K-Theory - source text Rørdam

a C^*-algebra A is a Banach algebra with an involution (adjoint operation) such that $\|a^*a\| = \|a\|^2$, $\forall a \in A$

A is unital if there exists $1 \in A$.
(Note $1^* = 1$: $(1^*a)^* = a^*1 = a^*$ so $1^*a = a$
Also $\|1\| = 1$)

The norm in A is <u>unique</u>.

If A has no unit, adjoin one. $\tilde{A} = A \oplus \mathbb{C} = \{(a,\alpha) : a \in A, \alpha \in \mathbb{C}\}$.
with $(a,\alpha)\cdot(b,\beta) = (ab + \alpha b + \beta a, \alpha\beta)$
$(a,\alpha)^* = (a^*, \bar{\alpha})$.

Identify A with $\underbrace{\{(a,0) : a \in A\}}_{\text{an ideal}} \subset \tilde{A}$

Define $\|(b,\beta)\| = \max\left[\left\{\sup\{\|\overbrace{(a,0)}^{\text{think of as in } A}(b,\beta)\| : \|a\| \le 1\}\right\}, |\beta|\right]$

$= \|ab + \beta a\|_A$

Ref: Rørdam, Exercise 1.3

The embedding $A \to \{(a,0) : a \in A\}$ preserves norms:
$$\|(b,0)\| = \sup_{\|a\| \le 1} \|ab\| \le \|b\|.$$

Other way: If $a_0 = \dfrac{b^*}{\|b\|}$, $\|a_0\| = 1$

note: $\|a_0 b\| = \left\| \dfrac{b^*b}{\|b\|} \right\| = \|b\| \le \|(b,0)\|$

$\Rightarrow \|(b,0)\| = \|b\|$.

Gelfand-Naimark Theorem: Every C^*-algebra A is an algebra of operators on a Hilbert space. i.e.: There exists a Hilbert space H and a faithful $*$-representation $\pi : A \to \mathcal{B}(H)$

faithful: 1-1 $*$-representation: homomorphism

refs. Conway, Op Th. p. 33
Murphy, p. 94

constructing K_0:

$p \in A$ is a projection if $p^2 = p$, $p^* = p$.

Let $\mathcal{P}(A)$ = set of all projections in A

Need 2 equivalence relations

(1) Murray - von Neumann equivalence

say projections p, q are M-von-N equivalent (write $p \sim q$) if there exists $v \in A$ with $p = v^* v$, $q = v v^*$.

(in $\mathcal{B}(H)$, p = projection onto $(\ker v)^\perp$, q = projection onto vH)
(v = partial isometry)

(2) unitary equivalence

$p \sim_u q$ if there exists u (unitary) in $\tilde{A}$ (so $u^* u = u u^* = 1 = 1_{\tilde{A}}$)
with $p = u q u^* = u q u^{-1}$.

If $p \sim q$ via v (so (1) holds) then:

$$v = qv$$
$$v = vp$$
$$v = qvp.$$

$M_n(A)$ = set of $n \times n$ matrices over A

typical element.

$$(*) \quad a = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & & & a_{nn} \end{pmatrix}, \quad \text{where } a_{ij} \in A.$$

for the norm: embed $A \subset \mathcal{B}(H)$

$$H^{(n)} = \underbrace{H \oplus \cdots \oplus H}_{n \text{ times}}$$

identify matrices (*) as being in $\mathcal{B}(H^{(n)})$ — has a C^*-norm.

$M_n(A)$ inherits this norm

$M_{m,n}(A)$ = all $m \times n$ matrices over A which are

$$a = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad (**)$$

Embed $A \subset B(H)$, then think of $(**)$ as being in $B(H^{(n)}, H^{(m)})$ and $a^* \in M_{n,m}(A)$.

$P_n(A)$ = set of all projections in $M_n(A)$

$P_\infty(A) = \bigcup_{n \geq 1} P_n(A)$.

Suppose $p \in P_m(A)$, $q \in P_n(A)$.

then $p \sim_0 q$ if there exists $x \in M_{m,n}(A)$ with

$$p = x^* x, \quad q = x x^*.$$ (variation on Murray-von Neumann equivalence)

as operators, x is a partial isometry with initial space $pH^{(n)}$ and final space $qH^{(m)}$.

to see that $\sim_0$ is an equivalence relation, the main thing to check is transitivity – we'll come back to this.

