On the level curves and conformal equivalence of analytic functions of a complex variable.

Trevor Richards

Washington and Lee University

richardst@wlu.edu

November 4, 2014

Definitions and Basic Facts.

Definition

A generalized finite Blaschke product is a function f, analytic on an open simply connected region $G \subset \mathbb{C}$, such that f pulls back to a finite Blaschke product on $\mathbb{D}$.

Definitions and Basic Facts.

Definition

A generalized finite Blaschke product is a function f, analytic on an open simply connected region $G \subset \mathbb{C}$, such that f pulls back to a finite Blaschke product on $\mathbb{D}$.

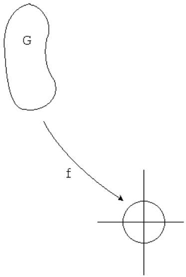

Figure : Generalized Finite Blaschke Product

Definitions and Basic Facts.

Definition

A generalized finite Blaschke product is a function f, analytic on an open simply connected region $G \subset \mathbb{C}$, such that f pulls back to a finite Blaschke product on $\mathbb{D}$.

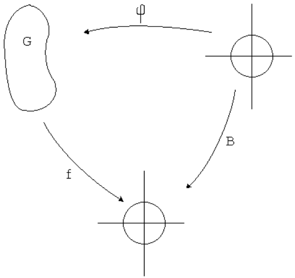

Figure : Generalized Finite Blaschke Product

Definitions and Basic Facts.

Equivalently, $f : G \to \mathbb{C}$ is a generalized finite Blaschke product if the following holds:

Equivalently, $f : G \to \mathbb{C}$ is a generalized finite Blaschke product if the following holds:

- f is analytic on $cl(G)$.

Equivalently, $f : G \to \mathbb{C}$ is a generalized finite Blaschke product if the following holds:

- f is analytic on $cl(G)$.
- $|f| = 1$ on ∂G.

Equivalently, $f : G \to \mathbb{C}$ is a generalized finite Blaschke product if the following holds:

- f is analytic on $cl(G)$.
- $|f| = 1$ on ∂G.
- $f' \neq 0$ on ∂G.

Definitions and Basic Facts.

Equivalently, $f : G \to \mathbb{C}$ is a generalized finite Blaschke product if the following holds:

- f is analytic on $cl(G)$.
- $|f| = 1$ on ∂G.
- $f' \neq 0$ on ∂G.

Let (f, G) denote such a generalized finite Blaschke product, and let $\mathcal{A}$ denote the collection of all such (f, G).

Critical level curves of (f, G).

Critical level curves of (f, G) are connected finite graphs Λ such that the following holds:

Critical level curves of (f, G).

Critical level curves of (f, G) are connected finite graphs Λ such that the following holds:

- Each edge of Λ is incident to the unbounded face of Λ.

Critical level curves of (f, G).

Critical level curves of (f, G) are connected finite graphs Λ such that the following holds:

- Each edge of Λ is incident to the unbounded face of Λ.
- Evenly many (and $\geqslant 4$) edges of Λ are incident to each vertex of Λ.

Critical level curves of (f, G).

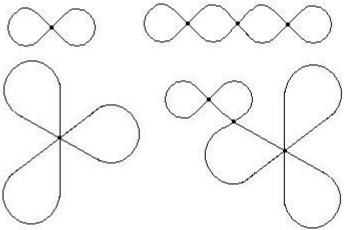

Figure : Admissible Graphs

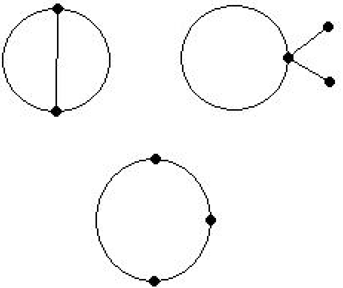

Figure : Non Admissible Graphs

Non-critical level curves of (f, G).

Consider the following diagram of critical level curves of the function $f(x) = x^2(x - \frac{2}{9})(x - \frac{11}{9})e^x$.

