NONCOMMUTATIVE L^p MODULES

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ABSTRACT. We construct classes of von Neumann algebra modules by considering "column sums" of noncommutative L^p spaces. Our abstract characterization is based on an $L^{p/2}$-valued inner product, thereby generalizing Hilbert C*-modules and representations on Hilbert space. While the (single) representation theory is similar to the L^2 case, the concept of L^p bimodule ($p \neq 2$) turns out to be nearly trivial.

0. INTRODUCTION

Noncommutative L^p spaces, by now, are standard objects in the theory of operator algebras. Starting with a von Neumann algebra $\mathcal{M}$, there are a variety of equivalent methods for producing the (quasi-)Banach space $L^p(\mathcal{M})$. If $\mathcal{M}$ is $L^\infty(X,\mu)$, the result is (isometric to) $L^p(X,\mu)$, so this can rightfully be thought of as a generalization to noncommutative measure spaces. When $\mathcal{M}$ is semifinite, the presence of a trace offers great simplification, but in general one needs modular theory [H].

These spaces have many aspects worthy of investigation. As Banach spaces, their isometries have been investigated by many authors [Ye], [W2]; others have used the matrix order [Sc] or operator space techniques [JNRX]. (For a more complete bibliography see [PX].) We focus here on the *module* structure. Indeed, the inclusion as left (or right) multipliers

$$\mathcal{M} \hookrightarrow \mathcal{B}(L^p(\mathcal{M}))$$

is isometric. If Hilbert space representations are (categorically) generated by $L^2(\mathcal{M})$, and self-dual C*-modules are generated by $L^\infty(\mathcal{M}) = \mathcal{M}$, where are the modules generated by $L^p(\mathcal{M})$? This paper sets out to describe the missing L^p representation theory.

Proceeding by analogy, our target is the class of "columns of $L^p(\mathcal{M})$". We show that a sufficient condition for an $\mathcal{M}$-module to belong to this class is the existence of an $L^{p/2}(\mathcal{M})$-*valued inner product*. The description which results is a natural generalization of the cases $p = 2$ (the usual decomposition for Hilbert space representations) and $p = \infty$ (see [Pa]). We employ a variety of methods, but perhaps the most notable direction is a consistent translation of Connes' L^2 spatial theory [C] to the L^p setting.

Building on results about the module structure of $L^p(\mathcal{M})$ which are interesting in their own right, we find that the L^p representation theory is largely analogous to the L^2 case, with a well-behaved sum and relative tensor product. It would therefore seem natural that there be a similarly rich bimodule category, i.e. a theory of L^p correspondences. But surprisingly, the category is nearly trivial: when $p \neq 2$, there

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is an L^p $\mathcal{M}$-$\mathcal{N}$ bimodule if and only if $\mathcal{M}$ and $\mathcal{N}$ are Morita equivalent. Modulo a possible degeneracy where both algebras are abelian, such bimodules naturally implement an equivalence of appropriate representation categories.

Only one application - to ultraproducts - of our theory is given. We plan to discuss further examples and development in future articles.

1. The module structure of $L^p(\mathcal{M})$

Throughout, $\mathcal{M}$, $\mathcal{N}$, etc. are von Neumann algebras; we frequently abbreviate $L^p(\mathcal{M})$ to L^p and understand $L^\infty(\mathcal{M})$ as $\mathcal{M}$. All weights are normal and semifinite, so we omit the adjectives for brevity. Unsubscripted $\mathfrak{H}$ denotes the separable infinite-dimensional Hilbert space, $s(\varphi)$ is the support of φ, and $s_\ell(x)$ (resp. $s_r(x)$) stands for the left (resp. right) support of x. Subscripts are occasionally used to represent an action: e.g. $\mathfrak{X}_\mathcal{M}$ indicates that $\mathfrak{X}$ is a right $\mathcal{M}$-module. But when the expressions are longer, we signify a bimodule by writing out the triple: an $\mathcal{M}$-$\mathcal{N}$ bimodule $\mathfrak{X}$ is $\mathcal{M}$-$\mathfrak{X}$-$\mathcal{N}$. The phrase "left (resp. right) action of" is freqently abbreviated to L (resp. R) for operators or entire algebras, so that we speak of $L(x)$ or $R(\mathcal{M})$. Finally, we often write M_∞ for $\mathcal{B}(\mathfrak{H})$ and $M_\infty(\mathcal{M})$ for $\mathcal{B}(\mathfrak{H})\otimes\mathcal{M}$. Note that in contrast to much of the literature, the results of this paper (except for Section 6) do not require that algebras be σ-finite or that $p \geq 1$.

We assume that the reader has some basic familiarity with noncommutative L^p spaces. Conceptually, one can think

$$L^p_+ = (L^1_+)^{1/p},$$

and we take this as a basis for our notation: the typical positive element is $\varphi^{1/p}$, where φ is a positive linear functional on $\mathcal{M}$. What this *means* is a matter of perspective, as there are many equivalent constructions of L^p, but we find the Haagerup construction [H] most useful. In this setting L^p is exactly the set of τ-measurable operators affiliated with the core $\widetilde{\mathcal{M}} \simeq \mathcal{M} \rtimes_\sigma \mathbb{R}$ which are $1/p$-scaled by the dual action: $\theta_s(T) = e^{-s/p}T$. (The operator we call φ is more commonly called h_φ. An unbounded weight corresponds to a positive operator satisfying all the above conditions except for τ-measurability.) Operator concepts like composition, positivity, left and right support, adjoint, and polar decomposition transfer directly into the L^p setting. Basic exposition can be found in [Te], and the reader is also referred to the elegant "coordinate-free" approaches in [Y] (more algebraic) and [FT] (more analytic). We use the Haagerup notation Tr for the evaluation functional on L^1:

$$\mathrm{Tr}(\omega) = \omega(1),$$

and recall that Tr implements the "tracial" duality between L^p and L^q:

$$< \xi, \eta >= \mathrm{Tr}(\xi\eta) = \mathrm{Tr}(\eta\xi), \qquad \xi \in L^p, \eta \in L^q, \ \frac{1}{p} + \frac{1}{q} = 1.$$

In this notation,

$$\psi^{it}\varphi^{-it} = (D\psi : D\varphi)_t; \quad \varphi^{it}x\varphi^{-it} = \sigma^\varphi_t(x), \qquad (t \in \mathbb{R})$$

whenever $s(\varphi)$ dominates $s(\psi), s_\ell(x), s_r(x)$. The cocycles or modular automorphism groups extend off the imaginary line exactly when the corresponding operator compositions do. For more discussion of negative powers of states, see [S2].

A fundamental fact for us is Kosaki's generalized Hölder inequality [K2]:

$$\|\xi\eta\|_r \leq \|\xi\|_p \|\eta\|_q, \quad \xi \in L^p,\ \eta \in L^q,\ \frac{1}{p}+\frac{1}{q}=\frac{1}{r}.$$

In particular, left or right multiplication by an element $x \in \mathcal{M}$ is bounded with norm $\leq \|x\|$. We will show a stronger fact momentarily, but first recall

$$\varphi^{1/p} \geq \psi^{1/p} \iff \varphi^{-\frac{1}{2p}}\psi^{\frac{1}{2p}} \text{ is a contraction in } s(\varphi)\mathcal{M}s(\varphi) \subset \mathcal{M}. \tag{1.1}$$

Then

$$\varphi^{1/p} \geq \psi^{1/p} \Rightarrow \|\psi^{1/p}\| = \|(\psi^{\frac{1}{2p}}\varphi^{-\frac{1}{2p}})\varphi^{1/p}(\varphi^{-\frac{1}{2p}}\psi^{\frac{1}{2p}})\| \leq \|\varphi^{1/p}\|. \tag{1.2}$$

Lemma 1.1. *Let $x \geq 0$, and consider the map $\xi \mapsto x\xi$ on L^p. We have*

$$\inf sp(x) = \inf_{\|\xi\|=1} \|x\xi\|, \quad \|x\| = \sup sp(x) = \sup_{\|\xi\|=1} \|x\xi\|.$$

Proof. We discuss $p < \infty$; $p = \infty$ only requires different wording.

Hölder's inequality is half of the last equation. To see the opposite inequality, choose ε and let φ be a state supported on the spectral projection $q = e([\|x\| - \varepsilon, \|x\|])$ of x. We use (1.2) above to get

$$\|x\varphi^{1/p}\|_p = \|\varphi^{1/p}qx^2q\varphi^{1/p}\|_{p/2}^{1/2} \geq \|\varphi^{1/p}q(\|x\|-\epsilon)^2q\varphi^{1/p}\|_{p/2}^{1/2} = (\|x\|-\varepsilon)\|\varphi^{1/p}\|_p.$$

The first equation is proven similarly. □

For an arbitrary element with polar decomposition $x = v|x|$, we have $\|x\xi\| = \||x|\xi\|$, and the proposition alters naturally by considering the spectrum of $|x|$.

This does not give us an "L^p spatial spectral theorem". A positive operator x generates a projection-valued decomposition of the identity, and the action on L^p is still "multiplication" (in an appropriate sense) by $\int \lambda de(\lambda)$. But for disjoint sets I and J, there is no simple norm relation between the L^p elements $e(I)\xi$, $e(J)\xi$ and their sum unless $p = 2$. This prevents us from using vectors to provide (pth roots of) measures, and we cannot write, say, $\|\xi\|^p = \int d\|e(\lambda)\xi\|^p$.

Now we turn to a full description of the intertwiner set $\mathrm{Hom}(L^p_{\mathcal{M}}, L^q_{\mathcal{M}})$. The next three lemmas facilitate the proofs; the second is a slight improvement of [J], Lemma 2.3.

Lemma 1.2. *For $\varphi \in \mathcal{M}_*^+$,*

$$\overline{\mathcal{M}\varphi^{1/p}} = L^p s(\varphi), \qquad \overline{\varphi^{1/p}\mathcal{M}} = s(\varphi)L^p.$$

Proof. This result is well-known for $p \geq 1$, but we present a full proof for completeness.

Let $p > 1$. Suppose there is $\xi \in s(\varphi)L^p \setminus \overline{\varphi^{1/p}\mathcal{M}}$. By Hahn-Banach separation we may find $\eta \in L^q$ (p, q conjugate exponents) with

$$\mathrm{Tr}(\eta\xi) > 0, \qquad 0 = \mathrm{Tr}(\eta\varphi^{1/p}y), \quad \forall y \in \mathcal{M}.$$

Then we must have $\eta\varphi^{1/p} = 0$, so $\eta s(\varphi) = 0$. But

$$0 < \mathrm{Tr}(\eta\xi) = \mathrm{Tr}(\eta s(\varphi)\xi) = 0,$$

a contradiction. By a symmetric argument we have $\overline{\mathcal{M}\varphi^{1/p}} = L^p s(\varphi)$.

Keep the same p, and assume that we have

$$\overline{\mathcal{M}\varphi^{n/p}} = L^{p/n} s(\varphi), \qquad \overline{\varphi^{n/p}\mathcal{M}} = s(\varphi)L^{p/n}$$

for a positive integer n. We compute

$$\begin{aligned}\overline{\varphi^{(n+1)/p}\mathcal{M}} &= \overline{\varphi^{1/p}\,\overline{\varphi^{n/p}\mathcal{M}}} = \overline{\varphi^{1/p}L^{p/n}} = \overline{\varphi^{1/p}\mathcal{M}L^{p/n}} \\ &= \overline{s(\varphi)L^p \cdot L^{p/n}} = s(\varphi)L^{p/(n+1)},\end{aligned}$$

where the first equality is justified by Hölder: if $\varphi^{n/p}x_j$ converges, so does $\varphi^{(n+1)/p}x_j$. The other equality is obtained similarly.

Since any positive number can be written as p/n with $p > 1$ and n a positive integer, the result follows by induction. □

Lemma 1.3. *Let $\varphi \in \mathcal{M}_*^+$, $p > 0$, and $\{x_\alpha\}$ be a bounded net in $\mathcal{M}$. If*

$$(*) \quad x_\alpha \varphi^{1/p} \to 0,$$

then $x_\alpha \xi \to 0$ for any ξ in $s(\varphi)L^q$, where q is any positive real. This implies that on bounded sets, the strong topology that $\mathcal{M}$ acquires from its action on L^p does not depend on p.

Proof. Suppose

$$(*) \quad x_\alpha \varphi^{\beta} \to 0$$

for $\beta = 1/p$. Then $(*)$ holds for $\beta > 1/p$, as

$$\|x_\alpha \varphi^\beta\| \le \|x_\alpha \varphi^{1/p}\| \|\varphi^{\beta - 1/p}\| \to 0.$$

We may also conclude that $(*)$ holds for $\beta = \frac{1}{2p}$ by

$$\|x_\alpha \varphi^{\frac{1}{2p}}\|^2 = \|x_\alpha \varphi^{1/p} x_\alpha^*\| \le \|x_\alpha \varphi^{1/p}\| \|x_\alpha^*\| \to 0,$$

since $\|x_\alpha^*\|$ is bounded. Together these two steps imply that $(*)$ holds for all positive β.

