② ⇒ ③: Assume there exists $u \in \mathcal{U}(A)$ with $q = upu^*$.

Let $v = up$, $w = u(1_A - p)$

Then
$$\begin{cases} vv^* = (up)(pu^*) = upu^* = q. \\ v^*v = pu^*up = p^2 = p \end{cases}$$

and
$$\begin{cases} ww^* = u(1_A - p)(1_A - p)^* u^* \\ \quad = u(1_A - p)u^* \\ \quad = uu^* - upu^* \\ \quad = 1_A - q \\ w^*w = (1_A - p)u^*u(1_A - p) \\ \quad = 1_A - p. \end{cases}$$

⇒ ③ holds.

③ ⇒ ①: Assume
$$\begin{cases} v^*v = p \\ vv^* = q \\ w^*w = 1_A - p \\ ww^* = 1_A - q. \end{cases}$$
v, w partial isometries

Recall: $f = 1_{\tilde{A}} - 1_A$, a projection, where $1_{\tilde{A}} = (0, 1)$

$z = v + w + f \in \tilde{A}$.

Then z is unitary and $zpz^* = q$ (so ① will hold).

Note: $v^*v + w^*w + f^*f = p + 1_A - p + 1_{\tilde{A}} - 1_A$
$$= 1_{\tilde{A}}$$

Similarly, $vv^* + ww^* + ff^* = q + 1_A - q + 1_{\tilde{A}} - 1_A$
$$= 1_{\tilde{A}}.$$

Apply Corollary (last time) to get $z \in \mathcal{U}(\tilde{A})$.

$$zpz^* = (v+w+f)p(v^*+w^*+f^*)$$
$$= vpv^* + vpw^* + \overset{0}{\cancel{vpf}} + wpv^* + wpw^* + \overset{0}{\cancel{wpf}}$$
$$+ \underset{0}{\cancel{fpv^*}} + \underset{0}{\cancel{fpw^*}} + \underset{0}{\cancel{fpf}}$$

since if $a \in A$, $af = 0$, $+ fa = 0$.

$$= vpv^* + vpw^* + wpv^* + wpw^*$$

Imagine $A \subseteq \mathcal{B}(\mathcal{H})$. $\operatorname{Ran} w^* = \operatorname{Ran} w^*w$
$= \operatorname{Ran}(1_A - p)$

$\Rightarrow$ $pw^* = 0$, and $wp = 0$ (take adjoints)

$\Rightarrow$ $zpz^* = vpv^*$ $\quad$ $\operatorname{Ran} v^* = \operatorname{Ran} p$
$= vv^*$ $\quad$ since $v^*v = p$.
$= q$.

$\Rightarrow$ ① holds. ∎

could also compute without appealing to Ran, doing a more algebraic proof.

<u>Lemma 2.2.3:</u> Let p be a projection in a unital C^*-algebra A. Let $a \in A$ be self-adjoint with $\|p-a\| = \delta < 1/2$. Then
$$\sigma(a) \subseteq [-\delta, \delta] \cup [1-\delta, 1+\delta]$$

note: $\sigma(p) = \{0,1\}$.

<u>proof:</u> Consider $t \in \mathbb{R}$ with $d = \operatorname{dist}(t, \{0,1\}) > \delta$.
It is enough to show that
$$(a - t1)^{-1}$$
exists. (Then $t \notin \sigma(a)$).

$$\|(p - t1)^{-1}\| = \max\left\{\frac{1}{|t|}, \frac{1}{|1-t|}\right\} = \frac{1}{d}.$$

$$\begin{aligned}\|\underbrace{(p - t1)^{-1}\cdot(a - t1)} - 1\| &= \|(p-t1)^{-1}(a-p+p-t1) - 1\| \\ &= \|(p-t1)^{-1}(a-p)\| \\ &\le \delta\left(\tfrac{1}{d}\right) \\ &< 1. \qquad \text{since } \delta < d.\end{aligned}$$

$\Rightarrow$ $(p-t1)^{-1}(a-t1)$ is invertible.

$\Rightarrow$ $(a-t1)$ is invertible. ∎

<u>Note</u>: If p, q are projections in a C^*-algebra A, then

$$\|p-q\| \le 1.$$

<u>To see</u>: Assume $A \subseteq \mathcal{B}(\mathcal{H})$.
If $x \in \mathcal{H}$, $\|x\|=1$, $\langle (p-q)x, x\rangle \le \langle px, x\rangle \le 1$
and similarly,
$-1 \le \langle -qx, x\rangle \le \langle (p-q)x, x\rangle$.

Since $p-q$ is self-adjoint, $\|p-q\| = \sup_{\|x\|=1} |\langle (p-q)x, x\rangle|$
$\le 1.$ ← C^*-algebra

<u>Proposition</u>: If p, q are projections in A and $\|p-q\| < 1$ then $p \sim_h q$ in $\mathcal{P}(A)$.

<u>proof</u>: Let $a_t = (1-t)p + tq$, $0 \le t \le 1$.
Then a_t is self-adjoint, gives continuous curve from p to q.

$$\|a_t - p\| = \|(1-t)p + tq) - p\| = t\|p-q\|.$$

and
$$\|a_t - q\| = (1-t)\|p-q\|.$$
So $\min\{\|a_t - p\|, \|a_t - q\|\} \leq \frac{1}{2}\|p-q\| < \frac{1}{2}$.

Let $\delta = \frac{1}{2}\|p-q\| < \frac{1}{2}$.

Let $K = [-\delta, \delta] \cup [1-\delta, 1+\delta]$.

Let $\Omega_K = \{a \in \tilde{A} : a$ is self-adjoint and $\sigma(a) \subseteq K\}$.
By lemma 1.2.5, we know that $f \in C(K)$ is fixed, the map $a \mapsto f(a)$ is continuous from $\Omega_K \to \tilde{A}$.

Define
$$f(t) = \begin{cases} 0, & -\delta \leq t \leq \delta \\ 1, & 1-\delta \leq t \leq 1+\delta \end{cases}$$
Then f is a projection in $C(K)$.

Hence $f(a_t)$ is a projection in $\tilde{A}$. But since $\tilde{A} = A \oplus \mathbb{C}$ and $a_0 = p_0 \oplus 0$, $a_1 = p_1 \oplus 0$ $\Rightarrow f(a_t)$ is a projection in A. (projections in $\mathbb{C}$ are $0, 1$)

Note: $f(a_0) = f(p) = p$,
$f(a_1) = f(q) = q$.

$\boxtimes$

Proposition 2.2.5: a, b are self adjoint in a unital A. Suppose $b = zaz^{-1}$ for some invertible z.
Let $z = u|z|$, u unitary in A.
then $b = uau^*$.

proof: $bz = za$

$\Rightarrow$ $z^* b = az^*$

So $|z|^2 a = z^* z a = z^* b z = a z^* z = a|z|^2$

$\Rightarrow$ a commutes with $|z|^2$ $\Rightarrow$ a commutes with any element of $C^*(1, |z|^2)$.
In particular, $a|z|^{-1} = |z|^{-1}a$, by an "easy approximation argument."

So $ua = z|z|^{-1}a$
$= z\, a\, |z|^{-1}$

$= b\, z\, |z|^{-1}$
$= bu.$

$\Rightarrow$ $b = uau^*$. $\boxtimes$

<u>Proposition 2.2.6</u>: Let $p, q \in \mathcal{P}(A)$. Then

$$p \sim_h q \text{ in } \mathcal{P}(A) \iff q = upu^* \text{ where } u \in \mathcal{U}_0(\tilde{A}).$$