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Expectation Operators & the Bergman Projection

joint w/ Michael Stessin

Defn: Let $(X, \mathcal{M}, \mu)$ be a measure space, $\mu > 0$.
Let $\mathcal{A} \subset \mathcal{M}$ be a σ-subalgebra.
For $h \in L^1(X, \mathcal{M}, \mu)$, the function $\mathcal{E}_{\mathcal{A}}(h)$ is defined by

$$\int_\Delta \mathcal{E}_{\mathcal{A}}(h)\, d\mu = \int_\Delta h \, d\mu, \qquad \forall \Delta \in \mathcal{A}$$

and Radon-Nikodým.

$\mathcal{E}_{\mathcal{A}}$ = conditional expectation operator

Remarks: For each $1 \le p \le \infty$, $\mathcal{E}_{\mathcal{A}}$ maps $L^p(X, \mathcal{M}, \mu)$ to $L^p(X, \mathcal{A}, \mu|_{\mathcal{A}})$, which is a subspace of $L^p(X, \mathcal{M}, \mu)$, and

$$\|\mathcal{E}_{\mathcal{A}}\|_{L^p(X,\mathcal{M},\mu) \to L^p(X, \mathcal{A}, \mu|_{\mathcal{A}})} = 1.$$

Motivation: circle setting
$\mathbb{T}$ = unit circle
m = normalized Lebesgue measure on $\mathbb{T}$
P = Riesz projection.

If $f: \mathbb{T} \to \mathbb{C}$ is Lebesgue measurable, let $\mathcal{A}(f)$ denote the smallest σ-algebra relative to which f is measurable.

Aleksandrov's Theorem (1986):
Let $\mathcal{A}$ be a σ-algebra of Lebesgue measurable subsets of $\mathbb{T}$.
Then
$\mathcal{E}_{\mathcal{A}} P = P \mathcal{E}_{\mathcal{A}}$ on $L^2(\mathbb{T}, m)$
iff $\mathcal{A} = \mathcal{A}(f)$ for some inner function f.

Connection to Clark Measures:

If $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic, then (by Herglotz) for every $\alpha \in \mathbb{T}$ there exists a positive measure μ_α such that

$$\int_{\mathbb{T}} \operatorname{Re} \frac{\zeta + z}{\zeta - z} \, d\mu_\alpha(\zeta) = \operatorname{Re} \frac{\alpha + \varphi(z)}{\alpha - \varphi(z)}$$

since the RHS is positive, harmonic in $\mathbb{D}$.

Aleksandrov (1985): For $f \in L^1(\mathbb{T}, m)$

$$\int_{\mathbb{T}} f(\zeta)\, dm(\zeta) = \int_{\mathbb{T}} \left(\int_{\mathbb{T}} f(\zeta)\, d\mu_\alpha(\zeta) \right) dm(\alpha)$$

Remark:

Let φ be inner with $\varphi(0) = 0$.

Let $T(f)(\alpha) = \int_{\mathbb{T}} f(\zeta) d\mu_\alpha(\zeta)$, $f \in L^1(\mathbb{T}, m)$.

Then

$$\mathcal{E}_{\mathcal{A}}(f) = (Tf) \circ \varphi, \quad \text{for } \mathcal{A} = \mathcal{A}(\varphi).$$

Vague Q: To what extent does Aleksandrov (1986) hold in disk setting?

Let $\mathbb{D}$ = unit disk, $d\sigma(z) = \frac{1}{\pi} dx dy$, and $\mathcal{P}$ = Bergman projection.

Note: for $1 < p < \infty$ $\mathcal{P} : L^p(\mathbb{D}, \sigma) \to A^p$ by

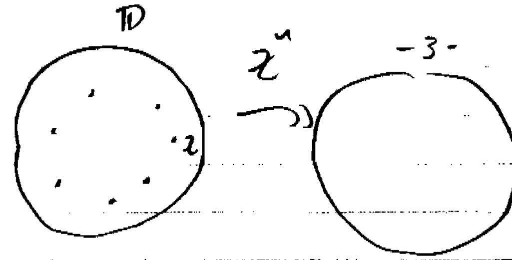
$$\mathcal{P}(h)(w) = \int_{\mathbb{D}} \frac{h(z)}{(1 - w\bar{z})^2} d\sigma(z), \quad h \in L^p(\mathbb{D}, \sigma), \ w \in \mathbb{D}.$$

Precise Q: For which σ-algebras $\mathcal{A}$ of Lebesgue area measurable subsets of $\mathbb{D}$ does the condition

$$\mathcal{E}_{\mathcal{A}} P = P \mathcal{E}_{\mathcal{A}}$$

hold on $L^2(\mathbb{D}, \sigma)$?

