SIMPLICITY OF C^*-ALGEBRAS USING UNIQUE EIGENSTATES

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Abstract. We consider a one-parameter family of operators constructed from a pair of isometries on Hilbert space with orthogonal ranges. For special values of the parameter, the operator plays a role in the representation theory of free groups and in free probability theory. For each parameter value, we identify the irreducible $*$-representations of the pair of isometries in which the operator has an eigenvalue. This yields a new technique for showing that certain C^*-algebras, including the C^*-algebra generated by the operator, are simple. We establish several other fundamental properties of this C^*-algebra and its generator.

1. Summary

The main result of this paper is Theorem 5.9, which establishes the simplicity of certain subalgebras of the Cuntz algebra $\mathcal{O}_2$. More precisely, let T_1 and T_2 be isometries of a Hilbert space into itself with orthogonal ranges, let their range projections be Q_1 and Q_2, and let $Q_0 = I - Q_1 - Q_2$ be the defect projection. For $q > 0$, consider the operator

$$X_q = T_1(Q_0 + Q_1 + qQ_2) + (Q_0 + qQ_1 + Q_2)T_2^* ,$$

so that X_q belongs to the Toeplitz extension $\mathcal{T}_2$ of $\mathcal{O}_2$. Of course, we actually have many unitarily inequivalent operators X_q, but the C^*-algebra generated by this operator depends only on q, regardless of whether or not Q_0 vanishes, as will emerge later. In particular, we may regard $C^*(X_q)$ as a subalgebra of the Cuntz algebra $\mathcal{O}_2$. Theorem 5.9 states that any C^*-subalgebra of $\mathcal{O}_2$ containing $C^*(X_q)$ and invariant under the automorphisms of $\mathcal{O}_2$ that switch the generating isometries or rotate them by unimodular scalar multiplication through given angles in opposite senses must be simple.

In Section 3, we uncover our original motivation for the definition of X_q by showing that, for integer $n \geq 2$, the operator $\sqrt{n}X_{\sqrt{(n-1)/n}}$ acting in a natural way on the Fock space of a two-dimensional Hilbert space is unitarily equivalent to the restriction of the sum of the n generators of the free group $\mathbb{F}_n$ to an invariant subspace of $\ell^2(\mathbb{F}_n)$. With some work, this connection yields a number of pleasant properties of $C^*(X_q)$ for general q, as the representation on Fock space

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makes sense for all $q > 0$. For example, we extend from the "group case" values of q to general q by a vanishing polynomial argument to show that the state of $C^*(X_q)$ corresponding to the vacuum vector is a faithful trace. Further calculation in the Fock space representation yields the norm and spectrum of the operators X_q, $X_q + X_q^*$, and $X_q^* X_q$, as well as the existence of non-trivial projections in $C^*(X_q)$ for some, but not all, q. In Section 4, the foundation is laid for the proof of our main result by the study of the states on $\mathcal{T}_2$ that kill $(X_q - \lambda)^*(X_q - \lambda)$ for values of λ in the spectrum of X_q. We explicitly compute all such states and show that they are uniquely determined by λ. Further, in the GNS representation of $\mathcal{T}_2$ corresponding to such a state, we exhibit an at most two-dimensional cyclic subspace for $C^*(X_q)$. Section 5 then shows that $C^*(X_q)$ escapes unscathed when taking the quotient by the ideal of compact operators, and provides the main simplicity result. Section 6 examines the two special cases where q is either one or zero. We end by showing that $C^*(X_q)$ is not nuclear and that the irreducible representations of $\mathcal{O}_2$ that arise in our investigation decompose into two inequivalent irreducible representations when restricted to $C^*(X_q)$, for λ in the interior of the spectrum of X_q, in Section 7.

2. Introduction

Let x be an element of a unital C^*-algebra A. For a complex number λ, a λ-eigenstate of A for x is a state of A with $x - \lambda$ in its left kernel – that is, a state that annihilates $(x-\lambda)^*(x-\lambda)$, and hence $A(x-\lambda)$. This definition is equivalent to the cyclic vector for the state in its GNS representation being a λ-eigenvector for x in that representation.

Let $\mathbb{F}_n$ be the free group on $n \geq 2$ generators $u_1, u_2, \ldots, u_n$. The complex group algebra $\mathbb{C}\mathbb{F}_n$ is the set of finite linear combinations of group elements. The norm closure of $\mathbb{C}\mathbb{F}_n$ in the left regular representation on $\ell^2(\mathbb{F}_n)$ is called the reduced group C^*-algebra, $C_r^*(\mathbb{F}_n)$. Paschke [17] has conjectured that there exist only finitely many pure zero-eigenstates of $C_r^*(\mathbb{F}_n)$ for each non-zero element of $\mathbb{C}\mathbb{F}_n$. While the conjecture is still open in general, the existence of a unique zero-eigenstate has been shown for a linear combination of the generators minus either its inner or outer spectral radius in a paper by Paschke [18], and for a self-adjoint linear combination of the generators and their inverses plus or minus its norm by Kuhn and Steger [15].

The story of this paper actually begins in an algebra seemingly remote from those just presented, where we find an element that acts just as the sum of the generators does in $C_r^*(\mathbb{F}_n)$. The algebra generated by n isometries $T_1, \ldots, T_n$ that satisfy

$$\sum_{i=1}^{n} T_i T_i^* < I$$

is called the Toeplitz algebra, $\mathcal{T}_n$. Cuntz [8] showed that the defect projection

$$P = 1 - \sum_{i=1}^{n} T_i T_i^*$$

generates the only ideal, which is isomorphic to the C*-algebra $\mathcal{K}$ of compact operators on a separable Hilbert space. The images $S_1, \ldots, S_n$ of the original isometries $T_1, \ldots, T_n$ in the quotient by $\mathcal{K}$ satisfy

$$\sum_{i=1}^{n} S_i S_i^* = I,$$

and generate the Cuntz algebra $\mathcal{O}_n$.

If $\mathcal{H}$ is an n-dimensional Hilbert space with orthonormal basis $e_1, \ldots, e_n$, the full Fock space of $\mathcal{H}$ is the Hilbert space

$$\mathbb{C}\Omega \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus \ldots .$$

The operator l_i that tensors on the left with e_i is called a left-creation operator, and $l_1, \ldots, l_n$ generate a representation of $\mathcal{T}_n$ (see Voiculescu, Dykema, and Nica [22] for an introduction).

Suppose that $\mathcal{H}$ is two-dimensional, and let l_1 and l_2 be the creation operator isometries generating $\mathcal{T}_2$ in the representation just mentioned. For any real number $q > 0$, define

$$X_q = l_1 \left(1 + (q-1) l_2 l_2^*\right) + \left((q-1) l_1 l_1^* + 1\right) l_2^*.$$

As it turns out (and is proved later), the action of $\sqrt{n} X_q$ on Fock space is exactly that of the sum of the free group generators in $C_r^*(\mathbb{F}_n)$ when $q = \sqrt{(n-1)/n}$ and $n \geq 2$.

In addition to the work towards the conjecture mentioned above, the search for eigenstates of particular elements of free group complex group algebras has also paid dividends in harmonic analysis on free groups in papers by Figà-Talamanca and Picardello [10] and Figà-Talamanca and Steger [11]. In this paper, we hope to present compelling evidence that looking for eigenstates can be a rewarding occupation, and not just in the free group setting. Specifically, we are able to compute all λ-eigenstates of $\mathcal{T}_2$ for X_q, show they are unique for each λ, and use information gained about the GNS representations of $\mathcal{T}_2$ corresponding to the λ-eigenstates during the computation to show the simplicity of a class of algebras containing $C^*(X_q)$. This suggests a new approach to identifying simple C*-algebras. Along the way, we uncover several other facts about the operator X_q and the algebra $C^*(X_q)$.

3. A Representation on Fock Space

For most of this paper, we shall be interested in the C*-algebra generated by X_q, $C^*(X_q)$. We begin by establishing that it is unital. Denote by $\ker(X)$ the kernel of an operator.

Lemma 3.1. *Let ρ be a representation of $\mathcal{T}_2$ on a Hilbert space $\mathcal{H}$. For every $q > 0$,*

$$\ker(\rho(X_q)) \cap \ker(\rho(X_q^*)) = (0) .$$

Proof. Let $\mathcal{T}_2$ in the given representation be generated by T_1 and T_2 with range projections Q_1 and Q_2 and defect projection $Q_0 = 1 - Q_1 - Q_2$. Suppose $\xi \in \mathcal{H}$ satisfies $\rho(X_q)\xi = 0$. Write $\xi_j = Q_j\xi$, so that $\xi = \xi_0 + \xi_1 + \xi_2$ and

$$\rho(X_q)\xi = T_1(\xi_0 + \xi_1 + q\xi_2) + (Q_0 + qQ_1 + Q_2)T_2^*\xi_2 = 0 .$$

Apply Q_1 to see that $T_1(\xi_0 + \xi_1 + q\xi_2) = -qQ_1T_2^*\xi_2$. Since T_1 is an isometry, the square of the norm on the left is $\|\xi_0\|^2 + \|\xi_1\|^2 + q^2\|\xi_2\|^2$. The square of the norm on the right, however, is at most $q^2\|\xi_2\|^2$. It follows that $\xi = \xi_2$, so ξ must lie in the range of T_2. If also $\rho(X_q^*)\xi = 0$, a similar calculation shows that ξ must lie in the range of T_1, forcing ξ to be the zero vector. □

Proposition 3.2. *The operator $X_qX_q^* + X_q^*X_q$ is invertible in $\mathcal{T}_2$ for all $q > 0$.*

Proof. Fix $q > 0$, and suppose that $X_qX_q^* + X_q^*X_q$ is not invertible in $\mathcal{T}_2$. Then there exists a state f of $\mathcal{T}_2$ such that $f(X_qX_q^* + X_q^*X_q) = 0$. Since $X_qX_q^*$ and $X_q^*X_q$ are both positive elements, $f(X_qX_q^*) = f(X_q^*X_q) = 0$. Thus f is a zero-eigenstate of $\mathcal{T}_2$ for both X_q and X_q^*. Let $(\rho, \mathcal{H}, \xi)$ be the GNS representation corresponding to f. Then $\rho(X_q)\xi = \rho(X_q^*)\xi = 0$, contradicting Lemma 3.1. □

Corollary 3.3. *The algebra $C^*(X_q)$ is unital for all $q > 0$.*

3.1. **Free Group Connections.** Let $\mathbb{S}$ be the free semigroup on two generators, with identity element 0, and elements written as finite strings from $\{+, -\}$. The Hilbert space $\ell^2(\mathbb{S})$ is spanned by the point-mass functions Δ_s, where $s \in \mathbb{S}$, and identifies with the full Fock space of a two-dimensional Hilbert space. The left-creation operators on the latter correspond to the operators $l(+)$ and $l(-)$ on $\ell^2(\mathbb{S})$ where $l(+)\Delta_s = \Delta_{+s}$ and $l(-)\Delta_s = \Delta_{-s}$.

Let δ_g represent the point-mass function in $\ell^2(\mathbb{F}_n)$ corresponding to the element $g \in \mathbb{F}_n$. As before, $\mathrm{C}^*_r(\mathbb{F}_n)$ is the norm closure of $\mathbb{C}\mathbb{F}_n$ in the left regular representation on $\ell^2(\mathbb{F}_n)$. We will also be interested in the von Neumann algebra of the left regular representation, $\mathrm{W}^*(\mathbb{F}_n)$.