Now suppose that $\xi \in s(\varphi)L^q = \overline{\varphi^{1/q}\mathcal{M}}$. Given $\varepsilon > 0$, choose $y \in \mathcal{M}$ so that

$$\|\xi - \varphi^{1/q} y\| \le \frac{\varepsilon}{2 \sup \|x_\alpha\|}.$$

Then

$$\begin{aligned}\|x_\alpha \xi\| &\le \|x_\alpha \xi - x_\alpha \varphi^{1/q} y\| + \|x_\alpha \varphi^{1/q} y\| \\ &\le (\sup \|x_\alpha\|)\, \|\xi - \varphi^{1/q} y\| + \|x_\alpha \varphi^{1/q}\| \|y\|,\end{aligned}$$

which is less than ε when α is so large that $\|x_\alpha \varphi^{1/q}\| < \frac{\varepsilon}{2\|y\|}$.

When $\mathcal{M}$ is σ-finite, this last step is the L^p version of the well-known fact that for a faithful state φ,

$$x \mapsto \varphi(x^* x)^{1/2}$$

implements the strong topology on bounded sets of $\mathcal{M}$. □

Lemma 1.4. *Let $\{\xi_\alpha\} \subset L^r \quad (r < \infty), \quad \{p_\alpha\} \subset \mathcal{P}(\mathcal{M})$ be nets such that*

$$\xi_\alpha = \xi_\alpha p_\alpha, \quad \xi_\beta p_\alpha = \xi_\alpha \text{ for } \alpha < \beta, \quad \sup_\alpha \|\xi_\alpha\| = C < \infty. \tag{1.3}$$

Then ξ_α converges in norm, say to ξ, and $\xi_\beta = \xi p_\beta$.

The idea is that adding columns (=increasing the right support) without exceeding an L^r bound implies convergence in L^r.

Proof. First we handle the case where $r > 2$. We have $\xi_\alpha \xi_\alpha^*$ increasing and norm-bounded; let $\varphi^{2/r}$ be the weak-* limit in the reflexive Banach space $L^{r/2}$ and write

$$\xi_\alpha \xi_\alpha^* = \varphi^{1/r} x_\alpha \varphi^{1/r} \text{ with } x_\alpha \le q = s(\varphi).$$

Using $L^r(q\mathcal{M}q) = qL^r(\mathcal{M})q$, weak convergence implies that

$$< \varphi^{1/r} x_\alpha \varphi^{1/r}, \psi^{1/s} > \to < \varphi^{2/r}, \psi^{1/s} >, \qquad \forall \psi \in qL^s q \quad (\frac{2}{r} + \frac{1}{s} = 1),$$

or

$$< x_\alpha, \varphi^{1/r} \psi^{1/s} \varphi^{1/r} > \to < q, \varphi^{1/r} \psi^{1/s} \varphi^{1/r} >;$$

that is, $x_\alpha = q x_\alpha q \nearrow q$ weakly in $\mathcal{M}$. Now

$$q \ge x_\alpha^{1/2} \ge x_\alpha \nearrow q \text{ weakly } \Rightarrow x_\alpha^{1/2} \nearrow q \text{ weakly}$$
$$\Rightarrow (q - x_\alpha^{1/2})^2 = q + x_\alpha - 2x_\alpha^{1/2} \searrow 0 \text{ weakly},$$

so $x_\alpha^{1/2} \nearrow q$ strongly. By the preceding lemma,

$$\varphi^{1/r} x_\alpha^{1/2} \to \varphi^{1/r},$$

and therefore

$$\xi_\alpha \xi_\alpha^* = (\varphi^{1/r} x_\alpha^{1/2})(\varphi^{1/r} x_\alpha^{1/2})^* \to \varphi^{2/r}.$$

Finally, for $\alpha < \beta$ the increasing right supports imply

$$\|\xi_\alpha - \xi_\beta\|^2 = \|\xi_\alpha \xi_\alpha^* + \xi_\beta \xi_\beta^* - \xi_\alpha \xi_\beta^* - \xi_\beta \xi_\alpha^*\|$$
$$= \|\xi_\alpha \xi_\alpha^* + \xi_\beta \xi_\beta^* - \xi_\alpha \xi_\alpha^* - \xi_\alpha \xi_\alpha^*\| = \|\xi_\beta \xi_\beta^* - \xi_\alpha \xi_\alpha^*\| \to 0.$$

If $r \le 2$, still $\xi_\alpha \xi_\alpha^*$ is increasing in $L^{r/2}$ and bounded. Choose $\gamma > 2/r$; $(\xi_\alpha \xi_\alpha^*)^{1/\gamma}$ is then norm-bounded and increasing in a reflexive Banach space. By the above argument it converges in norm, so the continuity of exponentiation (see [R], Lemma 3.2) implies $\xi_\alpha \xi_\alpha^* = ((\xi_\alpha \xi_\alpha^*)^{1/\gamma})^\gamma$ converges in $L^{r/2}$. The last computation of the previous paragraph again shows the convergence of ξ_α.

Finally, set $\xi = \lim_\alpha \xi_\alpha$ and use that right multiplication by p_β is continuous:

$$\xi p_\beta = (\lim_\alpha \xi_\alpha) p_\beta = \lim_\alpha (\xi_\alpha p_\beta) = \lim_\alpha \xi_\beta = \xi_\beta.$$

□

When $p = \infty$, Lemma 1.4 still holds. The same line of argument works, but instead of reflexivity one uses that von Neumann algebras are monotone closed.

The next theorem extends work of several authors and solves a problem stated in Yamagami [Y].

Theorem 1.5. *If $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$, then any bounded map in $Hom(L^p_{\mathcal{M}}, L^q_{\mathcal{M}})$ is left composition with some element of L^r.*

Proof. Let T be such a map. If $p = \infty$, this is easy: $T(x) = T(1)x$. So assume $p < \infty$, and for the moment assume $\mathcal{M}$ is σ-finite. Choose a faithful $\varphi \in \mathcal{M}_*^+$. With

$$T(\varphi^{1/p}) = v\psi^{1/q}$$

the polar decomposition, set

$$\rho^{2/p} = \varphi^{2/p} + \psi^{2/p}.$$

and write

$$\psi^{1/p} = y_1 \rho^{1/p}; \qquad \varphi^{1/p} = y_2 \rho^{1/p}$$

with y_1, y_2 contractive. The module property means that for any $x \in \mathcal{M}$,

$$T(y_2 \rho^{1/p} x) = v\psi^{1/q} x = v\psi^{1/r} y_1 \rho^{1/p} x.$$

By continuity of T we may conclude

$$T(y_2 \xi) = v\psi^{1/r} y_1 \xi$$

for all $\xi \in L^p$.

Now let $y_2 = |y_2^*|u$ be the polar decomposition and q_n be the spectral projection of $|y_2^*|$ corresponding to $[\frac{1}{n}, 1]$. Since

$$q_n = y_2 u^* |y_2^*|^{-1} q_n,$$

$$T(q_n \xi) = T(y_2 u^* |y_2^*|^{-1} q_n \xi) = (v\psi^{1/r} y_1 u^* |y_2^*|^{-1} q_n)(q_n \xi).$$

It follows from this that

$$\|v\psi^{1/r} y_1 u^* |y_2^*|^{-1} q_n\|_r \le \|T\|$$

for all n, and notice the q_n are increasing to 1 since y_2 is nonsingular.

If $r < \infty$, Lemma 1.4 allows us to conclude the convergence of this sequence; say

$$v\psi^{1/r} y_1 u^* |y_2^*|^{-1} q_n \to \eta.$$

Since T agrees with $L(\eta)$ on the dense set $\cup q_n L^p$, they are identical.

If $r = \infty$, then $\psi^{1/r}$ can be replaced with 1. The uniform bound implies that $v y_1 u^* |y_2^*|^{-1} q_n$ converges strongly to an operator z with $\|z\| \le \|T\|$. Again, T and $L(z)$ agree on $\cup q_n L^p$, so they are identical.

Now we remove the σ-finiteness assumption.

Let $r < \infty$. If s is a σ-finite projection in $\mathcal{M}$, we may find a state φ with $s(\varphi) = s$ and apply the same argument to conclude

$$T \mid_{sL^p} = L(\eta_s).$$

Then the η_s satisfy

$$\eta_s = \eta_s s, \qquad \eta_t s = \eta_s \text{ for } s < t, \qquad \|\eta_s\| \le \|T\|.$$

Lemma 1.4 tells us that η_s converges along the naturally-ordered net of σ-finite projections, say to η, and $\eta_s = \eta s$. Finally, if $\xi \in L^p$, $f = s_\ell(\xi)$ must be σ-finite, and

$$T(\xi) = T(f\xi) = \eta_f \xi = \eta f \xi = \eta \xi.$$

In case $r = \infty$, the vectors η_s, η are replaced by operators z_s, z. □

We single out the case $r = \infty$ as a separate corollary. Though basic, there does not seem to be a proof for general p in the literature. (Terp [Te] settled the case $p \ge 1$ by different methods.)

Corollary 1.6. *The left and right actions of $\mathcal{M}$ on L^p are commutants of each other.*

Notice that for $p \ge 1$, $(L^p)^* = L^{p'}$ can be identified with $\mathrm{Hom}(L^p_{\mathcal{M}}, L^1_{\mathcal{M}})$, with Tr implementing the duality as usual. It is known [W1] that $(L^p)^* = \{0\}$ when $p < 1$ and $\mathcal{M}$ has no minimal projection; compare that with

Corollary 1.7. *If $\mathcal{M}$ has no minimal projection and $p < q$,*

$$Hom(L^p_{\mathcal{M}}, L^q_{\mathcal{M}}) = \{0\}.$$

Proof. Choose a state φ. If T is a bounded morphism and $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$, set

$$\widetilde{T} : L^q_{\mathcal{M}} \to L^q_{\mathcal{M}} \text{ by } \widetilde{T}(\xi) \mapsto T(\varphi^{1/r}\xi).$$

This is a bounded module map, so by the preceding corollary there must be $x \in \mathcal{M}$ with

$$x\xi = \widetilde{T}(\xi) = T(\varphi^{1/r}\xi).$$

If $x \neq 0$, let $x = v|x|$ and $e = e(\varepsilon, \infty)$ be a nonzero spectral projection of $|x|$. For all $\xi \in L^p$, we have

$$\|e\xi\|_q = \|ve\xi\| = \|v|x|e|x|^{-1}e\xi\| = \|T(\varphi^{1/r}e|x|^{-1}e\xi)\| = \|T(\eta e\xi)\| \leq C\|\eta e\xi\|_p,$$

where $\eta = \varphi^{1/r}e|x|^{-1}e$. It remains to show that such a "reversed Hölder inequality" cannot hold.

Let f_n be a decreasing sequence of nonzero projections $\leq e$ and converging strongly to 0. (This is where nonatomicity is essential.) Then by Lemma 1.3, $\|\eta f_n\| \to 0$. Choose an element f with

$$\|\eta f\|_r < 1/C.$$

Now take a functional ρ with $s(\rho) = f$. It follows that

$$\|\rho^{1/q}\|_q \leq C\|\eta f\rho^{1/q}\|_p \leq C\|\eta f\|_r\|\rho^{1/q}\|_q < \|\rho^{1/q}\|_q,$$

which is impossible. So $x = 0$, which implies $T(\varphi^{1/r}\cdot)$ is the zero map. Since this holds for any choice of φ, T must also be the zero map. □

2. L^p MODULES

Now we turn to the development of an L^p representation theory. Note that this cannot mean representations on classical L^p spaces: $L^p(\mathcal{M})$ itself is not a classical L^p space unless $p = 2$ or $\mathcal{M}$ is commutative. We would like to build the category out of $L^p(\mathcal{M})$ in the same way that nondegenerate normal right Hilbert space representations are built out of $L^2(\mathcal{M})$.

Let us examine a countably generated Hilbert module $\mathfrak{H}_{\mathcal{M}}$. Following standard arguments, $\mathfrak{H}$ decomposes into a direct sum of cyclic representations $(\overline{\xi_n\mathcal{M}})_{\mathcal{M}}$, each of which is isomorphic to the GNS representation for the associated vector state, and all GNS representations are reductions of $L^2(\mathcal{M})$. So we have

$$\mathfrak{H}_{\mathcal{M}} \simeq (\bigoplus \overline{\xi_n\mathcal{M}})_{\mathcal{M}} \simeq (\bigoplus \mathfrak{H}_{\omega_{\xi_n}})_{\mathcal{M}} \simeq (\bigoplus q_nL^2(\mathcal{M}))_{\mathcal{M}}.$$

(In fact $q_n = s_\ell(\xi_n)$.)

Since this is a right module, it is natural to write vectors as columns with the nth entry in q_nL^2:

$$\mathfrak{H} \simeq \begin{pmatrix} q_1L^2(\mathcal{M}) \\ q_2L^2(\mathcal{M}) \\ \vdots \end{pmatrix} \simeq (\sum q_n \otimes e_{nn}) \begin{pmatrix} L^2(\mathcal{M}) \\ L^2(\mathcal{M}) \\ \vdots \end{pmatrix}.$$

Here e_{nn} are diagonal matrix units in M_∞, so $(\sum q_n \otimes e_{nn})$ is a diagonal projection in $\mathcal{M} \otimes \mathcal{B}(\mathfrak{H})$. The right action of $\mathcal{M}$ is, of course, matrix multiplication (by 1×1 matrices) on the right. Modules which are not countably generated can be represented by columns and projections of larger size, and non-diagonal projections work equally well - see Section 5.