Warmup example: Let $\mathcal{A} = \mathcal{A}(z^n)$ for some $n \in \mathbb{N}$. Then $\mathcal{E}_{\mathcal{A}} \mathcal{P} = \mathcal{P} \mathcal{E}_{\mathcal{A}}$ on $L^2(\mathbb{D}, \sigma)$.



<u>outline of proof:</u>

let $e_k = e^{2\pi i k/n}$

can show $\mathcal{E}_{\mathcal{A}}(h)(\zeta) = \frac{1}{n}\sum_{k=1}^{n} h(e_k\zeta)$, $\forall\, h \in L^2(\mathbb{D},\sigma)$, $\forall\, \zeta \in \mathbb{D}$.

then
$$P\mathcal{E}_{\mathcal{A}}(h)(w) = \int_{\mathbb{D}} \frac{\mathcal{E}_{\mathcal{A}}(h)(\zeta)}{(1-w\bar{\zeta})^2}\, d\sigma(\zeta)$$
$$= \frac{1}{n}\sum_{k=1}^{n}\int_{\mathbb{D}} \frac{h(e_k\zeta)}{(1-w\bar{\zeta})^2}\, d\sigma(\zeta)$$

change of variables
$z = e_k\zeta$
$d\sigma(z) = d\sigma(\zeta)$
$\bar{\zeta} = e_k\bar{z}$

$$= \frac{1}{n}\sum_{k=1}^{n}\int_{\mathbb{D}} \frac{h(z)}{(1-e_k w\bar{z})^2}\, d\sigma(z)$$
$$= \frac{1}{n}\sum_{k=1}^{n} P(h)(e_k w)$$
$$= \mathcal{E}_{\mathcal{A}} P(h)(w).$$

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<u>Theorem 1:</u> Let f be a finite Blaschke product, $f(0)=0$.
Let $\mathcal{A} = \mathcal{A}(f)$. Suppose $\mathcal{E}_{\mathcal{A}} P = P\mathcal{E}_{\mathcal{A}}$ on $L^2(\mathbb{D},\sigma)$.
Then $\mathcal{A} = \mathcal{A}(z^n)$ for some $n \in \mathbb{N}$.

(n will be the order of the Blaschke product)

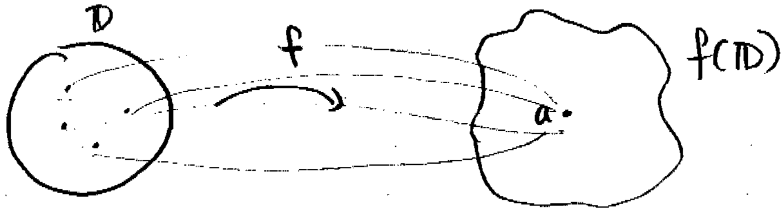
<u>Goal:</u> Discuss ingredients of proof + some generalizations.

<u>Rest of talk:</u>
Fix $f \in H^\infty(\mathbb{D})$ (not constant)
Let $\mathcal{A} = \mathcal{A}(f)$
Suppose $\mathcal{E}_{\mathcal{A}} P = P\mathcal{E}_{\mathcal{A}}$ on $L^2(\mathbb{D},\sigma)$

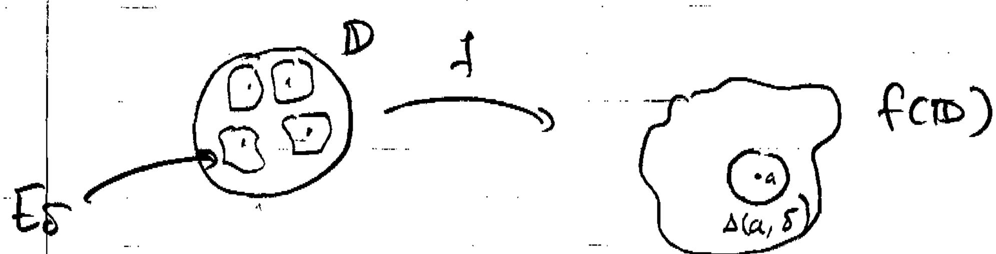
If $a \in f(\mathbb{D})$, let $E_a = \{z \in \mathbb{D} : f(z) = a\}$.

