

# Finite dimensional spaces

Note Title

1/18/2008

Recall  $\Theta$  inner

$$K_\Theta = H^2 \cap (\Theta H^2)^\perp$$

$P_\Theta = \text{ortho proj} : L^2 \rightarrow K_\Theta$

$$P_\Theta f = \langle f, k_\lambda \rangle$$

$$k_\lambda(z) = \frac{(1 - \overline{\Theta(\lambda)} \Theta(z))}{1 - \overline{\lambda} z} \quad (\text{r.k.})$$

For  $\varphi \in L^2$

$$A_\varphi : K_\Theta \rightarrow \text{Hol}(\mathbb{D})$$

$$(A_\varphi f)(\lambda) = \int \varphi f \overline{k_\lambda} \, d\mu$$

$$\text{If } \varphi \in L^\infty, \quad A_\varphi = P_\Theta T_\varphi|_{K_\Theta}$$

compression of Toep. op.  
 $\downarrow$   
 $K_\Theta$

[Q] When is  $A \in B(K_0)$  a truncated Toeplitz operator?

Sarason:  $\iff$

$$A = A_z^* A A_z + g_1 \otimes S^* \theta + S^* \theta \otimes g_2$$

Some  $g_1, g_2 \in K_\theta$

This condition is difficult to apply

There is a "Brauer-Halmos" theorem for  $T, T, 0$ .

Example  $\Theta(z) = z^N$

$$K_{\Theta} = \text{span} \{1, z, \dots, z^{N-1}\}$$

$T_{\varphi} \equiv$   
w.r.t.  $\{z^n\}_{n=0}^{\infty}$

$$a_n = \varphi(n)$$

$$\begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots \\ a_1 & & & \\ a_2 & & & \\ \vdots & & & \end{pmatrix}$$

(standard Toeplitz matrix)

( $N=5$ )

(upper left corner of Toeplitz matrix)

$$A_{\varphi} = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} \\ a_1 & a_0 & a_{-1} & a_{-2} & a_{-3} \\ a_2 & a_1 & a_0 & a_{-1} & a_{-2} \\ a_3 & a_2 & a_1 & a_0 & a_{-1} \\ a_4 & a_3 & a_2 & a_1 & a_0 \end{pmatrix}$$

Note  $K_{\Theta}$  is finite dim  $\iff \Theta$   
is a finite  $B$ -product

Assume  $B$  has zeros  $a_1, \dots, a_N$   
( $a_j \neq a_k$ )

$K_B$  has dim  $N$  and

$k_{a_j}(z) = \frac{1}{1 - \bar{a}_j z}$  ;  $1 \leq j \leq N$   
is a basis for  $K_B$

$\mathcal{T}_B = \{A_{\varphi} : \varphi \in L^{\infty}\}$  (since finite dim)

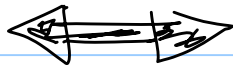
Sarason :  $\dim(\mathcal{T}_B) = 2N - 1$

Notice how this works for  
 $B(z) = z^N$  (see example)

(Q) When does  $A \in \mathcal{B}(K_B)$  belong to  $\mathcal{T}_B$ ?

Equivalently, if  $M_A$  is the matrix representation of  $A$  w.r.t.  $\{k_{a_1}, \dots, k_{a_N}\}$ . When is  $M_A = M_{A\varphi}$  for some  $\varphi$ ?

Thm (Cunha-Rwosen)  $(r_{ij}) \in M_{N \times N}(\mathbb{C})$  represents a  $\mathcal{T}_B$  o.p., w.r.t.  $\{k_{a_1}, \dots, k_{a_N}\}$



$$r_{ij} = \frac{\overline{\mathcal{B}(a_i)}}{\overline{\mathcal{B}(a_j)}} \left( \frac{r_{i,i} \overline{(a_i - a_i)} + r_{i,j} \overline{(a_j - a_i)}}{\overline{a_j - a_i}} \right)$$

(+)

$$1 \leq i, j \leq N, i \neq j$$

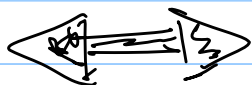
(This is a testable condition)

## Notice

1.  $(r_{i,j})$  depends only on the diagonal and first row. So  $\dim T_B = 2N-1$  (as in Sarason's Thm)