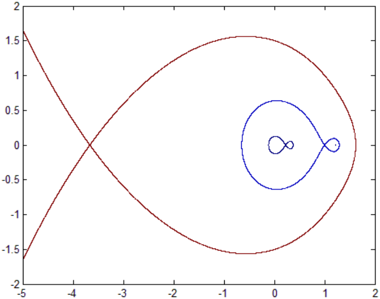

Figure : $f(x) = x^2(x - \frac{2}{9})(x - \frac{11}{9})e^x$.

Non-critical level curves of (f, G).

Consider the following diagram of critical level curves of the function $f(x) = x^2(x - \frac{2}{9})(x - \frac{11}{9})e^x$.

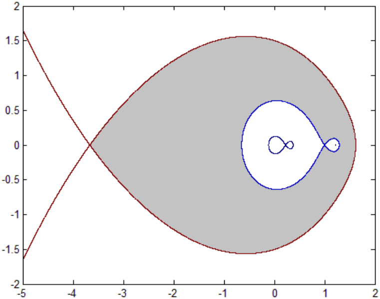

Figure : $f(x) = x^2(x - \frac{2}{9})(x - \frac{11}{9})e^x$.

Non-critical level curves of (f, G).

Consider the following diagram of critical level curves of the function $f(x) = x^2(x - \frac{2}{9})(x - \frac{11}{9})e^x$.

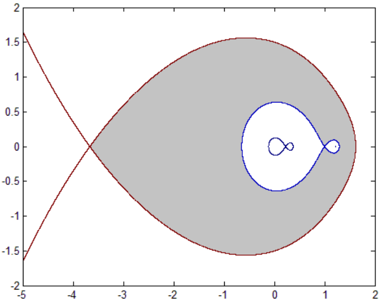

Figure : $f(x) = x^2(x - \frac{2}{9})(x - \frac{11}{9})e^x$.

- Therefore we hope to completely characterize a generalized finite Blaschke product by the configuration of its critical level curves only.

Construction of PC_a.

To define this "configuration of critical level curves":

Construction of PC_a.

To define this "configuration of critical level curves":

Start with one of the admissible graphs Λ, which will represent a critical level curve of an analytic function f. Then add auxilliary data to represent...

Construction of PC_a.

To define this "configuration of critical level curves":

Start with one of the admissible graphs Λ, which will represent a critical level curve of an analytic function f. Then add auxilliary data to represent...

- The value of $|f|$ on Λ.

Construction of PC_a.

To define this "configuration of critical level curves":

Start with one of the admissible graphs Λ, which will represent a critical level curve of an analytic function f. Then add auxilliary data to represent...

- The value of $|f|$ on Λ.
- The number of zeros of f in each bounded face of Λ.

Construction of PC_a.

To define this "configuration of critical level curves":

Start with one of the admissible graphs Λ, which will represent a critical level curve of an analytic function f. Then add auxilliary data to represent...

- The value of $|f|$ on Λ.
- The number of zeros of f in each bounded face of Λ.
- The points in Λ at which f takes positive real values.

Construction of PC_a.

To define this "configuration of critical level curves":

Start with one of the admissible graphs Λ, which will represent a critical level curve of an analytic function f. Then add auxilliary data to represent...

- The value of $|f|$ on Λ.
- The number of zeros of f in each bounded face of Λ.
- The points in Λ at which f takes positive real values.
- The value of $\arg(f)$ at each vertex of Λ.

Construction of PC_a.

To define this "configuration of critical level curves":

Start with one of the admissible graphs Λ, which will represent a critical level curve of an analytic function f. Then add auxilliary data to represent...

- The value of $|f|$ on Λ.
- The number of zeros of f in each bounded face of Λ.
- The points in Λ at which f takes positive real values.
- The value of $\arg(f)$ at each vertex of Λ.

All defined modulo composition with an orientation preserving homeomorphism of $\mathbb{C}$.

Construction of PC_a.

The collection of all such Λ, with all such choices of auxiliary data, we denote by P_a.

Construction of PC_a.