Our target class of modules is obtained by replacing the index 2 by p. Although this seems simple enough, the geometry of such spaces presents certain difficulties.

To start with, one cannot obtain the norm of a column via an ℓ^p (or ℓ^2) sum. The following example will serve as motivation.

Consider the right L^p $\mathcal{M}$-module

$$\mathfrak{X} = \left(\begin{smallmatrix} L^p \\ L^p \end{smallmatrix}\right).$$

This should be a left L^p $M_2(\mathcal{M})$-module, as it is

$$L^p(M_2(\mathcal{M}))e_{11} = \left(\begin{smallmatrix} L^p(\mathcal{M}) & 0 \\ L^p(\mathcal{M}) & 0 \end{smallmatrix}\right).$$

It is then a left submodule of $L^p(M_2(\mathcal{M}))$ and so inherits the norm:

$$\| \left(\begin{smallmatrix} \xi \\ \eta \end{smallmatrix}\right) \| = \| \left(\begin{smallmatrix} \xi & 0 \\ \eta & 0 \end{smallmatrix}\right) \|_{L^p(M_2(\mathcal{M}))} = \|\xi^*\xi + \eta^*\eta\|_{p/2}^{1/2},$$

which is in general not purely a function of the norms of ξ and η.

Norm-determining expressions of the form $\xi^*\xi$ recall inner products in Hilbert C*-modules. Based on this parallel, we make

Definition 2.1. *Let $\mathfrak{X}$ be a complex vector space which is a right $\mathcal{M}$-module and $p \in (0, \infty]$. By an $L^{p/2}$-**valued inner product** on $\mathfrak{X}$ we mean a sesquilinear mapping, conjugate linear in the first variable, from $\mathfrak{X} \times \mathfrak{X}$ to $L^{p/2}(\mathcal{M})$ which satisfies*

(i) $< \xi, \eta x >=< \xi, \eta > x$;
(ii) $< \xi, \eta >=< \eta, \xi >^*$;
(iii) $< \xi, \xi >\geq 0$; $\quad < \xi, \xi >= 0 \iff \xi = 0$.

Proposition 2.2.

$$< \xi, \eta >=< \xi, \xi >^{1/2} T < \eta, \eta >^{1/2} \tag{2.1}$$

for some $T \in \mathcal{M}$ with $\|T\| \leq 1$. So if we set

$$||\xi|| \triangleq || < \xi, \xi >^{1/2} ||_p,$$

then

$$\| < \xi, \eta > \|_{p/2} \leq \|\xi\|\|\eta\|. \tag{2.2}$$

We have that $\| \cdot \|$ is a norm when $p \geq 2$ and a $p/2$-norm when $p \leq 2$. (This is improved by the end of the next section.)

Proof. Most of this proof is standard. For $\xi, \eta \in \mathfrak{X}$, consider the matrix

$$A = \left(\begin{smallmatrix} <\xi,\xi> & <\xi,\eta> \\ <\eta,\xi> & <\eta,\eta> \end{smallmatrix}\right) \in M_2(L^{p/2}(\mathcal{M})) \simeq L^{p/2}(M_2(\mathcal{M})).$$

We claim that A is positive. If $\mathcal{M}$ is semifinite, we may choose a faithful semifinite trace τ and consider the L^p spaces to be spaces of τ-measurable operators. For $x, y \in L^2 \cap L^\infty$, which is dense in L^2,

$$\left\langle \left(\begin{smallmatrix} <\xi,\xi> & <\xi,\eta> \\ <\eta,\xi> & <\eta,\eta> \end{smallmatrix}\right) \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \right\rangle =< \xi x + \eta y, \xi x + \eta y > \geq 0,$$

and by density the matrix is positive.

If $\mathcal{M}$ is purely infinite, then so is $M_2(\mathcal{M})$; let v be a partial isometry in $M_2(\mathcal{M})$ with $vv^* = 1$, $v^*v = e_{11}$. Thus v is of the form

$$\left(\begin{smallmatrix} v_{11} & 0 \\ v_{12} & 0 \end{smallmatrix}\right).$$

We have

$$\begin{aligned} A = vv^*Avv^* &= v \left(\begin{smallmatrix} v_{11}^* & v_{12}^* \\ 0 & 0 \end{smallmatrix}\right) A \left(\begin{smallmatrix} v_{11} & 0 \\ v_{12} & 0 \end{smallmatrix}\right) v^* \\ &= v \left(\begin{smallmatrix} <\xi v_{11} + \eta v_{12}, \xi v_{11} + \eta v_{12}> & 0 \\ 0 & 0 \end{smallmatrix}\right) v^* \geq 0. \end{aligned}$$

A von Neumann algebra decomposes as a direct sum of semifinite and purely infinite summands, so we see that A is positive in general.

Now the usual matrix manipulations give (2.1), and (2.2) follows by Hölder's inequality.

When $p \leq 2$, use the inequality from [K2]

$$\|v+w\|_q^q \leq \|v\|_q^q + \|w\|_q^q, \quad v, w \in L^q, \quad q \leq 1$$

to write

(2.3)

$$\begin{aligned}\|\xi+\eta\|^p &= \| < \xi,\xi > + < \xi,\eta > + < \eta,\xi > + < \eta,\eta > \|_{p/2}^{p/2} \\ &\leq \| < \xi,\xi > \|_{p/2}^{p/2} + \| < \xi,\eta > \|_{p/2}^{p/2} + \| < \eta,\xi > \|_{p/2}^{p/2} + \| < \eta,\eta > \|_{p/2}^{p/2} \\ &\leq \|\xi\|^p + 2\|\xi\|^{p/2}\|\eta\|^{p/2} + \|\eta\|^p \\ &= (\|\xi\|^{p/2} + \|\eta\|^{p/2})^2.\end{aligned}$$

Therefore

$$\|\xi+\eta\|^{p/2} \leq \|\xi\|^{p/2} + \|\eta\|^{p/2}.$$

When $p \geq 2$, one starts (2.3) with $\|\xi+\eta\|^2$ and proves the triangle inequality via the same manipulations. □

It is worth noting that $\|\xi x\| \leq \|\xi\|\|x\|$, so the action of $\mathcal{M}$ is continuous.

Definition 2.3. *For $p < \infty$, a right $\mathcal{M}$-module $\mathfrak{X}$ is called a **right** L^p $\mathcal{M}$**-module** if it has an $L^{p/2}$-valued inner product and is complete in the inherited (quasi)norm. For $p = \infty$, we keep this condition (so $\mathfrak{X}$ is a Hilbert C^* module) and impose the additional requirement that the unit ball of $\mathfrak{X}$ be closed in the strong topology, i.e. the topology arising from the seminorms*

$$\xi \mapsto (\varphi(< \xi,\xi >))^{1/2}, \quad \varphi \in \mathcal{M}_*^+.$$

The set $\overline{< \mathfrak{X}, \mathfrak{X} >}$ is a closed self-adjoint sub-bimodule of $L^{p/2}$, which must have the form $zL^{p/2}$ for some central projection $z \in \mathcal{M}$. So $\mathfrak{X}$ is a *faithful* right L^p $z\mathcal{M}$-module.

Examples:

- Any classical L^p space is a right L^p module for the corresponding L^∞ algebra, with inner product

$$< f, g >= \bar{f} g,$$

.

- Any normal right representation of $\mathcal{M}$ on a Hilbert space $\mathfrak{H} = \mathfrak{X}$ admits a unique structure as L^2-module by setting the inner product

$$< \xi, \eta >_{\mathfrak{X}}$$

to be the state $\omega_{\xi,\eta}$ defined by

$$< \xi, \eta x >_{\mathfrak{H}} = \omega_{\xi,\eta}(x).$$

(For coherence, the inner product in $\mathfrak{H}$ should be linear in the second argument.) On the other hand, any L^2 $\mathcal{M}$-module $\mathfrak{X}$ is also a Hilbert space via

$$< \xi, \eta >_{\mathfrak{H}} \triangleq < \xi, \eta >_{\mathfrak{X}} (1) = \mathrm{Tr}(< \xi, \eta >_{\mathfrak{X}}),$$

where Tr denotes the Haagerup trace on L^1. Since the $\mathfrak{X}$-inner product is $\mathcal{M}_*$-valued, the Hilbert space representation is automatically normal.

- $L^p(\mathcal{M})$ is a right L^p module with inner product $< \xi, \eta >= \xi^* \eta$. Similarly for qL^p, where q is a projection in $\mathcal{M}$.

We wish to highlight a special class of right L^p $\mathcal{M}$-modules; call them 'principal' for the time being. If $\{q_\alpha\}_{\alpha \in I}$ are projections in $\mathcal{M}$, the set

$$\left\{ (\xi_\alpha) \mid \xi_\alpha \in q_\alpha L^p, \ \sum \xi_\alpha^* \xi_\alpha \in L^{p/2} \right\} \tag{2.4}$$

is a right L^p $\mathcal{M}$-module with

$$< (\xi_\alpha), (\eta_\alpha) >= \sum \xi_\alpha^* \eta_\alpha.$$

For $p = \infty$, Paschke [P] showed that the directed net of finite sums converges strongly; he called this construction an *ultraweak direct sum*.

For $p < \infty$, the limit (of finite sums) exists in norm. This follows from the Cauchy-Schwarz inequality, which can be proven directly as follows. Let $\mathfrak{H}_I$ be the Hilbert space with dimension $|I|$, and set

$$\tilde{\xi} = \sum \xi_\alpha \otimes e_{\alpha 1} \in L^p(\mathcal{M} \otimes \mathcal{B}(\mathfrak{H}_I)).$$

(So we are placing ξ along the first column of a matrix.) Then Kosaki's generalized Hölder inequality [K] guarantees that

$$\tilde{\xi}^* \tilde{\eta} \in L^{p/2}(\mathcal{M} \otimes \mathcal{B}(\mathfrak{H}_I))$$

and

$$|| < (\xi_\alpha), (\eta_\alpha) > ||_{p/2} = ||\tilde{\xi}^* \tilde{\eta}||_{p/2} \leq ||\tilde{\xi}||_p ||\tilde{\eta}||_p = \left\| (\sum \xi_\alpha^* \xi_\alpha)^{1/2} \right\|_p \left\| (\sum \eta_\alpha^* \eta_\alpha)^{1/2} \right\|_p$$
$$= \left\| (\sum \xi_\alpha^* \xi_\alpha) \right\|_{p/2}^{1/2} \left\| (\sum \eta_\alpha^* \eta_\alpha) \right\|_{p/2}^{1/2} = ||\xi|| ||\eta||.$$

We denote this module $\bigoplus_c q_\alpha L^p$ for *column* sum. Indeed, the reader should think of principal modules as columns with entries from L^p. Motivated by this, we make

Definition 2.4. *Let $\{\mathfrak{X}_\alpha\}$ be L^p modules. If (ξ_α) and (η_α) have finite support, set*

$$< (\xi_\alpha), (\eta_\alpha) >= \sum < \xi_\alpha, \eta_\alpha > .$$

*The **column sum**, $\bigoplus_c \mathfrak{X}_\alpha$, is the closure of the finitely supported vectors with respect to the (quasi)norm ($p < \infty$) or strong topology ($p = \infty$) coming from this inner product.*

We denote the countable column sum of $L^p(\mathcal{M})$ as $C^p(\mathcal{M})$, or simply C^p if the underlying algebra is clear.

Note: As above, it will turn out that $\bigoplus_c \mathfrak{X}_\alpha = \{(\xi_\alpha) | \sum_\alpha < \xi_\alpha, \xi_\alpha > \in L^{p/2}\}$.

We can now state one of our main results.

Theorem 2.5. *Any L^p module is isometrically isomorphic, as a module, to a principal L^p module.*

If $\mathfrak{X}$ is cyclic, this is easy. Take

$$\mathfrak{X} = \overline{\xi \mathcal{M}}$$

and consider the densely-defined isomorphism of L^p modules

$$\mathfrak{X} \leftrightarrow s(< \xi, \xi >^{1/2})L^p : \quad \xi x \leftrightarrow < \xi, \xi >^{1/2} x.$$

Since the inner product and the bounded action of $\mathcal{M}$ extend continuously to the completion, this is an isomorphism.

The whole difficulty of the proof lies in devising the column sum decomposition. This may be thought of as a generalization of the fact that Hilbert spaces have an orthonormal basis. (A version of this theorem was proven for a special type of L^p module in [J1, Prop. 2.8].)

3. Proof of Theorem 2.5

If $p = 2$, $\mathfrak{X}$ is a Hilbert space. The previously mentioned decomposition theorem gives

$$\mathfrak{X} \simeq \bigoplus q_\alpha L^2(\mathcal{M})$$

isometrically as modules. Now the right-hand side admits a unique L^1-valued inner product and so is a column sum in our sense; therefore $\mathfrak{X}$ is principal.

We consider the cases $p = \infty$, $p > 2$, and $p < 2$ separately.