Proposition 3.4. *For $q = \sqrt{(n-1)/n}$ and $n \geq 2$, $C^*(X_q)$ is isomorphic to the subalgebra of $C^*_r(\mathbb{F}_n)$ generated by $X = u_1 + \cdots + u_n$, where $u_1, \ldots, u_n$ are the generators of $\mathbb{F}_n$. The isomorphism sends X_q to $X/\sqrt{n}$.*

Proof. Recall that the element δ_e, where e is the identity element of $\mathbb{F}_n$, is a cyclic and separating trace vector for $\mathrm{W}^*(\mathbb{F}_n)$. Further, the trace given by δ_e is faithful (see, for example, Davidson [9]). Thus the restriction of $\mathrm{C}^*(X)$ to the Hilbert space $\overline{\mathrm{C}^*(X)\delta_e}$ is isomorphic to $\mathrm{C}^*(X)$. Define the sign of a generator of $\mathbb{F}_n$ to be positive, and the sign of the inverse of a generator to be negative. Extend this to any reduced word $g = g_1 \ldots g_n$ in the generators and their inverses as $\mathrm{sign}(g) = \mathrm{sign}(g_1) \ldots \mathrm{sign}(g_n)$. Let G_s be the set of group elements whose sign is s, and define the vector Δ_s in $\ell^2(\mathbb{F}_n)$ as

$$\Delta_s = \frac{\sum_{a \in G_s} \delta_a}{\left\| \sum_{a \in G_s} \delta_a \right\|} ,$$

with the convention that 0 is the empty string and $\Delta_0 = \delta_e$. Thus $\overline{\mathrm{C}^*(X)\delta_e}$ naturally identifies with $\ell^2(\mathbb{S})$. Let $\overline{\mathrm{X}} = X/\sqrt{n}$, and consider how $\overline{\mathrm{X}}$ acts on a Δ_s. For strings s and s' in $\mathbb{S}$, one may verify that

$$\overline{\mathrm{X}}\Delta_s = \begin{cases} \Delta_+ , & s = 0 ; \\ \Delta_{+s} , & s = +s' ; \\ \sqrt{\frac{n-1}{n}}\Delta_{+s} + \Delta_0 , & s = - ; \\ \sqrt{\frac{n-1}{n}}\Delta_{+s} + \Delta_{-s'} , & s = - - s' ; \\ \sqrt{\frac{n-1}{n}}\left(\Delta_{+s} + \Delta_{+s'}\right) , & s = - + s' , \end{cases} \tag{1}$$

which gives the same action as X_q on $\ell^2(\mathbb{S})$, and the desired result. Here are the details for the case $s = --s'$. The other cases are similar. For ease of notation, let

$$\sigma_s = \sum_{a \in G_s} \delta_a \ .$$

Then

$$\|\sigma_s\| = \sqrt{|G_s|}$$
$$\|\sigma_{\pm\pm s}\| = \sqrt{n}\,\|\sigma_{\pm s}\|$$

and

$$\|\sigma_{\pm\mp s}\| = \sqrt{(n-1)}\,\|\sigma_{\mp s}\| \ .$$

Using these equations,

$$\begin{aligned}
\overline{X}\Delta_{--s} &= \frac{u_1 + \cdots + u_n}{\sqrt{n}} \left(\frac{\sigma_{--s}}{\|\sigma_{--s}\|} \right) \\
&= \frac{\sum_{a \in G_{--s}} (\delta_{u_1 a} + \cdots + \delta_{u_n a})}{\sqrt{n}\,\|\sigma_{--s}\|} \\
&= \frac{1}{\sqrt{n}\,\|\sigma_{--s}\|} \sum_{\substack{i \neq j \\ b \in G_{-s}}} \delta_{u_i u_j^{-1} b} + \frac{1}{\sqrt{n}\,\|\sigma_{--s}\|} \sum_{\substack{i = j \\ b \in G_{-s}}} \delta_{u_i u_j^{-1} b} \\
&= \frac{\sigma_{+--s}}{\sqrt{n}\,\|\sigma_{--s}\|} + \frac{n\sigma_{-s}}{\sqrt{n}\,\|\sigma_{--s}\|} \\
&= \sqrt{\frac{n-1}{n}}\,(\Delta_{+--s}) + \Delta_{-s} \ .
\end{aligned}$$

□

The abelian von Neumann subalgebra of $\mathrm{W}^*(\mathbb{F}_n)$ generated by the sum of the generators and their inverses is called the radial algebra and was introduced and studied in papers by Cohen and Pytlik [6, 20]. From Proposition 3.4, $\mathrm{W}^*(X_q)$ contains the radial algebra of $\mathrm{W}^*(\mathbb{F}_n)$ when $q = \sqrt{(n-1)/n}$ and $n \geq 2$.

3.2. **A Trace.** From Proposition 3.4, $\mathrm{C}^*(X_q)$ comes equipped with a trace when $q = \sqrt{(n-1)/n}$, $n \geq 2$, given by the vector Δ_0. We would like to show that Δ_0 is a trace vector for $\mathrm{C}^*(X_q)$ for all values of q.

Lemma 3.5. *Let s be a string in $\{+,-\}$. For $q>0$, in the representation of $\mathcal{T}_2$ on $\ell^2(\mathbb{S})$, each Δ_s, $s\in\mathbb{S}$, can be written as $\Delta_s = Y_s\Delta_0$ for some Y_s in the unital algebra generated by X_q and X_q^*.*

Proof. Induct on the length, l, of s. For $l=1$, $\Delta_+ = X_q\Delta_0$ and $\Delta_- = X_q^*\Delta_0$. Given a string of length k, solving the appropriate equation from Equation 1, or the similar result for X_q^*, for the vector with highest string length and invoking the induction hypothesis completes the proof. □

Proposition 3.6. *The vector Δ_0 is a cyclic and separating trace vector for the von Neumann algebra $W^*(X_q)$ generated by X_q in the representation of $\mathcal{T}_2$ on $\ell^2(\mathbb{S})$ for all $q>0$.*

Proof. Using Proposition 3.4, $\tau(\cdot) = \langle\cdot\Delta_0,\Delta_0\rangle$ is a trace on $\mathrm{W}^*(X_q)$ when $q=\sqrt{(n-1)/n}$. To see that τ is a trace for general q, it is enough to show that $\tau(yz)=\tau(zy)$ when y and z are polynomials in X_q and X_q^*. Notice that τ applied to any polynomial in X_q and X_q^* yields a polynomial in q. Since τ is trace on $\mathrm{W}^*(\mathbb{F}_n)$ for $q=\sqrt{(n-1)/n}$, $n\geq 2$, $\tau(yz)-\tau(zy)$ is a polynomial in q with infinitely many zeros, meaning it is identically zero.

Lemma 3.5 shows that Δ_0 is cyclic for $\mathrm{W}^*(X_q)$, and a cyclic trace vector for a von Neumann algebra is always separating, since, for a, b, c in $\mathrm{W}^*(X_q)$, the trace property gives $\langle ab\Delta_0, c\Delta_0\rangle = \langle bc^*a\Delta_0,\Delta_0\rangle$, and so if $a\Delta_0=0$, cyclicity forces a to be the zero operator. □

When a von Neumann algebra $\mathcal{R}$ acts on a Hilbert space $\mathcal{H}$ with cyclic (and hence separating) trace vector ξ, the conjugate linear operator J defined (initially) on the dense subspace $\mathcal{R}\xi$ by $Ja\xi = a^*\xi$ is isometric and hence extends to a conjugate linear isometry of period two, called the modular conjugation operator for the pair $(\mathcal{R},\xi)$, taking $\mathcal{H}$ onto itself. The crucial fact in this situation (see, for instance, Jones and Sunder [13, Theorem 1.24]) is that $J\mathcal{R}J=\mathcal{R}'$.

We now identify the modular conjugation operator for $(\mathrm{W}^*(X_q),\Delta_0)$ on Fock space, and show that the resulting $*$-anti-isomorphism of $\mathrm{W}^*(X_q)$ and its commutant amounts to replacing left action by right action.

Proposition 3.7. *Let J be the modular conjugation operator defined for $(\mathrm{W}^*(X_q),\Delta_0)$. Then J acts on $\ell^2(\mathbb{S})$ by sending Δ_s to Δ_{s^*}, where s^* is the string in $\{+,-\}$ obtained from s by swapping $+$ and $-$ and reversing the order.*

Proof. Consider the modular conjugation operator $\hat{J}$ for $(\mathrm{W}^*(\mathbb{F}_n),\delta_0)$ on $\ell^2(\mathbb{F}_n)$. There, we have

$$\hat{J}\delta_g = \hat{J}L_g\delta_0 = (L_g)^*\delta_0 = L_{g^{-1}}\delta_0 = \delta_{g^{-1}}\ .$$

Using Proposition 3.4 to identify $\mathrm{W}^*(X_q)$ for $q = \sqrt{(n-1)/n}$ with a subgroup of $\mathrm{W}^*(\mathbb{F}_n)$, the modular conjugation operator J of $(\mathrm{W}^*(X_q), \Delta_0)$ is merely the restriction of $\hat{J}$ to the completion of $\mathrm{W}^*(X_q)\delta_0$, and so $J\Delta_s = \Delta_{s^*}$ for $\mathrm{W}^*(X_q)$ when $q = \sqrt{(n-1)/n}$.

We now establish the statement for all $q > 0$. From Lemma 3.5, we may find a unique element of the unital algebra generated by X_q and X_q^*, call it Y_s, such that $\Delta_s = Y_s\Delta_0$. By the definition of J, it is enough to show that $(Y_s)^* = Y_{s^*}$. Y_s can be written as a linear combination of monomials in X_q and X_q^* with coefficients that are Laurent polynomials in q. Then

$$\tau\left(\left(Y_{s^*} - (Y_s)^*\right)^* \left(Y_{s^*} - (Y_s)^*\right)\right)$$

is a Laurent polynomial in q. As noted above, this polynomial is zero for infinitely many values of q (the free group cases), and so must be zero for all values of q. □

Corollary 3.8. *The operator that acts on $\ell^2(\mathbb{S})$ on the right in the same way that X_q acts on the left generates the commutant $W^*(X_q)'$ of $W^*(X_q)$.*

3.3. **Projections and Spectra.** Because $\mathrm{C}_r^*(\mathbb{F}_n)$ contains no non-trivial projections (see, for example, Davidson [9]), we have from Proposition 3.4 that $\mathrm{C}^*(X_q)$ also contains no non-trivial projections for $q = \sqrt{(n-1)/n}$. We will now construct a projection in $\mathrm{C}^*(X_q)$ for $q < 1/\sqrt{2}$ by considering the spectrum of the element $X_q^*X_q$. We do not know whether $\mathrm{C}^*(X_q)$ contains non-trivial projections for other values of q. Denote by $\mathrm{Sp}(X)$ the spectrum of the operator X.

Theorem 3.9. *For $q \geq 1/\sqrt{2}$, $\|X_q\| = 2q$ and $\mathrm{Sp}(X_q^*X_q) = [0, 4q^2]$.*

For $0 < q < 1/\sqrt{2}$,

$$\mathrm{Sp}(X_q^*X_q) = [0, 4q^2] \cup \left\{\frac{1}{1-q^2}\right\}$$

and

$$\|X_q\| = \frac{1}{\sqrt{1-q^2}} \; .$$

Proof. Consider $\mathrm{C}^*(X_q)$ in the representation of $\mathcal{T}_2$ on $\ell^2(\mathbb{S})$ as previously described. Let $S = l(-)l(+)$ and let $\mathcal{K}$ be the Hilbert space $\overline{\mathrm{C}^*(S)\Delta_0}$. Then $\mathcal{K} \simeq l_2(\mathbb{N})$, and S acts as the unilateral shift on $\mathcal{K}$. We may restrict our attention to $\mathcal{K}$ because Δ_0 is a cyclic vector for $X_q^*X_q$ in this representation of $\mathcal{T}_2$ and $\mathcal{K}$ is invariant under $X_q^*X_q$. More precisely, $X_q^*X_q$ acts as $q^2(S^* + 2I + S)$ on the majority of $\mathcal{K}$ (that is, except for the subspace spanned by Δ_{-+} and Δ_0). Thus $X_q^*X_q$ is a perturbation of $q^2(S^* + 2I + S)$ by a compact operator.