The collection of all such Λ, with all such choices of auxiliary data, we denote by P_a.
We construct the possible level curve configurations (PC_a) from the members of P_a recursively. There are two parts to this.

Construction of PC_a.

The collection of all such Λ, with all such choices of auxiliary data, we denote by P_a.
We construct the possible level curve configurations (PC_a) from the members of P_a recursively. There are two parts to this.

- Which critical level curves lie in the faces of which.

Construction of PC_a.

The collection of all such Λ, with all such choices of auxiliary data, we denote by P_a.
We construct the possible level curve configurations (PC_a) from the members of P_a recursively. There are two parts to this.

- Which critical level curves lie in the faces of which.
- The orientation of each critical level curve with respect to the others.

Construction of PC_a.

The collection of all such Λ, with all such choices of auxiliary data, we denote by P_a.
We construct the possible level curve configurations (PC_a) from the members of P_a recursively. There are two parts to this.

- Which critical level curves lie in the faces of which.
- The orientation of each critical level curve with respect to the others.

The "level 0" members of PC_a are just single points (which represent zeros of f).

Construction of PC_a

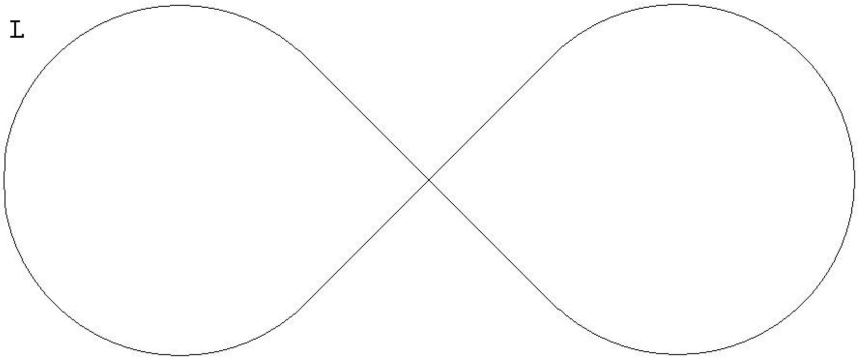

Figure : Bare admissible graph.

Construction of PC_a

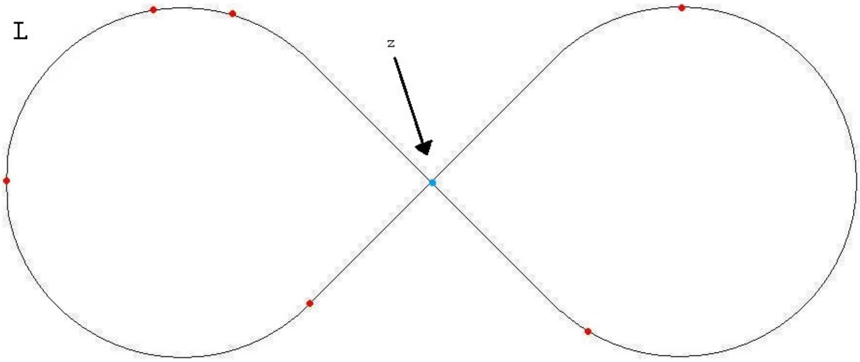

Figure : Add auxiliary data.

Construction of PC_a

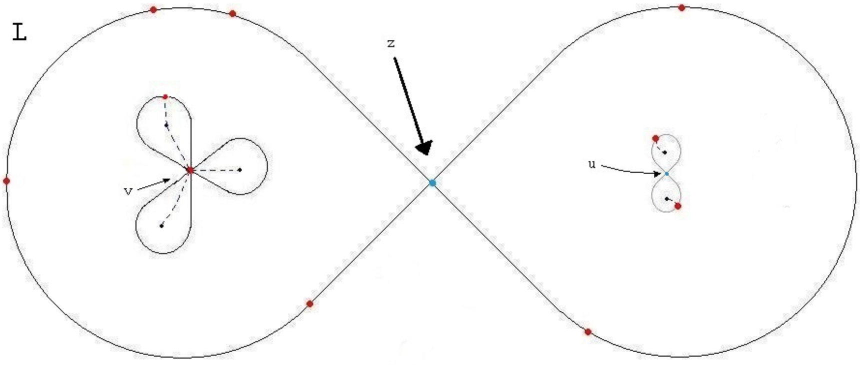

Figure : Recursive assignment.

