Area, Capacity, and Diameter Versions of the Schwarz Lemma 11/05/07

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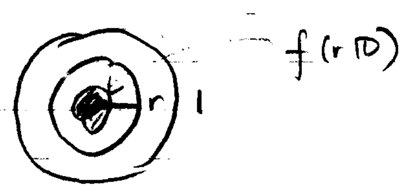
Schwarz Lemma: f analytic on $\mathbb{D}$, $f(0)=0$, and $|f(z)|<1$ $\forall z \in \mathbb{D}$.
Then $|f(z)| \le |z|$ $\forall z \in \mathbb{D}$ and
$|f(z)| = |z|$ iff $f(z) = e^{i\alpha} z$.
Also, $|f'(0)| \le 1$ and $|f'(0)| = 1$ iff $f(z) = e^{i\alpha} z$.



$$f(z) = f(0) + f'(0)z + \dots$$

$$\operatorname{Rad} f(r\mathbb{D}) = \sup_{z \in r\mathbb{D}} |f(z) - f(0)| \qquad (= \text{radius})$$

Then reformulate Schwarz Lemma:

$$|f'(0)| \swarrow \phi_{Rad}(r) = \frac{\operatorname{Rad} f(r\mathbb{D})}{r} \le 1.$$

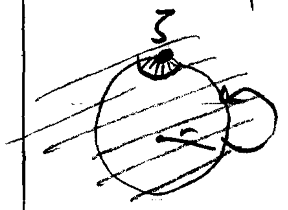
↑ Schwarz Lemma

Carathéodory (1907): first proof of Schwarz Lemma.
proof:
$f(z) - f(0) = z\, g(z)$, g analytic

$\phi_{Rad}(r) = \operatorname{Max}_{r\mathbb{D}} |g(z)|$. this max is increasing, by the Maximum Principle.
Actually strictly increasing and log-convex (by Haddamard 3-circle theorem) /

Landau - Toeplitz (1907): f is analytic in $\mathbb{D}$, $\operatorname{Diam} f(\mathbb{D}) = 2$ (Diameter). Then
a) $\operatorname{Diam} f(r\mathbb{D}) \le 2r$
b) $|f'(0)| \le 1$
c) equality in (a) or (b) iff f is linear ($f(z) = cz + b$, $|c|=1$.)

Meat of proof:



$g(\zeta) = \zeta.$

Then $g'(\zeta) > 0.$ (this observation is due to Hartogs)

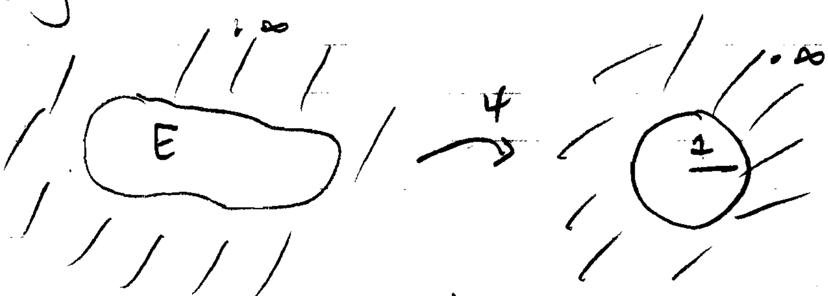
<u>n-diameter of $E \subset \mathbb{C}$:</u>

$$\sup_{(\zeta_1, \dots, \zeta_n) \in \underbrace{E \times \cdots \times E}_{n \text{ times}}} \left(\prod_{j<k} |\zeta_j - \zeta_k| \right)^{\frac{2}{n(n-1)}}$$

$= n\text{-diam}(E) \searrow \text{Cap}(E)$ as $n \to \infty$.

$n\text{-diam}(\mathbb{D}) = n^{\frac{1}{n-1}} \searrow 1 = \text{Cap}(\mathbb{D})$ Cap = logarithmic capacity

capacity:



$\varphi(z) = az + \cdots$
$a = \text{cap}(E)$

$$\Phi_{n\text{-diam}}(r) = \frac{n\text{-diam}(f(r\mathbb{D}))}{n\text{-diam}(r\mathbb{D})}$$

are these ratios strictly increasing + log convex? (except when f is linear)

proof of Landau-Toeplitz goes through for n-diam.

New proof shows n-diam $\searrow$ Cap.

<u>Theorem 1</u>: $\phi_{n\text{-diam}}(r)$ and $\phi_{cap}(r)$ are strictly increasing and log-convex except when f is linear.

Polya: E compact

$$\text{Area}(E) \le \pi\,(\text{cap}(E))^2 \le \pi \frac{(n\text{-diam}(E))^2}{n^{2/n-1}}$$

equality holds only for disks.

<u>Lemma 1</u>: $\phi_{n\text{-diam}}(r)$ is increasing & log-convex except where f is linear.

<u>proof</u>: Fix $w_1, \dots, w_n \in \overline{\mathbb{D}}$.

$$F_{w_1,\dots,w_n}(z) = C \prod_{j<k} \left(f(w_k z) - f(w_j z) \right), \quad C \text{ normalizing constant.}$$

$$= z^{\frac{n(n-1)}{2}} g(z)$$

$$\frac{\sup_{r\mathbb{D}} |F_{w_1,\dots,w_n}(z)|}{r^{\frac{n(n-1)}{2}}}$$ is increasing & log-convex, for fixed $w_1, \dots, w_n$.

then sup over w_j to get $\phi_{n\text{-diam}}(r)$ is increasing & log-convex. $\boxtimes$

<u>Lemma 2</u>: f is <u>not</u> linear then $\exists\, r_0$ such that $\forall\ 0<r<r_0$,

$$|f'(0)|^2 < \phi_{\text{Area}}(r) = \frac{\text{Area}(f(r\mathbb{D}))}{\text{Area}(r\mathbb{D})}$$ (Area with <u>no</u> multiplicity)

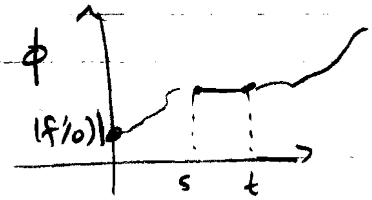
<u>proof</u>:
If $f'(0)=0$, clear.