Case 1: $p = \infty$

Choose $\xi \in \mathfrak{X}$ and set

$$\xi_n = \xi \left(< \xi, \xi > + \frac{1}{n} \right)^{-1/2} \in \mathfrak{X},$$

so $\|\xi_n\| < 1$.

We observe

$$\varphi(< \xi_m - \xi_n, \xi_m - \xi_n >) \to 0, \quad m, n \to 0.$$

Then ξ_n converges strongly, say to η_1, and apparently $< \eta_1, \eta_1 >$ is a projection q_1.

Consider a maximal set $\{\eta_\alpha\}$ with the property that

$$< \eta_\beta, \eta_\gamma >= \delta_{\beta\gamma} q_\beta.$$

If the strong closure of $\sum \eta_\alpha \mathcal{M}$ is not all of $\mathfrak{X}$, choose ξ outside this set and write

$$\xi = \sum \eta_\alpha < \eta_\alpha, \xi > + \left(\xi - \sum \eta_\alpha < \eta_\alpha, \xi > \right). \tag{3.1}$$

The first summand should be interpreted as a strong limit; existence follows from the Bessel-type inequality

$$0 \leq \left\langle \xi - \sum \eta_\alpha < \eta_\alpha, \xi >, \xi - \sum \eta_\alpha < \eta_\alpha, \xi > \right\rangle =< \xi, \xi > - \sum | < \eta_\alpha, \xi > |^2.$$

By assumption the second summand in (3.1) is nonzero. We can normalize it as above (which does not change orthogonality) and add it to our set $\{\eta_\alpha\}$ - but this violates the maximality of $\{\eta_\alpha\}$.

Therefore the strong closure of $\sum \eta_\alpha \mathcal{M}$ is $\mathfrak{X}$. Finally we have an isomorphism

$$\mathfrak{X} \ni \xi \leftrightarrow (< \eta_\alpha, \xi >) \in \bigoplus_c q_\alpha \mathcal{M}.$$

Essentially this is Paschke's result [P], but we have started with a topological condition instead of an algebraic one (self-duality). A Hilbert C*-module $\mathfrak{X}$ is called *self-dual* if $\mathfrak{X} \simeq \mathrm{Hom}(\mathfrak{X}_\mathcal{M}, \mathcal{M}_\mathcal{M})$ via $\xi \leftrightarrow < \xi, \cdot >$. Weaker than the strong topology we have defined is the *weak* topology on the unit ball, generated by the functionals

$$\xi \mapsto \varphi(< \eta, \xi >), \quad \varphi \in \mathcal{M}_*, \ \eta \in \mathfrak{X}.$$

We have arrived at

Theorem 3.1. *For a Hilbert C*-module $\mathfrak{X}$ over a von Neumann algebra $\mathcal{M}$, the following conditions are equivalent:*

(i) *the unit ball of $\mathfrak{X}$ is strongly closed;*
(ii) *$\mathfrak{X}$ is principal; or, to say the same thing, $\mathfrak{X}$ is an ultraweak direct sum of Hilbert C*-modules $q_\alpha\mathcal{M}$, for some projections q_α;*
(iii) *$\mathfrak{X}$ is self-dual;*
(iv) *the unit ball of $\mathfrak{X}$ is weakly closed.*

This theorem has consequences for an arbitrary Hilbert C*-module $\mathfrak{X}$ over a von Neumann algebra $\mathcal{M}$. Set $\bar{\mathfrak{X}}$ to be the strong closure; a straightforward argument shows that $\bar{\mathfrak{X}}$ is an L^∞ module for $\mathcal{M}$. Therefore $\mathfrak{X}$ is representable as a strongly dense submodule of a principal L^∞ module. This observation, and a similar discussion, are also found in [We].

Case 2: $p > 2$

Let $\{\xi_\alpha\}$ be a maximal orthogonal set (with no condition on $< \xi_\alpha, \xi_\alpha >$). Set

$$\mathfrak{X}_0 = \overline{\sum \overline{\xi\mathcal{M}}} \simeq \bigoplus_c q_\alpha L^p.$$

Any vector η in $\mathfrak{X}_0$ can be written as a limit, i.e.

$$\eta = \lim_n \sum \xi_\alpha x_{\alpha,n}.$$

But if this is Cauchy, the orthogonality of $\{\xi_\alpha\}$ implies

$$\begin{aligned} 0 &= \lim_{m,n} \| \sum \xi_\alpha x_{\alpha,n} - \sum \xi_\alpha x_{\alpha,m} \| \\ &\geq \lim_{m,n} \|\xi_\alpha (x_{\alpha,n} - x_{\alpha,m})\| \quad \text{for each } \alpha. \end{aligned}$$

Thus η has a unique representation as $\sum \eta_\alpha$, $\eta_\alpha \in q_\alpha L^p$, and we have an isomorphism of L^p modules

$$\mathfrak{X}_0 \ni \eta \leftrightarrow \bigoplus_c q_\alpha L^p.$$

So we just need to show that $\mathfrak{X}_0 = \mathfrak{X}$. Now $\mathfrak{X}$ is a Banach space since $p > 2$, and $\mathfrak{X}_0$ was seen to be reflexive in the last paragraph. Therefore $\mathfrak{X}_0$ is a proximinal subspace of $\mathfrak{X}$ ([Si], Cor. 2.1). This means that if $\xi \in \mathfrak{X} \setminus \mathfrak{X}_0$, there exists an element η_0 in $\mathfrak{X}_0$ with

$$\|\xi - \eta_0\| = \inf_{\eta \in \mathfrak{X}_0} \|\xi - \eta\|.$$

Then $\zeta = \xi - \eta_0$ has 0 as a best approximant. What does this say about ζ?

Fix α and $x \in \mathcal{M}$. By assumption, the function

$$\mathbb{R} \ni t \mapsto \|\zeta + t\xi_\lambda x\|$$

attains its minimum at $t = 0$. Set

$$< \zeta, \xi_\lambda > = < \zeta, \zeta >^{1/2} T_\lambda < \xi_\lambda, \xi_\lambda >^{1/2},$$

using (2.1), and observe

$$\begin{aligned}
&\|\zeta + t\xi_\lambda x\| \\
&\quad = \| < \zeta,\zeta > + 2t\mathrm{Re}\ (< \zeta,\zeta >^{1/2} T_\lambda < \xi_\lambda,\xi_\lambda >^{1/2} x) + t^2 x^* < \xi_\lambda,\xi_\lambda > x\| \\
&\quad \overset{o(t)}{\sim} \| < \zeta,\zeta > + 2t\mathrm{Re}\ (< \zeta,\zeta >^{1/2} T_\lambda < \xi_\lambda,\xi_\lambda >^{1/2} x) \\
&\qquad + t^2 x^* < \xi_\lambda,\xi_\lambda >^{1/2} T_\lambda^* T_\lambda < \xi_\lambda,\xi_\lambda >^{1/2} x\| \\
&\quad = \| < \zeta,\zeta >^{1/2} + tT_\lambda < \xi_\lambda,\xi_\lambda >^{1/2} x\| \triangleq f_{\lambda,x}(t).
\end{aligned}$$

Now $f_{\lambda,x}$ is differentiable since ζ was presumed nonzero and the norm in $L^p \setminus \{0\}$ is Fréchet differentiable. (The Clarkson inequalities imply L^p is uniformly smooth for $1 < p < \infty$ [QX].) It agrees up to $o(t)$ with a function which has a local minimum at $t = 0$, so $f'_{\lambda,x}(0) = 0$. Finally, it is convex by construction. It follows that $f_{\lambda,x}$ attains its absolute minimum ($= \| < \zeta,\zeta >^{1/2} \| = \|\zeta\|$) at 0.

Since this is true for all λ and x we get that in L^p

$$\mathrm{dist}(< \zeta,\zeta >^{1/2}, \overline{\sum T_\lambda < \xi_\lambda,\xi_\lambda >^{1/2} \mathcal{M}}) = \|\zeta\|.$$

By Hahn-Banach there is a norm one functional on L^p which annihilates the subspace $\overline{\sum T_\lambda < \xi_\lambda,\xi_\lambda >^{1/2} \mathcal{M}}$ and takes the value $\|\zeta\|$ at $< \zeta,\zeta >^{1/2}$. This functional must have the form $\mathrm{Tr}(v\rho^{1/q}\cdot)$ for some $v \in \mathcal{M}$, $\rho \in \mathcal{M}_*^+$. Then we fix λ and write out

$$\mathrm{Tr}(v\rho^{1/q} T_\lambda < \xi_\lambda,\xi_\lambda >^{1/2} \mathcal{M}) = 0$$

$$\Rightarrow s(\rho) \perp s_\ell(T_\lambda < \xi_\lambda,\xi_\lambda >^{1/2}). \tag{3.2}$$

Also

$$\| < \zeta,\zeta >^{1/2} \|_p = |\mathrm{Tr}(v\rho^{1/q} < \zeta,\zeta >^{1/2})| \le \|s(\rho) < \zeta,\zeta >^{1/2} \|_p \le \| < \zeta,\zeta >^{1/2} \|_p,$$

so these are equalities and in particular

$$s(\rho) = s(< \zeta,\zeta >^{1/2}). \tag{3.3}$$

Together (3.2) and (3.3) imply

$$< \zeta,\xi_\lambda > = < \zeta,\zeta >^{1/2} T_\lambda < \xi_\lambda,\xi_\lambda >^{1/2} = 0.$$

So the set $\{\xi_\lambda\}$ was not a maximal orthogonal set in $\mathfrak{X}$, a contradiction. This completes the proof for $p > 2$.

Case 3: $p < 2$

By restricting the algebra (see the discussion following Definition 2.3), we may assume that the module is faithful. We need two auxiliary constructions.

I. Let $\mathfrak{X}$ be an L^p module, and $1/p + 1/q = 1/r$. We write

$$\mathfrak{X} \otimes_{\mathcal{M}} L^q(\mathcal{M})$$

for the closure of the algebraic tensor product, modulo the null space, in the topology arising from the degenerate inner product

$$< \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 > = \eta_1^* < \xi_1,\xi_2 > \eta_2 \in L^{r/2}(\mathcal{M}).$$

It is easy to see that $\mathfrak{X} \otimes_{\mathcal{M}} L^q(\mathcal{M})$ satisfies the relation

$$\xi x \otimes \eta = \xi \otimes x\eta \tag{3.4}$$

and is an L^r module in our sense.

II. Let $\mathfrak{X}$ be an L^p module and φ be a fixed faithful *strictly semifinite* weight on $\mathcal{M}$. This means that $\varphi = \sum \varphi_\alpha$, where the φ_α are orthogonal and bounded. We will create an L^2 module with the same "shape" as $\mathfrak{X}$.

Lemma 3.2. *The following conditions on a vector $\xi \in \mathfrak{X}$ are equivalent:*

(i) $< \xi, \xi > \leq C\varphi^{2/p}$ *for some* C*;*
(ii) $< \xi, \xi >^{1/2} = y\varphi^{1/p}$ *for some* $y \in \mathcal{M}$*;*
(iii) $< \xi, \xi >^{1/2} = \varphi^{1/p} z$ *for some* $z \in \mathcal{M}$*.*

We denote the set of such vectors as $\mathcal{D}_\varphi$.

This is nothing but (1.1).

Lemma 3.3. *$\mathcal{D}_\varphi$ is dense in $\mathfrak{X}$.*

Proof. Given any $\xi \in \mathfrak{X}$, let $q = s(< \xi, \xi >)$ and densely define the L^p module isomorphism T by

$$T : qL^p \xrightarrow{\sim} \overline{\xi\mathcal{M}} \subset \mathfrak{X}, \qquad T(< \xi, \xi >^{1/2} x) = \xi x.$$

We need that elements of the form $y\varphi^{1/p}$ are dense in L^p. Because $\varphi^{1/p}$ is not necessarily τ-measurable, this is slightly more delicate than Lemma 1.2.

Set $q_\alpha = s(\varphi_\alpha)$, and let $\{r_\beta\}$ be the net of finite sums of the q_α (ordered naturally). Again by Lemma 1.4, the net $\{< \xi, \xi >^{1/2} r_\beta\}$ converges to $< \xi, \xi >^{1/2}$. Since r_β commutes with $\varphi^{1/p}$, we have

$$< \xi, \xi >^{1/2} r_\beta \in L^p r_\beta = \overline{\mathcal{M} r_\beta \varphi^{1/p}}.$$

Putting these two approximations together, we may find $\{y_n\} \subset \mathcal{M}$ with $y_n\varphi^{1/p} \to < \xi, \xi >^{1/2}$.

T is an isomorphism and preserves inner products, so

$$T(y_n\varphi^{1/p}) \to T(< \xi, \xi >^{1/2}) = \xi.$$

Also

$$< T(y_n\varphi^{1/p}), T(y_n\varphi^{1/p}) > = < y_n\varphi^{1/p}, y_n\varphi^{1/p} > = \varphi^{1/p} y_n^* y_n \varphi^{1/p} \leq \|y\|^2 \varphi^{2/p},$$

so by Lemma 3.2, $T(y_n\varphi^{1/p}) \in \mathcal{D}_\varphi$. Thus we have written ξ as a limit of vectors in $\mathcal{D}_\varphi$.

