<u>Examples</u> $\mathcal{U}(A)/\mathcal{U}_0(A)$

① $A = M_n(\mathbb{C})$
For $u \in \mathcal{U}(A)$, $\sigma(u)$ is finite
So $\sigma(u) \subsetneq \mathbb{T} \Rightarrow u \in \mathcal{U}_0(A)$

$\Rightarrow \mathcal{U}(A)/\mathcal{U}_0(A) = \{[0]\}$

② $A = C([0,1])$
let $f \in \mathcal{U}(A)$.
Then $|f| = 1 \Rightarrow f(x) = e^{ik(x)}$, where k is real-valued,

$f_t = e^{(1-t)k(x)i}$
where in particular
$f_0 = f(x)$

$f_1 = e^0 = 1$.

$\Rightarrow f \sim_h 1$ in $\mathcal{U}(A) \Rightarrow f \in \mathcal{U}_0(A)$

$\Rightarrow \mathcal{U}(A)/\mathcal{U}_0(A) = \{[0]\}$.

③ $A = C(\mathbb{T})$
$f \in \mathcal{U}(A) \Rightarrow |f| = 1 \Rightarrow f : \mathbb{T} \to \mathbb{T}$.

$f(e^{i\theta}) = e^{ih(\theta)}$, where $h(2\pi) = h(0) + 2k\pi$, $k \in \mathbb{Z}$
k = winding number of f around 0.

h real-valued, continuous.

<u>claim</u>: $f \sim_h z^k$

$f_t(e^{i\theta}) = e^{ikt\theta + (1-t)h(\theta)i}$

Need: $f_t(e^{i0}) = f_t(e^{i2\pi})$

$$f_t(e^{i\cdot 0}) = e^{(1-t)h(0)i}$$

$$f_t(e^{i2\pi}) = e^{i2\pi kt + (1-t)(h(0)+2k\pi)i}$$

$$= e^{(1-t)h(0)i + 2k\pi i}$$

$\Rightarrow \quad f \sim_h z^k$ //

$z^k \not\sim_h z^l$ if $k \neq l$

$\Rightarrow \quad \mathcal{U}(A)/\mathcal{U}_0(A) \cong \mathbb{Z}$.

④ $A = C(X)$, X = compact Hausdorff space

$\pi^1(X)$ = first cohomotopy group of X

= group of homotopy classes of continuous maps from X to the circle group $\mathbb{T}$ with pointwise multiplication

$\mathcal{U}(A)/\mathcal{U}_0(A) \cong \pi^1(X)$

<u>ref</u>: Douglas Banach Algebra Techniques in Operator Theory

⑤ $A = \mathcal{B}(\ell^2)$

let $u \in \mathcal{B}(\ell^2)$.

<u>case 1</u>: $\sigma(u) \subsetneq \mathbb{T} \Rightarrow u \in \mathcal{U}_0(A)$.

using Borel functional calculus

<u>case 2</u>: $\sigma(u) = \mathbb{T}$, then

$$\frac{\mathrm{Log}(u)}{i}$$

real-valued on $\mathbb{T}$, jump disc at -1.

$$u = e^{i\left(\frac{\mathrm{Log}\, u}{i}\right)}$$

note: $\frac{\mathrm{Log}\, z}{i}$ is real-valued on $\mathbb{T}$

$\Rightarrow$ $\frac{\mathrm{Log}(u)}{i}$ is self-adjoint.

So $u \in \mathcal{U}_0(A)$.

$\Rightarrow \mathcal{U}(A)/\mathcal{U}_0(A) = \{[0]\}$.

⑥ $A = \mathbb{C}I + \mathcal{K}$, $\mathcal{K}$ = compacts in $\mathcal{B}(\ell^2)$.

Let $u \in \mathcal{U}(A)$.
$u = \lambda + k$, $\lambda \in \mathbb{C}$, $k \in \mathcal{K}$.

$$I = uu^* = (\lambda + k)(\bar{\lambda} + k^*) = |\lambda|^2 + \underbrace{\bar{\lambda} k + \lambda k^* + kk^*}_{\in \mathcal{K}}$$

$\Rightarrow I - |\lambda|^2 I \in \mathcal{K}$
$\Rightarrow |\lambda| = 1$.

$$I = u^* u = (\bar{\lambda} + k^*)(\lambda + k) = |\lambda|^2 + \bar{\lambda} k + \lambda k^* + k^* k.$$

$\Rightarrow kk^* = k^* k$ (so k is a compact normal operator).

So $\sigma(k)$ is either finite or a countable set converging to 0.

$\sigma(\lambda + k)$ is still either finite or countable
$\Rightarrow \sigma(\lambda + k)$ is not all of $\mathbb{T}$

$\Rightarrow u = \lambda + k \sim_h 1$. $\Rightarrow \mathcal{U}(A)/\mathcal{U}_0(A) = \{[0]\}$.

