1. Evaluate the following limits or determine that the limit does not exist

a.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{0},\mathbf{0})} \frac{\mathbf{x^2+y^2-4}}{\mathbf{x+y+2}}$

We can simply plug in $(x, y) = (0, 0)$ and we obtain

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2-4}{x+y+2} = \frac{-4}{2} = -2$$

b.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{1},\mathbf{1})} \frac{\mathbf{xy-y-2x+2}}{\mathbf{x-1}}$

Unlike in **a.)**, if we simply plug in (1,1) for (x, y) we get an undefined value. This is because of the $x - 1$ in the denominator. Thus, we need to remove this discontinuity. To do this we need to factor the top. Notice that we have,

$$xy - y - 2x + 2 = (xy - 2x) - (y - 2) = x(y - 2) - (y - 2) = (y - 2)(x - 1)$$

Thus the limit becomes

$$\lim_{(x,y)\to(1,1)} \frac{xy-y-2x+2}{x-1} = \lim_{(x,y)\to(1,1)} \frac{(y-2)(x-1)}{x-1} = \lim_{(x,y)\to(1,1)} y - 2 = -1$$

c.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{2},\mathbf{2})} \frac{\mathbf{x+y-4}}{\sqrt{\mathbf{x+y}}-\mathbf{2}}$

Let $r = \sqrt{x+y}$, then notice that we have

$$\frac{x+y-4}{\sqrt{x+y}-2} = \frac{r^2-2^2}{r-2} = \frac{(r-2)(r+2)}{r-2} = r + 2 = \sqrt{x+y} + 2$$

Thus we can evaluate the limit as

$$\lim_{(x,y)\to(2,2)} \frac{x+y-4}{\sqrt{x+y}-2} = \lim_{(x,y)\to(2,2)} \sqrt{x+y} + 2 = \sqrt{4} + 2 = 4$$

d.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{0},\mathbf{0})} \frac{\sin(\mathbf{x^2+y^2})}{\mathbf{x^2+y^2}}$

Let $r = x^2 + y^2$, then we obtain

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0} \frac{\sin(r)}{r} = {}^{1}1$$

e.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{1},\mathbf{1})} \frac{\sin(\mathbf{x^2-y^2})}{\mathbf{x-y}}$

Here we can't use the same trick as before. However, we see that we have $x^2 - y^2 = (x - y)(x + y)$ in the numerator, which could cancel the denominator if we didn't have the sine. Note that we could use a Taylor series trick here. Recall that

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots$$

[1]Note: This limit can be easily found now that you know Taylor series. Just let $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$, then $\sin(x)/x$ as $x \to 0$ is clearly 1

so we obtain

$$\sin(x^2 - y^2) = \sin((x-y)(x+y)) = [(x-y)(x+y)] - \frac{[(x-y)(x+y)]^3}{3!} + \frac{[(x-y)(x+y)]^5}{5!} + \cdots$$

and so our limit is

$$\begin{aligned}\frac{\sin(x^2 - y^2)}{x-y} &= \frac{(x-y)(x+y)}{x-y} - \frac{[(x-y)(x+y)]^3}{3!(x-y)} + \frac{[(x-y)(x+y)]^5}{5!(x-y)} + \cdots \\ &= [(x+y)] - \frac{(x-y)^2(x+y)^3}{3!} + \frac{(x-y)^4(x+y)^5}{5!}\end{aligned}$$

and taking the limit we see that the only non-zero term is the first term, thus

$$\lim_{(x,y)\to(1,1)} \frac{\sin(x^2-y^2)}{x-y} = \lim_{(x,y)\to(1,1)} x+y = 2$$

f.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{1},-\mathbf{1})} \arcsin\left(\frac{\mathbf{xy}}{\sqrt{\mathbf{x^2+y^2}}}\right)$

We can just plug into this one

$$\lim_{(x,y)\to(1,-1)} \arcsin\left(\frac{xy}{\sqrt{x^2+y^2}}\right) = \arcsin\left(\frac{-1}{\sqrt{2}}\right) = \arcsin\left(\frac{-\sqrt{2}}{2}\right) = -\frac{\pi}{4}$$

g.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{0},\mathbf{0})} \frac{\mathbf{x^3}}{\mathbf{x^4+y^3}}$

