Finite dimensional spaces

Note Title 1/18/2008

Recall Θ inner

$$K_\Theta = H^2 \cap (\Theta H^2)^\perp$$

P_Θ = ortho proj : $L^2 \to K_\Theta$

$$P_\Theta f = \langle f, k_\lambda \rangle$$

$$k_\lambda(z) = \frac{1 - \overline{\Theta(\lambda)}\,\Theta(z)}{1 - \bar{\lambda} z}. \quad \text{(r.k.)}$$

For $\varphi \in L^2$

$$A_\varphi : K_\Theta \to Hol(\mathbb{D})$$

$$(A_\varphi f)(\lambda) = \int \varphi f \overline{k_\lambda}\, dm$$

If $\varphi \in L^\infty$, $A_\varphi = P_\Theta T_\varphi | K_\Theta$

compression of Toep. op.
to K_Θ

$\boxed{Q:}$ When is $A \in B(K_\theta)$ a truncated Toep. op.?

Sarason: $\Longleftrightarrow$

$$A = A_z^* A A_z + g_1 \otimes S^*\theta + S^*\theta \otimes g_2$$

some $g_1, g_2 \in K_\theta$

This condition is difficult to apply

Thm is a "Brown-Halmos" theorem for T.T.O.

Example $\Theta(z) = z^N$

$$K_\Theta = \text{span}\{1, z, \ldots, z^{N-1}\}$$

$$T_\varphi \cong \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \ddots \\ a_1 & \ddots & \ddots & \ddots \\ a_2 & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

w.r.t. $\{z^n\}_{n=0}^\infty$

$a_n = \hat{\varphi}(n)$

(standard Toeplitz matrix)

$(N=5)$

$$A_\varphi = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} \\ a_1 & a_0 & a_{-1} & a_{-2} & a_{-3} \\ a_2 & a_1 & a_0 & a_{-1} & a_{-2} \\ a_3 & a_2 & a_1 & a_0 & a_{-1} \\ a_4 & a_3 & a_2 & a_1 & a_0 \end{pmatrix}$$

(upper left corner of Toeplitz matrix)

<u>Note</u> K_Θ is finite dim $\Leftrightarrow$ Θ is a finite B-product

Assume B has zeros $a_1, \ldots, a_N$
($a_j \neq a_k$)

K_B has dim N and

$$k_{a_j}(z) = \frac{1}{1-\bar{a}_j z} \; ; \; 1 \leq j \leq N$$

is a <u>basis for K_B</u>

(since finite dim)

$$\mathcal{T}_B = \{ A_\varphi : \varphi \in L^\infty \}$$

<u>Sarason:</u> $\dim(\mathcal{T}_B) = 2N-1$

Notice how this works for $B(z) = z^N$ (see Example)

$\boxed{Q}$ When does $A \in \mathcal{B}(K_B)$ belong to T_B?

Equivalently, if M_A is the matrix representation of A w.r.t. $\{k_{a_1}, \ldots, k_{a_N}\}$, when is $M_A = M_{A_\varphi}$ for some φ?

<u>Thm</u> (Cima–Ross–Wogen) $(r_{ij}) \in M_{n \times n}(\mathbb{C})$ represents a T_B op. w.r.t. $\{k_{a_1}, \ldots, k_{a_n}\}$

$\Longleftrightarrow$

$$r_{ij} = \left(\frac{\overline{B'(a_1)}}{B'(a_i)}\right)\left(\frac{r_{1,i}\,\overline{(a_1 - a_i)} + r_{1,j}\,\overline{(a_j - a_1)}}{\overline{a_j - a_i}}\right) \qquad (\dagger)$$

$$1 \le i, j \le N, \quad i \ne j$$

(This is a testable condition)

Notice

1. $(r_{i,j})$ depends only on the diagonal and first row. So $\dim T_B = 2N-1$ (as in Sarason's Thm)

2. When $N=2$

$\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$ represents a T_B op. w.r.t $\{k_{a_1}, k_{a_2}\}$

$$\Longleftrightarrow$$

$$\overline{B'(a_1)}\, r_{1,2} = \overline{B'(a_2)}\, r_{2,1}$$

Proof (Some assembly required)

$$L^2 = H^2 + \overline{H^2}$$

$$= (K_B + BH^2) + (\overline{K_B} + \overline{BH^2})$$

So if $\varphi \in L^2$, then

$$\varphi = \overline{\psi_1} + \psi_2 + \overline{\eta_1} + \eta_2$$

$$\psi_1, \psi_2 \in K_B$$

$$\eta_1, \eta_2 \in BH^2$$

and so

$$A_\varphi = A_{\overline{\psi_1} + \psi_2} + \underbrace{A_{\overline{\eta_1} + \eta_2}}_{\equiv 0 \text{ (from last time)}}$$