2. When  $N=2$

$\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$  represents a  $T_B$  op. wrt  $\{h_{a_1}, h_{a_2}\}$



$$\overline{B'(a_1)} r_{1,2} = \overline{B'(a_2)} r_{2,1}$$

Proof (Same assembly required)

$$L^2 = H^2 + \overline{H^2}$$

$$= (K_B + B H^2) + (\overline{K_B} + \overline{B H^2})$$

So if  $\varphi \in L^2$ , then

$$\varphi = \overline{\psi_1} + \psi_2 + \overline{\eta_1} + \eta_2$$

$$\psi_1, \psi_2 \in K_B$$

$$\eta_1, \eta_2 \in B H^2$$

and so

$$A\varphi = A\overline{\psi_1} + \psi_2 + \underbrace{A\overline{\eta_1} + \eta_2}_{\equiv 0 \text{ (from last time)}}$$

Thus

$$T_B = \{ A \cdot \bar{\psi}_1 + \psi_2 : \psi_1, \psi_2 \in K_B \}$$

$$\begin{aligned}\psi_1 &= \sum c_j \left( \frac{B}{z-a_j} \right) \\ \psi_2 &= \sum d_j \left( \frac{B}{z-a_j} \right)\end{aligned}$$

↖ ↙  
(Basis  
elements  
for  $K_B$ )

We can take  $\varphi$  to be

$$\varphi = \sum c_j \left( \frac{B}{z-a_j} \right) + \sum d_j \left( \frac{B}{z-a_j} \right)$$

(some constants  $c_j, d_j, 1 \leq j \leq N$ )



$$(\text{Sarasen}) \quad k_{a_j} \otimes \widetilde{k}_{a_j} = A \left( \frac{\overline{B}}{z - a_j} \right)$$

$$\widetilde{k}_{a_j}(z) = \frac{B(z)}{z - a_j}$$

See this from  
Cauchy Residue  
Theorem  
Apply to  $k_{a_j}$

$$A \overline{\psi_1} = \sum_{j=1}^n c_j k_{a_j} \otimes \widetilde{k}_{a_j}$$

$$A \psi_2 = \sum_{j=1}^n d_j \widetilde{k}_{a_j} \otimes k_{a_j}$$

So  $T_B$  consists of

$$\sum_j c_j k_{a_j} \otimes \widetilde{k}_{a_j} + \sum_j d_j \widetilde{k}_{a_j} \otimes k_{a_j}$$

$$c_j, d_j \in \mathbb{C}$$

Useful inner product formulas

$$1) \langle ka_i, ka_j \rangle = \frac{1}{1 - \bar{a}_i a_j}$$

$$2) \langle \tilde{ka}_i, ka_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ B'(a_j) & i = j \end{cases}$$

$$3) \langle \tilde{ka}_i, \tilde{ka}_j \rangle = \frac{1}{1 - \bar{a}_j a_i}$$

$$4) \tilde{ka}_j = \sum_{s=1}^N \frac{1}{B'(a_s)} \frac{1}{1 - \bar{a}_s a_j} ka_s$$

use (2) + (3) to get this

Use these to compute

$$A_p \quad || \quad ka_p \quad (p = 1, 2, \dots, N)$$

$$x_{1,p} ka_1 + x_{2,p} ka_2 + \dots + x_{N,p} ka_N$$

Note  $\therefore A_p = \sum_j c_j ka_j \otimes \tilde{ka}_j$   
 $+ \sum_j d_j \tilde{ka}_j \otimes ka_j$

Use the above "orthogonality" relations to compute.

After gathering up terms, we get

$$(M_{A_p})_{s,p} = \overline{c_p B'(a_p)} \delta_{s,p} + \frac{1}{\overline{B'(a_s)}} \sum_{j=1}^N \frac{d_j}{(1 - \overline{a_s} a_j)(1 - \overline{a_p} a_j)}$$

From here the conditions are "easy" to check — Lots of partial fractions.

Thus  $M_{A_p}$  has entries satisfying (+)

The other direction is basic linear algebra + Sarason's fact that  $\dim(T_B) = 2N - 1$ .

Indeed, let  $V$  be the vector space of matrices satisfying (+). One can see that  $\dim(V) = 2N - 1$ .

Notice that

$$M_{\tilde{T}_B} \in V \text{ by (+)}$$

$$\xrightarrow{\text{matrix rep of } T_B \text{ w.r.t. } \{h_{a_1}, \dots, h_{a_n}\}}$$

and  $\dim(M_{T_B}) = 2N - 1$  (Sarason).

Thus  $M_{T_B} = V$   $\square$