$a \in M_m(A)$, $b \in M_n(A)$, call $a \oplus b = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in M_{m+n}(A)$.

<u>Lemma:</u> Let $p, q, p', q', r \in P_\infty(A)$. Then
(i) $p \sim_0 p \oplus 0_n$ for all $n \geq 1$. $\qquad 0_n$ = $n \times n$ zero matrix

<u>proof (i):</u> Let $p \in P_m(A)$, 0_n be given.
Let

$x = \begin{pmatrix} p \\ 0_{n,m} \end{pmatrix} \in M_{m+n,m}(A)$

then $x^*x = \begin{pmatrix} p & 0_{m,n} \end{pmatrix} \begin{pmatrix} p \\ 0_{n,m} \end{pmatrix} = p^*p + 0_m^2 = p.$

$(p^* = p)$

+ $xx^* = \begin{pmatrix} p \\ 0_{n,m} \end{pmatrix} \begin{pmatrix} p & 0_{m,n} \end{pmatrix} = \begin{pmatrix} p & 0_{m,n} \\ 0_{n,m} & 0_{n} \end{pmatrix} = p \oplus 0_n.$

$\Rightarrow p \sim_0 p \oplus 0_n.$ ∎

(ii) If $p \sim_0 p'$, and $q \sim_0 q'$, then $p \oplus q \sim_0 p' \oplus q'$.

<u>proof:</u> know $p = x^*x$, $p' = xx^*$ for some x
$q = y^*y$, $q' = yy^*$ for some y.

let $w = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$. Check $w^*w = p \oplus q$, $ww^* = p' \oplus q'$. ∎

(iii) $p \oplus q \sim_0 q \oplus p$.

<u>proof:</u> let $u = \begin{pmatrix} 0_{m,n} & p \\ q & 0_{n,m} \end{pmatrix} \in M_{n+m}(A)$.
Check $u^*u = p \oplus q$, $uu^* = q \oplus p$. ∎

(iv) if $p, q \in P_n(A)$ and $pq = 0$ (so $qp = 0$) then $p+q \in P_n(A)$ and $p+q \sim_0 p \oplus q \in M_{2n}(A)$.

<u>proof:</u> observe that $p+q$ is self-adjoint, idempotent
$(p+q)^2 = p^2 + pq + qp + q^2 = p^2 + q^2 = p+q$, so get a projection.

let $v = \begin{pmatrix} p \\ q \end{pmatrix} \in M_{2n,n}(A)$. check $v^*v = \begin{pmatrix} p & q \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = p+q$

$vv^* = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$ ∎

(v) $(p \oplus q) \oplus r = p \oplus (q \oplus r)$

(easy to check).

09/11/07

<u>Background + Motivation for K-theory</u>

Grothendieck (1950s) (Algebraic Geometry)
"invented" K-theory K comes from Klasses.

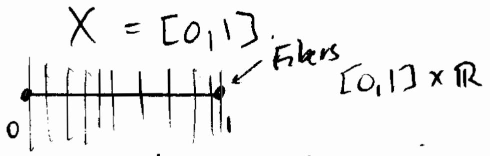
Atiyah-Hirzebruch: topological K-theory
Atiyah's book 1967 - classical reference

idea of topological K-theory:

X = topological space
look at vector bundles over X
To get $K^0(X)$:
take the isomorphism classes of finitely generated vector bundles over X endowed with $\oplus$

- this structure gives a monoid.
To get a group, adjoin inverses.
This gives $K^0(X)$.

