DIAGONALITY AND IDEMPOTENTS, JASPER'S FRAME THEORY PROBLEM, AND SCHUR-HORN THEOREMS

JIREH LOREAUX AND GARY WEISS*
ANALYSIS SEMINARS: UNIVERSITY OF VIRGINIA, US NAVAL ACADEMY, OCTOBER 7,10 2014

1. Diagonality

Diagonality - term coined by the authors for the study of:

• Properties that the diagonal sequences can possess for a fixed operator and in all bases.
And, if possible, characterize those sequences.

• • Properties that the diagonal sequences can possess for a class of operators and in all bases.
And, if possible, characterize those sequences.

Such information is used ubiquitously throughout operator theory.
Goal: to unify the various diagonal studies.
This talk focuses mainly on • but some results also have the flavor of • •.

Focus: finite matrices/rank, projections, idempotents, s.a., normal, positive, compact, positive compact.
Easy questions, hard answers.
From basic to current active areas of research: some examples that pervade our work.

(i) Which numbers can appear on the diagonal of an operator? Answer: its numerical range.
Which operators have only positive diagonal entries? Answer: positive operators.

(ii) Well-known highly useful diagonality examples:

Every trace class operator in every basis has an absolutely summable diagonal sequence (and those sums are invariant);

Likewise compact operators have diagonal sequences tending to zero in every basis.
In contrast, finite rank operators fail to always have finite rank diagonals:
witness any nonzero rank one projection $\xi \otimes \xi$, $\xi \in \ell^2$ of infinite support.

Generalization - diagonal invariance for $B(H)$-ideals $\mathcal{I}$: $E_{\mathfrak{e}}(\mathcal{I}) \subset \mathcal{I}$ for every basis $\mathfrak{e}$
Diagonal invariance is equivalent to $\mathcal{I}$ being arithmetic mean-closed (${}_a(\mathcal{I}_a) = \mathcal{I}$, am-closed for short)[1].

The converses (tests for membership in $\mathcal{I}$) seem to us to be less well-known:
In every basis an operator's diagonal sequence is absolutely summable $\implies$ operator is trace class;
and likewise - In every basis an operator's diagonal sequence $\to 0 \implies$ operator is compact.

This phenomenon is totally general. I.e., a sufficient test for membership in an arbitrary ideal $\mathcal{I}$ is:

$$E_{\mathfrak{e}}(T) \in \mathcal{I}, \forall \mathfrak{e} \implies T \in \mathcal{I}.$$

(Contrapositive - consider the real and imaginary parts of T.)

[1] For details see [Kaftal-W, IUMJ, Theorem 4.5], but for now, $\mathcal{I}_a$ and ${}_a\mathcal{I}$ are the arithmetic mean and pre-arithmetic ideals generated respectively by: operators with s-numbers the arithmetic means of the s-numbers of operators from ideal $\mathcal{I}$, and operators whose arithmetic means of their s-numbers are s-numbers of operators in $\mathcal{I}$.

(iii) What diagonal sequences can arise for a **specific operator**? I.e., the study of $\bullet$.

Schur-Horn theory (our expanded view, although an almost century old Schur-Horn theory studies diagonals of selfadjoint operators that continues today and extends recently into operator algebras)

Much work focuses on **diagonals of positive compact operators.**
Fundamental tool: majorization theory, including new types of majorization such as
∞- and approximate ∞-majorization defined using p- and approximate p-majorization.

Convexity also plays a central role. Also orthogonal matrices (hard study like Hadamard studies).

Some 1923-1964 contributors: Schur 1923, Horn 1954, Markus 1964, Gohberg-Markus 1964.
Last 10 years — Arveson-Kadison 2006, Kaftal-W 2010 and Loreaux 2014.
Others for operator algebra Schur-Horn theory — most recently Ravichandran 2012 and
Skoufranis 2014 (informal communications).

Schur-Horn theory for finite spectrum selfadjoint operators studied extensively by
Kadison 2002 (The Carpenter Problem for projections $\equiv$ 2-point spectrum for normal operators),
Arveson 2007 (a necessary condition on diagonals of certain finite spectrum normal operators),
Jasper 2013 (3-point spectrum selfadjoint operators),
and Bownik-Jasper 2013 (finite spectrum selfadjoint operators)
- all 4 only non-compact operator work known to the authors.