Construction of PC_a

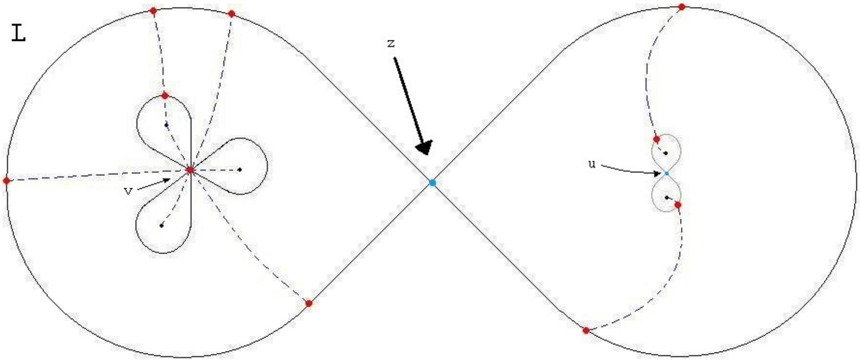

Figure : Choice of orientations.

Construction of PC_a

Let PC_a denote the collection of all such possible critical level curve configurations.

Construction of PC_a

Let PC_a denote the collection of all such possible critical level curve configurations.
The critical level curves of a generalized finite Blaschke product (f, G) naturally forms a member of PC_a. We define the map

$$\Pi : \text{GFBPs} \to PC_a.$$

Topological equivalence and conformal equivalence.

Theorem

[R.] Given any $(f_1, G_1), (f_2, G_2) \in \mathcal{A}$*,*

$$\Pi(f_1, G_1) = \Pi(f_2, G_2) \Leftrightarrow (f_1, G_1) \sim (f_2, G_2).$$

Topological equivalence and conformal equivalence.

Theorem

[R.] Given any $(f_1, G_1), (f_2, G_2) \in \mathcal{A}$*,*

$$\Pi(f_1, G_1) = \Pi(f_2, G_2) \Leftrightarrow (f_1, G_1) \sim (f_2, G_2).$$

For generalized finite Blaschke products...

Topological equivalence and conformal equivalence.

Theorem

[R.] Given any $(f_1, G_1), (f_2, G_2) \in \mathcal{A}$*,*

$$\Pi(f_1, G_1) = \Pi(f_2, G_2) \Leftrightarrow (f_1, G_1) \sim (f_2, G_2).$$

For generalized finite Blaschke products...

Conformal equivalence of functions.	$\Longleftrightarrow$	Geometric equivalence of level curve configurations

Proof of Theorem 2.

Recall the example from earlier.

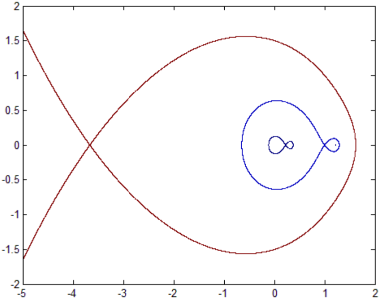

Figure : $f(x) = x^2(x - \frac{2}{9})(x - \frac{11}{9})e^x$.

Proof of Theorem 2.

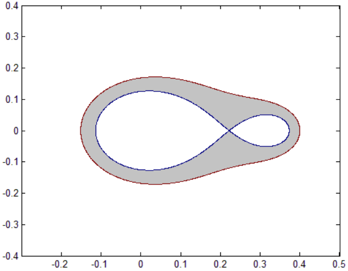

Figure : $f(x) = x^2(x - \frac{2}{9})(x - \frac{11}{9})e^x$.

