0$ is clearly 1

h.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{0},\mathbf{0})} \frac{\mathbf{xy}}{\mathbf{x^2+y^2}}$

Similar to above, consider the direction when $x = y$, then we have

$$\lim_{y\to 0} \frac{y^2}{2y^2} = \frac{1}{2}$$

and when $x = 0$, we have

$$\lim_{y\to 0} \frac{0}{y^2} = 0$$

thus the limit does not exist.

i.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{0},\mathbf{0})} \frac{\mathbf{xy^2}}{\mathbf{x^2+y^4}}$

Same as above, but let's compare $x = y^2$ with $x = 0$.

$x = y^2$:

$$\lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2}$$

$x = 0$:

$$\lim_{y\to 0} \frac{0}{y^4} = 0$$

thus the limit does not exist.

j.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{2},-\mathbf{2})} \frac{\mathbf{4-xy}}{\mathbf{4+xy}}$

$x = y$:

$$\lim_{y\to -2} \frac{4-y^2}{4+y^2} = 0$$

$x = 0$:

$$\lim_{y\to -2} \frac{4}{4} = 1$$

thus the limit does not exist.

k.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{0},\mathbf{0})} \left(\mathbf{1+3xy^2}\right)^{\frac{\mathbf{2}}{\mathbf{xy^2}}}$

This looks like the goofy e^x limits of the past. Let $u = \frac{2}{xy^2}$, then we have

$$\left(1+3xy^2\right)^{2/xy^2} = \left(1+\frac{3}{u/2}\right)^u = \left(1+\frac{6}{u}\right)^u$$

Moreover, notice that

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{2} = \lim_{u\to\infty} \frac{2}{u}$$

thus we have

$$\lim_{(x,y)\to(0,0)} \left(1+3xy^2\right)^{\frac{2}{xy^2}} = \lim_{u\to\infty}\left(1+\frac{6}{u}\right)^u = e^6$$

l.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{0},\mathbf{0})} \frac{\mathbf{xy^2}}{\mathbf{x^2+y^4}}$

Same as above, but let's compare $x = y^2$ with $x = 0$.

$x = y^2$:

$$\lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2}$$

$x = 0$:

$$\lim_{y\to 0} \frac{0}{y^4} = 0$$

thus the limit does not exist.

m.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{1},-\mathbf{2})} \frac{\mathbf{(x-1)^2+3(y+2)^2}}{\mathbf{x-1+(y+2)^2}}$

Let $y = -2$:

$$\lim_{x\to 1} \frac{(x-1)^2}{(x-1)} = \lim_{x\to 1} x-1 = 0$$

Let $x = 1$:

$$\lim_{y\to -2} \frac{3(y+2)^2}{(y+2)^2} = \lim_{y\to -2} 3 = 3$$

Thus the limit does not exist.

n.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{1},\mathbf{2})} \frac{\mathbf{xy+2x-y-2}}{\mathbf{xy-y+3x-3}}$