The operator $S^* + S$ is known to have spectrum $[-2, 2]$, so that the spectrum of $X_q^* X_q$ consists of the interval $[0, 4q^2]$ and perhaps some isolated eigenvalues (see, for example, Conway [7, Theorem 4.6]). For convenience, let $L = X_q^* X_q / q^2 - 2I$, which has the form

$$\begin{bmatrix} \frac{1}{q^2} - 2 & \frac{1}{q} & 0 & 0 & 0 & \dots \\ \frac{1}{q} & 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & 0 & 1 & \ddots \\ 0 & 0 & 0 & 1 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix},$$

relative to the standard basis for $\ell^2(\mathbb{N})$. Suppose $\lambda > 2$ is an eigenvalue of L with eigenvector $\vec{v} = \langle v_i \rangle_{i=0}^{\infty}$. The equation $L\vec{v} = \lambda\vec{v}$ yields the following simultaneous equations for the components of $\vec{v}$:

$$\begin{cases} \left(1/q^2 - 2\right) v_0 + (1/q) v_1 = \lambda v_0 \\ (1/q) v_0 + v_2 = \lambda v_1 \\ v_{n-1} + v_{n+1} = \lambda v_n \quad \text{for } n \geq 2. \end{cases}$$

Except for the first two equations, this is a recurrence relation. Because $\lambda > 2$, the two-by-two matrix for the linear recurrence has two real eigenvalues, the first greater than one, and the other, namely $r = (\lambda - \sqrt{\lambda^2 - 4})/2$, in the open unit interval. Because $\vec{v} \in \ell^2(\mathbb{N})$, the only eigenvalue that contributes to $\vec{v}$ is r; that is, we must have $v_n = v_1 r^{n-1}$ for $n \geq 1$. In particular, $v_2 = r v_1$. This and the equations above specifying λv_0 and λv_1, along with $1 = r(\lambda - r)$ readily yield $v_1 = (r/q) v_0$ and $\lambda = q^{-2} - 2 + r q^{-2}$ (else $\vec{v}$ vanishes). With $\mu = \lambda + 2$, this makes $\mu - \sqrt{\mu(\mu - 4)} = 2q^2\mu$, and hence $\mu = q^{-2}(1 - q^2)^{-2}$. Thus

$$\lambda = \frac{1}{q^2\left(1 - q^2\right)} - 2$$

for q less than $1/\sqrt{2}$, and if $q \geq 1/\sqrt{2}$, no such λ exists. $\square$

Corollary 3.10. *The algebras $C^*(X_q)$ are not all isomorphic.*

Proof. For q between zero and $1/\sqrt{2}$, by Theorem 3.9, the spectrum of $X_q^* X_q$ is not connected so that $\mathrm{C}^*(X_q)$ contains a non-trivial (spectral) projection. On the other hand, for $q = \sqrt{(n-1)/n}$, $n \geq 2$, $\mathrm{C}^*(X_q)$ is projectionless. $\square$

We also analyze $X_q + X_q^*$:

Proposition 3.11. *For* $0 < q < 1$,

$$\mathrm{Sp}\left(X_q + X_q^*\right) = \left[-2\sqrt{1+q^2}, 2\sqrt{1+q^2}\right]$$

and

$$\left\|X_q + X_q^*\right\| = 2\sqrt{1+q^2}\ .$$

Proof. Leaving the calculations to the reader, the subspace $\overline{(X_q + X_q^*)\Delta_0}$ is spanned by the orthonormal vectors $\{w_n\}$, where $w_0 = \Delta_0$, $w_1 = (\Delta_+ + \Delta_-)/\sqrt{2}$, and

$$w_k = \frac{\sum_{|s|=k} q^{\mu(S)}\Delta_s}{\sqrt{2(1+q)^{k-1}}}$$

for $k \geq 2$, where $\mu(s_1 \dots s_k)$ is the number of j for which $s_j \neq s_{j+1}$. $X_q + X_q^*$ acts on the w_k by

$$\left(X_q + X_q^*\right) w_0 = \sqrt{2}w_1\ ,$$
$$\left(X_q + X_q^*\right) w_1 = \sqrt{2}w_0 + \sqrt{1+q^2}w_2\ ,$$

and

$$\left(X_q + X_q^*\right) w_k = \sqrt{1+q^2}\left(w_{k-1} + w_{k+1}\right)\ ,$$

for $k \geq 2$.

Thus, with respect to this basis,

$$\frac{X_q + X_q^*}{\sqrt{1+q^2}} = \begin{bmatrix} 0 & \sqrt{\frac{2}{1+q^2}} & 0 & 0 & 0 & \dots \\ \sqrt{\frac{2}{1+q^2}} & 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & 0 & 1 & \ddots \\ 0 & 0 & 0 & 1 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

If λ is eigenvalue for this operator with $\lambda > 2$, the entries v_0, v_1, and v_2 of an ℓ^2-eigenvector must satisfy $sv_1 = \lambda v_0,\ sv_0 + v_2 = \lambda v_1$ (where $s = \sqrt{2/(1+q^2)}$), and $v_2 = rv_1$ (where r is as in the proof of Theorem 3.9). These three equations make $\lambda = rs^2$, so $\lambda < 2r < \lambda$, a contradiction. We likewise rule out $\lambda < -2$ using the recurrence matrix eigenvalue $(\lambda + \sqrt{\lambda^2 - 4})/2$ in place of r. ☐

Finally, we are able to determine the spectrum of X_q.

Proposition 3.12. *The spectrum of X_q is the closed unit disc for all $q > 0$.*

Proof. Paschke [18] has shown that the spectrum of the normalized sum of the generators in $C^*_r(F_n)$ is the unit disc. Since

$$(P_0 + P_1 + qP_2)\, X_q \left(P_0 + P_1 + \frac{1}{q} P_2 \right) = T_1 + T_2^* \ ,$$

the X_q for $q > 0$ are all similar. The result then follows from Proposition 3.4. ☐

4. Eigenstates

Fix $q > 0$. The goal of this section is to gather what information we can about the λ-eigenstates of $\mathcal{T}_2$ for X_q. If the unital C*-algebra A has a faithful trace, then for x in A, there is a λ-eigenstate of A for x for every λ in the spectrum of x. This is because one-sided inverses in A are automatically two-sided inverses. Conversely, of course, the existence of a λ-eigenstate puts λ in the spectrum. Because of Proposition 3.12, we may assume that $|\lambda| \leq 1$. After some preliminaries, we are forced to treat the cases $|\lambda| = 1$ and $|\lambda| < 1$ separately.

First, we establish some notation. Let f be a λ-eigenstate of $\mathcal{T}_2$ for X_q and let $(\rho, \mathcal{H}, \xi)$ be the GNS representation corresponding to f. Let T_1 and T_2 be the partial isometries generating $\mathcal{T}_2$, let $P_i = T_i T_i^*$ for $i = 1$ and 2, and let $P_0 = 1 - T_1 T_1^* - T_2 T_2^*$ be the defect projection. Let $V_i = \rho(T_i)$ and $E_i = \rho(P_i)$ for $i = 0, 1, 2$.

Using the Wold decomposition of $\mathcal{H}$ with respect to V_2, we may write

$$\mathcal{H} = \mathcal{H}_{-1} \oplus \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \dots$$

where V_2 acts unitarily on $\mathcal{H}_{-1}$, $\mathcal{H}_0 = \ker(V_2^*)$, and $\mathcal{H}_{i+1} = V_2 \mathcal{H}_i$ for $i \geq 0$. Note that $\mathcal{H}_0 = (E_1 + E_0)\mathcal{H}$ so that $V_1 \mathcal{H} \subseteq \mathcal{H}_0$.

Write $\xi = \xi_{-1} + \xi_0 + \xi_1 + \dots$, where $\xi_i \in \mathcal{H}_i$. Then $\rho(X_q)\xi$ can be decomposed as

$$\begin{aligned}\rho(X_q)\xi &= V_1(E_0 + E_1 + qE_2)\xi + (E_0 + qE_1 + E_2)V_2^*\xi \\ &= V_2^*\xi_{-1} + (V_1(E_0 + E_1 + qE_2)\xi + (E_0 + qE_1)\, V_2^*\xi_1) + V_2^*\xi_2 + V_2^*\xi_3 + \dots\end{aligned}$$

since $V_2^*\xi_0 = 0$, $(E_0 + E_1)V_2^*\xi_i = 0$ for all $i \neq 1$, and the range of V_1 is contained in $\mathcal{H}_0$.

For notational purposes, let $\eta = V_1(E_0 + E_1 + qE_2)\xi \in \mathcal{H}_0$ so that

$$\rho(X_q)\xi = V_2^*\xi_{-1} + (\eta + (E_0 + qE_1)\,V_2^*\xi_1) + V_2^*\xi_2 + V_2^*\xi_3 + \dots \,. \tag{2}$$

By definition, $\rho(X_q)\xi = \lambda\xi$ so that (2) gives the equations

$$\begin{aligned} \lambda\xi_{-1} &= V_2^*\xi_{-1} \\ \lambda\xi_0 &= \eta + (E_0 + qE_1)V_2^*\xi_1 \\ \lambda\xi_i &= V_2^*\xi_{i+1} \end{aligned} \tag{3}$$

for $i \geq 1$.

Lemma 4.1. *In the above discussion, suppose that $|\lambda| < 1$. Then*

- $\xi_1 = 0 = E_0V_2^*\xi_1$*;*
- *setting $\nu = V_2^*\xi_1$, $\mu = V_2(1 - \lambda V_2)^{-1}\xi_1$, and*

$$a = \frac{-q\overline{\lambda}}{1 - |\lambda|^2}\,,$$

we have $\nu \in V_1\mathcal{H}$ (so $\mu \perp \nu$) and $\xi = a\nu + \mu$; and

- *The span of μ and ν is invariant under V_1^* and V_2^*. Indeed,*

$$V_1^*\mu = 0 = V_2^*\nu,$$

$$V_2^*\mu = \nu + \lambda\mu,$$

and

$$V_1^*\nu = \overline{\lambda}\nu - (1 - |\lambda|^2)\mu.$$

Proof. Using the notation established above, because $\lambda\xi_{-1} = V_2^*\xi_{-1}$, and V_2^* is isometric on $\mathcal{H}_{-1}$, we have $\xi_{-1} = 0$. Also, for $i \geq 1$, $\xi_{i+1} = \lambda^i V_2^i\xi_1$ so that

$$\|\xi_{i+1}\| = |\lambda|^i\|\xi_1\|\,. \tag{4}$$

Let $\phi = E_0V_2^*\xi_1$ and $\nu = E_1V_2^*\xi_1$ so that

$$V_2^*\xi_1 = \phi + \nu\,. \tag{5}$$

V_2^* is also isometric on $\mathcal{H}_1$ so that

$$\|V_2^*\xi_1\|^2 = \|\xi_1\|^2 = \|\phi\|^2 + \|\nu\|^2 \,. \tag{6}$$

Now, using (4) in the orthogonal decomposition of ξ, we have

$$\begin{aligned} 1 = \|\xi\|^2 &= \|\xi_0\|^2 + \|\xi_1\|^2 \left(1 + |\lambda|^2 + |\lambda^4| + \dots\right) \\ &= \|\xi_0\|^2 + \frac{1}{1-|\lambda|^2}\|\xi_1\|^2 \,. \end{aligned} \tag{7}$$

From the definition of η, using $\lambda\xi_0 = \eta + (E_0 + qE_1)V_2^*\xi_1$ from (3), and that V_1 acts isometrically on $\mathcal{H}$, we have

$$\begin{aligned} \|\eta\|^2 &= \|V_1(E_0 + E_1 + qE_2)\xi\|^2 \\ &= \|(E_0 + E_1 + qE_2)\xi\|^2 \\ &= \|\xi_0 + q\,(\xi_1 + \xi_2 + \dots)\|^2 \\ &= \|\xi_0\|^2 + \frac{q^2}{1-|\lambda|^2}\|\xi_1\|^2 \,. \end{aligned} \tag{8}$$

Because $\eta \in E_1\mathcal{H}$ and $\phi \in E_0\mathcal{H}$, we have $\langle \phi, \eta \rangle = 0$, and so

$$\begin{aligned} \|\eta\|^2 &= \|\eta + \phi\|^2 - \|\phi\|^2 \\ &= \|\lambda\xi_0 - q\nu\|^2 - \|\phi\|^2 \\ &= |\lambda|^2\|\xi_0\|^2 - 2q\,\mathrm{Re}\,\lambda\langle\xi_0,\nu\rangle + q^2\|\nu\|^2 - \|\phi\|^2 \,. \end{aligned} \tag{9}$$

Together, (6), (8), and (9) give

$$\begin{aligned} 0 &= \left(1-|\lambda|^2\right)\|\xi_0\|^2 + \left(\frac{q^2}{1-|\lambda|^2}\right)\|\xi_1\|^2 + 2q\,\mathrm{Re}\,\lambda\langle\xi_0,\nu\rangle - q^2\|\nu\|^2 + \|\phi\|^2 \\ &= \left(1-|\lambda|^2\right)\|\xi_0\|^2 + \frac{q^2}{1-|\lambda|^2}\left(\|\phi\|^2 + \|\nu\|^2\right) + 2q\,\mathrm{Re}\,\lambda\langle\xi_0,\nu\rangle - q^2\|\nu\|^2 + \|\phi\|^2 \\ &= \left(1-|\lambda|^2\right)\|\xi_0\|^2 + 2q\,\mathrm{Re}\,\lambda\langle\xi_0,\nu\rangle + \frac{q^2|\lambda|^2}{1-|\lambda|^2}\|\nu\|^2 + \left(1 + \frac{q^2}{1-|\lambda|^2}\right)\|\phi\|^2 \\ &= \left\|\sqrt{1-|\lambda|^2}\xi_0 + \frac{q\overline{\lambda}}{\sqrt{1-|\lambda|^2}}\nu\right\|^2 + \left(1 + \frac{q^2}{1-|\lambda|^2}\right)\|\phi\|^2 \,. \end{aligned}$$

Thus

$$\|\phi\| = 0 = \left\|\sqrt{1-|\lambda|^2}\xi_0 + \frac{q\overline{\lambda}}{\sqrt{1-|\lambda|^2}}\nu\right\|$$

so that

$$\phi = E_0 V_2^* \xi_1 = 0$$

$$\nu = E_1 V_2^* \xi_1 = V_2^* \xi_1$$

and $\xi_0 = a\nu$.