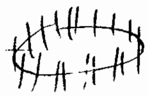
<u>Example</u>: $X = [0,1]$

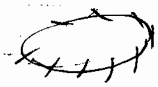


Up to isomorphism, the dimension n vector bundle is unique.
$n \geq 1$

The vector bundles (up to isomorphism) with $\oplus$ are isomorphic to $(\{0\} \cup \mathbb{N}, +)$ (which is not a group).
Add on inverses to get $K^0([0,1]) = \mathbb{Z}$

<u>Example:</u> $X = \mathbb{T}$

 trivial bundle (identify as cylinder in $\mathbb{R}^3$)

 half-twist (Möbius)

$\left(\text{vector bundles over } \mathbb{T}/\sim, \oplus\right) \cong \{0\} \cup (\mathbb{N} \times \mathbb{Z}_2)$

twist or not

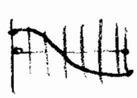
make a group: $K^0(\mathbb{T}) = \mathbb{Z} \times \mathbb{Z}_2$

K^0: [topological space] $\to$ [abelian groups]

contravariant functor (covariant functors have the superscript as a subscript)

<u>Note</u>: K-theory is very sensitive to the scalar field. (all the pictures above are $\mathbb{R}$)

If E is a vector bundle over X, let $\Gamma(E)$ be the vector space of <u>sections</u>.



$\Gamma(E)$ is a $C(X)$-module.

{finitely generated vector bundles over X} $\leftrightarrow$ {finitely generated projective $C(X)$ modules}

<u>Algebraic K-theory</u> (what we want) 1960s
Bass, 1968: book on algebraic K-theory

R is a ring.

equivalence relationship is isomorphism as R-modules

$G\left(\left(\text{finitely generated projective } R\text{-modules}/\sim\right), \oplus\right) = K_0(R)$

Here: K_0 is a covariant functor. [rings] → [abelian groups].

Note: $K_0(C(X)) = K^0(X)$.

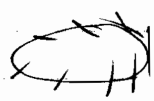
Note: when R is commutative (like $C(X)$), $\otimes$ makes $K_0(R)$ a ring.

a free left module over R $\begin{pmatrix} R \\ R \\ \vdots \\ R \end{pmatrix}$

a projective left module is a direct summand of a free module $\begin{pmatrix} R \\ R \\ \vdots \\ R \end{pmatrix}(P)$

P is an idempotent in $M_n(R)$

Example:

 E $\begin{pmatrix} C(\mathbb{T}) \\ C(\mathbb{T}) \end{pmatrix}(P)$

where $P \in M_2(C(\mathbb{T})) = C(\mathbb{T} \to M_2)$
function which gives 1-dim'l function at every point of $\mathbb{T}$, not constant 1-dim'l projection

For C^*-algebras A: can take $P = P^* = P^2$ and rephrase in terms of idempotents.

A-modules / isomorphism ⟷ projections in some $M_n(A)/\sim$

these are the equivalence relations from last time, at least includes the Murray-von Neumann equivalences

$\oplus$ ⟷ sum of orthogonal representatives of classes of projections

A C^*-algebra $\quad$ to get $K_0(A)$ $\downarrow$

A

$M_n(A)$

$\mathrm{proj}\left(\bigcup_n M_n(A)\right)$

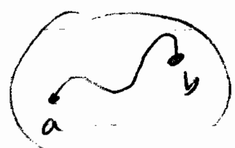
$$Gp\left(\mathrm{proj}\left(\bigcup_n M_n(A)\right)/\sim\right) = K_0(A)$$

Recall: topological space X, $\quad a, b \in X$

a, b are homotopic if there exists $v : [0,1] \to X$ with $v(0) = a$, $v(1) = b$. (v is continuous)

Write $a \sim_h b$ in X

or $a \sim_h b$ (if X is understood))



Let A be a unital C^*-algebra.

$\mathcal{U}(A)$ = set of unitaries u: $uu^* = 1$, $u^*u = 1$.

$\mathcal{U}(A)$ is a group. (since $u^* = u^{-1}$ + $\mathcal{U}(A)$ is closed under products)

If $u \in \mathcal{U}(A)$, $\sigma(u) \subset \mathbb{T}$.

$\mathcal{U}_0(A) = \{u \in \mathcal{U}(A) : \ u \sim_h 1 \ (\text{in } \mathcal{U}(A))\}$

If $u_1 \sim_h v_1$ and $u_2 \sim_h v_2$, then $u_1 u_2 \sim_h v_1 v_2$.

Lemma:

(i) If $h \in A$ is self-adjoint ($h = h^*$) then $e^{ih} \in \mathcal{U}_0(A)$.