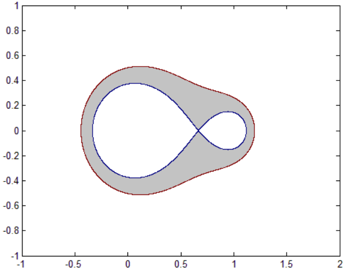

Figure : $q(x) = Cx^2(x - 1)$.

Proof of Theorem 2.

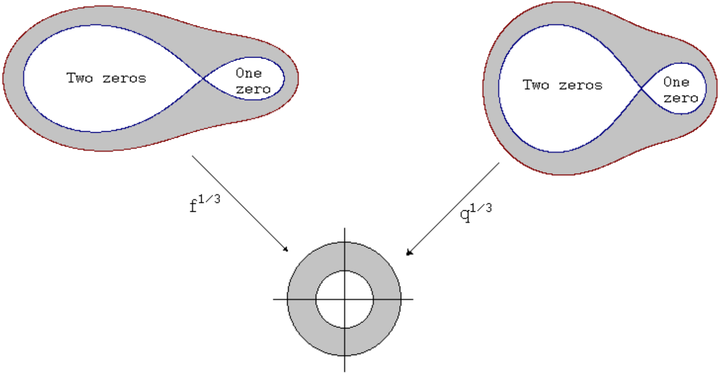

Figure : Construction of ϕ.

Proof of Theorem 2.

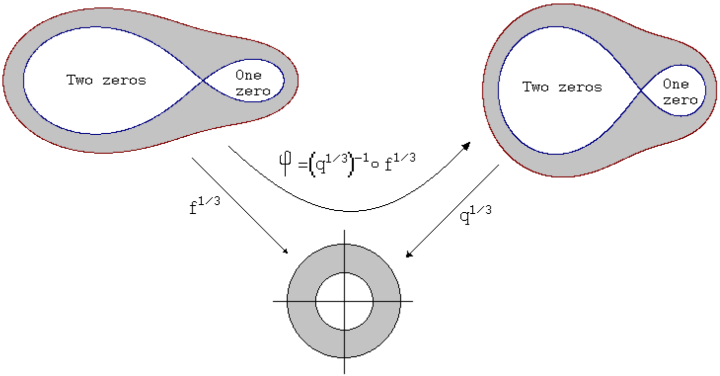

Figure : Construction of ϕ.

Proof of Theorem 2.

On D_2, $q = \left(q^{1/3}\right)^3$. Therefore on D_1,

$$q \circ \phi =$$

Proof of Theorem 2.

On D_2, $q = \left(q^{1/3}\right)^3$. Therefore on D_1,

$$q \circ \phi = \left[q^{1/3}\left(\left(q^{1/3}\right)^{-1} \circ \left(f^{1/3}\right)\right)\right]^3 =$$

Proof of Theorem 2.

On D_2, $q = \left(q^{1/3}\right)^3$. Therefore on D_1,

$$q \circ \phi = \left[q^{1/3}\left(\left(q^{1/3}\right)^{-1} \circ \left(f^{1/3}\right)\right)\right]^3 = \left(f^{1/3}\right)^3 =$$

Proof of Theorem 2.

On D_2, $q = \left(q^{1/3}\right)^3$. Therefore on D_1,

$$q \circ \phi = \left[q^{1/3}\left(\left(q^{1/3}\right)^{-1} \circ \left(f^{1/3}\right)\right)\right]^3 = \left(f^{1/3}\right)^3 = f.$$

Proof of Theorem 2.

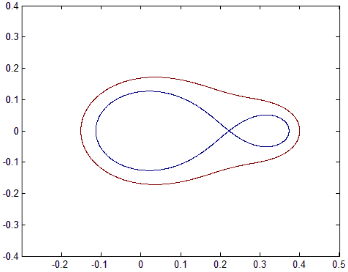

Figure : $f(x) = x^2(x - \frac{2}{9})(x - \frac{11}{9})e^x$.

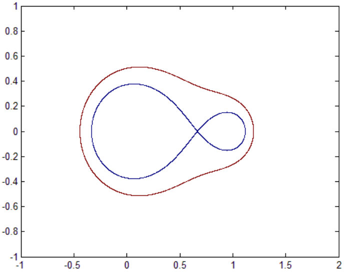

Figure : $q(x) = Cx^2(x - 1)$.