Thus

$$T_B = \{ A\overline{\psi_1} + \psi_2 : \psi_1, \psi_2 \in K_B \}$$

$$\psi_1 = \sum c_j \left(\frac{B}{z-a_j}\right)$$

$$\psi_2 = \sum d_j \left(\frac{B}{z-a_j}\right)$$

(Basis Elements for K_θ)

We can take φ to be

$$\varphi = \overline{\sum c_j \left(\frac{B}{z-a_j}\right)} + \sum d_j \left(\frac{B}{z-a_j}\right)$$

(some constants c_j, d_j, $1 \le j \le N$)

(Sarason) $k_{a_j} \otimes \tilde{k}_{a_j} = A_{\left(\frac{\overline{B}}{z - a_j}\right)}$

$$\tilde{k}_{a_j}(z) = \frac{B(z)}{z - a_j}$$

$\boxed{\begin{array}{l}\text{See this from Cauchy Residue Theorem} \\ \hline \text{Apply to } k_{a_j}\end{array}}$

$$A_{\overline{\psi_1}} = \sum_{j=1}^{N} c_j \, k_{a_j} \otimes \tilde{k}_{a_j}$$

$$A_{\psi_2} = \sum_{j=1}^{N} d_j \, \tilde{k}_{a_j} \otimes k_{a_j}$$

So T_B consists of

$$\sum_j c_j \, k_{a_j} \otimes \tilde{k}_{a_j} + \sum_j d_j \, \tilde{k}_{a_j} \otimes k_{a_j}$$

$$c_j, d_j \in \mathbb{C}$$

Useful inner product formulas

1) $\langle k_{a_i}, k_{a_j} \rangle = \dfrac{1}{1 - \overline{a_i} a_j}$

2) $\langle \tilde{k}_{a_i}, k_{a_j} \rangle = \begin{cases} 0 & \text{if } i \neq j \\ B'(a_j) & i = j \end{cases}$

3) $\langle \tilde{k}_{a_i}, \tilde{k}_{a_j} \rangle = \dfrac{1}{1 - \overline{a_j} a_i}$

4) $\underbrace{\tilde{k}_{a_j} = \sum_{s=1}^{N} \dfrac{1}{\overline{B'(a_s)}} \dfrac{1}{1 - \overline{a_s} a_j} k_{a_s}}_{\text{use (2) + (3) to get this}}$

Use these to compute

$$A\varphi\, ka_p \qquad (p = 1, 2, \ldots, N)$$

$$\parallel$$

$$x_{1,p}\, ka_1 + x_{2,p}\, ka_2 + \cdots + x_{N,p}\, ka_N$$

<u>Note</u>: $A\varphi = \sum_j c_j\, ka_j \otimes \widetilde{ka_j}$

$$+ \sum_j d_j\, \widetilde{ka_j} \otimes ka_j$$

Use the above "orthogonality" relation ~~to~~ to compute.

After gathering up terms, we get

$$(M_{A_\varphi})_{s,p} = c_p \overline{B'(a_p)}\, \delta_{s,p} + \frac{1}{\overline{B'(a_s)}} \sum_{j=1}^{N} \frac{d_j}{(1-\overline{a_s} a_j)(1-\overline{a_p} a_j)}$$

From here the condition are "easy" to check — $\underline{\underline{\text{Lots}}}$ of partial fractions.

Thus M_{A_φ} has entries satisfying $(+)$

The other direction is basic linear algebra + Sarason's fact that $\dim(T_B) = 2N-1$.

Indeed, let V be the vector space of matrices satisfying $(+)$. One can see that $\dim(V) = 2N-1$.

Notice that

$$M_{T_B} \subseteq V \quad \text{by } (+)$$

matrix rep of T_B w.r.t. $\{ha_1, \ldots, ha_n\}$

and $\dim(M_{T_B}) = 2N-1$ (Sarason).

Thus $M_{T_B} = V$ $\boxed{}$