(ii) If $u \in \mathcal{U}(A)$ and $\sigma(u) \subsetneq \mathbb{T}$, then $u \in \mathcal{U}_0(A)$

(iii) If $u, v \in \mathcal{U}(A)$ and $\|u - v\| < 2$, then $u \sim_h v$.

(note: $\|u - v\| \le \|u\| + \|v\| = 2$)

proof:

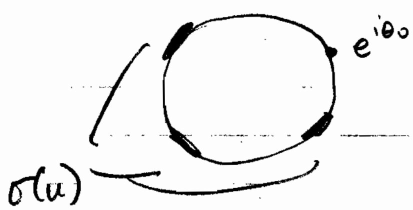
(i) $(e^{ih})^* = e^{-ih}$ (via continuous functional calculus)

so

$$(e^{ih})^* \cdot e^{ih} = e^{-ih} \cdot e^{ih} = e^0 = 1$$

+ similarly $e^{ih}(e^{ih})^* = 1$.

(ii) Fix $e^{i\theta_0} \in \mathbb{T} \setminus \sigma(u)$



Define $\varphi(e^{it}) = t$, $\theta_0 < t < \theta_0 + 2\pi$

$\varphi : \sigma(u) \to \mathbb{R}$

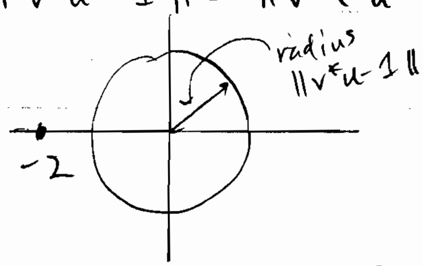
Then $e^{i\varphi(e^{it})} = e^{it}$, so $\varphi(u)$ make sense and

$e^{i\varphi(u)} = u$. Note $\varphi(u) = \varphi(u)^*$ since φ is real-valued.

Let $h = \varphi(u)$ be the self-adjoint operator sought.

(iii) Suppose $u, v \in \mathcal{U}(A)$ and $\|u - v\| < 2$

$\|v^* u - 1\| = \|v^*(u - v)\| = \|u - v\| < 2$.



so $-2 \notin \sigma(v^* u - 1)$

or $-1 \notin \sigma(v^* u)$.

But $v^* u$ is unitary $\Rightarrow$ $v^* u = e^{ih}$ where h is self-adjoint.

$\Rightarrow$ $v^* u \sim_h 1$

$\Rightarrow$ $u \sim_h v$.

[If h is self-adjoint, then $e^{ih} \in \mathcal{U}_0(A)$. The curve is e^{ith}, $0 \le t \le 1$] ∎

<u>Corollary</u>: $\mathcal{U}(M_n(\mathbb{C}))$ is connected (i.e. $\mathcal{U} = \mathcal{U}_0$ for $M_n(\mathbb{C})$).

(since $\sigma(u)$ is finite $\subset \mathbb{T}$).

<u>Lemma</u> (Whitehead): Let A be a unital C^*-algebra, $u, v \in \mathcal{U}(A)$. Then $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \sim_h \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} vu & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}$ in $\mathcal{U}(M_2(A))$

Thus: $\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

<u>proof</u>: consider $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(A)$

Note: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, its self adjoint
$\Rightarrow$ is in $\mathcal{U}(M_2(A))$

Also: $\sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \{-1, 1\}$.

So $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \sim_h \begin{pmatrix} uv & 0 \\ 0 & 1 \end{pmatrix}$$

Also: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} v & 0 \\ 0 & u \end{pmatrix} = \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sim_h \begin{pmatrix} vu & 0 \\ 0 & 1 \end{pmatrix}$$

∎

number in Rørdam

Proposition 2.1.6: Let A be a unital C^*-algebra.

(i) $\mathcal{U}_0(A)$ is a normal subgroup in $\mathcal{U}(A)$

(ii) $\mathcal{U}_0(A)$ is open and closed in $\mathcal{U}(A)$

(iii) $u \in A$ is in $\mathcal{U}_0(A) \Leftrightarrow u = e^{ih_1}e^{ih_2}\cdots e^{ih_n}$ and each h_j is self-adjoint.