(7) $A = \mathcal{B}(\ell^2)/\mathcal{K}$. Calkin Algebra
idea: Fredholm index

$A \in \mathcal{B}(\ell^2)$ is Fredholm iff $A + \mathcal{K}$ is invertible in $\mathcal{B}(\ell^2)/\mathcal{K}$.

Fredholm index

$\operatorname{ind} A = \dim \ker A - \dim \ker A^*$.

- constant under compact perturbation.
$\operatorname{ind}(A) = \operatorname{ind}(A+k)$, $k \in \mathcal{K}$.

$\Rightarrow \operatorname{ind}(A+\mathcal{K}) = \operatorname{ind}(A)$ is well-defined.

Define $\Phi : \mathcal{U}(A)/\mathcal{U}_0(A) \to \mathbb{Z}$
by
$$\Phi((u+\mathcal{K})+\mathcal{U}_0) = \operatorname{ind}(u).$$

-well-defined since Fredholm index is continuous.

Fredholm index is a group homomorphism

claim: Φ is onto.

Let S be the forward shift
$S(x_0, x_1, x_2, \dots) = (0, x_0, x_1, x_2, \dots)$

S^* = backward shift

$$S^*(x_0, x_1, x_2, \dots) = (x_1, x_2, \dots)$$

$S^m + \mathcal{K}$, $S^{*m} + \mathcal{K}$ are unitaries, $m \in \mathbb{N}$:

(note: $\Phi((I+\mathcal{K})+\mathcal{U}_0) = \operatorname{ind}(I) = 0$.)

$$(S^m + k)(S^{m*} + k^*) = \underbrace{S^m S^{m*}}_{= I - \text{finite rank operator}} + \underbrace{S^m k^* + k S^{m*} + k k^*}_{\in \mathcal{K}}$$

$\Rightarrow \quad (S^m + K)(S^{m*} + K^*) = I + \mathcal{K}.$

similarly for $(S^{*m} + \mathcal{K})(S^m + \mathcal{K})$

$\operatorname{ind}(S^m) = \dim\ker S^m - \dim\ker S^{m*}$
$= 0 - m = -m$

$\operatorname{ind}(S^{m*}) = m.$

$\Rightarrow \Phi$ is onto. ✓ //

(still need to show Φ is 1-1)

Then $\mathcal{U}(A)/\mathcal{U}_0(A) \cong \mathbb{Z}.$

$$\begin{array}{ccccc} \mathcal{B}(\ell^2) & \longrightarrow & \mathcal{B}(\ell^2)/\mathcal{K} & \xrightarrow{\text{Fredholm index}} & \mathbb{Z} \\ \cup\!\mid & & \cup\!\mid & & \nearrow_{\text{winding number}} \\ C^*(S) & \longrightarrow & C^*(S)/\mathcal{K} & \cong & C(\mathbb{T}) \end{array}$$

- For $A = \mathcal{B}(\ell^2)$, $\mathcal{U}(A)$ is contractible

- Connection between $\mathcal{U}(A)/\mathcal{U}_0(A)$ and $K_1(A)$

always have homomorphism
$\mathcal{U}(A)/\mathcal{U}_0(A) \longrightarrow K_1(A)$

if A is abelian, $K_1 \cong \mathcal{U}(A)/\mathcal{U}_0(A) \oplus \ker\Delta$,
where Δ is a homomorphism.

<u>Proposition 2.1.11</u>: A is a unital C^*-algebra $a \in GL(A)$, $b \in A$, with $\|b - a\| < \dfrac{1}{\|a^{-1}\|}$

then

(i) $b \in GL(A)$ and

$$\frac{1}{\|b^{-1}\|} \geq \frac{1}{\|a^{-1}\|} - \|a - b\|.$$

(ii) $b \sim_h a$ in $GL(A)$.

<u>proof</u>: (i) familiar geometric series argument.

(ii) $c_t = (1-t)a + tb, \quad 0 \leq t \leq 1.$
$c_t \in$ Ball also $\subset GL(A)$. ∎

<u>Recall</u>: $\mathcal{P}(A)$ = projections in A (A may not be unital)
Murray-von Neumann equivalence:
$p \sim q$ <u>if</u> $p = v^*v$, $q = vv^*$ for some $v \in A$
(such v are necessarily partial isometries).

unitary equivalence:
$p \sim_u q$ if $q = zpz^*$ for some $z \in \mathcal{U}(\tilde{A})$

$\tilde{A}$ = unitization of A

<u>Exercise</u>. Let $p_1, \ldots, p_n \in A$ be projections, where A = unital C^*-alg.
TFAE:
① $p_1, p_2, \ldots, p_n$ are mutually orthogonal
② $p_1 + p_2 + \cdots + p_n$ is a projection
③ $p_1 + p_2 + \cdots + p_n \leq 1$. (using the ordering on self-adjoint operators)

<u>Corollary</u>: If $v_1, \dots, v_n$ are partial isometries and

$$\sum_{i=1}^{n} v_i^* v_i = 1_A = \sum_{i=1}^{n} v_i v_i^*$$

then
$$\sum_{i=1}^{n} v_i \in \mathcal{U}(A).$$

<u>proof</u>: $v_1 v_1^*, \dots, v_n v_n^*$ are projections
$v_1^* v_1, \dots, v_n^* v_n$ are projections.