A. Neumann 1999 - an *approximate* Schur-Horn type theorem for general selfadjoint operators,
i.e., characterized the ℓ^∞-closure of the diagonal sequences of a selfadjoint operator.
In contrast, Kaftal-W, Loreaux-W, Kadison, Jasper and Bownik-Jasper are all *exact*,
i.e., characterize precisely the diagonals of certain classes of selfadjoint operators.

(iv) What diagonal sequences can arise for a **class of operators**? I.e., the study of $\bullet\bullet$.

Diverse subject - Works we know of:

Horn (same 1954 paper) characterizes the diagonals of the class of rotation matrices in $M_n(\mathbb{C})$.
Applications: characterizes the diagonals of the **classes** of orthogonal matrices and unitary matrices.

Fong 1986 - any bounded sequence of complex numbers appears as the diagonal of a nilpotent operator in $B(H)$ of order four ($N^4 = 0$), hence automatically for nilpotent and quasinilpotent operators.

Here Fong remarks - a finite complex-valued sequence
appears as the diagonal of a nilpotent matrix in $M_n(\mathbb{C})$ if and only if its sum is zero.

More recently, Giol, Kovalev, Larson, Nguyen and Tener 2011 (because of relevance to frame theory)
- classified the diagonals of idempotent matrices in $M_n(\mathbb{C})$ to be:
sum is a positive integer less than n or $\langle 0, \dots, 0\rangle$ or $\langle 1, \dots, 1\rangle$

(v) A similar J. Jasper frame theory question to us: Characterize inner products of dual-frame pairs?
Evolved into questions on diagonal sequences of idempotents (operators for which $D^2 = D$):

Idempotent operator questions
(a) If an idempotent has a basis in which its diagonal is absolutely summable, is it finite rank?
(b) If an idempotent has a basis in which its diagonal consists solely of zeros, is it finite rank?
(Zero-diagonal operator in the terminology of Fan 1984.)

This talk is on (iii)-(v), Schur-Horn and Jasper's idempotent problems and diagonality

(v) (a)-(b) Yes for Selfadjoint idempotents (i.e., projections)
since they are positive operators hence trace preserved by unitary conjugation (i.e., change of bases).

In fact, for projections, having an absolutely summable (or even summable) diagonal is a characterization of those projections with finite rank since in general $\operatorname{rank} P = \operatorname{Tr} P$.
Moreover, the only projection with a zero diagonal is the zero operator for this same reason.

A negative answer for the entire class of idempotents would be a notable departure versus projections.
Hence the classification of their diagonals is potentially harder than one might naïvely expect.

Perspective:
$\exists$ lots of work on diagonals of selfadjoints.

To study diagonals of idempotents, diagonals of projections (selfadjoint idempotents) are relevant.
Characterized by Kadison 2002.

Straddles for us (iii) and (iv): although the projection class characterization adapts easily to a fixed projection characterization.

Reason: two projection $P, P' \in B(H)$ are unitarily equivalent
if and only if $\operatorname{Tr} P = \operatorname{Tr} P'$ and $\operatorname{Tr}(1-P) = \operatorname{Tr}(1-P')$.
These quantities are precisely the sum of the diagonal entries d_k and the sum of $1-d_k$, respectively, so the 3 finite versus infinite case combinations with some additional work provides a fixed projection characterization.

Theorem 1.1 (Kad-2002)**.** *Given an infinite sequence $\langle d_k \rangle \in [0,1]^{\mathbb{N}}$ with*

$$a = \sum_{d_k < \frac{1}{2}} d_k \quad \text{and} \quad b = \sum_{d_k \geq \frac{1}{2}} (1 - d_k),$$

then there is a projection $P \in B(H)$ (i.e., $P^2 = P = P^*$) with diagonal $\langle d_n \rangle$ if and only if one of the following mutually exclusive conditions holds:

1. *either a or b is infinite;*
2. *$a, b < \infty$ and $a - b \in \mathbb{Z}$.*

2. 0-DIAGONALITY

An operator having zero diagonal in some basis is one such diagonality property.

