PROPERTIES AND GENERALIZATIONS OF TSIRELSON'S SPACE

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A major program in the isomorphic theory of Banach spaces is to describe the geometry of the subspaces a general Banach space. The geometry of subspaces and complemented subspaces of the classical sequence spaces ℓ_p and c_0 have been understood since the 1940s. For Hilbert space ℓ_2 every infinite dimensional subspace is isomorphic (i.e. linearly homeomorphic) to the whole space and for ℓ_p with $p \neq 2$ and c_0 all (infinite dimensional) complemented subspaces are isomorphic to the whole space and every space contains a further subspace which is complemented. Given this nice structure, in the early days of Banach space theory there was some hope that the isomorphic theory for general Banach spaces could be described in terms of which classical sequences space they contain and that, in particular, a classical sequence space could be found as a subspace of each infinite dimensional Banach space. In the early 1970s Joram Lindenstrauss, in his address to the International Congress of Mathematicians, asked if there was an infinite dimensional Banach space which does not contain a isomorphic copy of any ℓ_p or c_0.

In 1974 this problem was solved with the construction of a space by Boris Tsirelson not containing any ℓ_p or c_0. Tsirelson's method for constructing his norm was inspired by the forcing method of Cohen. The norm itself is the limit of increasing sequence of norms on the space of finite sequences, c_{00}, and, quite unexpectedly, satisfies and the following implicit formula:

$$\|x\| = \max\{\|x\|_\infty, \sup \theta \sum_{i=1}^{n} \|E_i x\| : n \in \mathbb{N}, n \leqslant E_1 < E_2 < \cdots < E_n\} \tag{1}$$

Tsirelson's space has been called the first truly non-classical Banach space. His method for constructing Banach spaces has been continuously adapted in the past four decades to solve many of the most famous long-standing conjectures in Banach space theory. In particular W.T. Gowers' seminal contributions to this area resulted in him being awarded the Fields Medal in 1998. In addition to Gowers, B. Maurey, E. Odell, Th. Schlumprecht and S.A Argyros have made significant contributions.

Contructing Tsirelson-type spaces to solve problems in Banach spaces theory remains a very active area of mathematics. There are still many open problems related to Tsirelson space itself and its many variants. In a series of lectures I will present some of the many interesting theorems and open problems related to Tsirelson space like constructions. Some of the material I will present is well-known to experts whereas others topics are based on

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recent and ongoing work (some with Washington and Lee undergraduates). The material will be accessible to anyone with some (but not necessarily a lot) of knowledge of functional analysis. I will not assume you know much Banach space theory and will do my best to recall all relevent theorems in Banach space theory that are unfamilar to non-experts. Here is a tentative list of topics:

Lecture Topic 1: Present the construction of Tsirelson's space T, the proof that T does not contain any ℓ_p or c_0, say a few words about distortion in Banach spaces and the open problems regarding the distortion of T.

Lecture Topic 2: Prove that T is a reflexive and that the unit ball of any reflexive space has uncountably many extreme points. Currently there is no explicit description of an uncountable set of extreme points of the ball of T. We will present recent progress towards this problem.

Lecture Topic 3: Present the python program that computes the Tsirelson norm of a finitely supported vector. Tsirelson's norm $\|\cdot\|$ is the maximum of the countable set of increasing norms $(\|\cdot\|_k)_{k=1}^{\infty}$. Let $j(n)$ be the smallest integer so that if x is a vector of the length n then $\|x\| = \|x\|_{j(n)}$. Optimal bounds are $j(n)$ are not known. Recent work gives a lower bound on the order of $\log_2(n)$ and an upper bound on the order of $\sqrt{n}$. We will present these proofs and discuss strategies to improve these bounds.

Lecture Topic 4: We will present a fascinating outgrowth of Tsirelson's method for constructing norms. For each $p \in (1, \infty)$ there are norms satisfying an implicit equation that are equivalent to ℓ_p. We will also discuss how these norms could potentially be used to show that these classical sequence spaces are distortable.

Lecture Topic 5: In 2009, Gowers suggested a Polymath project regarding Tsirelson space entitled: "Does every 'explicitly defined' Banach space contain ℓ_p or c_0?" We will discuss the (limited) progress on this problem. In particular we will discuss the difficulty in defining the term 'explicitly defined.'

Lecture Topic 6: We will present a new modification of Tsirelson's space called Tsirelson spaces under contraints. Here is the motivation: From the definition of T it is easy to see that the only ℓ_p that is block finitely representable in T is ℓ_1. We present a clever modification of T so that for any $p, q \in [1, \infty]$ the only two ℓ_p that are block finitely representable in T are ℓ_p and ℓ_q. There are several unanswered questions related to these spaces.

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