□

With $\frac{1}{2} + \frac{1}{r} = \frac{1}{p}$, we define an L^1-valued inner product on $\mathcal{D}_\varphi$ by

$$< \xi, \eta >_{\mathcal{D}_\varphi} \triangleq \varphi^{-1/r} < \xi, \eta >_{\mathfrak{X}} \varphi^{-1/r}. \tag{3.5}$$

By (2.1) and Lemma 3.2, $\varphi^{1/p}$ factors out of $< \xi, \eta >$ on both the left and the right, and (3.5) is justified. The nontrivial fact that composition with $\varphi^{-1/r}$ is the inverse of composition with $\varphi^{1/r}$ is found in [S2].

We now describe the module action. Clearly the previous $\mathcal{M}$-action is not compatible with the new inner product (and $\mathcal{D}_\varphi$ is not a submodule of $\mathfrak{X}$). Instead we need to work with $\mathcal{M}_a^\varphi$, the operators in $\mathcal{M}$ for which

$$t \mapsto \sigma_t^\varphi(x) = \varphi^{it} x \varphi^{-it}$$

extends off the real line to an entire $\mathcal{M}$-valued function.

The action must be

$$\eta \cdot x = \eta\varphi^{-1/r} x \varphi^{1/r};$$

then

$$\begin{aligned} < \xi, \eta \cdot x >_{\mathcal{D}_\varphi} &=< \xi, \eta\varphi^{-1/r} x \varphi^{1/r} >_{\mathcal{D}_\varphi} = \varphi^{-1/r} < \xi, \eta\varphi^{-1/r} x\varphi^{1/r} >_{\mathfrak{X}} \varphi^{-1/r} \\ &= \varphi^{-1/r} < \xi, \eta >_{\mathfrak{X}} \varphi^{-1/r} x \varphi^{1/r}\varphi^{-1/r} =< \xi, \eta >_{\mathcal{D}_\varphi} x. \end{aligned}$$

As we noted before, an L^1-valued inner product composed with Tr is a usual inner product; therefore the closure of $\mathcal{D}_\varphi$ in the inner product norm is a Hilbert space $\mathfrak{H}_{\mathfrak{X},\varphi}$. The *-algebra $\mathcal{M}_a^\varphi$ is represented isometrically on it - in fact it is a *-representation:

$$\begin{aligned} < \xi, \eta \cdot x^* >_{\mathfrak{H}_{\mathfrak{X},\varphi}} &= \mathrm{Tr}(< \xi, \eta \cdot x^* >_{\mathcal{D}_\varphi}) = \mathrm{Tr}(< \xi, \eta >_{\mathcal{D}_\varphi} x^*) \\ &= \mathrm{Tr}(x^* < \xi, \eta >_{\mathcal{D}_\varphi}) = \mathrm{Tr}(< \xi \cdot x, \eta >_{\mathcal{D}_\varphi}). \end{aligned}$$

We need to show that the von Neumann closure of $\mathcal{M}_a^\varphi$ is exactly $\mathcal{M}$. A dense set of vector states in this representation is

$$x \mapsto < \xi, \xi \cdot x >_{\mathfrak{H}_{\mathfrak{X},\varphi}}, \quad \xi \in \mathcal{D}_\varphi, \tag{3.6}$$

and these are identical to the linear functionals

$$x \mapsto \mathrm{Tr}(< \xi, \xi >_{\mathcal{D}_\varphi} x), \quad \xi \in \mathcal{D}_\varphi.$$

Deducing further and using Lemma 3.2,

$$\begin{aligned} \{< \xi, \xi >_{\mathcal{D}_\varphi} | \ \xi \in \mathcal{D}_\varphi\} &= \{\varphi^{-1/r}\psi^{2/p}\varphi^{-1/r} \mid C\varphi^{2/p} \geq \psi^{2/p} \in L^{p/2}\} \\ &= \{\varphi^{1/2}|y|^2\varphi^{1/2} \mid y\varphi^{1/p} \in L^p\}. \end{aligned}$$

Now we need another double approximation argument, and we are brief. Since φ is semifinite, any element of $\mathcal{M}_*^+$ is a norm limit of elements $\varphi^{1/2}|y_n|^2\varphi^{1/2}$, where $y_n \in \mathfrak{N}_\varphi$, the definition ideal of φ. Each of these can be approximated by an element

$$r_\beta\varphi^{1/2}|y|^2\varphi^{1/2}r_\beta = \varphi^{1/2}r_\beta|y|^2r_\beta\varphi^{1/2}$$

(r_β are as in the proof of Lemma 3.3), and these belong to the sets above since $yr_\beta\varphi^{1/p} \in L^p$.

The upshot of all this is that the vector states in (3.6) form a dense set in $\mathcal{M}_*^+$. Thus the strong topology in this representation agrees with the strong topology in the representation of $\mathcal{M}_a^\varphi$ on $L^2(\mathcal{M})$. Happily, $\mathcal{M}_a^\varphi$ is dense in $\mathcal{M}$ in the latter topology, so the von Neumann closure is $\mathcal{M}$.

The reader can check that the extensions of the $\mathcal{M}$-action and L^1-valued inner product to $\mathfrak{H}_{\mathfrak{X},\varphi}$ do make it into an L^2 module for $\mathcal{M}$.

Now consider the L^p module

$$\mathfrak{H}_{\mathfrak{X},\varphi} \otimes_{\mathcal{M}} L^r.$$

We will make two observations: that it is principal, and that it is isomorphic to $\mathfrak{X}$.

$\mathfrak{H}_{\mathfrak{X},\varphi}$ is an L^2 module and so of the form $\bigoplus_c q_\alpha L^2$. It is not hard to see that the functor "$\otimes_{\mathcal{M}} L^r$" commutes with column sums; i.e.

$$\mathfrak{H}_{\mathfrak{X},\varphi} \otimes_{\mathcal{M}} L^r = \left(\bigoplus_c q_\alpha L^2\right) \otimes_{\mathcal{M}} L^r \simeq \bigoplus_c \left(q_\alpha L^2 \otimes_{\mathcal{M}} L^r\right) = \bigoplus_c (q_\alpha L^p),$$

which is principal.

Consider the dense submodule

$$\mathcal{D}_\varphi \otimes_{\mathcal{M}} (\varphi^{1/r}\mathcal{M} \cap L^r) \subset \mathfrak{H}_{\mathfrak{X},\varphi} \otimes_{\mathcal{M}} L^r.$$

For elements of this subset, we have

$$< \xi\otimes\varphi^{1/r}x, \eta\otimes\varphi^{1/r}y >= x^*\varphi^{1/r} < \xi, \eta >_{\mathfrak{H}} \varphi^{1/r}y = x^* < \xi, \eta >_{\mathfrak{X}} y =< \xi x, \eta y >_{\mathfrak{X}} .$$

So the correspondence

$$\xi \otimes \varphi^{1/r}x \leftrightarrow \xi x$$

densely defines an L^p module isomorphism

$$\mathfrak{H}_{\mathfrak{X},\varphi} \otimes L^r \simeq \mathfrak{X}.$$

As before, the $\mathcal{M}$-action and inner product must agree on the closure, and the proof is complete.

Since any L^p module is principal, we see that 1) $\|\cdot\|$ is a norm for $p \geq 1$ and a p- (not just $p/2$-) norm for $p < 1$; and 2)

$$\bigoplus_c \mathfrak{X}_\alpha = \{(\xi_\alpha) | \sum_\alpha < \xi_\alpha, \xi_\alpha > \in L^{p/2}\};$$

as were mentioned in Section 2.

It also follows from the proof that for any set $S \subset \mathfrak{X}$ an L^p module,

$$S^{\perp\perp} = \overline{S\mathcal{M}}. \tag{3.7}$$

So if S is already an L^p module,

$$\mathfrak{X} = S \oplus_c S^\perp;$$

that is, right L^p submodules are necessarily column summands.

4. An application to ultraproducts

Here we give a nontrivial application of Theorem 2.5.

Fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. For a Banach space X, we define the ultrapower $X_\mathcal{U}$ by

$$\ell^\infty(X)/\mathcal{N}_\mathcal{U}, \text{ where } \mathcal{N}_\mathcal{U} = \{(x_n) | \lim_{n\to\mathcal{U}} \|x_n\| = 0\}.$$

We will need the following result of Raynaud:

Theorem 4.1 (R)**.** *Let $\mathcal{M}$ be a von Neumann algebra. Set*

$$\mathcal{N} = ((\mathcal{M}_*)_\mathcal{U})^*.$$

Then $\mathcal{N}$ is a von Neumann algebra and

$$L^p(\mathcal{M})_\mathcal{U} \simeq L^p(\mathcal{N}).$$

In fact $\mathcal{M}_\mathcal{U}$ is strongly dense in $\mathcal{N}$.

Now take $p \geq 2$ for simplicity, and consider the Banach space $C^p(\mathcal{M})_\mathcal{U}$. While $C^p(\mathcal{M})_\mathcal{U}$ naturally contains $C^p(\mathcal{N})$, they are not equal; the reader should think of a sequence of unit vectors in $C^p(\mathcal{M})$ where the support wanders off to infinity. (Evaluating the limit componentwise is a projection from $C^p(\mathcal{M})_\mathcal{U}$ onto $C^p(\mathcal{N})$.) As huge as $C^p(\mathcal{M})_\mathcal{U}$ is, we can still gain some control over it via

Proposition 4.2. *$C^p(\mathcal{M})_\mathcal{U}$ is a right L^p $\mathcal{N}$-module.*

Proof. We explain the L^p module structure. Let $x \in \mathcal{M}_{\mathcal{U}} \subset \mathcal{N}$ have representing sequence (x_n), and let $\xi, \eta \in C^p(\mathcal{M})_{\mathcal{U}}$ have representing sequences $(\oplus_{c,k=1}^{\infty} \xi_k^n)$, $(\oplus_{c,k=1}^{\infty} \eta_k^n)$. We naturally define ξx by the representing sequence $(\oplus_c \xi_k^n x_n)$; it is easy to see that this does not depend on the initial choices. Similarly, we set

$$< \xi, \eta >= \lim_{n \to \mathcal{U}} < \oplus_{c,k=1}^{\infty} \xi_k^n, \oplus_{c,k=1}^{\infty} \eta_k^n > \in L^{p/2}(\mathcal{M})_{\mathcal{U}} = L^{p/2}(\mathcal{N}).$$

It is clear that this inner product generates the norm and is compatible with the module action.

Finally, we show that the module action extends naturally from $\mathcal{M}_{\mathcal{U}}$ to $\mathcal{N}$. The strong topology that $\mathcal{M}_{\mathcal{U}}$ inherits from its action on $C^p(\mathcal{M})_{\mathcal{U}}$ is generated by seminorms of the form

$$\begin{aligned}(x_n) = x \mapsto \|\xi x\| &= \lim_{n \to \mathcal{U}} \|x_n^* < \oplus_c \xi_k^n, \oplus_c \xi_k^n > x_n\|^{1/2} \\ &= \lim_{n \to \mathcal{U}} \|x_n^* \sum_k < \xi_k^n, \xi_k^n > x_n\|^{1/2} = \lim_{n \to \mathcal{U}} \|(\sum_k < \xi_k^n, \xi_k^n >)^{1/2} x_n\| \\ &= \lim_{n \to \mathcal{U}} \|\varphi_n^{1/p} x_n\| = \|\varphi^{1/p} x\|,\end{aligned}$$

where the $\varphi_n^{1/p} \in L^p(\mathcal{M})$ form a representing sequence for $\varphi^{1/p} \in L^p(\mathcal{N})$. By Lemma 1.3, these are exactly the seminorms which generate the strong topology on $\mathcal{M}_{\mathcal{U}}$ inside $\mathcal{N}$. This completes the proof. □

By Theorem 2.5, we know that any L^p module can be written as a column sum. One can think of $C^p(\mathcal{M})_{\mathcal{U}}$ as containing countably many copies of $L^p(\mathcal{N})$ from componentwise limits, plus uncountably many more from all the directions in which support might wander. Perhaps it is more natural to think of $C^p(\mathcal{M})_{\mathcal{U}}$ as a continuous column integral of $L^p(\mathcal{N})$ over a *very* large space; the adventurous reader may want to consider how to make this statement more precise.

5. Commutants and categorical properties

Consider a countably generated right L^p module $\mathfrak{X}$ for a σ-finite von Neumann algebra $\mathcal{M}$. By Theorem 2.5, there are projections $\{q_n\}$ with

$$\mathfrak{X} \simeq \bigoplus_c q_n L^p \simeq (\sum q_n \otimes e_{nn}) C^p = qC^p.$$

If two such projections $q_1, q_2 \in M_\infty(\mathcal{M})$ are Murray-von Neumann equivalent via a partial isometry v, we have a module isomorphism (even isometric) $q_1 C^p \simeq q_2 C^p$ via left multiplication by v. We will obtain the converse after proving

Proposition 5.1. *On qC^p, the right action of $\mathcal{M}$ (= $R(\mathcal{M})$) and the left action of $qM_\infty(\mathcal{M})q$ (= $L(qM_\infty(\mathcal{M})q)$) are commutants of each other.*

Proof. Once again we may assume that the module is faithful for the right action of $\mathcal{M}$. Now note that the two actions mentioned are commuting and bounded: boundedness of $L(qM_\infty(\mathcal{M})q)$ follows by viewing it as a subalgebra of $M_\infty(\mathcal{M})$ and using the Hölder inequality. Finally, by the remark before the proposition, we may assume that $q^\perp \sim 1$.

