Math 21B
Disussion Sheet 2 - Key
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Note: These answers are not endorsed by Dr. Gravner and may be incorrect!

Find $\mathbf{F'(x)}$ if

a) $\mathbf{F(x) = \int_{\sqrt{x}}^{\pi} e^{2t^2} \ dt}$
First, using the reversal property of integrals, we get

$$F(x) = \int_{\sqrt{x}}^{\pi} e^{2t^2} \ dt = -\int_{\pi}^{\sqrt{x}} e^{2t^2} \ dt$$

Then we can simply apply the Fundamental theorem of Calculus with chain rule and get

$$F'(x) = e^{2(\sqrt{x})^2} \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}} e^{2x}$$

b) $\mathbf{F(x) = \int_{-x}^{x} \frac{1}{3+t^2} \ dt}$

Here, since we don't know how to handle functions of x in both sides of the interval, we break this integral up. For any number x, we know that 0 is in $[-x, x]$, so we can break this integral up into

$$F(x) = \int_{-x}^{x} \frac{1}{3+t^2} \ dt = \int_{-x}^{0} \frac{1}{3+t^2} \ dt + \int_{0}^{x} \frac{1}{3+t^2} \ dt$$

Then we can flip the first integral as in **a)**, and we get

$$F(x) = \int_{-x}^{0} \frac{1}{3+t^2} \ dt + \int_{0}^{x} \frac{1}{3+t^2} \ dt = -\int_{0}^{-x} \frac{1}{3+t^2} \ dt + \int_{0}^{x} \frac{1}{3+t^2} \ dt$$

Thus

$$F'(x) = -\left(\frac{1}{3+(-x)^2}(-1)\right) + \left(\frac{1}{3+x^2}\right) = \frac{2}{3+x^2}$$

2. Compute

$$\lim_{\mathbf{x \to 0}} \frac{\int_{\mathbf{2}}^{\mathbf{2+5x}} \mathbf{e^{t^2} \ dt}}{\int_{\mathbf{1}}^{\mathbf{1+x}} \mathbf{e^{-t^2} \ dt}}$$

There are a couple things you need to recognize to do this problem. First, these integrals are unsolvable because they are Gaussians (e^{x^2}), and second, an integral with the same start and end points is zero. Thus we have an limit of the form 0/0, thus we can do L'Hopitale's rule, and we obtain

$$\lim_{x \to 0} \frac{\int_{2}^{2+5x} e^{t^2} \ dt}{\int_{1}^{1+x} e^{-t^2} \ dt} \overset{\mathcal{L}'\mathcal{H}}{=} \lim_{x \to 0} \frac{5e^{x^2}}{e^{-x^2}} = \lim_{x \to 0} 5e^{2x^2} = 5$$

3. Let $\mathbf{f(x) = x + 1/x}$. For which interval $\mathbf{I = [a, a+2], a > 0}$, is the average of f over I minimal?

So we define the average of a function by

$$\frac{1}{b-a}\int_a^b f(x)\ dx$$

Thus, the average of f over $[a, a+2]$ is

$$\frac{1}{2}\int_a^{a+2} x + \frac{1}{x}\ dx$$

To find which a makes this the smallest, we do the usual "set the derivative equal to 0" steps from MAT 21A. However, we need to recognize that we are varying a in this case. Thus our derivative is with respect to a. Thus, by the Fundamental Theorem of Calculus,

$$\frac{d}{da}\frac{1}{2}\int_a^{a+2} x+\frac{1}{x}\ dx = \frac{d}{da}\frac{1}{2}\int_a^{c} x+\frac{1}{x}\ dx+\frac{d}{da}\frac{1}{2}\int_c^{a+2} x+\frac{1}{x}\ dx = -\frac{1}{2}\left(a+\frac{1}{x}\right)+\frac{1}{2}\left((a+2)+\frac{1}{a+2}\right) = 1+\frac{1/2}{a-2}-\frac{1/2}{a}$$

where $c \in [a, a+2]$. Thus setting this equal to zero we have

$$0 = 1 + \frac{1/2}{a-2} - \frac{1/2}{a} \iff 0 = a(a-2) + \frac{1}{2}a - \frac{1}{2}a + 1 \iff 0 = a^2 - 2a + 1$$

Which has the solution $a = 1$ or $a = -1$. However, the problem states that $a > 0$, so the answer is $a = 1$.