From (5), $V_2^*\xi_1 = \nu$. Because $\xi_1 \in V_2\mathcal{H}_0$, it follows that $E_2\xi_1 = \xi_1$, so that we have $\xi_1 = V_2\nu$, $\xi_i = \lambda^{i-1}V_2^i\nu$, and $\xi = a\nu + \mu$ where $\mu = \left(V_2\nu + \lambda V_2^2\nu + \dots\right)$. Also, $\langle \nu, \mu\rangle = 0$, both ν and μ are non-zero, and $V_2^*\mu = \nu + \lambda\mu$.

Consider the action of $\rho(X_q)$ on ξ:

$$\begin{aligned}\lambda(a\nu + \mu) = \lambda\xi = \rho(X_q)\xi = \rho(X_q)(a\nu + \mu) &= V_1(a\nu + q\mu) + (E_0 + qE_1 + E_2)(\nu + \lambda\mu)\\ &= V_1(a\nu + q\mu) + q\nu + \lambda\mu\end{aligned}$$

so that

$$V_1(a\nu + q\mu) + q\nu = \lambda a\nu$$

and

$$a\nu + q\mu = (\lambda a - q)\, V_1^*\nu \ . \tag{10}$$

Since

$$\lambda a - q = \lambda\left(\frac{-q\overline{\lambda}}{1-|\lambda|^2}\right) - q = \frac{\left(-q|\lambda|^2\right) - q\left(1-|\lambda|^2\right)}{1-|\lambda|^2} = \frac{-q}{1-|\lambda|^2}\ ,$$

for $q > 0$, we have that $\lambda a - q \neq 0$ and we may solve (10) for $V_1^*\nu$ to get

$$V_1^*\nu = \frac{a}{\lambda a - q}\nu + \frac{q}{\lambda a - q}\mu\ . \tag{11}$$

With a little more effort, we have

$$\frac{a}{\lambda a - q} = \frac{\frac{-q\overline{\lambda}}{1-|\lambda|^2}}{\frac{-q}{1-|\lambda|^2}} = \overline{\lambda}$$

and

$$\frac{q}{\lambda a - q} = \frac{q}{\frac{-q}{1-|\lambda|^2}} = |\lambda|^2 - 1\ ,$$

so that (11) becomes

$$V_1^*\nu = \overline{\lambda}\nu - (1-|\lambda|^2)\mu\ .$$

It is immediate from the definitions of μ and ν that $V_1^*\mu = 0 = V_2^*\nu$ and that $V_2^*\mu = \nu + \lambda\mu$. Thus $\mathrm{Span}\{V_\alpha^*\xi\} = \mathbb{C}\nu + \mathbb{C}\mu$. $\square$

The subspace spanned by μ and ν will play a major role in what follows, so we dub it $\mathcal{H}_*$. The next two propositions, which come up naturally here, are needed for the simplicity result in the next section rather than the uniqueness result of the present one.

Proposition 4.2. *In the situation of Lemma 4.1, let $\xi' = \nu + q\lambda\mu$. Then ξ' is a $\overline{\lambda}$-eigenvector for $\rho(X_q^*)$. The vectors ξ and ξ' are orthogonal and hence span $\mathcal{H}_*$.*

Proof. Calculate as follows:

$$\begin{aligned}
\rho(X_q^*)(\nu + q\lambda\mu) &= V_2\left(E_0 + qE_1 + E_2\right)(\nu + q\lambda\mu) + \left(E_0 + E_1 + qE_2\right)V_1^*\left(\nu + q\lambda\mu\right) \\
&= V_2\left(q\nu + q\lambda\mu\right) + \left(E_0 + E_1 + qE_2\right)V_1^*\left(\nu\right) \\
&= qV_2\nu + q\lambda V_2\mu + \left(E_0 + E_1 + qE_2\right)\left(\overline{\lambda}\nu + \left(|\lambda|^2 - 1\right)\mu\right) \\
&= qV_2\nu + q\left(\mu - V_2\nu\right) + \left(\overline{\lambda}\nu + q\left(|\lambda|^2 - 1\right)\mu\right) \\
&= \overline{\lambda}\nu + q|\lambda|^2\mu \\
&= \overline{\lambda}\left(\nu + q\lambda\mu\right) .
\end{aligned}$$

Set $\xi' = \nu + q\lambda\mu$. Then

$$\begin{aligned}
\|\nu\|^2 &= \|\xi_1\|^2 , \\
\|\mu\|^2 &= \|\xi_1 + \lambda V_2\xi_1 + \lambda^2 V_2^2\xi_1 + \ldots\|^2 = \frac{1}{1 - |\lambda|^2}\|\xi_1\|^2 ,
\end{aligned} \tag{12}$$

and so

$$\langle \xi, \xi' \rangle = a\|\nu\|^2 + q\overline{\lambda}\|\mu\|^2 = (a - a)\|\xi_1\|^2 = 0 .$$

It is immediate from the definitions of μ and ν that $V_1^*\mu = 0 = V_2^*\nu$, and that $V_2^*\mu = \nu + \lambda\mu$. $\square$

We will show in Section 7 that the subspaces $C^*(\rho(X_q))\xi$ and $C^*(\rho(X_q))\xi'$ are, in fact, orthogonal.

Proposition 4.3. *In the situation of Proposition 4.2, the subspace $\mathcal{H}_*$ is cyclic for $C^*(\rho(X_q))$. That is,*

$$\mathcal{H} = \overline{C^*(\rho(X_q))\mathcal{H}_*} .$$

Proof. Recall that if $\alpha = \alpha_1 \dots \alpha_l$ is a word in $\{1,2\}$, the convention is to write $V_\alpha = V_{\alpha_1} \dots V_{\alpha_l}$. With $\mathcal{H}_* = \text{Span}\{\nu, \mu\} = \text{Span}\{\xi, \xi'\}$, let $\tilde{\mathcal{H}} = \overline{C^*(\rho(X_q))\mathcal{H}_*}$. We wish to show that $\mathcal{H} \subseteq \tilde{\mathcal{H}}$. Recall that in the GNS construction, the vector ξ is cyclic for the GNS representation ρ, so that $\mathcal{H} = \overline{C^*(V_1, V_2)\xi}$. Since we may write $\xi = a\nu + V_2(1 - \lambda V_2)^{-1}\nu$, we have that $\mathcal{H} = \overline{C^*(V_1, V_2)\nu}$, and it suffices to show that the dense subspace of $\mathcal{H}$ spanned by elements of the form $V_\alpha V_\beta^* \nu$ is contained in $\tilde{\mathcal{H}}$. Further, because $\mathcal{H}_*$ is invariant under both V_1^* and V_2^*, we need only show $V_\alpha \mathcal{H}_* \subseteq \tilde{\mathcal{H}}$ for all α.

Proceed by induction on the length of α. For $|\alpha| = 1$, we have

$$V_1\nu = \rho(X_q)\nu$$

$$V_2\mu = \rho(X_q^*)\mu$$

$$V_2\nu = \mu - \lambda V_2\mu = \mu - \lambda\rho(X_q^*)\mu$$

and since

$$\rho(X_q)\mu = \left(V_1\left(E_1 + qE_2\right) + \left(qE_1 + E_2\right)V_2^*\right)\mu$$
$$= qV_1\mu + (qE_1 + E_2)\left(\nu + \lambda V_2\nu + \lambda^2 V_2^2\nu + \dots\right) = qV_1\mu + q\nu + \lambda\mu\ ,$$

we have

$$V_1\mu = q^{-1}\left(\rho(X_q)\mu - q\nu + \lambda\mu\right)\ .$$

Now, assume that $|\alpha| > 1$. If α begins with two ones so that $\alpha = 11\alpha'$, then we may write $V_\alpha = \rho(X_q)V_{1\alpha'}$. By the induction hypothesis, $V_{1\alpha'}\mathcal{H}_* \subseteq \tilde{\mathcal{H}}$, and $\rho(X_q)\tilde{\mathcal{H}} \subseteq \tilde{\mathcal{H}}$ by definition. If $\alpha = 12\alpha'$, then $V_\alpha = q^{-1}\left(\rho(X_q)V_{2\alpha'} - cV_{\alpha'}\right)$, where c is either 1 or q depending on what α' begins with. Again, apply the induction hypothesis. If α begins with a two, use a similar argument replacing $\rho(X_q)$ with $\rho(X_q^*)$. □

Theorem 4.4. *For each $q > 0$ and $|\lambda| < 1$, there exists a unique λ-eigenstate of $\mathcal{T}_2$ for X_q.*

Proof. Retain the notation of Proposition 4.3. Recall that elements of the form $T_\alpha T_\beta^*$, where α and β are words in $\{1,2\}$, span a dense subspace of $\mathcal{T}_2$. For the state f,

$$f\left(T_\alpha T_\beta^*\right) = \langle\rho(T_\alpha T_\beta^*)\xi, \xi\rangle = \langle V_\alpha V_\beta^*\xi, \xi\rangle = \langle V_\beta^*\xi, V_\alpha^*\xi\rangle\ .$$

Since we know how V_1^* and V_2^* act on the span of μ and ν, inner products of the form $\langle V_\beta^* \xi, V_\alpha^* \xi \rangle$ may be calculated from $\|\nu\|$, $\|\mu\|$, and $\langle \nu, \mu \rangle$. From the definitions, $\langle \nu, \mu \rangle = 0$. We now show that $\|\nu\|$ and $\|\mu\|$ depend algebraically on q and λ.

Because $\xi = a\nu + \mu$, we have

$$|a|^2 \|\nu\|^2 + \|\mu\|^2 = 1 \; . \tag{13}$$

Further, $\mu = \left(V_2 \nu + \lambda V_2^2 \nu + \dots\right)$ so that

$$\|\mu\|^2 = (1 + |\lambda| + \dots)\, \|\nu\|^2 = \frac{\|\nu\|^2}{1 - |\lambda|^2} \; . \tag{14}$$

Together, (13) and (14) yield

$$\|\nu\|^2 = \frac{(1 - |\lambda|^2)^2}{q^2|\lambda|^2 + 1 - |\lambda|^2}$$

and

$$\|\mu\|^2 = \frac{1 - |\lambda|^2}{q^2|\lambda|^2 + 1 - |\lambda|^2} \; .$$

□

We now consider the case $|\lambda| = 1$, where an essential change occurs in the behavior of the representations.

Proposition 4.5. *In the discussion at the beginning of this section, let $|\lambda| = 1$. Then $\mathcal{H} = \overline{C^*(\rho(X_q))\xi}$.*

Proof. Since $|\lambda| = 1$, we have $\xi_i = 0$ for all $i \geq 1$, and

$$\lambda \xi_0 = V_1 \left(q \xi_{-1} + \xi_0\right)$$

and

$$V_2^* \xi_{-1} = \lambda \xi_{-1} \; .$$

Thus $\|\xi_0\|^2 = q^2 \|\xi_{-1}\|^2 + \|\xi_0\|^2$, and so $\xi_{-1} = 0$ as well, leaving $\xi = \xi_0$. In particular, we have

$$V_1 \xi_0 = \lambda \xi_0$$

and

$$V_1^* \xi_0 = \overline{\lambda} \xi_0$$

so that

$$\rho(X_q^*)\xi = \overline{\lambda}\xi + qV_2\xi \ .$$

Thus the V_i and their adjoints act on ξ as follows:

$$\begin{aligned} V_1^*\xi &= \overline{\lambda}\xi, & V_2^*\xi &= 0, \\ V_1\xi &= \lambda\xi, & V_2\xi &= q^{-1}(\rho(X_q^*) - \overline{\lambda})\xi, \end{aligned}$$

so that Span$\{\xi\}$ is invariant under the V_1^* and V_2^*. Thus, as in the proof of Proposition 4.3, it suffices to show that $V_\alpha\xi \in \overline{C^*(\rho(X_q))\xi}$ for all strings α of ones and twos and the argument to do so, by induction on the length of α, is exactly the same as that given there. $\square$

Theorem 4.6. *For each q and λ with $q > 0$ and $|\lambda| = 1$, there exists a unique λ-eigenstate of $\mathcal{T}_2$ for X_q.*

Proof. Retaining the notation of Proposition 4.5, for $Z \in \mathcal{T}_2$, $f(Z) = \langle \rho(Z)\xi_0, \xi_0\rangle$. Again, elements of the form $T_\alpha T_\beta^*$ are a dense subset of $\mathcal{T}_2$. Since $\xi = \xi_0$ is in the range of V_1, $V_2^*\xi = 0$. Then, as above,

$$\langle \rho(T_\alpha T_\beta^*)\xi, \xi\rangle = \langle V_\alpha V_\beta^*\xi, \xi\rangle = \langle V_\beta^*\xi, V_\alpha^*\xi\rangle$$

and

$$f(T_\alpha T_\beta^*) = \begin{cases} 0\ , & \text{if “2” occurs in } \alpha \text{ or } \beta; \\ \lambda^{|\alpha|-|\beta|}\ , & \text{otherwise,} \end{cases}$$

uniquely determines f on $\mathcal{T}_2$. $\square$

Proposition 4.7. *For each λ in the spectrum of X_q, the unique λ-eigenstate f of $\mathcal{T}_2$ for X_q is pure, as is the restriction of f to $C^*(X_q)$.*

Proof. Let f be the unique λ-eigenstate of $\mathcal{T}_2$ for X_q, and suppose that f can be written as a non-trivial convex linear combination of two states g and h. By definition, f kills the positive element $(X_q^* - \overline{\lambda})(X_q - \lambda)$, but then so must both g and h. Thus g and h are both λ-eigenstates of $\mathcal{T}_2$ for X_q, and we must have $f = g = h$ by the uniqueness of f.