So let T be a bounded operator on qC^p commuting with the right action of $\mathcal{M}$, and then set $T' = T \circ L(q)$. T' is bounded and commutes with $R(\mathcal{M})$ on all of C^p.

Since T' acts on column vectors, it has a matrix representation as (T'_{ij}), where each T'_{ij} operates on $L^p(\mathcal{M})$. Fix a single T'_{ij}. For any $\xi \in L^p(\mathcal{M})$ and $x \in \mathcal{M}$ we

may consider the vector $\bar{\xi}$ in C^p with ξ in the jth position and 0 elsewhere. Since T' commutes with $R(\mathcal{M})$,

$$(T'_{ij}\xi)x = ((T'\bar{\xi})x)_i = (T'(\bar{\xi}x))_i = T'_{ij}(\xi x).$$

By Corollary 1.6 we know that $T'_{ij} = L(y_{ij})$ for some $y_{ij} \in \mathcal{M}$. Considering the kernel and range, we deduce that $y_{ij} \in q_i\mathcal{M}q_j$. Then $T' = L((y_{ij}))$ for some bounded operator $(y_{ij}) \in qM_\infty(\mathcal{M})q$, and this representation is the restriction T as well.

Instead of trying to check that any operator commuting with $L(qM_\infty(\mathcal{M})q)$ must lie inside $R(\mathcal{M})$, we give a small argument involving projections in order to invoke symmetry. In the σ-finite algebra $M_\infty \otimes M_\infty \otimes \mathcal{M}$, the projections

$$I \otimes q \text{ and } e_{11} \otimes I \otimes I_{\mathcal{M}}$$

are both properly infinite and therefore equivalent ([KR], Corollary 6.3.5). They remain so after subtracting their common subprojection $e_{11} \otimes q$ (we assumed $q^\perp \sim I \otimes I_{\mathcal{M}}$), allowing us to find a partial isometry v between them which fixes $e_{11} \otimes q$. Conjugation by v gives an isomorphism

$$\begin{aligned} M_\infty(qM_\infty(\mathcal{M})q) &= (I \otimes q)(M_\infty \otimes M_\infty \otimes \mathcal{M})(I \otimes q) \\ &\simeq (e_{11} \otimes I \otimes I_{\mathcal{M}})(M_\infty \otimes M_\infty \otimes \mathcal{M})(e_{11} \otimes I \otimes I_{\mathcal{M}}) \\ &= M_\infty(\mathcal{M}). \end{aligned} \tag{5.1}$$

Now let $r = v^*(e_{11} \otimes e_{11} \otimes 1_{\mathcal{M}})v$ be the projection in the first algebra which corresponds to $e_{11} \otimes 1_{\mathcal{M}}$ in the last, and notice $e_{11} \otimes q$ is the "outer" matrix unit e'_{11} for $M_\infty(qM_\infty(\mathcal{M})q)$. Via the isomorphism above, we have the isomorphic bimodule presentations

$$\begin{aligned} q(M_\infty(\mathcal{M}))q \,-\, qL^p(M_\infty(\mathcal{M}))e_{11} \,-\, \mathcal{M} \quad \simeq \\ q(M_\infty(\mathcal{M}))q \,-\, e'_{11}L^p(M_\infty(q(M_\infty(\mathcal{M}))q))r \,-\, r(M_\infty(q(M_\infty(\mathcal{M}))q))r. \end{aligned} \tag{5.2}$$

(The point is to observe that module and commutant are written as reduced amplifications of the *left* algebra.) Now applying the first argument finishes the proof. □

That q be diagonal, i.e. of the form $\sum q_n \otimes e_{nn}$, is actually unnecessary. For any projection q in $M_\infty(\mathcal{M})$, the L^p module qC^p inherits its structure from C^p.

Corollary 5.2. *If the L^p modules q_1C^p, q_2C^p are isomorphic, then the projections q_1, q_2 are Murray von-Neumann equivalent.*

Proof. The proof is no different than the L^2 case. If S is the isomorphism, extend it to

$$\widetilde{S} : q_1C^p \oplus_c q_1^\perp C^p = C^p \to C^p;$$
$$\widetilde{S}(\xi \oplus_c \eta) = S(\xi).$$

$\widetilde{S}$ is clearly bounded, so by Proposition 5.1, it is given by left composition with some $y \in M_\infty(\mathcal{M})$. By considering the kernel and range of $\widetilde{S}$, we see that $s_r(y) = q_1$ and $s_\ell(y) = q_2$. □

By virtually the same argument we obtain

Corollary 5.3. *If $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$,*

$$Hom(q_1C^p_{\mathcal{M}}, q_2C^q_{\mathcal{M}}) = L(q_2L^r(M_\infty(\mathcal{M}))q_1).$$

Remark: Proposition 5.1 and its corollaries still hold without the assumptions that the algebra is σ-finite and the module is countably generated. (This requires either a direct limit argument or a more subtle calculation with projections.) In the general case, the typical module is $qL^p(M_J(\mathcal{M}))e_{11}$ for some cardinal J and projection $q \in M_J(\mathcal{M})$; we will not need the full result in the sequel and so opted for clarity.

Now we continue to investigate the category of isomorphism classes of countably generated right L^p $\mathcal{M}$-modules, with intertwiners as morphisms, which we call *Right* $L^pMod(\mathcal{M})$. These are submodules of C^p; from the foregoing discussion we may conclude that they are parameterized by Murray von-Neumann equivalence classes of projections in $M_\infty(\mathcal{M})$, which is $V(M_\infty(\mathcal{M}))$ in the language of K-theory [W-O]. It should be clear that this is an additive category, with addition being the column sum of orthogonal representatives. This actually gives us monoidal equivalence with $V(M_\infty(\mathcal{M}))$:

$$q_1C^p \oplus_c q_2C^p = (q_1 + q_2)C^p, \quad q_1 \perp q_2$$

corresponds exactly to

$$[q_1] + [q_2] = [q_1 + q_2], \quad q_1 \perp q_2.$$

It follows that $C^p \oplus_c qC^p \simeq C^p$, which is the L^p version of Kasparov's stabilization theorem for Hilbert C*-modules ([L], Theorem 6.2). In case $\mathcal{M}$ is a II_1 factor, we can make the correspondence with $V(M_\infty(\mathcal{M})) \simeq [0, \infty]$ explicit with the natural definition

$$\dim_{\mathcal{M}}(qC^p) = \tau_{M_\infty(\mathcal{M})}(q). \tag{5.3}$$

Clearly $\dim_{\mathcal{M}}(\oplus_c \mathfrak{X}_i) = \sum \dim_{\mathcal{M}} \mathfrak{X}_i$.

All of this is identical to the L^2 case, but we recall the difference at the vector level: the norm in a column sum ($p \neq 2$)

$$\|\xi \oplus_c \eta\| = \|\xi^*\xi + \eta^*\eta\|^{1/2}$$

is not, in general, a function of the norms in each component. Now it may occur to the reader to try a "diagonal" sum $\left(\begin{smallmatrix} \mathfrak{X} & 0 \\ 0 & \mathfrak{Y} \end{smallmatrix}\right)$, as is done for operator spaces. This *is* an ℓ^p direct sum, but no compatibility is required or retained: the diagonal sum of a right L^p $\mathcal{M}_1$-module and a right L^p $\mathcal{M}_2$-module is a right L^p $(\mathcal{M}_1 \oplus \mathcal{M}_2)$-module. If $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$, the diagonal sum is algebraically an $\mathcal{M}$-module, but not necessarily an L^p module in our sense: the inner product would naturally be $L^{p/2}(\mathcal{M} \oplus \mathcal{M})$-valued. The difference is already apparent in the simplest possible case:

$$\mathcal{M} = \mathfrak{X} = \mathfrak{Y} = \mathbb{C}.$$

As modules,

$$\mathbb{C} \oplus_c \mathbb{C} = \left(\begin{smallmatrix} \mathbb{C} \\ \mathbb{C} \end{smallmatrix}\right)_{\mathbb{C}}, \quad \|\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\| = (|a|^2 + |b|^2)^{1/2}, \tag{5.4}$$

$$\mathbb{C} \oplus_d \mathbb{C} = \left(\begin{smallmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{smallmatrix}\right)_{\mathbb{C}I_2 \subset M_2}, \quad \|\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)\| = (|a|^p + |b|^p)^{1/p}.$$

$\mathbb{C} \oplus_d \mathbb{C}$ cannot be a right L^p $\mathbb{C}$-module, since it is apparently not isometrically isomorphic to the only two-dimensional right L^p $\mathbb{C}$-module, $\mathbb{C} \oplus_c \mathbb{C}$. (Instead it is a right L^p $(\mathbb{C} \oplus \mathbb{C})$-module.)

Since there are many equivalent constructions of $L^p(\mathcal{M})$, it should not be surprising that there are other ways to build the class of L^p modules. We do not reproduce

the details from [S1] but simply note that the class of countably generated right L^p modules, modulo spatial isomorphism, can also be described as

- a minimal class of complete right $\mathcal{M}$-modules which contains $L^p(\mathcal{M})$ and is closed under taking submodules and forming countable column sums (recall equation (3.7));
- a class of spaces of "column" operators which satisfy a $-1/p$-homogeneity condition in the sense of Connes-Hilsum [Hi];
- a class of interpolation spaces, following Kosaki [K1].

In the sequel we will frequently be concerned with left actions. Of course, the theory of left L^p modules is entirely analogous. The counterparts to column sums, C^p, and *Right* L^pMod we call *row sums*, R^p, and $Left\ L^pMod$. There is a 1-1 correspondence between left and right L^p $\mathcal{M}$-modules given by the *contragredient* ${}_{\mathcal{M}}\bar{\mathfrak{X}}$ of $\mathfrak{X}_{\mathcal{M}}$: $\bar{\mathfrak{X}}$ is conjugate linearly isomorphic to $\mathfrak{X}$, with left action $x \cdot \bar{\xi} = \overline{\xi x^*}$ and inner product $< \bar{\xi}, \bar{\eta} >=< \xi, \eta >$. Of course, one may similarly take the contragredient of a left L^p module; $\bar{\bar{\mathfrak{X}}}$ is canonically isomorphic to $\mathfrak{X}$. It is easy to see that when $\mathfrak{X}$ is represented as a principal L^p module, the contragredient corresponds to the operator adjoint.

Definition 5.4. *An L^p $\mathcal{M}-\mathcal{N}$* ***bimodule*** *is an $\mathcal{M}-\mathcal{N}$ bimodule (meaning that the actions commute) which is simultaneously a left L^p $\mathcal{M}$-module and a right L^p $\mathcal{N}$-module. We denote the category of isomorphism classes, with intertwiners as morphisms, by $L^pBimod(\mathcal{M},\mathcal{N})$.*

Notice, by Proposition 5.1 and (5.2), that every left or right L^p $\mathcal{M}$-module is an L^p bimodule, with opposite action coming from the commutant. We will explore this more fully in the next section.

Our final observation of this section concerns the relative tensor product, a sort of "multiplication" for Hilbert modules. The original arguments are due to Connes and Sauvageot (and found in [P] and [Sa]); the informed reader will recognize our L^p formulation as a minor modification. As explained in [S3], on the module level the relative tensor product only "sees" the projections (more precisely, the elements of $V(M_\infty(\mathcal{M}))$) which determine the modules. The densities of the modules - all $1/2$ in the usual case - are irrelevant, and so we may choose any p, q, r we please. In the following definition the notations $\mathcal{L}(\mathfrak{X}_{\mathcal{M}})$ and $\mathcal{L}({}_{\mathcal{M}}\mathfrak{Y})$ stand for commutants.

Definition 5.5. *By an $(\mathcal{M}, p, q, r)$-**relative tensor product** we mean a functor, covariant in both variables,*

$$Right\ L^pMod(\mathcal{M}) \times Left\ L^qMod(\mathcal{M}) \to L^rBimod(\mathcal{L}(\mathfrak{X}_{\mathcal{M}}), \mathcal{L}({}_{\mathcal{M}}\mathfrak{Y}));$$

$$(\mathfrak{X}, \mathfrak{Y}) \mapsto \mathfrak{X} \otimes_{\mathcal{M},p,q,r} \mathfrak{Y}$$

which satisfies

$$L^p(\mathcal{M}) \otimes_{\mathcal{M},p,q,r} L^q(\mathcal{M}) \simeq L^r(\mathcal{M})$$

as bimodules.

We remind the reader that the sums in these categories (which the relative tensor product must distribute, by functoriality) are not direct. So, for example,

$$(\oplus_c L^p) \otimes_{\mathcal{M},p,q,r} L^q \simeq \oplus_c (L^r).$$

By decomposition and functoriality, it is simple to see that such functors exist and are unique up to unitary equivalence. One has the following representation result:

Proposition 5.6. *Let $\mathfrak{X} \simeq q_1 C^p \in RightL^pMod(\mathcal{M}), \mathfrak{Y} \simeq R^q q_2 \in LeftL^qMod(\mathcal{M})$ for some $q_1, q_2 \in \mathcal{P}(M_\infty(\mathcal{M}))$. Then*

$$\mathfrak{X} \otimes_{\mathcal{M},p,q,r} \mathfrak{Y} \simeq q_1 L^r(M_\infty(\mathcal{M}))q_2$$

with natural action of the commutants.