Jasper's challenge to us: example of a zero-diagonal idempotent?

$$\text{Observe } D = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \text{ NO,}$$

but among those completely general D, because when $H = kerD \oplus ker^{\perp}D$, D looks like

$$D = \begin{pmatrix} I & B \\ 0 & 0 \end{pmatrix}, \text{ Who knows? (We now do.)}$$

Jasper's motivation: Analog of Theorem 1.1-Equivalent to the characterization of inner products of dual frame pairs in finite dimensional Hilbert Space.
and motivated by the infinite dimensional case: idempotent = projection, where the answers are easily affirmative as just described.

Jasper's Goal: to characterize the diagonals of idempotents for infinite dimensional Hilbert space, equivalently, characterize the inner products of dual frame pairs on infinite dimensions.

Next - one counterexample using Fan and Fong (related to work of Fan-Fong-Herrero) 1980's on 0-diagonability. Leads to a characterization of their diagonals. (Thanks to D. Beltita for citations suggestion.)

The counterexample: Claim: Idempotent $D = \begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix}$ is 0-diagonalable.

Main tool: Fan-Fong

D is zero-diagonal if and only if for all θ, $Tr\, Re^{+}(e^{i\theta}D) = Tr\, Re^{-}(e^{i\theta}D)$ and likewise for Im.

Proof.

$$2Re(e^{i\theta}D) = \begin{pmatrix} 2cos\theta I & e^{i\theta}I \\ e^{-i\theta}I & 0 \end{pmatrix} \cong \sum^{\oplus} \begin{pmatrix} 2cos\theta & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix} \cong \sum^{\oplus} \begin{pmatrix} cos\theta + \sqrt{cos^2\theta + 1} & 0 \\ 0 & cos\theta - \sqrt{cos^2\theta + 1} \end{pmatrix}$$

From which it is clear

$$Tr\, Re^{+}(e^{i\theta}D) = Tr\, Re^{-}(e^{i\theta}D) = \infty \quad \forall\theta.$$

□

Recall again, **Idempotents**: operators $D = D^2$, and via $H = RangeD \oplus Range^{\perp}D$, gives cannonical form

$$D = \begin{pmatrix} I & B \\ 0 & 0 \end{pmatrix}$$

General Idempotent Theorem. If Range(Tr D) denotes the set of finite traces in all bases, TFAE

(i) D is not a Hilbert-Schmidt perturbation of a projection.
(ii) B is not Hilbert-Schmidt.
(iii) Range(Tr D) is the plane.
(iv) D has a zero diagonal.
(v) D has an absolutely summable diagonal.
(vi) Range(Tr D) is nonempty.

Proof outline: Previous proof for $\begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix}$ + Borel functional calculus to handle the non-atomic case for B.

A main tool for the previous idempotent theorem is Fan-Fong-Herrero's generalized trace theorem:

B(H) partitions into 4 distinct diagonality classes in that

Range(Tr T) =
(i) empty (where all diagonals fail to converge, e.g., $I_\infty \oplus 0$)
(ii) a point (e.g., any trace class or positive operator)
(iii) a line (e.g., any self-adjoint compact operator with non-trace class negative and positive parts)
(iv) the whole complex plane (e.g., any compact normal with real and imaginary part not trace class)

3. Schur-Horn II

Review: What diagonal sequences can arise for a fixed operator?

For finite selfadjoint matrices the classical Schur-Horn Theorem is:

- For a selfadjoint $N \times N$ matrix A having eigenvalue list η, i.e., singular number sequence $\lambda(A) = \eta$,

  there is a basis in which A has diagonal entries ξ if and only if $\xi \preccurlyeq \eta$.

$\xi \preccurlyeq \eta$ is classical majorization (geometrically **area majorization** in contrast to **height majorization**):
$\sum_1^m \lambda_k(A) \leq \sum_1^m \eta_k \ \forall m \in [1, N]$ & equality at the end ($\sum_1^N \lambda_k(A) = \sum_1^N \eta_k$)

The question studied by Gohberg-Markus, Arveson-Kadison, Kaftal-W, and others:

Characterize the possible diagonal sequences of
the unitary orbit operators of a positive compact operator?