Results of Theorem 2.

The backwards direction of Theorem 2 (that if $(f_1, G_1) \sim (f_2, G_2)$, then $\Pi(f_1, G_1) = \Pi(f_2, G_2)$) implies that Π is well defined when viewed as acting on $\mathcal{A}/\sim$.

Results of Theorem 2.

The backwards direction of Theorem 2 (that if $(f_1, G_1) \sim (f_2, G_2)$, then $\Pi(f_1, G_1) = \Pi(f_2, G_2)$) implies that Π is well defined when viewed as acting on $\mathcal{A}/\sim$.

The forward direction implies that $\Pi : \mathcal{A}/\sim \to PC_a$ is injective.

Results of Theorem 2.

Theorem (R.)

$\Pi(\mathcal{A}/\sim) = PC_a$.

Proof.

Results of Theorem 2.

Theorem (R.)

$\Pi(\mathcal{A}/\sim) = PC_a$.

Proof.

Fix for the moment some list of $n-1$ critical values.

Results of Theorem 2.

Theorem (R.)

$\Pi(\mathcal{A}/\sim) = PC_a$.

Proof.

Fix for the moment some list of $n-1$ critical values.

- It is known (Beardon, Carne, and Ng 2002)that there are exactly n^{n-3} polynomials with these critical values.

Results of Theorem 2.

Theorem (R.)

$\Pi(\mathcal{A}/\sim) = PC_a$.

Proof.

Fix for the moment some list of $n-1$ critical values.

- It is known (Beardon, Carne, and Ng 2002)that there are exactly n^{n-3} polynomials with these critical values.
- By counting directly, there are exactly n^{n-3} different members of PC_a with a given list of critical values.

Results of Theorem 2.

Theorem (R.)

$\Pi(\mathcal{A}/\sim) = PC_a$.

Proof.

Fix for the moment some list of $n-1$ critical values.

- It is known (Beardon, Carne, and Ng 2002)that there are exactly n^{n-3} polynomials with these critical values.
- By counting directly, there are exactly n^{n-3} different members of PC_a with a given list of critical values.
- Since Π is injective, Π must also be surjective onto the members of PC_a with these critical values.

$\square$

Putting the Pieces Together.

- In the argument above, we only counted equivalence classes of $\mathcal{A}/\sim$ which contained a polynomial.

Putting the Pieces Together.

- In the argument above, we only counted equivalence classes of $\mathcal{A}/\sim$ which contained a polynomial.
- Therefore every equivalence class of $\mathcal{A}/\sim$ must contain a polynomial.

Putting the Pieces Together.

- In the argument above, we only counted equivalence classes of $\mathcal{A}/\sim$ which contained a polynomial.
- Therefore every equivalence class of $\mathcal{A}/\sim$ must contain a polynomial.
- Therefore every finite Blaschke product is conformally equivalent to a polynomial.

Fingerprints

Fingerprints

- Weaknesses:

Fingerprints

- Weaknesses:
  - Non-constructive (although there are computer programs which approximate a curve from its fingerprint).

Fingerprints

- Weaknesses:
  - Non-constructive (although there are computer programs which approximate a curve from its fingerprint).
  - Does not seem to extend in any way to general functions which are analytic on $cl(\mathbb{D})$.

Fingerprints

- Weaknesses:
  - Non-constructive (although there are computer programs which approximate a curve from its fingerprint).
  - Does not seem to extend in any way to general functions which are analytic on $cl(\mathbb{D})$.
- Stregths:

Fingerprints

- Weaknesses:
  - Non-constructive (although there are computer programs which approximate a curve from its fingerprint).
  - Does not seem to extend in any way to general functions which are analytic on $cl(\mathbb{D})$.
- Stregths:
  - Makes elegant use of Kirillov's theorem.