Let f' be the restriction of f to $C^*(X_q)$. Suppose f' can be written as a non-trivial convex linear combination of states g' and h'. Since f', g', and h' all kill $(X_q^* - \overline{\lambda})(X_q - \lambda)$, by the uniqueness of f extending any one to $\mathcal{T}_2$ must give f. $\square$

We remark that by Proposition 4.7 and Kadison's transitivity theorem [14], we do not need to take closures in the various cyclic subspaces encountered in this section. We have $\mathcal{H} = \rho(\mathcal{T}_2)\xi$ because ρ is an irreducible representation of $\mathcal{T}_2$. The subspaces $\rho(\mathrm{C}^*(X_q))\xi$ and (for $|\lambda| < 1$) $\rho(\mathrm{C}^*(X_q))\xi'$ are also closed, being the GNS spaces of pure states of $\mathrm{C}^*(X_q)$. When $|\lambda| = 1$, we have $\rho(\mathrm{C}^*(X_q))\xi = \mathcal{H}$. By contrast, it is reasonable to surmise that $\mathcal{H}$ is the orthogonal sum of $\rho(\mathrm{C}^*(X_q))\xi$ and $\rho(\mathrm{C}^*(X_q))\xi'$ when $|\lambda| < 1$. We will prove this in Section 7 by establishing that when $|\lambda| < 1$, the representations of $\mathrm{C}^*(X_q)$ on $\mathrm{C}^*(X_q)\xi$ and $\mathrm{C}^*(X_q)\xi'$ are unitarily inequivalent.

5. Simplicity

In the proof of our simplicity result, we will wish to use the simplicity of $\mathcal{O}_2$, and therefore need to begin this section by identifying $\mathrm{C}^*(X_q)$ as a subalgebra of $\mathcal{O}_2$. In what follows, we will frequently make use of the observation that the kernel of the GNS representation ρ of a state f of a C^*-algebra $\mathcal{A}$ is $\{a \in \mathcal{A} \colon f(\mathcal{A}a\mathcal{A}) = 0\}$. In particular, if f annihilates an ideal $\mathcal{J}$ of $\mathcal{A}$, then so does ρ.

5.1. **Modulo the Compacts.** Notice that when we represent $\mathcal{T}_2$ on Fock space, the ideal $\mathcal{K}$ generated by the defect projection P_0 gets mapped to the ideal of compact operators on Fock space. We will use the same customary letter for both ideals.

Proposition 5.1. *Let $q > 0$ and $|\lambda| \leq 1$. The unique λ-eigenstate f of $\mathcal{T}_2$ for X_q kills the ideal $\mathcal{K}$ of $\mathcal{T}_2$ generated by the defect projection.*

Proof. Let V_1 and V_2 generate $\mathcal{T}_2$ in the GNS representation corresponding to f. Let $E_i = V_i V_i^*$ for $i = 1$ and 2, and let $E_0 = 1 - E_1 - E_2$. Finally, let μ and ν be as in Lemma 4.1. Recall that elements of the form $T_\alpha P_0 T_\beta^*$ span a dense subspace of $\mathcal{K}$. If $|\lambda| < 1$, then

$$f\left(T_\alpha P_0 T_\beta^*\right) = \langle V_\alpha E_0 V_\beta^* \xi, \xi \rangle \ .$$

From the definitions, $\xi = a\nu + \mu$ and

$$E_1\nu = \nu, \qquad E_1\mu = 0,$$
$$E_2\nu = 0, \qquad E_2\mu = \mu \ .$$

It follows that

$$E_0\left(\mathbb{C}\nu + \mathbb{C}\mu\right) = \left(1 - E_1 - E_2\right)\left(\mathbb{C}\nu + \mathbb{C}\mu\right) = 0 \ .$$

Finally, since $V_\beta^*\xi \in \mathbb{C}\nu + \mathbb{C}\mu$, $V_\alpha E_0 V_\beta^* \xi = 0$.

If $|\lambda| = 1$, then

$$\begin{aligned} V_\alpha E_0 V_\beta^* \xi_0 &= V_\alpha \left(1 - E_1 - E_2\right) V_\beta^* \xi_0 \\ &= V_\alpha V_\beta^* \xi_0 - V_\alpha E_1 V_\beta^* \xi_0 - V_\alpha E_2 V_\beta^* \xi_0 \\ &= \lambda^{-|\beta|} \left(V_\alpha \xi_0 - V_\alpha E_1 \xi_0 - V_\alpha E_2 \xi_0\right) \ . \end{aligned}$$

Since $E_1\xi_0 = \xi_0$ and $E_2\xi_0 = 0$, $V_\alpha E_0 V_\beta^* \xi = 0$ for any α, β. □

We have shown for the unique λ-eigenstate f of $\mathcal{T}_2$ for X_q, $f(\mathcal{K}) = 0$ where $\mathcal{K}$, the ideal generated by the defect projection P_0, is isomorphic to the compact operators on a separable Hilbert space. Thus these eigenstates are actually eigenstates of $\mathcal{O}_2$ for the image of X_q modulo the compacts.

Proposition 5.2. *$W^*(X_q) \cap \mathcal{K} = 0$.*

Proof. Suppose for the sake of eventual contradiction that $\mathrm{W}^*(X_q) \cap \mathcal{K} \neq 0$. Then there exists a non-zero finite dimensional projection $P \in \mathrm{W}^*(X_q)$. Let J be the modular conjugation operator for $(\mathrm{W}^*(X_q), \Delta_0)$ on Fock space. The projection JPJ, which commutes with $\mathrm{W}^*(X_q)$, must also be finite-dimensional. Let θ be the representation of $\mathrm{W}^*(X_q)$ obtained by restriction to the finite-dimensional invariant subspace $JP\mathcal{H}$. Then $\theta(X_q)$ has an eigenvector, implying a finite-dimensional GNS representation for an eigenstate. In our determination of the unique eigenstates of X_q, however, we found the GNS representations to always be infinite-dimensional. □

Because $\mathrm{W}^*(X_q)$ misses the compacts, so does $\mathrm{C}^*(X_q)$, and the latter is thus isomorphic to its image in $\mathcal{O}_2$ modulo the compacts. Specifically, viewing X_q as an element of $\mathcal{O}_2$, we may write

$$X_q = S_1(P_1 + qP_2) + (qP_1 + P_2)S_2^*$$

where $P_i = S_i S_i^*$ for $i = 1$ and 2.

5.2. **Some Simple Algebras.** In this section, we use the detailed information we now possess about the eigenstates of $\mathrm{C}^*(X_q)$ to show those algebras are simple for all $q > 0$. From above, we may treat $\mathrm{C}^*(X_q)$ as a subalgebra of $\mathcal{O}_2$. Further, composing any state with the quotient map from $\mathcal{T}_2$ to $\mathcal{O}_2$ shows that eigenstates of $\mathcal{O}_2$ for X_q must also be unique.

Let σ be the automorphism of $\mathcal{O}_2$ given by $\sigma(S_i) = S_{3-i}$, and α_z the automorphism of $\mathcal{O}_2$ defined by sending $S_1 \mapsto zS_1$ and $S_2 \mapsto \overline{z}S_2$, where z is a complex number of modulus one. Note that $\sigma(X_q) = X_q^*$ and $\alpha_z(X_q) = zS_1 + (\overline{z}S_2)^* = zX_q$.

For what follows, let $\mathcal{D}$ be a C*-subalgebra of $\mathcal{O}_2$ that contains $C^*(X_q)$. We will show that any such $\mathcal{D}$ that is also invariant under σ and each α_z is simple. We remark that this does not provide a new proof of the simplicity of $\mathcal{O}_2$, since we use this fact in the first lemma below.

If λ is a complex number in the spectrum of X_q, denote by f_λ the unique λ-eigenstate of $\mathcal{O}_2$ for X_q with corresponding GNS representation $(\rho_\lambda, \mathcal{H}_\lambda, \xi_\lambda)$. Further, let $\mathcal{J}_\lambda$ be the kernel of the GNS representation, γ_λ, of the restriction of f_λ to $\mathcal{D}$. The main thing we will wish to show, when all the hypotheses on $\mathcal{D}$ are in force, is that $\mathcal{J}_\lambda$ is (0) for all $|\lambda| \leq 1$.

Lemma 5.3. *If $C^*(X_q) \subseteq \mathcal{D} \subseteq \mathcal{O}_2$ and $|\lambda| = 1$ then $\mathcal{J}_\lambda = (0)$.*

Proof. By Proposition 4.3, we have $\overline{\mathcal{D}\xi} = \mathcal{H}$, so the representation γ_λ is the restriction of ρ_λ to $\mathcal{D}$. The representation ρ_λ is faithful because $\mathcal{O}_2$ is simple. □

Lemma 5.4. *Let $\mathcal{D}$ be as in Lemma 5.3, and suppose $\mathcal{J}$ is a two-sided closed ideal of $\mathcal{D}$. Let θ be the quotient map*

$$\theta\colon \mathcal{D} \to \mathcal{D}/\mathcal{J}\ .$$

If the spectral radius of $\theta(X_q)$ is one, then $\mathcal{J} = (0)$.

Proof. Let $\mathcal{A} := \mathcal{D}/\mathcal{J}$. Suppose $\lambda \in \mathrm{Sp}(\theta(X_q))$ and $|\lambda| = 1$. Then, since $\theta(X_q) - \lambda$ is not invertible, either $\theta(X_q) - \lambda$ has no left inverse and $\mathcal{A}(\theta(X_q) - \lambda)$ is not all of $\mathcal{A}$, or $\theta(X_q) - \lambda$ has no right inverse and $(\theta(X_q) - \lambda)\mathcal{A}$ is not all of $\mathcal{A}$.

Consider first the case where $\theta(X_q) - \lambda$ has no left inverse and hence $\mathcal{A}(\theta(X_q) - \lambda)$ is not all of $\mathcal{A}$. Recall that in any unital Banach algebra, the norm closure of a proper left or right (algebraic) ideal is also a proper ideal. Since $\mathcal{A}(\theta(X_q) - \lambda)$ is a proper left ideal of $\mathcal{A}$, $\overline{\mathcal{A}(\theta(X_q) - \lambda)}$ is also a proper left ideal of $\mathcal{A}$. Thus $(\theta(X_q) - \lambda)^*(\theta(X_q) - \lambda)$ is not invertible, and is therefore killed by a state ϕ. By definition, $\theta(X_q) - \lambda$ is in the left kernel of ϕ, which makes ϕ a λ-eigenstate of $\mathcal{A}$ for $\theta(X_q)$, and $\phi \circ \theta$ a λ-eigenstate of $\mathcal{D}$ for X_q.

Extending $\phi \circ \theta$ to a state of $\mathcal{O}_2$ yields a λ-eigenstate of $\mathcal{O}_2$ for X_q, and must therefore give the unique λ-eigenstate of $\mathcal{O}_2$ for X_q, f_λ. Thus $\mathcal{J}_\lambda$ is the kernel of the GNS representation of $\mathcal{D}$

corresponding to $\phi \circ \theta$, and, since $|\lambda| = 1$, by Lemma 5.3, $\mathcal{J}_\lambda = (0)$. Since $\phi \circ \theta$ annihilates $\mathcal{J}$, we have $\mathcal{J} \subset \mathcal{J}_\lambda$, and hence $\mathcal{J} = 0$.

If $\theta(X_q) - \lambda$ has a left inverse but has no right inverse, then $(\theta(X_q) - \lambda)\mathcal{A}$ is not all of $\mathcal{A}$. Applying the involution, we have $\mathcal{A}^*(\theta(X_q)^* - \overline{\lambda})$ is also not all of $\mathcal{A}^* = \mathcal{A}$. Interchanging ones and twos, we know that there is a unique $\overline{\lambda}$-eigenstate of $\mathcal{O}_2$ for X_q^*, so that the proof in this case may proceed as in the previous case by replacing λ with $\overline{\lambda}$ and X_q with X_q^*. □

Proposition 5.5. *If $\mathcal{D}$ is a C^*-subalgebra of $\mathcal{O}_2$ that contains $C^*(X_q)$ and further satisfies $\sigma(\mathcal{D}) = \mathcal{D}$, (0) is a prime ideal of $\mathcal{D}$.*

Proof. Suppose that $(0) = \mathcal{J}_1 \cap \mathcal{J}_2$ for ideals $\mathcal{J}_1$ and $\mathcal{J}_2$.