It is also possible to give an element-wise construction of the relative tensor product based on a fixed faithful state (or weight) φ. The usual construction is

$$\xi \otimes_\varphi \eta = \xi\varphi^{-1/2}\eta$$

for a suitable dense set of ξ, η, and the (p, q, r)-relative tensor product requires

$$\xi \otimes_\varphi \eta = \xi\varphi^{\frac{1}{r}-\frac{1}{p}-\frac{1}{q}}\eta.$$

Both of these identities are discussed in [S2], and in [S3] the preclosedness of this relative tensor map is investigated in full. The reader will notice that the auxiliary constructions introduced to prove the $p < 2$ case of Theorem 2.5 are nothing but relative tensor products.

6. L^p BIMODULES

The theory of L^2 bimodules, which contains that of subfactors, is one of the most fruitful fields in the study of von Neumann algebras. But for the L^p analogues with $p \neq 2$, the lack of Hilbert space symmetry makes for a much more restrictive theory. One deficit which is apparent from the outset is that L^p bimodules do not add: row and column sums preserve one algebra only. We will see that there are other significant limitations. *In this section we simplify the discussion by assuming $1 < p < \infty$, $p \neq 2$, all algebras to be σ-finite, and all L^p modules to be countably generated and faithful.*

The structure theorems proven so far show that every left or right L^p $\mathcal{M}$-module is an L^p bimodule, with opposite action coming from the commutant. But of course the commutant is the "largest" choice, so an $\mathcal{M} - \mathcal{N}$ L^p bimodule gives injective homomorphisms of each algebra into an amplification of the other.

Lemma 6.1. *Let $\mathfrak{X}$ be an $\mathcal{M} - \mathcal{N}$ L^p bimodule, and suppose that $\mathfrak{X} \simeq R^p(\mathcal{M})q$ via an isomorphism T. Then there is an injective normal *-homomorphism*

$$\pi : \mathcal{N} \hookrightarrow qM_\infty(\mathcal{M})q$$

such that

$$T(\xi n) = T(\xi)\pi(n).$$

Proof. The only things to show are that π is normal and *-preserving. For normality, note that by Lemma 1.3 the strong topologies on the unit balls of $\mathcal{N}$ and $\pi(\mathcal{N})$ are both given by the module actions, which are identical.

A variant of Lemma 1.1 says that the norms are also generated by the module actions. Since (orthogonal) projections are exactly idempotents of norm one, π takes projections to projections. By approximating with projections, we see π takes self-adjoint elements to self-adjoint elements. Finally,

$$\pi(x^*) = \pi(\text{Re } x) - i\pi(\text{Im } x) = \pi(x)^*.$$

□

One is tempted to follow the theory of correspondences [P] and guess that L^p bimodules are equivalent to normal unital *-homomorphisms, but this is asking too much. For example, $L^p(\mathcal{M})$ is an $\mathcal{M}-\mathcal{M}$ L^p bimodule, and naturally $\mathbb{C}\subset\mathcal{M}$. But if $L^p(\mathcal{M})$ is a right L^p $\mathbb{C}$-module, then $L^p(\mathcal{M})\simeq qC^p(\mathbb{C})$ for some q, and this last object is actually a Hilbert space. Then the norm in $L^p(\mathcal{M})$ follows the parallelogram law; if φ, ψ are states with orthogonal supports,

$$4=2(\|\varphi^{1/p}\|^2+\|\psi^{1/p}\|^2)=\|\varphi^{1/p}+\psi^{1/p}\|^2+\|\varphi^{1/p}-\psi^{1/p}\|^2=2\cdot 2^{2/p}. \tag{6.1}$$

Since this is false when $p\neq 2$, such states cannot exist. Thus $L^p(\mathcal{M})$ is a right L^p $\mathbb{C}$-module iff $\mathcal{M}=\mathbb{C}$.

The cause of such a phenomenon is clear: noncommutative L^p spaces remember their generating algebras (except for amnesiac $p=2$). The existence of an L^p $\mathcal{M}-\mathcal{N}$ bimodule therefore implies a relationship between $\mathcal{M}$ and $\mathcal{N}$, and the remainder of this section is devoted to a precise description of this relationship.

A major tool will be the following result of Raynaud and Xu (relying heavily on Kosaki's papers [K2], [K3], where a subcase is proved).

Theorem 6.2. *[RX] When $p\neq 2$, two elements ξ,η of a noncommutative L^p space satisfy $s_\ell(\xi)\perp s_\ell(\eta), s_r(\xi)\perp s_r(\eta)$ iff they satisfy*

$$\|\xi+\eta\|^p+\|\xi-\eta\|^p=2\left(\|\xi\|^p+\|\eta\|^p\right). \tag{6.2}$$

Proposition 6.3. *Let $\mathfrak{X}\simeq R^p(\mathcal{M})q$ be an L^p $\mathcal{M}-\mathcal{N}$ bimodule. Then the centers of $\mathcal{M}$ and $\mathcal{N}$ are isomorphic and act identically on $\mathfrak{X}$.*

Proof. We will use the classical L^p notion of *L^p-projection* [B]: an idempotent E on a Banach space satisfying

$$\|\xi\|^p=\|E\xi\|^p+\|\xi-E\xi\|^p \tag{6.3}$$

for all elements ξ. Commutative L^p spaces are characterized by having sufficiently many L^p-projections, but the same is not true in the noncommutative setting. We will show that the L^p-projections on $\mathfrak{X}$ can be identified spatially with the central projections of either $\mathcal{M}$ or $\mathcal{N}$.

That a central projection is an L^p-projection is clear. So let E be an L^p-projection, and make the identification $\mathfrak{X}\simeq R^p(\mathcal{M})q$. For any two vectors $\xi\in\mathrm{Ran}E$, $\eta\in\ker E$, (6.3) implies (6.2) (since $-\eta\in\ker E$ also). Then ξ and η have orthogonal left and right supports, and we have the decomposition in $\mathcal{M}$

$$\bigvee_{\xi\in\mathrm{Ran}E} s_\ell(\xi)+\bigvee_{\eta\in\ker E} s_\ell(\eta)=1.$$

Now the left action of the first projection above is apparently E, which must also equal the right action of $\bigvee_{\xi\in\mathrm{Ran}E} s_r(\xi)\in(\mathcal{M}')^{\mathrm{op}}=qM_\infty(\mathcal{M})q$. This is only possible if the projection in $\mathcal{M}$ is central.

Thus each central projection in $\mathcal{N}$, being an L^p-projection, is identified spatially with a central projection in $\mathcal{M}$. It follows that the centers of $\mathcal{M}$ and $\mathcal{N}$ are isomorphic.

□

At this point, our original approach was to decompose $\mathfrak{X}$ into a direct integral of L^p bimodules between factors. This requires a significant detour into measure theory, and we have opted to omit these arguments (which may appear elsewhere) and deal with central projections.

We will use the following two results, which characterize isometries between noncommutative L^p spaces under certain circumstances.

Theorem 6.4 (Ye)**.** *Let $\{\mathcal{M}, \tau_{\mathcal{M}}\}, \{\mathcal{N}, \tau_{\mathcal{N}}\}$ be semifinite von Neumann algebras with given traces. Let T be a linear isometry from $L^p(\mathcal{M}, \tau_{\mathcal{M}})$ to $L^p(\mathcal{N}, \tau_{\mathcal{N}})$ $(p \neq 2)$, where we view these as spaces of $\tau_{\mathcal{M}}$ or $\tau_{\mathcal{N}}$-measurable operators. Then there exist, uniquely, a partial isometry $w \in \mathcal{N}$, an injective normal Jordan *-homomorphism J of $\mathcal{M}$ into $\mathcal{N}$, and a positive unbounded operator B affiliated with $J(\mathcal{M})' \cap \mathcal{N}$, all satisfying*

$$w^*w = J(1) = s(B);$$
$$\tau_{\mathcal{M}}(x) = \tau_{\mathcal{N}}(B^p J(x)), \qquad \forall x \in \mathcal{M}_+;$$
$$T(x) = wBJ(x), \qquad \forall x \in \mathcal{M} \cap L^p(\mathcal{M}, \tau_{\mathcal{M}}).$$

It is worth explaining here that a map between von Neumann algebras is *Jordan* if it preserves the Jordan product $x \circ y = (1/2)(xy + yx)$. An injective normal Jordan *-homomorphism is the sum of a *-homomorphism and a *-antihomomorphism; the two supports, which are central projections, have sum ≥ 1, and the two ranges are orthogonal. A surjective Jordan *-isomorphism is necessarily normal and so can be centrally decomposed, in both domain and range, into a *-isomorphism and a *-antiisomorphism. (See [HaS] for details. The only ambiguity in these decompositions arises from abelian summands.)

Theorem 6.5 (S4)**.** *Suppose that $T : L^p(\mathcal{M}) \to L^p(\mathcal{N})$ is a surjective isometry, $1 < p < \infty$, $p \neq 2$. Then there are a surjective Jordan *-isomorphism $J : \mathcal{M} \to \mathcal{N}$ and a unitary $u \in \mathcal{N}$ such that*

$$T(\varphi^{1/p}) = u(\varphi \circ J^{-1})^{1/p}, \qquad \forall \varphi \in \mathcal{M}_*^+. \tag{6.4}$$

We have $T(\xi x) = T(\xi)J(x)$ (resp. $T(\xi x) = uJ(x)u^*T(\xi)$) when ξx is supported on a central summand for which J is multiplicative (resp. antimultiplicative).

Proposition 6.6. *Let $\mathfrak{X} \simeq R^p(\mathcal{M})q$ be an L^p $\mathcal{M}-\mathcal{N}$ bimodule with $q \preccurlyeq e_{11}$. Then the inclusion*

$$\pi : \mathcal{N} \hookrightarrow q\mathcal{M}q$$

is surjective.

Proof. If necessary, implement an isomorphism so that $q \leq e_{11}$. The hypotheses mean that we have bimodules

$$\mathcal{M} - L^p(\mathcal{M})q - q\mathcal{M}q, \tag{6.5}$$
$$q'M_\infty(\mathcal{N})q' - q'C^p(\mathcal{N}) - \mathcal{N},$$

where $q' \in \mathcal{P}(M_\infty(N))$ and the bimodules are isometrically isomorphic as Banach spaces. By Lemma 6.1 we have inclusions consistent with the module actions:

$$\mathcal{M} \subset q'M_\infty(\mathcal{N})q', \quad \mathcal{N} \overset{\pi}{\subset} q\mathcal{M}q. \tag{6.6}$$

So via its left action, the projection $q \in \mathcal{M}$ is identified with a projection $q'' \in q'M_\infty(\mathcal{N})q'$. If we implement $L(q) \leftrightarrow L(q'')$, we get subbimodules

$$q\mathcal{M}q \;-\; L^p(q\mathcal{M}q) \;-\; q\mathcal{M}q, \tag{6.7}$$
$$q''M_\infty(\mathcal{N})q'' \;-\; q''C^p(\mathcal{N}) \;-\; \mathcal{N}.$$

Again the modules themselves are isometrically isomorphic, and we still have the same inclusion $\mathcal{N} \overset{\pi}{\subset} q\mathcal{M}q$ from the right actions.

Our next step will be to show that $q'' \sim e_{11}^{\mathcal{N}}$, so that $q''C^p(\mathcal{N})$ may be replaced with $L^p(\mathcal{N})$ in (6.7). By implementing projections from the common center of $\mathcal{M}$ and $\mathcal{N}$, we may consider separately the cases where $\mathcal{N}$ is finite or properly infinite. If $\mathcal{N}$ is properly infinite, then the given center-preserving inclusions $\mathcal{N} \hookrightarrow q\mathcal{M}q \hookrightarrow q''M_\infty(\mathcal{N})q''$ imply that q'' is properly infinite, and so $q'' \sim e_{11}^{\mathcal{N}}$.