I.e., the cannonical conditional expectation range over unitary orbit ($E(U^*AU)$)

$$U^* \begin{pmatrix} \eta_1 & 0 & 0 & \cdots \\ 0 & \eta_2 & 0 & \ddots \\ 0 & 0 & \eta_3 & \ddots \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix} U = \begin{pmatrix} ? & * & * & \cdots \\ * & ? & * & \ddots \\ * & * & ? & \ddots \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Clearly the answer must depend only on the η eigenvalue sequence!

For finite selfadjoint matrices the classical Schur-Horn Theorem just mentioned gives a characterization.

For infinite rank, historical focus on **positive** **compact operators** produced answers by

- Gohberg-Markus + Loreaux-W lemma (used first by Kaftal-W):
  The "Exact" Schur-Horn theorem holds as stated for positive trace class 0-kernel operators
  (i.e., majorization with equality at the end $N = \infty$)

- Arveson-Kadison and others:
  Various kinds of approximate Schur-Horn theorems for compact and trace class operators.

- Kaftal-W (JFA 2010): Exact Schur-Horn theorem for non trace class compact operators holds as stated
  without the equality at the end condition needed

BUT ONLY FOR A with 0 DIMENSIONAL KERNEL-
The case infinite dimensional kernel
for us requires new kinds of majorizations.

The rest of this talk is devoted to:
How complicated is that unitary U? And what are the new kinds of majorizations and consequences?

Theorem: Infinite products of T-transforms. Describe.
Kaftal-W modified slightly then matrix modeled the Markus construction used to prove
$\xi \prec \eta \quad \Leftrightarrow \quad \xi = P\eta$ for some substochastic P.

- How does this work?

Getting started: $\xi \prec \eta \Rightarrow \xi_1 \leq \eta_1$
So $\xi_1 \in (\eta_k, \eta_{k+1}]$ so $\xi_1 = t\eta_k + (1-t)\eta_{k+1}$ (unique !! cannonical k, unlike Markus $[\eta_k, \eta_{k+1}]$) yields:

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ t\eta_k + (1-t)\eta_{k+1} \\ (1-t)\eta_k + t\eta_{k+1} \\ \vdots \\ \eta_n \\ \vdots \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \xi_1 \\ (1-t)\eta_k + t\eta_{k+1} \\ \vdots \\ \eta_n \\ \vdots \end{pmatrix} \prec \eta \qquad \text{and better}$$

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \end{pmatrix} \prec \begin{pmatrix} \xi_1 \\ \eta_1 \\ \eta_2 \\ \vdots \\ (1-t)\eta_k + t\eta_{k+1} \\ \vdots \\ \eta_n \\ \vdots \end{pmatrix} \prec \eta$$

Notice density at play. Middle vector lost monotonicity, BUT truncated vectors are monotone!!

$$\begin{pmatrix} \xi_2 \\ \vdots \end{pmatrix} \prec \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ (1-t)\eta_k + t\eta_{k+1} \\ \vdots \\ \eta_n \\ \vdots \end{pmatrix} \qquad \text{Now iterate}$$

Next-building the U's that transform diag η into unitarily equivalent matrix with diagonal ξ.

T-transforms: For every integer $m \geq 1$ and $0 < t \leq 1$, define the $m+1 \times m+1$ orthogonal matrix $\oplus I$

(1)

$$V(m,t) := \begin{pmatrix} 0 & 0 & \dots & 0 & \sqrt{t} & -\sqrt{1-t} \\ 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & \sqrt{1-t} & \sqrt{t} \end{pmatrix} \oplus I, \quad Q(m,t) := \begin{pmatrix} 0 & 0 & \dots & 0 & t & 1-t \\ 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1-t & t \end{pmatrix} \oplus I$$

Orthogonal (unitary with real entries)   Orthostochastic (Schur-square of orthogonal)

Example: Simplest case $m_k = 1$ for all k-simpler to find the forms $W(\{m_k, t_k\})$, $Q(\{m_k, t_k\})$.