Let

$$\theta_1 \colon \mathcal{D} \to \mathcal{D}/\mathcal{J}_1$$

and

$$\theta_2 \colon \mathcal{D} \to \mathcal{D}/\mathcal{J}_2$$

be the two quotient maps. Since $\mathcal{J}_1 \cap \mathcal{J}_2 = (0)$, then

$$\theta_1 \oplus \theta_2 \colon \mathcal{D} \to \mathcal{D}/\mathcal{J}_1 \oplus \mathcal{D}/\mathcal{J}_2\,,$$

defined by $d \mapsto \theta_1(d) \oplus \theta_2(d)$ is an injective $*$-homomorphism of $\mathcal{D}$. Since $\theta_1 \oplus \theta_2$ is injective, $(\theta_1 \oplus \theta_2)(X_q)$ has spectral radius one, and thus either $\theta_1(X_q)$ or $\theta_2(X_q)$ must have spectral radius one.

Suppose that $\theta_1(X_q)$ has spectral radius one. Then by Lemma 5.4, $\mathcal{J}_1 = (0)$. If $\theta_2(X_q)$ has spectral radius one, then by Lemma 5.4, $\mathcal{J}_2 = (0)$. Thus (0) is a prime ideal. □

Corollary 5.6. *If $\mathcal{D}$ is a C^*-subalgebra of $\mathcal{O}_2$ that contains $C^*(X_q)$ and further satisfies $\sigma(\mathcal{D}) = \mathcal{D}$, $\mathcal{J}$ is a closed two-sided ideal of $\mathcal{D}$, and $\mathcal{J} \cap \sigma(\mathcal{J}) = (0)$, then $\mathcal{J} = (0)$.*

Proof. First, note that because $\sigma(\mathcal{D}) = \mathcal{D}$, $\sigma(\mathcal{J})$ is an ideal of $\mathcal{D}$. Now apply Corollary 5.6 to see that either $\mathcal{J} = (0)$ or $\sigma(\mathcal{J}) = (0)$. In either case, $\mathcal{J} = (0)$ since $\sigma(\mathcal{J}) \simeq \mathcal{J}$. □

Lemma 5.7. *If $\sigma(\mathcal{D}) = \mathcal{D}$ and r is a real number that satisfies $|r| < 1$, then $\mathcal{J}_r = 0$.*

Proof. The r-eigenstate f_r of $\mathcal{O}_2$ for X_q satisfies

$$f_r \circ \sigma\left((X_q^* - r)^*(X_q^* - r)\right) = f_r\left((X_q - r)^*(X_q - r)\right) = 0,$$

so that $f_r \circ \sigma$ is the unique $\overline{\lambda}$-eigenstate of $\mathcal{O}_2$ for X_q^*. Notice that for a in $\mathcal{D}$, we have

$$f_r\left(\mathcal{D}\sigma(a)\mathcal{D}\right) = f_r \circ \sigma\left(\mathcal{D}a\mathcal{D}\right)$$

because $\mathcal{D}$ is σ-invariant. Thus, $\sigma(\mathcal{J}_r)$ is the kernel of the GNS representation of the restriction of $f_r \circ \sigma$ to $\mathcal{D}$. Let ξ_r' be the r-eigenvector, normalized to have length one, of $\rho_r(X_q^*)$ exhibited in Proposition 4.2. Then $f_r \circ \sigma = \langle \rho_r(\cdot)\xi_r', \xi_r' \rangle$. The state vectors ξ_r and ξ_r' correspond to kernels of GNS representations as follows:

$$\mathcal{J}_r = \{a \in \mathcal{D} \colon \rho_r(a)\mathcal{D}\xi_r = 0\}\ ,$$

and

$$\sigma(\mathcal{J}_r) = \{a \in \mathcal{D} : \rho_r(a)\mathcal{D}\xi_r' = 0\}\ .$$

By Proposition 4.3, only the zero operator can annihilate both $\mathcal{D}\xi_r$ and $\mathcal{D}\xi_r'$ and the representation ρ_r of $\mathcal{O}_2$ is faithful, so $\mathcal{J}_r \cap \sigma(\mathcal{J}_r) = 0$. That $\mathcal{J}_r = 0$ now follows from Corollary 5.6. ☐

Lemma 5.8. *If $\mathcal{D}$ is a C^*-subalgebra of $\mathcal{O}_2$ that contains $C^*(X_q)$ and further satisfies $\sigma(\mathcal{D}) = \mathcal{D}$ and $\alpha_z(\mathcal{D}) = \mathcal{D}$ for any complex z of modulus one, then $\mathcal{J}_\lambda = (0)$ for any λ in the closed unit disc.*

Proof. Write $\lambda = rz$ for $|z| = 1$ and $0 \leq r \leq 1$. Then

$$f_r \circ \alpha_z\left((X_q - \lambda)^*(X_q - \lambda)\right) \ = \ f_r\left((zX_q - \lambda)^*(zX_q - \lambda)\right) \ = \ f_r\left(|z|^2(X_q - r)^*(X_q - r)\right) \ = \ 0\ .$$

We conclude from eigenstate uniqueness that $f_\lambda = f_r \circ \alpha_z$. It then follows, as in the proof of Lemma 5.7 and using α_z-invariance of $\mathcal{D}$, that α_z maps $\mathcal{J}_\lambda$ to $\mathcal{J}_r$, which we know is (0). ☐

Theorem 5.9. *Let $\mathcal{D}$ be a C^*-subalgebra of $\mathcal{O}_2$ containing $C^*(X_q)$ for some $q > 0$ such that $\sigma(\mathcal{D}) = \mathcal{D}$ and $\alpha_z(\mathcal{D}) = \mathcal{D}$ for all complex z of modulus one. Then $\mathcal{D}$ is simple. In particular, $C^*(X_q)$ is simple for $q > 0$.*

Proof. Let $\mathcal{J}$ be a proper two-sided closed ideal of $\mathcal{D}$, θ the quotient map $\theta \colon \mathcal{D} \to \mathcal{D}/\mathcal{J}$. Reasoning as before, if $\mathcal{A} = \mathcal{D}/\mathcal{J}$ and $\lambda \in \mathrm{Sp}(\theta(X_q))$, $\theta(X_q) - \lambda$ is not invertible, so either $\mathcal{A}(\theta(X_q) - \lambda)$ or $(\theta(X_q) - \lambda)\mathcal{A}$ is not all of $\mathcal{A}$. As before, we may assume that $\mathcal{A}(\theta(X_q) - \lambda)$ is not all of $\mathcal{A}$, since the proof in the other case is similar, so that $(\theta(X_q) - \lambda)^*(\theta(X_q) - \lambda)$ is not invertible, and there exists a state ϕ on $\mathcal{A}$ for which $\theta(X_q) - \lambda$ is in the left kernel. Then $\phi \circ \theta$ is a λ-eigenstate of $\mathcal{D}$ for X_q and $\mathcal{J} \subseteq \mathcal{J}_\lambda = (0)$ by Lemma 5.8. ☐

5.3. Examples.

Theorem 5.10. *Let $\mathcal{F}$ be the family of elements of the form $S_\alpha S_\alpha^*$. Then the subalgebra of $\mathcal{O}_2$ generated by $\mathcal{F}$ and X_q, for fixed $q > 0$, is proper and simple.*

Proof. To show that the subalgebra $\mathcal{D}$ of $\mathcal{O}_2$ generated by $\mathcal{F}$ and X_q is proper, we will exhibit a representation of $\mathcal{O}_2$ in which $\mathcal{D}$ has a non-trivial closed invariant subspace that is not invariant under S_1 or S_2. Let $\mathfrak{F}$ be the full Fock space, with vacuum vector Ω, of $\mathbb{C}^2$ with orthonormal basis $\{e_1, e_2\}$, and let $\mathfrak{F}'$ be the full Fock space, with vacuum vector Ω' of $\mathbb{C}^2$ with orthonormal basis $\{e_1', e_2'\}$. Attached distinguished vectors ω and ω' to $\mathfrak{F}$ and $\mathfrak{F}'$ respectively to yield $\mathfrak{H} = \mathfrak{F} \oplus \mathbb{C}\omega$ and $\mathfrak{H}' = \mathfrak{F}' \oplus \mathbb{C}\omega'$. Define isometries S_1 and S_2 on $\mathcal{H} \oplus \mathfrak{H}'$ as follows: S_j is the left creation operator $l(e_j)$ on $\mathfrak{F}$, the left creation operator $l(e_j')$ on $\mathfrak{F}'$, and

$$S_1\omega = \omega', \qquad S_1\omega' = \Omega',$$

$$S_2\omega = \Omega, \qquad S_2\omega' = -\omega.$$

One checks readily that $S_1S_1^* + S_2S_2^* = 1$, so that this gives a representation of $\mathcal{O}_2$. Then

$$X_q\omega = 0, \qquad X_q^*\omega' = 0,$$

$$X_q\Omega = qe_1 + \omega, \qquad X_q\omega' = \Omega',$$

$$X_q^*\Omega = e_2 \qquad X_q^*\Omega', = qe_2' + \omega'.$$

Thus the subspaces $\mathfrak{H}$ and $\mathfrak{H}'$ are invariant under both X_q and X_q^*, as well as under all of the monomials $S_\alpha S_\alpha^*$, but not invariant under S_1 or S_2. Thus $\mathcal{D}$ is properly contained in $\mathcal{O}_2$. In fact, the representation of $\mathcal{O}_2$ described above is the GNS representation of the zero-eigenstate of $\mathcal{O}_2$ for X_q.

To see that $\mathcal{D}$ properly contains $C^*(X_q)$, observe that $X_qS_1S_1^* = S_{11}S_1^*$. This element of $\mathcal{D}$ – call it W – satisfies $WW^* = S_{11}S_{11}^*$ and $W^*W = S_1S_1^*$. Since $W^*W - WW^*$ is a non-zero projection, the C^*-algebra $\mathcal{D}$ cannot have a faithful trace. □

Theorem 5.11. *Let $p \neq q$ be positive real numbers. Then $C^*(X_q, X_p)$ is a proper simple subalgebra of $\mathcal{O}_2$.*

Proof. First, note that a suitable linear combination of X_q and X_p for $p \neq q$ yields any $X_{p'}$. Thus the $C^*(X_q, X_p)$ are all isomorphic. They are strictly larger than any $C^*(X_q)$ by Corollary 3.10.

$C^*(X_q, X_p)$ cannot be all of $\mathcal{O}_2$, since the GNS representation of the zero-eigenstate of $\mathcal{O}_2$ for X_q and non-trivial closed X_q-invariant and X_q^*-invariant subspace described in the proof of Theorem 5.10 is not dependent on q. $C^*(X_q, X_p)$ is simple by Theorem 5.9. □

Our unique eigenstate technique can also be used to give a proof of the simplicity of $C_r^*(\mathbb{F}_n)$ for $n \geq 2$. We present this not with the intention of improving on the elegant and economical original argument of Powers [19], but to show that our methods extend somewhat beyond the setting of the present paper.

The point of entry is the fact that there is a unique one-eigenstate of $C_r^*(\mathbb{F}_n)$ for u_1, namely the one whose restriction to $\mathbb{F}_n$ is the indicator function of G_1, the cyclic subgroup of $\mathbb{F}_n$ generated by u_1. The proof of this is in a paper of Paschke [17], but we can give a self-contained argument using Theorem 3.9 above, as follows: for any s in $\mathbb{F}_n \setminus G_1$, it is not difficult to show that the double coset $G_1 s G_1$ contains an infinite free set. For $t_1, t_2, \ldots, t_k$ in this free set, our calculation of the norm of X_q, with $q = \sqrt{(k-1)/k}$, gives the well-known formula (see, for instance, Akemann and Ostrand [1]) $\|t_1 + \cdots + t_k\| = 2\sqrt{k-1}$.

Dividing by k makes the norm of the resulting average arbitrarily small. If f is a one-eigenstate of $C_r^*(\mathbb{F}_n)$ for u_1 (and hence for u_1^*, because u_1 is normal), by the Cauchy-Schwarz inequality it is invariant under left and right multiplication by u_1, and thus constant on the closed convex hull of $G_1 s G_1$. We have just seen that this set contains zero, so $f(s) = 0$. Thus the restriction of f to $\mathbb{F}_n$ is the indicator function of G_1, as claimed.