$v_1 v_1^*, \dots, v_n v_n^*$ are mutually orthogonal (by exercise).

<u>claim</u>: If $j \neq k$, $v_j^* v_k = 0$.

that is,
$$\underset{\displaystyle\text{range } v_k v_k^*}{\overset{}{\text{range } v_k}} \overset{?}{\subseteq} \ker v_j^* = \underset{\displaystyle(\text{range } v_j v_j^*)^\perp}{(\text{range } v_j)^\perp}$$

with $\text{range } v_k = \text{range } v_k v_k^*$ and $(\text{range } v_j)^\perp = (\text{range } v_j v_j^*)^\perp$.

But $\text{range } v_k v_k^* \subseteq (\text{range } v_j v_j^*)^\perp$ since they have orthogonal ranges.
$\Rightarrow$ $v_j^* v_k = 0$.

<u>Note</u>:
$$\Big(\sum_{j=1}^{n} v_j\Big)^* \Big(\sum_{k=1}^{n} v_k\Big) = \sum_{j=1}^{n} v_j^* v_j + \underbrace{\sum_{j \neq k} v_j^* v_k}_{=0.}$$
$$= \sum_{j=1}^{n} v_j^* v_j$$
$$= 1 \quad \text{(by hypothesis).}$$

$\Rightarrow \displaystyle\sum_{j=1}^{n} v_j$ is <u>isometric</u>. Similarly, $\displaystyle\sum_{j=1}^{n} v_j^*$ is isometric.

$\Rightarrow \displaystyle\sum_{j=1}^{n} v_j$ is unitary. ∎

<u>Proposition 2.2.2:</u> let A be unital $p, q \in \mathcal{P}(A)$.

TFAE:

① $p \sim_u q$ (need $A \subseteq \tilde{A}$)

② $q = upu^*$ for some $u \in \mathcal{U}(A)$.

③ $p \sim q$ <u>and</u> $1_A - p \sim 1_A - q$

<u>proof:</u> $f = 1_{\tilde{A}} - 1_A$

If $a \in A$, $af = (1_{\tilde{A}} - 1_A) = a - a = 0$
Also $fa = 0$.

<u>note:</u> $\tilde{A} = A \oplus \mathbb{C}$, $1_{\tilde{A}} = (0, 1)$

$(a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha\beta)$

so $1_A = (1_A, 0)$

$\Rightarrow f = (-1_A, 1)$

let $b \in \tilde{A}$. $b = (a, \alpha) = (a, 0) + (0, \alpha)$
$= (a, 0) + \alpha(0, 1)$
$= a + \alpha 1_{\tilde{A}}$

Then $b = a + \alpha 1_{\tilde{A}}$

$= a + \alpha(1_{\tilde{A}} - 1_A) + \alpha 1_A$

$= \underbrace{a + \alpha 1_A}_{\in A} + \alpha f$

$\Rightarrow \tilde{A} = A + \mathbb{C}f$ (as a vector sum) note also that f is a projection.

$f^2 = (-1_A, 1)(-1_A, 1) = (-1_A, 1) = f.$

$f^* = f$

$\Rightarrow$ f is a projection.

① $\Rightarrow$ ②: assume $p \sim_u q$.
So $q = zpz^*$, $z \in \mathcal{U}(\tilde{A})$.

write $z = u + \alpha f$, $\alpha \in \mathbb{C}$, $u \in A$.

$$
\begin{aligned}
1_{\tilde{A}} = z^* z &= (u^* + \bar{\alpha} f)(u + \alpha f) \\
&= u^* u + |\alpha|^2 f \\
&= u^* u + |\alpha|^2 (1_{\tilde{A}} - 1_A)
\end{aligned}
$$

$$\Rightarrow \underbrace{(1 - |\alpha|^2) 1_{\tilde{A}}}_{\text{not in } A \text{ unless } |\alpha| = 1.} = \underbrace{u^* u - |\alpha|^2 1_A}_{\in A}$$

So $u^* u = 1_A$. Similarly, $uu^* = 1_A$

$\Rightarrow$ $u \in \mathcal{U}(A)$.

$$
\begin{aligned}
q = zpz^* &= (u + \alpha f) p (u^* + \bar{\alpha} f) \\
&= upu^* + \underbrace{\alpha f p u^*}_{= 0 \text{ since } fa = 0 \ \forall a \in A} + \underbrace{\bar{\alpha} u p f}_{= 0} + \underbrace{|\alpha|^2 f p f}_{= 0.} \\
&= upu^*
\end{aligned}
$$

$\Rightarrow$ ② holds.