- **Infinite products of $ad_{I_n \oplus W \oplus I}$'s and corresponding Q's (i.e., matrix modeling Markus)**

$$W(\{1, t_k\}) =$$
$$\begin{pmatrix} \sqrt{t_1} & -\sqrt{1-t_1} & 0 & 0 & \dots \\ \sqrt{t_2(1-t_1)} & \sqrt{t_2 t_1} & -\sqrt{1-t_2} & 0 & \dots \\ \sqrt{t_3(1-t_2)(1-t_1)} & \sqrt{t_3(1-t_2)t_1} & \sqrt{t_3 t_2} & -\sqrt{1-t_3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sqrt{t_k \prod_{i=1}^{k-1}(1-t_i)} & \sqrt{t_k t_1 \prod_{i=2}^{k-1}(1-t_i)} & \sqrt{t_k t_2 \prod_{i=3}^{k-1}(1-t_i)} & \sqrt{t_k t_3 \prod_{i=4}^{k-1}(1-t_i)} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$Q(\{1, t_k\}) = \begin{pmatrix} t_1 & 1-t_1 & 0 & 0 & \dots \\ t_2(1-t_1) & t_2 t_1 & 1-t_2 & 0 & \dots \\ t_3(1-t_2)(1-t_1) & t_3(1-t_2)t_1 & t_3 t_2 & 1-t_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t_k \prod_{i=1}^{k-1}(1-t_i) & t_k t_1 \prod_{i=2}^{k-1}(1-t_i) & t_k t_2 \prod_{i=3}^{k-1}(1-t_i) & t_k t_3 \prod_{i=4}^{k-1}(1-t_i) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

- **When is the infinite product of $ad_{I_n \oplus W \oplus I}$'s unitary?**

So likewise infinite product of Q's orthostochastic? (Schur-squares of these unitaries)

Theorem 3.1. *Let $\xi \prec \eta$ for some $\xi, \eta \in c_o^*$ with $\xi_n > 0$ for every n. Then TFAE.*
(i) $\xi \preccurlyeq \eta$.
(ii) $\sum\{t_k \mid m_k = 1\} = \infty$.
(iii) $Q(\xi, \eta)$ *is orthostochastic.*
(iv) $W(\xi, \eta)$ *is orthogonal.*

- **Which is what happens when η is strictly positive,**

i.e., the range projection of B is identity (recall condition $R_B = I$).

- **In summary, $U =$ an Infinite product of what Arveson-Kadison call T-transforms.**

They turned out to be our matrix model building blocks of Markus's Lemma
connecting majorization to stochastic matrices (even more mysterious orthostochastics)
and then to their Schur-square roots which are unitary (even more mysterious orthogonals)

New kinds of majorization are playing increasing roles in ideals, commutators, traces on ideals, and Schur-Horn theorems. (A taste of a growing majorization industry.)

- Uniform Hardy-Littlewood submajorization of Kalton-Sukochev-Zanin:

$$\exists \mu \text{ for which } \sum_{\mu a}^{m} \lambda_k(A) \leq \sum_{a}^{m} \eta_k \ \forall m \in [\mu a, \infty)$$

- Kaftal-W p-majorization:

$$\sum_{1}^{m+p} \lambda_k(A) \leq \sum_{1}^{m} \eta_k \text{ eventually in } m \text{ (p=0 is classical majorization)}$$

- Loreaux-W ∞-majorization and approximate p-majorization:

$$\forall \epsilon > 0, \sum_{1}^{m+p} \lambda_k(A) \leq \sum_{1}^{m} \eta_k + \epsilon \eta_{m+1} \text{ eventually in } m \text{ (p=0 is classical majorization)}$$

∞-majorization: all p- or equivalently all approximately p-majorization.

Our main contribution. Kaftal-W left open the case when A has infinite rank and nonzero kernel. We attempt here to close this gap. In particular, we characterize $E(\mathcal{U}(A))$ when A has infinite dimensional kernel. When A has finite dimensional kernel, we give a necessary condition for membership in $E(\mathcal{U}(A))$ (which we conjecture is also sufficient), and we give a sufficient condition for membership in $E(\mathcal{U}(A))$ (which we know not to be necessary when $0 < \operatorname{Tr} R_A^{\perp} < \infty$. Both of these membership conditions involve new kinds of majorization, which here we call p-majorization and herein our brand new approximate p-majorization (for $0 \leq p \leq \infty$. There is a natural hierarchy of these new types of majorization which the following diagram describes succinctly. All of these implications are natural except the two corresponding to the dashed arrows, which we prove and are only applicable when both sequences in question are in $c_0^+ \setminus \ell^1$. A linear interpretation of the diagram in Figure 1 is presented below in Figure 2.