I'm guessing this doesn't exist because the function $1/x$ doesn't exist as $x \to 0$. Thus we need to approach the limit from two directions that yield different values. Consider the case when $y = 0$, then from $x \to 0^+$ we have

$$\lim_{x\to0^+} \frac{x^3}{x^4} = \lim_{x\to0^+} \frac{1}{x} \to \infty$$

and from the left we have

$$\lim_{x\to0^-} \frac{x^3}{x^4} = \lim_{x\to0^-} \frac{1}{x} \to -\infty$$

thus, that is enough to show the limit does not exist.

h.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{0},\mathbf{0})} \frac{\mathbf{xy}}{\mathbf{x^2+y^2}}$

Similar to above, consider the direction when $x = y$, then we have

$$\lim_{y\to0} \frac{y^2}{2y^2} = \frac{1}{2}$$

and when $x = 0$, we have

$$\lim_{y\to0} \frac{0}{y^2} = 0$$

thus the limit does not exist.

i.) $\lim_{(\mathbf{x},\mathbf{y})\to(\mathbf{0},\mathbf{0})} \frac{\mathbf{xy^2}}{\mathbf{x^2+y^4}}$

Same as above, but let's compare $x = y^2$ with $x = 0$.

$x = y^2$:

$$\lim_{y \to 0} \frac{y^4}{2y^4} = \frac{1}{2}$$

$x = 0$:

$$\lim_{y \to 0} \frac{0}{y^4} = 0$$

thus the limit does not exist.

j.) $\lim_{(\mathbf{x},\mathbf{y}) \to (\mathbf{2},-\mathbf{2})} \frac{\mathbf{4-xy}}{\mathbf{4+xy}}$

$x = y$:

$$\lim_{y \to -2} \frac{4 - y^2}{4 + y^2} = 0$$

$x = 0$:

$$\lim_{y \to -2} \frac{4}{4} = 1$$

thus the limit does not exist.

k.) $\lim_{(\mathbf{x},\mathbf{y}) \to (\mathbf{0},\mathbf{0})} \left(\mathbf{1 + 3xy^2}\right)^{\frac{\mathbf{2}}{\mathbf{xy^2}}}$

This looks like the goofy e^x limits of the past. Let $u = \frac{2}{xy^2}$, then we have

$$\left(1 + 3xy^2\right)^{2/xy^2} = \left(1 + \frac{3}{u/2}\right)^u = \left(1 + \frac{6}{u}\right)^u$$

Moreover, notice that

$$\lim_{(x,y) \to (0,0)} \frac{xy^2}{2} = \lim_{u \to \infty} \frac{2}{u}$$

thus we have

$$\lim_{(x,y) \to (0,0)} \left(1 + 3xy^2\right)^{\frac{2}{xy^2}} = \lim_{u \to \infty} \left(1 + \frac{6}{u}\right)^u = e^6$$

l.) $\lim_{(\mathbf{x},\mathbf{y}) \to (\mathbf{0},\mathbf{0})} \frac{\mathbf{xy^2}}{\mathbf{x^2+y^4}}$

Same as above, but let's compare $x = y^2$ with $x = 0$.

$x = y^2$:

$$\lim_{y \to 0} \frac{y^4}{2y^4} = \frac{1}{2}$$

$x = 0$:

$$\lim_{y \to 0} \frac{0}{y^4} = 0$$

thus the limit does not exist.

m.) $\lim_{(\mathbf{x},\mathbf{y}) \to (\mathbf{1},\mathbf{-2})} \frac{\mathbf{(x-1)^2+3(y+2)^2}}{\mathbf{x-1+(y+2)^2}}$

Let $y = -2$:

$$\lim_{x \to 1} \frac{(x-1)^2}{(x-1)} = \lim_{x \to 1} x - 1 = 0$$

Let $x = 1$:

$$\lim_{y \to -2} \frac{3(y+2)^2}{(y+2)^2} = \lim_{y \to -2} 3 = 3$$

Thus the limit does not exist.

n.) $\lim_{(\mathbf{x},\mathbf{y}) \to (\mathbf{1},\mathbf{2})} \frac{\mathbf{xy+2x-y-2}}{\mathbf{xy-y+3x-3}}$