Let $E_\delta = f^{-1}(\Delta(a,\delta))$ Note: $E_\delta \in \mathcal{A}(f)$. Δ = Euclidean disk centered at a, radius δ

Let $v_\delta = \dfrac{\sigma|_{E_\delta}}{\sigma(E_\delta)}$ (probability measure)



that collection of points is E_a



<u>Lemma 1</u>: $\mathcal{E}_{\mathcal{A}}(A^2) \subset A^2$

<u>proof</u>: Let $h \in A^2$. then $\mathcal{E}_{\mathcal{A}}(h) = \mathcal{E}_{\mathcal{A}} P(h) = P\mathcal{E}_{\mathcal{A}}(h) \in A^2$. ∎

<u>Lemma 2</u>: Let $h \in A^2$, $\zeta \in E_a$. then

$\int_{\mathbb{D}} h \, dv_\delta \to \mathcal{E}_{\mathcal{A}}(h)(\zeta)$, as $\delta \to 0$.

<u>Main Idea</u>: $\int_{\mathbb{D}} h \, dv_\delta = \frac{1}{\sigma(E_\delta)} \int_{E_\delta} h(z)\, d\sigma(z)$

since $E_\delta \in \mathcal{A}(f)$ $= \frac{1}{\sigma(E_\delta)} \int_{E_\delta} \mathcal{E}_{\mathcal{A}}(h)(z)\, d\sigma(z)$

$\approx \frac{1}{\sigma(E_\delta)} \int_{E_\delta} \mathcal{E}_{\mathcal{A}}(h)(\zeta)\, d\sigma(z)$

$= \mathcal{E}_{\mathcal{A}}(h)(\zeta)$.

This requires proving that $|\mathcal{E}_{\mathcal{A}}(h)(z) - \mathcal{E}_{\mathcal{A}}(h)(\zeta)|$ is small for $z \in E_\delta$. ∎

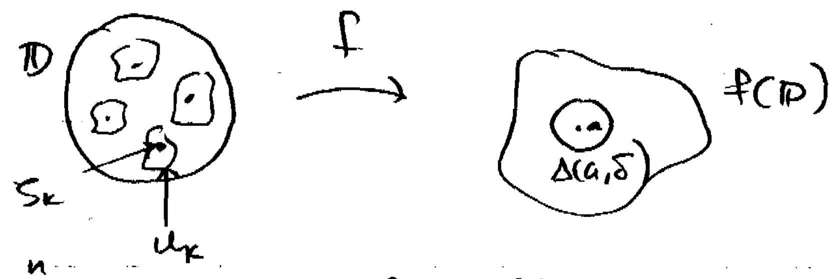
<u>theorem 2</u>: Suppose $f \in A(\mathbb{D})$ with finite multiplicity. let $a \in f(\mathbb{D}) \setminus f(\mathbb{T})$. Let $E_a = \{\zeta_1, \zeta_2, \ldots, \zeta_n\}$ and suppose $E_a \cap Z(f') = \phi$.

zero set

then, for $h \in A^2$ and $\zeta \in E_a$,

$$\mathcal{E}_f(h)(\zeta) = \frac{\sum_{k=1}^{n} h(\zeta_k) \frac{1}{|f'(\zeta_k)|^2}}{\sum_{k=1}^{n} \frac{1}{|f'(\zeta_k)|^2}}$$

<u>Main Idea</u>: For small δ, here's the picture:



$E_\delta = \bigcup_{k=1}^{n} U_k$ (disjoint). $f_k := f|_{U_k}$ maps U_k to $\Delta(a,\delta)$ bijectively.

then

$$\frac{1}{\delta^2}\int_{E_\delta} h\,d\sigma = \frac{1}{\delta^2}\sum_{k=1}^{n}\int_{U_k} h\,d\sigma$$

$$= \frac{1}{\delta^2}\sum_{k=1}^{n}\int_{\Delta(a,\delta)} h \circ f_k^{-1} \frac{1}{|f' \circ f_k^{-1}|^2}\,d\sigma$$

$$\to \sum_{k=1}^{n} h(\zeta_k)\frac{1}{|f'(\zeta_k)|^2}$$

For $h=1$, $\frac{1}{\delta^2}\sigma(E_\delta) \to \sum_{k=1}^{n}\frac{1}{|f'(\zeta_k)|^2}$