$$\begin{tikzcd}
 & \precsim_1 \arrow[dl] & \precsim_2 \arrow[l] \arrow[ddl] & \cdots \arrow[l] \arrow[ddl] & \precsim_\infty \arrow[l] \arrow[ddl] \arrow[dd, leftrightarrow] & \\
\prec & & & & & \prec \text{ and } \not\precsim \arrow[ul, dashed] \arrow[dl, dashed] \\
 & \prec_1 \arrow[ul] \arrow[uu] & \prec_2 \arrow[l] \arrow[uu] & \cdots \arrow[l] \arrow[uu] & \prec_\infty \arrow[l] &
\end{tikzcd}$$

Figure 1. Hierarchy of majorization

$$\prec \longleftarrow \precsim_1 \longleftarrow \prec_1 \longleftarrow \precsim_2 \longleftarrow \cdots \longleftarrow \precsim_\infty \longleftrightarrow \prec_\infty \dashleftarrow (\prec \text{ and } \not\precsim)$$

Figure 2. Linear hierarchy of majorization

• Main New Theorems:

Theorem 3.2.
Let $A, B \in K(H)^+$, $B \in \mathcal{D}$ and $\operatorname{Tr}(R_B^\perp) \leq \operatorname{Tr}(R_A^\perp)$ (necessary condition for membership $B \in E(\mathcal{U}(A))$).
If for some $0 \leq p \leq \infty$, $s(B) \prec_p s(A)$ and $\operatorname{Tr}(R_A^\perp) \leq \operatorname{Tr}(R_B^\perp) + p$, ***then*** *$B \in E(\mathcal{U}(A))$.*

Our main theorem on necessity for membership in $E(\mathcal{U}(A))$ depends on approximate p-majorization:

Theorem 3.3. *Suppose $A \in K(H)^+$ and $B \in E(\mathcal{U}(A))$. If*

$$p = \min\{n \in \mathbb{N} \cup \{0, \infty\} \mid \operatorname{Tr}(R_A^\perp) \leq \operatorname{Tr}(R_B^\perp) + n\},$$

then $s(B) \precsim_p s(A)$.

One of our main results is Corollary 3.4 which, in the rather general setting where A has infinite rank and infinite dimensional kernel, we obtain a precise characterization of $E(\mathcal{U}(A))$ in terms of majorization and ∞-majorization.

Corollary 3.4.
Suppose $A \in K(H)^+$ has infinite rank and infinite dimensional kernel ($\operatorname{Tr} R_A = \infty = \operatorname{Tr} R_A^\perp$). *Then*

$$E(\mathcal{U}(A)) = E(\mathcal{U}(A))_{fk} \sqcup E(\mathcal{U}(A))_{ik},$$

the members of $E(\mathcal{U}(A))$ with finite dimensional kernel and infinite dimensional kernel, respectively, are characterized by

$$E(\mathcal{U}(A))_{fk} = \{B \in \mathcal{D} \cap K(H)^+ \mid s(B) \prec_\infty s(A) \quad \textit{and} \quad \operatorname{Tr} R_B^\perp < \infty\}$$

and

$$E(\mathcal{U}(A))_{ik} = \{B \in \mathcal{D} \cap K(H)^+ \mid s(B) \prec s(A) \quad \textit{and} \quad \operatorname{Tr} R_B^\perp = \infty\}.$$

• Hard fought convexity results:

Theorem 3.5 (KW Corollary 6.7 (2010 JFA))**.** *Let $A \in K(H)^+$.*
Then
(i) $E(\mathcal{V}(A))$ is convex.
(ii) If $R_A = I$ or A has finite rank, then $E(\mathcal{U}(A))$ is convex.

Corollary 3.6. *If $A \in K(H)^+$ and $\operatorname{Tr}(R_A) = \infty = \operatorname{Tr}(R_A^\perp)$, then $E(\mathcal{U}(A))$ is convex.*