2.) Compute $\mathbf{z_x}$ and $\mathbf{z_y}$ for each of the following functions

a.) $\mathbf{z = xy^2 + \ln x + e^y + 5}$

$$z_x = \frac{\partial z}{\partial x} = y^2 + \frac{1}{x}$$
$$z_y = \frac{\partial z}{\partial y} = 2xy + e^y$$

b.) $\mathbf{z = xe^{2y} \arctan x}$

$$z_x = \frac{\partial z}{\partial x} = e^2 y \arctan(x) + xe^{2y} \frac{1}{1+x^2}$$
$$z_y = \frac{\partial z}{\partial y} = 2x \arctan xe^{2y}$$

c.) $\mathbf{z = \sqrt{x - y^2}}$

$$z_x = \frac{\partial z}{\partial x} = \frac{1}{\sqrt{x-y^2}}$$
$$z_y = \frac{\partial z}{\partial y} = \frac{-2y}{\sqrt{x-y^2}}$$

d.) $\mathbf{z = \frac{x^3}{y^2} + \sin(xy)}$

$$\begin{aligned} z_x &= \frac{\partial z}{\partial x} = \frac{3x^2}{y^2} + y\cos(xy) \\ z_y &= \frac{\partial z}{\partial y} = -\left(\frac{x}{y}\right)^3 + x\cos(xy) \end{aligned}$$

e.) $\mathbf{z = \frac{x+4}{x^2+y^2}}$

$$\begin{aligned} z_x &= \frac{\partial z}{\partial x} = \frac{x^2+y^2-2x^2-8x}{(x^2+y^2)^2} \\ z_y &= \frac{\partial z}{\partial y} = -\frac{2y(x+4)}{(x^2+y^2)^2} \end{aligned}$$

f.) $\mathbf{z = \left[e^{x^2y} + \tan(3y+4x)\right]^5}$

$$\begin{aligned} z_x &= \frac{\partial z}{\partial x} = 5\left[e^{x^2y} + \tan(3y+4x)\right]^4\left[2xye^{x^2y} + 4\sec(3y+4x)\right] \\ z_y &= \frac{\partial z}{\partial y} = 5\left[e^{x^2y} + \tan(3y+4x)\right]^4\left[x^2e^{x^2y} + 3\sec(3y+4x)\right] \end{aligned}$$

g.) $\mathbf{z = y^{1+x^3}}$

$$\begin{aligned} z_x &= \frac{\partial z}{\partial x} = 3x^2y^{1+x^3}\ln y \\ z_y &= \frac{\partial z}{\partial y} = (1+x^3)y^{x^3} \end{aligned}$$

3.) Show that $\mathbf{z = \ln(1+x^2+y^2)}$ satisfies the equation $\mathbf{z_{xy} + z_x z_y = 0}$

We are going to show this by showing that $z_{xy} = -z_x z_y$. Let $z = \ln(1+x^2+y^2)$, then

$$\begin{aligned} z_{xy} &= \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{2y}{1+x^2+y^2}\right) = \frac{-4xy}{(1+x^2+y^2)^2} \\ -z_x z_y &= -\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) = -\left(\frac{2x}{1+x^2+y^2}\right)\left(\frac{2y}{1+x^2+y^2}\right) = \frac{-4xy}{(1+x^2+y^2)^2} \end{aligned}$$

Thus we conclude that $z_{xy} + z_x z_y = 0$.

4. Verify that $\mathbf{w_{xy} = w_{yx}}$ for $\mathbf{w = y + \frac{x}{y}}$

Since this function is a combination of continuous functions it is also continuous. Thus this must be true. To verify, notice that

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(1 - \frac{x}{y^2}\right) = -\frac{1}{y^2}$$
$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{1}{y}\right) = -\frac{1}{y^2}$$

Thus $w_{xy} = w_{yx}$

5. Determine functions z whose partial derivatives are given, or state that this is impossible

a.) $\mathbf{z_x = 2x}$ **and** $\mathbf{z_y = 3y^2 + 1}$

$$\int z_x\ dx = x^2 + g(y) \int z_y\ dy \qquad\qquad = y^3 + y + f(x)$$

Thus we can define a function of the form

$$z = x^2 + y + y^3 + c$$

b.) $\mathbf{z_x = xy^2 - y}$ **and** $\mathbf{z_y = x^2y - x}$

$$\int z_x\ dx = \frac{1}{2}x^2y^2 - xy + g(y) \int z_y\ dy \qquad\qquad = \frac{1}{2}x^2y^2 - xy + f(x)$$