Fingerprints

- Weaknesses:
  - Non-constructive (although there are computer programs which approximate a curve from its fingerprint).
  - Does not seem to extend in any way to general functions which are analytic on $cl(\mathbb{D})$.
- Stregths:
  - Makes elegant use of Kirillov's theorem.
  - A nice use of conformal welding.

Stregths and Weaknesses: Fingerprints.

Fingerprints

- Weaknesses:
  - Non-constructive (although there are computer programs which approximate a curve from its fingerprint).
  - Does not seem to extend in any way to general functions which are analytic on $cl(\mathbb{D})$.
- Stregths:
  - Makes elegant use of Kirillov's theorem.
  - A nice use of conformal welding.
  - Generalizes to ratios of finite Blaschke products on $\mathbb{D}$.

Stregths and Weaknesses: Interpolating Rational Functions.

Interpolating Rational Functions.

Stregths and Weaknesses: Interpolating Rational Functions.

Interpolating Rational Functions.

- Weaknesses:

Stregths and Weaknesses: Interpolating Rational Functions.

Interpolating Rational Functions.

- Weaknesses:
    - Non-constructive.

Interpolating Rational Functions.

- Weaknesses:
  - Non-constructive.
  - Gives no visual/geometric interpretation, which would be appropriate for a proof involving conformal equivalence.

Interpolating Rational Functions.

- Weaknesses:
    - Non-constructive.
    - Gives no visual/geometric interpretation, which would be appropriate for a proof involving conformal equivalence.
- Strengths:

Stregths and Weaknesses: Interpolating Rational Functions.

Interpolating Rational Functions.

- Weaknesses:
  - Non-constructive.
  - Gives no visual/geometric interpretation, which would be appropriate for a proof involving conformal equivalence.
- Strengths:
  - No need to introduce new definitions, attacks f directly via Hermite interpolation.

Interpolating Rational Functions.

- Weaknesses:
  - Non-constructive.
  - Gives no visual/geometric interpretation, which would be appropriate for a proof involving conformal equivalence.
- Strengths:
  - No need to introduce new definitions, attacks f directly via Hermite interpolation.
  - Completely general, applies to meromrophic functions on multiply connected regions.

Level Curves Configurations.

Level Curves Configurations.

- Weaknesses:

Level Curves Configurations.

- Weaknesses:
  - Non-constructive.

Level Curves Configurations.

- Weaknesses:
  - Non-constructive.
  - Cumbersome definitions and notation.

Level Curves Configurations.

- Weaknesses:
    - Non-constructive.
    - Cumbersome definitions and notation.
    - At present does not handle multiply connected regions at all.

Level Curves Configurations.

- Weaknesses:
  - Non-constructive.
  - Cumbersome definitions and notation.
  - At present does not handle multiply connected regions at all.
- Strengths:

Level Curves Configurations.

- Weaknesses:
  - Non-constructive.
  - Cumbersome definitions and notation.
  - At present does not handle multiply connected regions at all.
- Strengths:
  - Generalizes to analytic functions on $cl(\mathbb{D})$.

Level Curves Configurations.

- Weaknesses:
  - Non-constructive.
  - Cumbersome definitions and notation.
  - At present does not handle multiply connected regions at all.
- Strengths:
  - Generalizes to analytic functions on $cl(\mathbb{D})$.
  - May generalize to meromorphic functions on multiply connected regions.

Level Curves Configurations.

- Weaknesses:
  - Non-constructive.
  - Cumbersome definitions and notation.
  - At present does not handle multiply connected regions at all.
- Strengths:
  - Generalizes to analytic functions on $cl(\mathbb{D})$.
  - May generalize to meromorphic functions on multiply connected regions.
  - Gives clear geometric and visual interpretation of conformal equivalence.

Level Curves Configurations.

- Weaknesses:
  - Non-constructive.
  - Cumbersome definitions and notation.
  - At present does not handle multiply connected regions at all.
- Strengths:
  - Generalizes to analytic functions on $cl(\mathbb{D})$.
  - May generalize to meromorphic functions on multiply connected regions.
  - Gives clear geometric and visual interpretation of conformal equivalence.

THE END