Now observe that for any t in $\mathbb{F}_n \setminus \{e\}$, we have

$$\lim_{m \to \infty} f(u_2^{-m} t u_2^m) = 0\ .$$

Thus, the trace $\langle \cdot \delta_e, \delta_e \rangle$ on $C_r^*(\mathbb{F}_n)$ is the w^*-limit of vector states in the GNS representation of f. Since the trace is faithful on $C_r^*(\mathbb{F}_n)$, it follows that the GNS representation of f must be faithful.

Composing with automorphisms of $C_r^*(\mathbb{F}_n)$ that spin u_1, we see that for any unimodular complex number z, there is a unique z-eigenstate f_z of $C_r^*(\mathbb{F}_n)$ for u_1, and that the GNS representation of f_z is faithful. Now let J be a proper closed ideal of $C_r^*(\mathbb{F}_n)$, and let W be the image of u_1 in the quotient $C_r^*(\mathbb{F}_n)/J$. Pick z in the spectrum of the unitary element W. There is a z-eigenstate g of the quotient for W. Its composition with the quotient map must be f_z. Thus f_z annihilates J. We conclude that J must be (0).

6. Special Cases

6.1. **The Case $\mathbf{q} = \mathbf{1}$.** When q is one, the algebra $C^*(X_1)$ is isomorphic to the algebra $C^*(s(\mathbb{C}^2))$ introduced by Voiculescu [23]; we thus know rather more about this case than we do for arbitrary positive q. To see the isomorphism, let T_1 and T_2 be the isometries on Fock space defined in Section 3 above. Voiculescu's paper makes it plain that $C^*(s(\mathbb{C}^2)) = C^*(T_1 + T_1^*, T_2 + T_2^*)$. Notice that

$$W_1 = \frac{1}{\sqrt{2}}(T_1 + iT_2)$$

and

$$W_2 = \frac{1}{\sqrt{2}}(T_1 - iT_2)$$

are isometries of Fock space with orthogonal ranges and a non-zero defect space. There is thus an isometry θ of $\mathcal{T}_2$ taking T_j to W_j. Since $X_1 = T_1 + T_2^*$, we have

$$\theta(X_1) = \frac{1}{\sqrt{2}}(T_1 + T_1^* + i(T_2 + T_2^*)) ,$$

so θ maps $C^*(X_1)$ onto $C^*(T_1 + T_1^*, T_2 + T_2^*)$.

It now follows from results by Voiculescu [23] (for the two-dimensional case of his construction) that $C^*(X_1)$ has a unique tracial state, and that the von Neumann algebra that it generates in the GNS representation of that state (that is, the representation on Fock space) is isomorphic to the von Neumann algebra generated by $\mathbb{F}_2$ in its left regular representation. Voiculescu exhibits $C^*(T_1 + T_1^*, T_2 + T_2^*)$ as the reduced free product (see the book by Voiculescu, Dykema, and Nica [22] for an introduction) of $C^*(T_1 + T_1^*)$ and $C^*(T_2 + T_2^*)$ with respect to the pair of states obtained by restricting the trace to these subalgebras. He also shows that each of these algebras contains a unitary, all of whose non-zero powers are annihilated by the trace. The simplicity of the free product then follows from a result of Avitzour [2], so that we have a second proof that $C^*(X_1)$ is simple.

6.2. **The Limit Case $\mathbf{q} = \mathbf{0}$.** The value of zero for q has so far been excluded from our discussion because from time to time we have had to divide by q. If we define X_0 in $\mathcal{O}_2$ by $X_0 = S_1P_1 + P_2S_2^*$, where $P_i = S_iS_i^*$, we soon discover that $C^*(X_0)$ differs significantly from any $C^*(X_q)$ with $q > 0$. Indeed, notice that S_1P_1 and S_2P_2 are partial isometries with orthogonal domain and range projections that both sum to the identity, and so $C^*(X_0)$ is isomorphic to the subalgebra of

$C^*(S) \oplus C^*(S)$ generated by $S \oplus S^*$, where S is a unilateral shift, by Coburn's Theorem [5]. From this, we have the exact sequence

$$0 \to \mathcal{K} \oplus \mathcal{K} \to C^*(X_0) \to C(\mathbb{T}) \oplus C(\mathbb{T}) \to 0.$$

Because the class of nuclear C*-algebras is closed under extensions (see, for example, Blackadar's book [4]), and $\mathcal{K}$ and $C(\mathbb{T})$ are nuclear, $C^*(X_0)$ is also nuclear. Finally, the presence of a non-trivial projection in $C^*(X_0)$ implies by spectral theory that $C^*(X_q)$ has a non-trivial projection for sufficiently small q, independently of the more precise assertion in Section 3.

7. Further Results

7.1. **Non-Nuclearity.** The assumption of nuclearity has played a large role in work on the classification of C*-algebras. We now show that while $C^*(X_q)$ is exact for $q > 0$, it is not nuclear. While this prohibits trying to fit $C^*(X_q)$, $q > 0$, into the current classifiable C*-algebras, it also suggests $C^*(X_q)$ has an interesting structure.

Proposition 7.1. *The algebras $C^*(X_q)$ are exact for all $q \geq 0$.*

Proof. By a result of Kirchberg [21, 6.3.11], a separable C*-algebra $\mathcal{A}$ is exact if and only if there is an injective $*$-homomorphism $i\colon \mathcal{A} \to \mathcal{O}_2$. Combine this with Proposition 5.2. □

For a C*-algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$, a *hypertrace* on $\mathcal{A}$ is a state ϕ on $\mathcal{B}(\mathcal{H})$ that contains $\mathcal{A}$ in its centralizer. That is, if $X \in \mathcal{A}$, then $\phi(XY) = \phi(YX)$ for all $Y \in \mathcal{B}(\mathcal{H})$. If $\mathcal{A}$ is a C*-algebra and π is a non-degenerate representation of $\mathcal{A}$, π is *hypertracial* if $\pi(A)$ has a hypertrace. The algebra $\mathcal{A}$ is *weakly-hypertracial* if there exists a faithful non-degenerate representation of $\mathcal{A}$ that is hypertracial [3, Corollary 2.3].

Theorem 7.2. *For $q > 0$, $C^*(X_q)$ is not weakly-hypertracial.*

Proof. Suppose to the contrary that $C^*(X_q)$ is weakly-hypertracial for some $q > 0$. Consider the image of $C^*(X_q)$ in the representation of $\mathcal{T}_2$ on $\mathcal{H} = \ell^2(\mathbb{S})$. Because $C^*(X_q)$ is simple by 5.9, this is a faithful, non-degenerate representation of $C^*(X_q)$. By our assumption, there exists a hypertrace Φ for $C^*(X_q)$ on $\mathcal{B}(\mathcal{H})$.

Consider now the image of $\mathcal{T}_2$ in $\mathcal{B}(\mathcal{H})$, where we let $P_i = T_iT_i^*$, $P_0 = 1 - P_1 - P_2$, $Q_1 = P_0 + P_1 + qP_2$, and $Q_2 = P_0 + qP_1 + P_2$, so that $X_q = T_1Q_1 + Q_2T_2^*$. Then

$$\begin{aligned}\Phi(T_1Q_1T_1Q_1) &= \Phi(X_q(T_1Q_1)) \\ &= \Phi((T_1Q_1)X_q) \\ &= \Phi(T_1Q_1T_1Q_1 + T_1Q_1Q_2T_2^*)\ ,\end{aligned}$$

so that

$$\Phi(T_1Q_1Q_2T_2^*) = 0\ , \tag{15}$$

and by symmetry,

$$\Phi(T_2Q_2Q_1T_1^*) = 0\ . \tag{16}$$

Also, we have

$$\begin{aligned}&\Phi(T_1Q_1T_1Q_1 + T_1Q_1Q_2T_2 + T_2Q_2Q_1T_1 + T_2Q_2T_2Q_2 + Q_1^2 + Q_2^2) \\ &\quad = \Phi((X_q + X_q^*)(T_1Q_1 + T_2Q_2)) \\ &\quad = \Phi((T_1Q_1 + T_2Q_2)(X_q + X_q^*)) \\ &\quad = \Phi(T_1Q_1T_1Q_1 + T_1Q_1Q_2T_2^* + T_1Q_1Q_1T_1^* + T_1Q_1T_2Q_2 \\ &\qquad + T_2Q_2T_2Q_2 + T_2Q_2Q_1T_1^* + T_2Q_2Q_2T_2^* + T_2Q_2T_1Q_1)\end{aligned}$$

so that

$$\Phi(Q_1^2 + Q_2^2) = \Phi(T_1Q_1Q_2T_2^* + T_1Q_1^2T_1^* + T_2Q_2^2T_2^* + T_2Q_2Q_1T_1^*)$$

and

$$\Phi(Q_1^2 + Q_2^2) = \Phi(T_1Q_1^2T_1^* + T_2Q_2^2T_2^*) \tag{17}$$

from (15) and (16).

Now $Q_1^2 = P_0 + P_1 + q^2P_2 = 1 - (1-q^2)P_2$ and likewise $Q_2^2 = 1 - (1-q^2)P_1$, so that, $Q_1^2 + Q_2^2 = (1+q^2) + (1-q^2)P_0$, and (17) becomes

$$1 + q^2 + (1-q^2)\Phi(P_0) = \Phi(T_1T_1^* + T_2T_2^* - (1-q^2)T_1P_2T_1^* - (1-q^2)T_2P_1T_2^*)\ ,$$

and

$$1 + q^2 + (1 - q^2)\Phi(P_0) = 1 - \Phi(P_0) - (1 - q^2)\Phi(T_1 P_2 T_1^* + T_2 P_1 T_2^*) .$$

Finally, we get

$$q^2 = (q^2 - 2)\Phi(P_0) + (q^2 - 1)\Phi(T_1 P_2 T_1^* + T_2 P_1 T_2^*) . \tag{18}$$

Because P_0 is a projection, $0 \leq E(P_0) \leq 1$, and adding $\Phi(P_0)$ to the right side of (18) gives

$$q^2 \leq (q^2 - 1)\Phi(P_0 + T_1 P_2 T_1^* + T_2 P_1 T_2^*) . \tag{19}$$

In fact, $P_0 + T_1 P_2 T_1^* + T_2 P_1 T_2^*$ is a projection, so that

$$0 \leq \Phi(P_0 + T_1 P_2 T_1^* + T_2 P_1 T_2^*) \leq 1 ,$$

so that (19), for $q > 0$, becomes

$$q^2 \leq q^2 - 1 ,$$

which is impossible and contradicts our original assumption. □

Corollary 7.3. *For $q > 0$, $C^*(X_q)$ is not nuclear.*

Proof. Suppose to the contrary that $C^*(X_q)$ is nuclear for some $q > 0$. Consider again the image of $C^*(X_q)$ in the representation of $\mathcal{T}_2$ on $\mathcal{H} = \ell^2(\mathbb{S})$. We now invoke a theorem of Lance [16], that states that for every representation ρ of a nuclear C*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, there exists a conditional expectation E from $\mathcal{L}(\mathcal{H})$ to the weak closure $\overline{\rho(\mathcal{A})}$. Thus there exists a conditional expectation $E\colon \mathcal{B}(\mathcal{H}) \to W^*(X_q)$. Because τ is a trace on $W^*(X_q)$, for any $a \in W^*(X_q)$,

$$\tau \circ E(ab) = \tau(aE(b)) = \tau(E(b)a) = \tau \circ E(ba)$$

for all $b \in \mathcal{B}(\mathcal{H})$, so that $\tau \circ E$ is a hypertrace on $W^*(X_q)$, and thus also on $C^*(X_q)$, contradicting 7.2. □

Corollary 7.4. *For $q > 0$, $C^*(X_q)$ is not quasidiagonal.*

Proof. Any quasidiagonal C*-algebra is weakly hypertracial [3, Remark 2.4]. □

7.2. **More on the Eigenstate Representation When $|\boldsymbol{\lambda}| < 1$.** For the sake of completeness, we end the paper by sharpening Theorem 4.4. Fix q and λ with $q > 0$ and $|\lambda| < 1$. Regard X_q as an operator in $\mathcal{O}_2$, and let the latter act on the Hilbert space $\mathcal{H}$ via the GNS representation of the λ-eigenstate for X_q. Let the vectors ν, μ, ξ, and ξ' be as in Section 4. Thus, the unit λ-eigenvector for X_q is

$$\xi = \frac{q\overline{\lambda}}{|\lambda|^2 - 1}\nu + \mu \ ,$$

and the non-zero vector $\xi' = \nu + q\lambda\mu$ is a $\overline{\lambda}$-eigenvector for X_q^*. The vector states of $\mathrm{C}^*(X_q)$ corresponding to ξ and $\xi'/||\xi'||$ are pure. The subspaces $\mathrm{C}^*(X_q)\xi$ and $\mathrm{C}^*(X_q)\xi'$ are closed by Kadison's transitivity theorem [14], and their sum is dense in $\mathcal{H}$.