Thus

$$z = \frac{1}{2}(xy)^2 - xy + c$$

c.) $\mathbf{z_x = e^y - 1}$ **and** $\mathbf{z_x = e^x - x}$

$$\int z_x\ dx = xe^y - x + g(y) \int z_y\ dy \qquad\qquad = ye^x - xy + g(x)$$

This can't be solved because the the difference between $\int z_x\ dx$ and $\int z_y\ dy$ are not functions of purely x or y.

d.) $\mathbf{ye^x \cos(xy) + e^x \sin(xy) - 2}$ **and** $\mathbf{z_y = xe^x \cos(xy) + 1}$

$$\int z_x\ dx = \int z_y\ dy \qquad\qquad =$$

6. Consider the function

$$\mathbf{f(x,y)} = \begin{cases} \frac{\sin(x^3+y)}{x^2+y^2}, & \text{if}(x,y) \neq (0,0) \\ 0, & \text{if}(x,y) = (0,0) \end{cases}$$

a.) Determine $\mathbf{f_x(x,y)}$ when $\mathbf{(x,y) \neq (0,0)}$.

$$f_x(x,y) = \frac{3x^2\cos(x^3+y) - 2x\sin(x^3+y)}{(x^2+y^2)^2}$$

b.) Determine $\mathbf{f_x(0,0)}$ (Use limit definition of partial derivatives).

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{\sin(h^3)}{h^2}}{h} = \lim_{h\to 0} \frac{\sin(h^3)}{h^3} = 1$$

c.) Determine $\mathbf{f_y(0,0)}$ (Use limit definition of partial derivatives).

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{\sin(h)}{h^2}}{h} = \lim_{h\to 0} \frac{\sin(h)}{h}\frac{1}{h^2} = \text{DNE}$$

7. Plane A, parallel to the xz-plane, and plane B, parallel to the yz-plane, pass through the surface determined by the equation $\mathbf{z = xy^2 - x^3 + 7}$. Both planes include the point (1,0,6), which lies on the surface.

Let $p = (x_0, y_0, z_0$, then the equation of A is $y = y_0$, B is $x = x_0$, and the tangent plane at p is

$$z_x(x_0,y_0)(x-x_0) + z_y(x_0,y_0)(y-y_0) + z_0$$

a.) Determine the slope ofthe line tangent to the surface at the point (1,0,6) if the line lies in

i.) Plane A

We want to find the intersection of the A and the tangent plane, thus x, y, z that satisfy

$$y = y_0 = 0$$
$$z = z_x(x_0,y_0)(x-x_0) + z_y(x_0,y_0)(y-y_0) + z_0 = -3(x-1) + 0(y-0) + 6 = -3x + 9$$

The line is given by this system of equations.

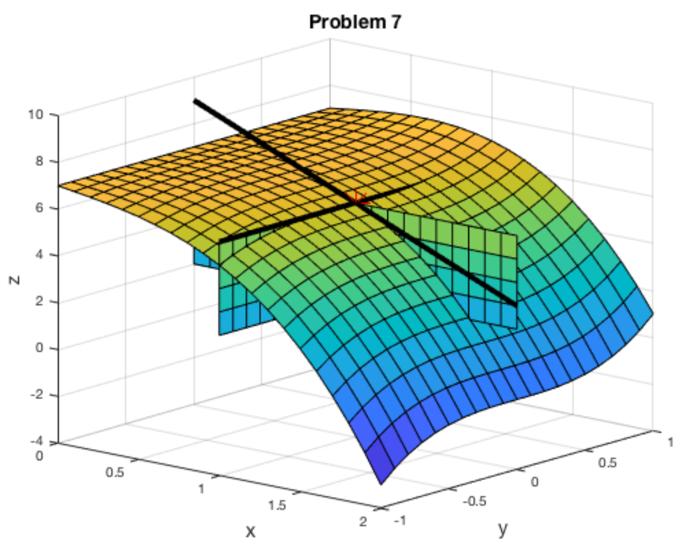
i.) Plane B

b.) Determine an equation of the plane tangent to the surface at the point (1,0,6).