<u>Take ratio</u>: $\mathcal{E}_f(h)(\zeta) = \int_{\mathbb{D}} h\,d\nu_\delta = \dfrac{\int_{E_\delta} h\,d\sigma}{\sigma(E_\delta)} \to \dfrac{\sum h(\zeta_k)\frac{1}{|f'(\zeta_k)|^2}}{\sum \frac{1}{|f'(\zeta_k)|^2}}$ ∎

Theorem 2 $\Rightarrow$ Theorem 1, (later)

Disintigration of Area Measure:

Since each ν_δ is a probability measure, $\exists$ a measure μ_a with support in $\overline{\mathbb{D}}$ such that some subsequence of the ν_δ converge weak-* to μ_a.

Easy fact: Let P be a polynomial, $\zeta \in E_a$. then

$$\int_{\overline{\mathbb{D}}} P\,d\mu_a = \mathcal{E}_A(P)(\zeta).$$

(Approximate both sides with $\int_{\mathbb{D}} P\,d\nu_\delta$, small δ)

Obvious Q: Can P be replaced with $h \in A^2$?

Key steps to answer Q:

Lemma 3: Let $1 \le p \le 2$. then $\exists$ $C = C(p,a) > 0$ so that

$$\left|\int_{\overline{\mathbb{D}}} P\,d\mu_a\right| \le C\|P\|_p, \quad \forall \text{ polynomial } P.$$

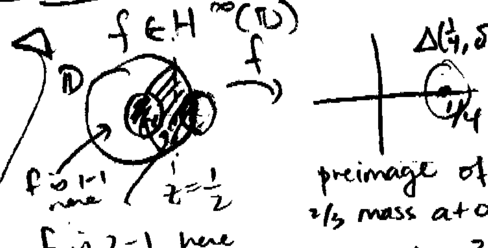
proof: Let $\zeta \in E_a$. Then $\exists$ $C > 0$ such that

$$|h(\zeta)| \le C\|h\|_p, \quad \forall h \in A^p.$$

Then

$$\left|\int_{\overline{\mathbb{D}}} P\,d\mu_a\right| = |\mathcal{E}_A(P)(\zeta)| \le C\|\mathcal{E}_A(P)\|_p \le C\|P\|_p. \quad \blacksquare$$

not obvious! Ex: $f(z) = (z - 1/2)^2$:



$f(0) = f(1) = \frac{1}{4}$

preimage of $\Delta(\frac{1}{4}, \delta)$ has $2/3$ mass at 0, $1/3$ at 1

so $\mu_{1/4}(0) = \frac{2}{3}$, $\mu_{1/4}(1) = \frac{1}{3}$.

Lemma 4: $\mu_a|_{\mathbb{T}} \perp m$.

proof: Egoroff ■

Lemma 5: $\mu_a|_{\mathbb{T}} = 0$.

proof: Lemma 4 + Lemma 3, $(p = 1)$. ■

Theorem 3: Let $h \in A^2$, $\zeta \in E_a$. Then $\int_{\mathbb{D}} h \, d\mu_a = \mathcal{E}_A(h)(\zeta)$.

proof: $P \longmapsto \int_{\mathbb{D}} P \, d\mu_a$ bounded on subspace of $L^2(\mathbb{D}, \sigma)$ consisting of polynomials (Lemma 3)

Same as $P \longmapsto \int_{\mathbb{D}} P \, d\mu_a$ (Lemma 5)

Functional extends boundedly to A^2 by Hahn-Banach.
If $P_n \to h$ in A^2, then

$$\int_{\mathbb{D}} h \, d\mu_a \approx \int_{\mathbb{D}} P_n \, d\mu_a = \mathcal{E}_A(P_n)(\zeta) \approx \mathcal{E}_A(h)(\zeta). \quad ■$$

(get pointwise convergence since everything is analytic)

Can extend Thm 2 to any bounded analytic function f.
(with all the other assumptions)
Further:

- Let $f \in H^\infty(\mathbb{D})$, $A = A(f)$, $a \in f(\mathbb{D})$. Don't assume $\mathcal{E}_A P = P \mathcal{E}_A$.
  If $\mu_a|_{\mathbb{T}} = 0$, then

$$\mu_a = \frac{\sum_{\zeta_k \in E_a} \frac{1}{|f'(\zeta_k)|^2} \delta_{\zeta_k}}{\sum_{\zeta_k \in E_a} \frac{1}{|f'(\zeta_k)|^2}}$$

point mass