4. Compute

a.) $\int \frac{\mathbf{x^2+1}}{\mathbf{(x-1)^3}}\ \mathbf{dx}$

Let $u = (x-1)$, then $du = dx$ and we get

$$\begin{aligned}\int \frac{x^2+1}{(x-1)^3}\ dx &= \int \frac{(u+1)^2+1}{(u)^3}\ du \\ &= \int \frac{u^2+2u+2}{u^3}\ du \\ &= \int \frac{1}{u} + \frac{2}{u^2} + \frac{2}{u^3}\ du \\ &= \ln|(|\,u) - \frac{2}{u} - \frac{1}{u^2} + C \\ &= \ln|(|\,x-1) - \frac{2}{x-1} - \frac{1}{(x-1)^2} + C\end{aligned}$$

b.) $\int_\mathbf{2}^\mathbf{3} \frac{\mathbf{x^2+1}}{\mathbf{(x-1)^3}}\ \mathbf{dx}$

Just plug in 2 and 3 to the above equation.

c.) $\int \frac{\mathbf{1}}{\mathbf{(x^2+1)\arctan x}}\ \mathbf{dx}$

Let $u = \arctan x$, the $du = \frac{1}{1+x^2}\ dx$, thus we get

$$\int \frac{1}{(x^2+1)\arctan x}\ dx = \int \frac{1}{u}\ du = \ln|u| + C = \ln|\arctan x| + C$$

d.) $\int_0^{\pi/2} \frac{\sin\theta}{1+\cos^2\theta}\ \mathbf{dx}$

Let $u = \cos\theta$, then $du = -\sin\theta\ d\theta$ and we obtain

$$\begin{aligned}\int_0^{\pi/2} \frac{\sin\theta}{1+\cos^2\theta}\ dx &= \int_{\cos 0}^{\cos\pi/2} \frac{1}{1+u^2}\ du \\ &= \arctan u\big|_{u=1}^{0} = -\frac{pi}{4}\end{aligned}$$

e.) $\int_0^1 (\mathbf{x^2}+\mathbf{1})^5\mathbf{x^3}\ \mathbf{dx}$
Just use Binomial Theorem to get

$$\begin{aligned}\int_0^1 (x^2+1)^5x^3\ dx &= \int_0^1 \left[(x^2)^5 + 5(x^2)^4 + 10(x^2)^3 + 10(x^2)^2 + 5(x^2) + 1\right] x^3\ dx \\ &= \int_0^1 x^{13} + 5x^{11} + 10x^9 + 10x^7 + 5x^5 + x^3\ dx \\ &= \frac{1}{14}x^{14} + \frac{5}{12}x^{12} + \frac{10}{10}x^{10} + \frac{10}{8}x^8 + \frac{5}{6}x^6 + \frac{1}{4}x^4\Big|_{x=0}^{1} \\ &= frac114 + \frac{5}{12} + \frac{10}{10} + \frac{10}{8} + \frac{5}{6} + \frac{1}{4}\end{aligned}$$

f.) $\int_0^{\pi/2} \sin\mathbf{x}\cos\mathbf{x}\sqrt{\mathbf{1}-\cos\mathbf{x}}\ \mathbf{dx}$

Let $u = 1-\cos(x)$ then $du = \sin x\ dx$ and we get

$$\begin{aligned}\int_0^{\pi/2} \sin x\cos x\sqrt{1-\cos x}\ dx &= \int_{1-\cos(0)}^{1-\cos(pi/2)} (u+1)\sqrt{u}\ du \\ &= \int_0^1 u^{1/2} + u^{3/2}\ du \\ &= \frac{2}{3}u^{3/2} + \frac{2}{5}u^{5/2}\Big|_{u=0}^{1} = \frac{2}{3} + \frac{5}{2}\end{aligned}$$