We want to find the intersection of the A and the tangent plane, thus x, y, z that satisfy

$$x = x_0 = 1$$
$$z = z_x(x_0,y_0)(x-x_0) + z_y(x_0,y_0)(y-y_0) + z_0 = -3(x-1) + 0(y-0) + 6 = -3x + 9$$

The line is given by this system of equations.



1. Plotting the planes A and B with the intersecting tangent line at the point p.

8. Compute $\mathbf{z_x}$ and $\mathbf{z_y}$ for each of the following functions

a.) $\mathbf{z = x^3y + y^4 - 2x + 5}$

$$z_x = 3x^2y - 2$$
$$z_y = x^3 + 4y^3$$

b.) $\mathbf{z = f(x) + g(y)}$

$$z_x = f_x(x)$$
$$z_y = g_y(y)$$

c.) $\mathbf{z = f(x^3) + g(4y)}$

$$z_x = 3x^2 f_x(x)$$
$$z_y = 4g_y(y)$$

d.) $\mathbf{z = f(x^2 + y^3) + g(xy^2)}$

$$z_x = 2xf_x(x^2 + y^3) + y^2 g_x(xy^2)$$
$$z_y = 3y^2 f_x(x^2 + y^3) + 2xyg_y(xy^2)$$

e.) $\mathbf{y^2 + z^2 + \sin(xz) = 4}$

$$2zz_x + (xz_x + z)\cos(xz) = 0 \iff z_x(2z + x\cos(xz)) = -z\cos(xz) \iff z_x = -\frac{z\cos(xz)}{2z + x\cos(xz)}$$
$$2y + 2zz_y + xz_y\cos(xz) = 0 \iff z_y = -\frac{2y}{2z + x\cos(xz)}$$

f.) $\mathbf{z = f(u, v)}$ where $\mathbf{u = \ln(x - y)}$ and $\mathbf{v = e^{xy}}$

$$z_x = f_u u_x + f_v v_x = \frac{1}{x-y} f_u + \frac{1}{x-y} f_v$$
$$z_y = f_u u_y + f_v v_y = -\frac{1}{x-y} f_u - \frac{1}{x-y} f_v$$

9. Find $\frac{\partial \mathbf{w}}{\partial \mathbf{t}}$ and $\frac{\partial \mathbf{w}}{\partial \mathbf{s}}$ if $\mathbf{w = f(4t^2 - 3s)}$ and $\mathbf{f'(x) = \ln x}$

Let $x = 4t^2 - 3s$, then $\frac{dx}{dt} = 8t \iff dx = 8tdt$. Then

$$\frac{\partial w}{\partial t} = \frac{\partial f(4t^2 - 3s)}{\partial t} = 8t\frac{\partial f(x)}{\partial x} = 8t\ln(x) = 8t\ln(4t^2 - 3s)$$

Let $x = 4t^2 - 3s$, then $\frac{dx}{ds} = 3 \iff dx = 3ds$. Then

$$\frac{\partial w}{\partial s} = \frac{\partial f(4t^2 - 3s)}{\partial s} = 3\frac{\partial f(x)}{\partial x} = 3\ln(x) = 3\ln(4t^2 - 3s)$$

10. Assume that f is a differentiable function of one variable with $\mathbf{z = xf(xy)}$. Show that $\mathbf{xz_x - yz_y = z}$.

$$xz_x = x(f(xy) + xf_x(xy)) = z + xyf_x(xy))$$
$$yz_y = y(xf_y(xy))) = xyf_y(xy)$$

since f is a differntiable function of one variable, $f' := f_y = f_x$ and thus

$$xz_x - yz_y = z + xyf' - xyf' = z$$

11. Assume that f and g are twice differentiable functions of one variable. SHow that $\mathbf{u = f(x + at) + g(x - at)}$ satisfies $\mathbf{a^2 \frac{\partial^2 u}{\partial x^2}} = \frac{\partial^2 u}{\partial t^2}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial f_x(x + at) + g_x(x - at)}{\partial x} = f_{xx}(x + at) + g_{xx}(x - at)$$
$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial af_t(x + at) - g_t(x - at)}{\partial t} = a^2 f_{tt}(x - at) + a^2 g_{tt}(x - at)$$