If $f'(0) \neq 0$, then f is analytic and 1-1 near 0.

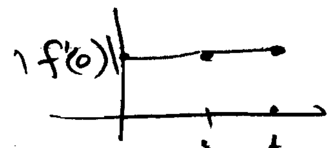
$\Rightarrow$ $\text{Area}\, f(r\mathbb{D}) = \pi \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \quad \left(= \int |f'|^2 dA \right)$ ■

<u>proof of Theorem 1</u>: (for $\Phi_{cap}(r)$ - same proof for $\Phi_{n\text{-diam}}(r)$)

Suppose not strictly increasing.



This is not a good picture because of log-convexity. log-convexity implies

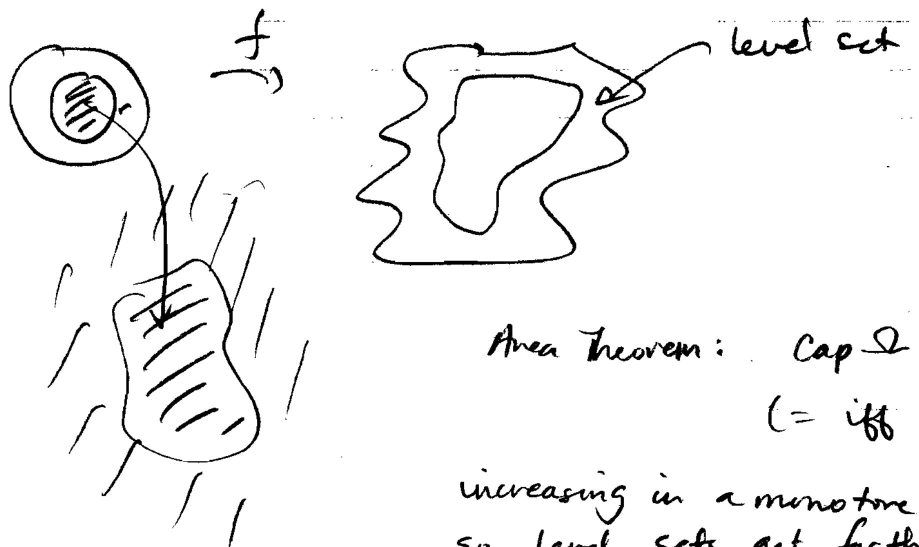


Lemma 2 ⟹ Polya

$$|f'(0)|^2 < \Phi_{Area}(r) \overset{\text{Polya}}{\leq} (\Phi_{cap}(r))^2 = |f'(0)|^2.$$

Contradiction ⟹ f must be linear. ∎

If f univalent



$\frac{1}{z}$ version of Ω

Area Theorem: $\operatorname{Cap}\Omega \cdot \operatorname{cap}\hat{\Omega} \geq 1$

($=$ iff $\Omega = \mathbb{D}$)

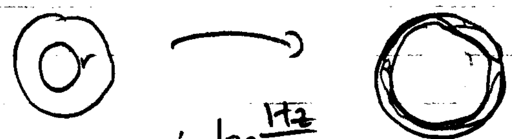
increasing in a monotone fashion, so level sets get farther away from being disks.

Rad $\mathbb{D} = 1$
Diam $\mathbb{D} = 2$
Cap $\mathbb{D} = 1$
Area $\mathbb{D} = \pi$ ⟹ Q: is there a result w/ ratios of Areas?

<u>Theorem 2</u>: $\phi_{Area}(r) = \dfrac{Area(f(r\mathbb{D}))}{Area(r\mathbb{D})} = \dfrac{A(r)}{\pi r^2}$

is strictly increasing unless f is linear but not log-convex in general. (is log-convex if f is 1-1)

<u>ex</u>



$e^{ic}\log\frac{1+z}{1-z}$

images of circles of radius r squeezed into annulus eventually meet & are not log-convex there.

z small enough

<u>proof</u>:

need to show $\phi' > 0$

$$\phi' = \frac{1}{\pi r^2}\left(A' - \frac{2A}{r}\right)$$

or $A' \geq \frac{2A}{r}$

It is enough to show $A' \geq \frac{L(r)}{2\pi r} = \dfrac{\text{length of boundary of image}}{2\pi r}$

isoperimetric ineq.

$4\pi A(r) \leq L(r)^2$

Find measurable domain $E \subset \mathbb{R}$ where $f|_E$ is 1-1.
(use co-Area Formula) ∎

Further directions

<u>ex</u>: principle frequency (first eigenvalue of Laplacian)

$$\Lambda^{-2}(r) = \sup_{u \in W_0^{1,2}(r)} \frac{\iint u^2\,dA}{\iint |\nabla u|^2\,dA}$$

Polya-Szego: $\dfrac{\frac{1}{\Lambda}(f(r\mathbb{D}))}{\frac{1}{\Lambda}(r\mathbb{D})} \geq |f'(0)|$

is this ratio strictly increasing? log-convex?
increasing?

Cor to Thm 2:

If $\quad \text{Area } f(r\mathbb{D}) = \pi$
$\quad \phi_{\text{Area}}(1) = 1.$

$\Rightarrow \quad |f'(0)| \leq 1.$

(reinterpret this as lower bound for hyperbolic density of plane)