Theorem 7.5. *The two subspaces $C^*(X_q)\xi$ and $C^*(X_q)\xi'$ are orthogonal. Their sum is $\mathcal{H}$, and the irreducible representations of $\mathrm{C}^*(X_q)$ obtained by restriction to these subspaces are unitarily inequivalent.*

Proof. Our argument depends crucially on previous results with a similar flavor by Paschke [18] for representations of $\mathbb{F}_n$. Fix $n \geq 2$ and let $\mathbb{F}_n$, with generators $u_1, \ldots, u_n$ act on $\ell^2(\mathbb{F}_n)$ via the left regular representation. Let $X = n^{-1/2}(u_1 + \ldots + u_n)$. There is a pure λ-eigenstate ϕ of $\mathrm{C}^*_r(\mathbb{F}_n)$ for X such that $\phi(w)$ depends only on $\mathrm{sign}(w)$ for w in $\mathbb{F}_n$ [17, 18]. Switching generators to their inverses gives a pure $\overline{\lambda}$-eigenstate ϕ' for X^*. Also the GNS representations of $\mathrm{C}^*_r(\mathbb{F}_n)$ for ϕ and ϕ' are unitarily inequivalent [18, Proposition 5.4(c)].

Let $q = \sqrt{(n-1)/n}$. We have seen in Section 3 that restriction to $\overline{\mathrm{C}^*(X)\delta_e}$ gives an isomorphism of $\mathrm{C}^*(X)$ with $\mathrm{C}^*(X_q)$ sending X to X_q. It follows that the restriction of ϕ to $\mathrm{C}^*(X)$ is the unique λ-eigenstate (call it f) of $\mathrm{C}^*(X)$ for X, and similarly for ϕ', which restricts to the unique $\overline{\lambda}$-eigenstate f' of $\mathrm{C}^*(X)$ for X^*. We claim that compression to $\overline{\mathrm{C}^*(X)\delta_e}$ gives a conditional expectation $E\colon \mathrm{C}^*_r(\mathbb{F}_n) \to \mathrm{C}^*(X)$, and that $\phi = f \circ E$, $\phi' = f' \circ E$. To see this, take w in $\mathbb{F}_n$ and let a be the average of all v in $\mathbb{F}_N$ with the same sign as w. The argument in Lemma 3.5 above shows that a is a polynomial in X and X^*, so in particular belongs to $\mathrm{C}^*(X)$. For a sign s, let Δ_s be as in Section 3, so Δ_s is a multiple of the indicator function of the set of group elements with sign s, and the Δ_s form an orthonormal basis for $\overline{\mathrm{C}^*(X)\delta_e}$. For any signs s and t, and any v in $\mathbb{F}_n$ with the same sign as w, we have $\langle v\Delta_s, \Delta_t\rangle = \langle w\Delta_s, \Delta_t\rangle$. Averaging over all such v shows that $\langle a\Delta_s, \Delta_t\rangle = \langle w\Delta_s, \Delta_t\rangle$. We conclude that the compression of w to $\overline{\mathrm{C}^*(X)\delta_e}$ is the restriction to this subspace of the average a. Since $a \in \mathrm{C}^*(X)$ and since the group elements w span a dense subspace of $\mathrm{C}^*_r(\mathbb{F}_n)$, we have the desired conditional expectation E. Consider now the state $f \circ E$ of $\mathrm{C}^*_r(\mathbb{F}_n)$. This state is a λ-eigenstate for X, and a glance at the state formula [18] shows that on group elements it depends only on the sign, just like ϕ. By a straightforward induction argument, using the properties that

$f \circ E(e) = 1$, that $f \circ E(w) = \overline{f \circ E(w^{-1})}$ for w in $\mathbb{F}_n$, and that

$$f \circ E((u_1 + \ldots + u_n)w) = \sqrt{n}\lambda\, f \circ E(w)$$

for w in $\mathbb{F}_n$, it follows that ϕ and $f \circ E$ must coincide. Likewise $\phi' = f' \circ E$.

It is a simple consequence of the unitary version of Kadison's transitivity theorem [12] that two pure states of a C^*-algebra have unitarily equivalent GNS representations if and only if the states themselves are unitarily equivalent. Suppose that there exists a unitary U in $C^*(X)$ such that $f'(Y) = f(UYU^*)$ for all Y in $C^*(X)$. Then for any y in $C^*_r(\mathbb{F}_n)$, we would have

$$\phi'(y) = f'(E(y)) = f(UE(y)U^*) = f(E(UyU^*)) = \phi(UyU^*)\ .$$

This would contradict the known inequivalence of the GNS representations of ϕ and ϕ'. We conclude that the GNS representations of $C^*(X)$ coming from f and f' are unitarily inequivalent, which means that the representations of $C^*(X_q)$ (for our special value of q) on $C^*(X_q)\xi$ and $C^*(X_q)\xi'$ are inequivalent. This in turn implies that $C^*(X_q)\xi \perp C^*(X_q)\xi'$ since the restriction to the first subspace of the orthogonal projection of $\mathcal{H}$ on the second intertwines two inequivalent irreducible representations, and hence must be zero. We have proved the theorem in the special case $q = \sqrt{(n-1)/n}$ for integer $n \geq 2$.

We deduce orthogonality of the subspaces for arbitrary q as follows: define vectors e_1 and e_2 by $e_1 = \nu/\|\nu\|$ and $e_2 = \mu/\|\mu\|$. By Equation 12 and Lemma 4.1, we have

$$V_1^* e_1 = \overline{\lambda} e_1 - \sqrt{1 - |\lambda|^2} e_2\ ,$$

and

$$V_2^* e_2 = \sqrt{1 - |\lambda|^2} e_1 + \lambda e_2\ ,$$

where $\{V_1, V_2\}$ is the pair of Cuntz isometries used to define X_q. Furthermore $V_1^* e_2 = 0 = V_2^* e_1$. It follows that when powers of V_1^* and V_2^* are restricted to the span of e_1 and e_2, the resulting matrix entries depend only on λ. From this in turn we conclude that the linear functionals $\langle \cdot\, e_i, e_j \rangle$ on $\mathcal{O}_2$ are independent of q. Consider now η and η' defined by

$$\eta = q\overline{\lambda} e_1 - \sqrt{1 - |\lambda|^2} e_2\ ,$$

and

$$\eta' = \sqrt{1-|\lambda|^2}e_1 + q\lambda e_2\ .$$

By Equation 9, these vectors are non-zero scalar multiples of ξ and ξ' respectively, namely

$$\eta = -\frac{\sqrt{1-|\lambda^2|}}{||\mu||}\xi\ ,$$

and

$$\eta' = \frac{1}{||\mu||}\xi'\ .$$

The net effect of these manipulations is to make $\langle Y\eta, \eta'\rangle$ a polynomial in q for any given monomial Y in X_q and X_q^*. We know that this polynomial vanishes for the infinitely many "group case" values of q, so it vanishes for all q. Thus $\mathrm{C}^*(X_q)\xi \perp \mathrm{C}^*(X_q)\xi'$.

Finally, unitary equivalence of the two representations would imply the existence of a unit vector $\tilde{\xi}$ in $\mathrm{C}^*(X_q)\xi'$ such that $X_q\tilde{\xi} = \lambda\tilde{\xi}$. By uniqueness of the λ-eigenstate of $\mathcal{O}_2$ for X_q, we would then have $\langle T\xi, \xi\rangle = \langle T\tilde{\xi}, \tilde{\xi}\rangle$ for all T in $\mathcal{O}_2$ and hence, by irreducibility of the representation of $\mathcal{O}_2$ on $\mathcal{H}$, for all bounded operators T on $\mathcal{H}$. This is impossible, because $\tilde{\xi} \perp \xi$. □

We remark that the formulas in the proof above for the matrix entries of the restrictions of V_1^* and V_2^* to the span of e_1 and e_2 show that the representations of $\mathcal{O}_2$ that we are dealing with depend only on λ (up to unitary equivalence). This is because these representations are, for instance, the GNS representations of the q-independent state $\langle \cdot e_1, e_1\rangle$.

References

1. Charles A. Akemann and Phillip A. Ostrand, *Computing norms in group C^*-algebras*, Amer. J. Math. **98** (1976), no. 4, 1015–1047. MR 0442698 (56 #1079)
2. Daniel Avitzour, *Free products of C^*-algebras*, Trans. Amer. Math. Soc. **271** (1982), no. 2, 423–435. MR 654842 (83h:46070)
3. Erik Bédos, *Notes on hypertraces and C^*-algebras*, J. Operator Theory **34** (1995), no. 2, 285–306. MR MR1373325 (96m:46111)
4. Bruce Blackadar, *K-theory for operator algebras*, second ed., Mathematical Sciences Research Institute Publications, vol. 5, Cambridge University Press, Cambridge, 1998. MR 1656031 (99g:46104)
5. L. A. Coburn, *The C^*-algebra generated by an isometry*, Bull. Amer. Math. Soc. **73** (1967), 722–726. MR 0213906 (35 #4760)

6. Joel M. Cohen, *Operator norms on free groups*, Boll. Un. Mat. Ital. B (6) **1** (1982), no. 3, 1055–1065. MR 683492 (85d:22011)
7. John B. Conway, *A course in functional analysis*, second ed., Graduate Texts in Mathematics, vol. 96, Springer-Verlag, New York, 1990. MR 1070713 (91e:46001)
8. Joachim Cuntz, *Simple C^*-algebras generated by isometries*, Comm. Math. Phys. **57** (1977), no. 2, 173–185. MR 0467330 (57 #7189)
9. Kenneth R. Davidson, *C^*-algebras by example*, Fields Institute Monographs, vol. 6, American Mathematical Society, Providence, RI, 1996. MR 1402012 (97i:46095)
10. Alessandro Figà-Talamanca and Massimo A. Picardello, *Harmonic analysis on free groups*, Lecture Notes in Pure and Applied Mathematics, vol. 87, Marcel Dekker Inc., New York, 1983. MR 710827 (85j:43001)
11. Alessandro Figà-Talamanca and Tim Steger, *Harmonic analysis for anisotropic random walks on homogeneous trees*, Mem. Amer. Math. Soc. **110** (1994), no. 531, xii+68. MR 1219707 (95a:22003)
12. James G. Glimm and Richard V. Kadison, *Unitary operators in C^*-algebras*, Pacific J. Math. **10** (1960), 547–556. MR 0115104 (22 #5906)
13. V. Jones and V. S. Sunder, *Introduction to subfactors*, London Mathematical Society Lecture Note Series, vol. 234, Cambridge University Press, Cambridge, 1997. MR 1473221 (98h:46067)
14. Richard V. Kadison, *Irreducible operator algebras*, Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 273–276. MR 0085484 (19,47e)
15. M. Gabriella Kuhn and Tim Steger, *Paschke's conjecture for the endpoint anisotropic series representations of the free group*, J. Aust. Math. Soc. **74** (2003), no. 2, 173–183. MR 1957877 (2003k:22007)
16. Christopher Lance, *On nuclear C^*-algebras*, J. Functional Analysis **12** (1973), 157–176. MR 0344901 (49 #9640)
17. William L. Paschke, *Pure eigenstates for the sum of generators of the free group*, Pacific J. Math. **197** (2001), no. 1, 151–171. MR 1810213 (2001m:46122)
18. ———, *Some irreducible free group representations in which a linear combination of the generators has an eigenvalue*, J. Aust. Math. Soc. **72** (2002), no. 2, 257–286. MR 1887136 (2003e:46100)
19. Robert T. Powers, *Simplicity of the C^*-algebra associated with the free group on two generators*, Duke Math. J. **42** (1975), 151–156. MR 0374334 (51 #10534)
20. T. Pytlik, *Radial functions on free groups and a decomposition of the regular representation into irreducible components*, J. Reine Angew. Math. **326** (1981), 124–135. MR 622348 (84a:22017)
21. M. Rørdam, *Classification of nuclear, simple C^*-algebras*, Classification of nuclear C^*-algebras. Entropy in operator algebras, Encyclopaedia Math. Sci., vol. 126, Springer, Berlin, 2002, pp. 1–145. MR 1878882 (2003i:46060)
22. D. V. Voiculescu, K. J. Dykema, and A. Nica, *Free random variables*, CRM Monograph Series, vol. 1, American Mathematical Society, Providence, RI, 1992, A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups. MR 1217253 (94c:46133)
23. Dan Voiculescu, *Symmetries of some reduced free product C^*-algebras*, Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983), Lecture Notes in Math., vol. 1132, Springer, Berlin, 1985, pp. 556–588. MR 799593 (87d:46075)

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