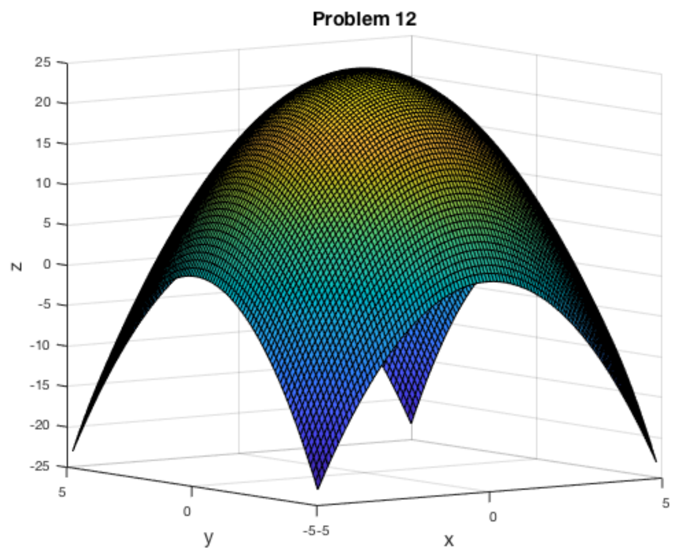
Since f and g are twice differentiable of one variable $f'' := f_{xx} = f_{tt}$ and $g'' := g_{xx} = g_{tt}$ and

$$a^2\frac{\partial^2 u}{\partial x^2} = a^2\left(f_{xx}(x + at) + g_{xx}(x - at)\right) = a^2 f_{tt}(x - at) + a^2 g_{tt}(x - at) = \frac{\partial^2 u}{\partial t^2}$$

Therefore we conclude that $a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$.

12. Consider the paraboloid given by $\mathbf{f(x,y) = 25 - x^2 - y^2}$

a.) Sketch the surface



2. Paraboloid given by $z = 25 - x^2 - y^2$

b.) Let $\mathbf{P} = (\mathbf{2}, -\mathbf{2})$. Compute the derivative of the function f at the point P in the direction

The derivative of f at the point P is given by

$$df = \nabla f(P) \cdot \hat{u}$$

where $\hat{u} = \frac{\vec{u}}{|\vec{u}|}$. Thus

$$dF = \langle -2(2), -2(-2) \rangle \cdot \hat{u} = \langle -4, 4 \rangle \cdot \hat{u}$$

i.) $\overrightarrow{A} = \overrightarrow{\langle -3, 4 \rangle}$

$$dF = \langle -4, 4 \rangle \cdot \hat{A} = \langle -4, 4 \rangle \cdot \frac{1}{5} \langle -3, 4 \rangle = \frac{1}{5}(12 + 16) = \frac{28}{5}$$

ii.) $\overrightarrow{A} = \overrightarrow{\langle 3, -4 \rangle}$

$$dF = \langle -4, 4 \rangle \cdot \hat{A} = \langle -4, 4 \rangle \cdot \frac{1}{5} \langle 3, -4 \rangle = \frac{1}{5}(-12 - 16) = -\frac{28}{5}$$

iii.) $\overrightarrow{A} = \overrightarrow{\langle 1, 0 \rangle}$

$$dF = \langle -4, 4 \rangle \cdot \hat{A} = \langle -4, 4 \rangle \cdot \langle 1, 0 \rangle = -4$$

iv.) $\overrightarrow{A} = \overrightarrow{\langle 0, -1 \rangle}$

$$dF = \langle -4, 4 \rangle \cdot \hat{A} = \langle -4, 4 \rangle \cdot \langle 0, -1 \rangle = -4$$

c.) In what directions is the derivative of f at point $\mathbf{P = (2, -2)}$ equal to zero?

$$dF = \langle -4, 4 \rangle \cdot \hat{A} = \langle 0, 0 \rangle \iff -4a_1 + 4a_2 = 0 \iff a_1 = a_2$$

thus any vector where $a_1 = a_2$.

d.) In what directions is the derivative of f at point $\mathbf{P = (-1, 1)}$ equal to 2?

$$df = \nabla f(P) \cdot \hat{u} = \langle -2(-1), -2(1) \rangle \cdot \hat{u} = 2u_1 - 2u_2 = 2 \iff u_1 = u_2 + 1$$

Thus any vector where $u_1 = u_2 + 1$.