If $\mathcal{N}$ is finite, first we argue that $q\mathcal{M}q$ must be finite. For otherwise $L^p(q\mathcal{M}q)$ contains an isometric copy of the Schatten class $S^p (= L^p(\mathcal{B}(\mathfrak{H}), \mathrm{tr}))$, and so we have that S^p embeds isometrically in $q''C^p(\mathcal{N})$, where $\mathcal{N}$ is finite. Letting τ be a faithful normal trace on $\mathcal{N}$, we show that this is impossible. Indeed, if $p > 2$, it is easily seen that $C^p(\mathcal{N})$ is a subspace of the intersection $X = L^p(M_\infty(\mathcal{N}), \tau \otimes \mathrm{tr}) \cap L^2(M_\infty(\mathcal{N}), \tau \otimes \mathrm{tr})$. According to [J2], X embeds into $L^p(\widetilde{\mathcal{M}})$ for some finite $\widetilde{\mathcal{M}}$. This yields an embedding of S^p into $L^p(\widetilde{\mathcal{M}})$, which is absurd in view of the result of Sukochev [Su, Theorem 3.1]. For $1 < p < 2$, we replace X by the sum $L^p(M_\infty(\mathcal{N}), \tau \otimes \mathrm{tr}) + L^2(M_\infty(\mathcal{N}), \tau \otimes \mathrm{tr})$. Again by [J2] X embeds into $L^p(\widetilde{\mathcal{M}})$ for some finite $\widetilde{\mathcal{M}}$ and thus an embedding of S^p in $C^p(\mathcal{N})$ provides an embedding of S^p into $L^p(\widetilde{\mathcal{M}})$. In this case, we may refer to the main result of [HRS] for the fact that this is impossible. So $q\mathcal{M}q$ is finite with faithful normal trace τ', and we may apply Theorem 6.4 to the L^p isometry

$$T : L^p(q\mathcal{M}q, \tau') \simeq q''C^p(\mathcal{N}) \hookrightarrow L^p(M_\infty(\mathcal{N}), \tau \otimes \mathrm{tr}).$$

With $T = wBJ(\cdot)$, the conditions of the theorem imply that $q'' = s_\ell(w) \sim s_r(w) = e_{11}^{\mathcal{N}}$, as desired.

This means that we may replace the bottom line of (6.7) by $\mathcal{N} - L^p(\mathcal{N}) - \mathcal{N}$. We still have $\pi : \mathcal{N} \hookrightarrow q\mathcal{M}q$, and we set $S : L^p(\mathcal{N}) \to L^p(q\mathcal{M}q)$ to be the isometric isomorphism of Banach spaces. Applying Theorem 6.5 to S we find the underlying pair $u \in q\mathcal{M}q$, $J : \mathcal{N} \to q\mathcal{M}q$; let $z, z^\perp$ be central projections of $\mathcal{N}$ which divide J into multiplicative and antimultiplicative parts. The intertwining relation between S and π gives

$$\begin{aligned} (6.8) \quad u(\varphi \circ J^{-1})^{1/p}\pi(x) &= S(\varphi^{1/p})\pi(x) = S(\varphi^{1/p}x) \\ &= u(\varphi \circ J^{-1})^{1/p}J(xz) + uJ(xz^\perp)(\varphi \circ J^{-1})^{1/p}, \qquad \varphi \in \mathcal{N}_*^+,\ x \in \mathcal{N}. \end{aligned}$$

We observe that J and π both identify the centers, so we may multiply on the left by $\pi(z^\perp)u^*$ to get

$$(\varphi \circ J^{-1})^{1/p}\pi(xz^\perp) = J(xz^\perp)(\varphi \circ J^{-1})^{1/p}, \qquad \varphi \in \mathcal{N}_*^+,\ x \in \mathcal{N}.$$

Then $R(\pi(xz^\perp)) = L(J(xz^\perp))$ is central for any x, so that $\mathcal{N}z^\perp$ is abelian, and in fact J is multiplicative. Now (6.8) shows that $\pi = J$. Since we know that J is surjective, this finishes the proof.

□

Return now to the general situation of an L^p $\mathcal{M}$-$\mathcal{N}$ bimodule $\mathfrak{X} \simeq R^p(\mathcal{M})q \simeq q'C^p(\mathcal{N})$, and identify the centers of $\mathcal{M}$ and $\mathcal{N}$. Use the comparability theorem to find the largest central projections z, z' satisfying $zq \preccurlyeq ze_{11}^{\mathcal{M}}, z'q' \preccurlyeq z'e_{11}^{\mathcal{N}}$. With $z'' = z \vee z'$, Proposition 6.6 tells us that $z''\mathcal{M}$ and $(z''\mathcal{N})^{\mathrm{op}}$ are commutants on $z''\mathfrak{X}$. On every central summand of the complement, both p and q are strictly larger than $e_{11}^{\mathcal{M}}$ and $e_{11}^{\mathcal{N}}$, respectively. It follows that $z''^\perp\mathcal{M}$ and $z''^\perp\mathcal{N}$ are finite; we will show that in fact they are abelian.

Proposition 6.7. *Let $\mathcal{M}$ and $\mathcal{N}$ be finite algebras, and assume that $\mathcal{M}$ has no abelian central summand. If $q \in \mathcal{P}(M_\infty(\mathcal{M}))$ and $q' \in \mathcal{P}(M_\infty(\mathcal{N}))$ are projections such that $qz \succsim e_{11}^{\mathcal{M}} z$ and $q'z' \succsim e_{11}^{\mathcal{N}} z'$ for all central projections $z \in M_\infty(\mathcal{M})$, $z' \in M_\infty(\mathcal{N})$, then there is no $\mathcal{M}-\mathcal{N}$ L^p bimodule $\mathfrak{X} \simeq R^p(\mathcal{M})q \simeq q'C^p(\mathcal{N})$.*

Proof. Seeking a contradiction, let $\mathfrak{X}$ be such a bimodule. Choose finite traces $\tau_{\mathcal{M}}, \tau_{\mathcal{N}}$ and consider the L^p elements to be measurable operators. As before, we may assume that $q \geq e_{11}^{\mathcal{M}}$ and $q' \geq e_{11}^{\mathcal{N}}$. Let T be the isometric isomorphism from $R^p(\mathcal{M})q$ to $q'C^p(\mathcal{N})$; the domain naturally contains $R^p e_{11}^{\mathcal{M}} \simeq L^p(\mathcal{M})$ to give the isometric restriction

$$T_1 : L^p(\mathcal{M}, \tau_{\mathcal{M}}) \cap \mathcal{M} \ni x \mapsto T((\, x \; 0 \; \cdots \,)) \in q'C^p(\mathcal{N}) \subset L^p(M_\infty(\mathcal{N}), \tau_{\mathcal{N}} \otimes \mathrm{tr}).$$

The vector $\xi = (\, 1 \; 0 \; \cdots \,) \in R^p(\mathcal{M})q$ has full left support, so by Theorem 6.2 it satisfies equation (6.2) for no nonzero $\eta \in \mathfrak{X}$. Since T is an isometric identification, the same is true for $T(\xi) \in q'C^p(\mathcal{N})$. Now our assumption on the size of q' means that $T(\xi)$ can have full left support on no central summand, so by Theorem 6.2 again we must have $s_r(T(\xi)) = e_{11}^{\mathcal{N}} = 1_{\mathcal{N}}$.

Theorem 6.4 tells us that $T_1(x) = wBJ(x)$. *A priori* these operators are affiliated with $M_\infty(\mathcal{N})$, but the conditions in the theorem imply

$$wB = T_1(1) \in L^p(M_\infty(\mathcal{N}))e_{11} \Rightarrow J(1) = s_r(w) = s(B) = e_{11}^{\mathcal{N}}.$$

Thus we see that B is affiliated with, and $J(\mathcal{M})$ are elements in, $e_{11}M_\infty(\mathcal{N})e_{11}$. We naturally identify the latter algebra with $\mathcal{N}$, so that J is unital.

Choose $y \in R^p(\mathcal{M})(q - e_{11})$ with $s_\ell(y)$ strictly between 0 and 1 on all central summands. (Recall that $\mathcal{M}$ has no abelian central summands.) So whenever $x \in \mathcal{M}, s_\ell(x) \perp s_\ell(y)$, (6.2) gives

$$\begin{aligned} 2(\|wBJ(x)\|^p + \|T(y)\|^p) &= 2(\|(\, x \; 0 \; \cdots \,)\|^p + \|y\|^p) \\ &= \|(\, x \; 0 \; \cdots \,) + y\|^p + \|(\, x \; 0 \; \cdots \,) - y\|^p \\ &= \|wBJ(x) + T(y)\|^p + \|wBJ(x) - T(y)\|^p, \end{aligned}$$

and this implies the *right* supports of $J(x)$ and $T(y)$ are orthogonal. Write $J = J_1 + J_2$ for the unique decomposition into multiplicative and antimultiplicative *-homomorphisms, with orthogonal ranges in $\mathcal{N}$. So

$$\begin{aligned} s_\ell(x) \leq s_\ell(y)^\perp &\Rightarrow s_r(T(y)) \perp [s_r(J_1(x)) + s_r(J_2(x))] \\ &\Rightarrow s_r(T(y)) \perp [J_1(s_r(x)) + J_2(s_\ell(x))]. \end{aligned}$$

Since $s_r(x)$ can be any projection subequivalent to $s_\ell(y)^\perp$, and J is unital, it follows in particular that $s_r(T(y)) \leq J_2(1)$. This inequality passes to the closed linear span of all y under discussion, which is $R^p(\mathcal{M})(q - e_{11}^{\mathcal{M}})$ by an easy Hahn-Banach argument. We obtain

$$\begin{aligned} q'C^p(\mathcal{N}) &= T(R^p(\mathcal{M})q) \\ &= T(R^p(\mathcal{M})e_{11}^{\mathcal{M}}) + T(R^p(\mathcal{M})(q - e_{11}^{\mathcal{M}})) \\ &\subset \overline{wBJ(\mathcal{M})} + q'C^p(\mathcal{N})J_2(1) \\ &\subset s_\ell(w)C^p(\mathcal{N}) + q'C^p(\mathcal{N})J_2(1). \end{aligned}$$

Now $s_\ell(w)$ is equivalent to $e_{11}^{\mathcal{N}}$, so $(q' - s_\ell(w))$ has full central support. Multiplying the containment above by $(q' - s_\ell(w))$ on the left, we get

$$(q' - s_\ell(w))C^p(\mathcal{N}) \subset (q' - s_\ell(w))C^p(\mathcal{N})J_2(1).$$

This is only possible if $J_2(1) = 1_{\mathcal{N}}$, and so $J = J_2$ is antimultiplicative.

Finally, let v be any partial isometry between orthogonal projections in $\mathcal{M}$. With $\pi : \mathcal{N} \hookrightarrow qM_\infty(\mathcal{M})q$ as before, we have

$$T((\, v^*v \; 0 \; \cdots)) = wBJ(v^*v) = (wBJ(v))J(v^*) = T((\, v \; 0 \; \cdots)(\pi(J(v^*))).$$

Since T is one-to-one, $(\, v^*v \; 0 \; \cdots) = (\, v \; 0 \; \cdots)(\pi(J(v^*)))$. But look at the left supports in $\mathcal{M}$ of these vectors; the first is v^*v and the second is $\leq s_\ell(v) = vv^*$. This contradiction finishes the proof.

□

The only case remaining is an abelian central summand of $\mathcal{M}$ and $\mathcal{N}$. Because column and row sums of $L^p(\mathbb{C}) = \mathbb{C}$ are identical, we cannot control the sizes of the commutants.

Proposition 6.8. *Let $\mathcal{A} = L^\infty(X, \mu)$ be an abelian von Neumann algebra. The L^p $\mathcal{A}-\mathcal{A}$ bimodules are exactly the p-direct integrals of measurable fields of Hilbert spaces over (X, μ).*

By a p-direct integral we mean exactly the same construction as a direct integral of a measurable field of Hilbert spaces, except that the norm is

$$\|\xi(\cdot)\| = \left(\int \|\xi(\omega)\|^p d\mu(\omega) \right)^{1/p}.$$

Proof. By Proposition 6.3, we may identify the left and right actions of $\mathcal{A}$. (If presentations of $\mathcal{A}$ are given, this may involve an algebraic isomorphism.) Subject to this, we claim that a right L^p $\mathcal{A}$-module admits a *unique* structure as an L^p $\mathcal{A}-\mathcal{A}$ bimodule: left action of $\mathcal{A}$ given by $f \cdot \xi = \xi f$ and left inner product by $< \xi, \eta >_L = < \eta, \xi >_R$.

For suppose we are given an L^p $\mathcal{A}-\mathcal{A}$ bimodule. That $f \cdot \xi = \xi f$ is automatic from the assumption; we further have, for any measurable set $E \subset X$,

$$\begin{aligned} \|\chi_E < \xi, \xi >_L \| = \| < \chi_E\xi, \chi_E\xi >_L \| &= \|\chi_E\xi\|^2 \\ &= \|\xi\chi_E\|^2 = \| < \xi\chi_E, \xi\chi_E >_R \| = \| < \xi, \xi >_R \chi_E\|. \end{aligned} \tag{6.9}$$

Both $< \xi, \xi >_L$ and $< \xi, \xi >_R$ are positive functions in $L^{p/2}(X, \mu)$. Taking E in (6.9) to be the set where one dominates the other, we deduce that $< \xi, \xi >_L = < \xi, \xi >_R$ μ-a.e.. By polarization,

$$4 < \xi, \eta >_L = \sum_{k=1}^{4} i^k < \xi + i^k\eta, \xi + i^k\eta >_L = \sum_{k=1}^{4} i^k < \xi + i^k\eta, \xi + i^k\eta >_R = 4 < \eta, \xi >_R .$$

So it is the same problem to describe the right L^p $\mathcal{A}$-modules. Any random projection $q : X \to \mathcal{P}(\mathcal{B}(\mathfrak{H}))$ gives a bimodule of the form qC^p, and since the isomorphism class of qC^p depends only on the Murray-von Neumann equivalence class of q, we may assume that

$$q = \sum p_n \otimes \chi_{X_n} \in M_\infty(\mathcal{A}), \qquad \left(p_n = \sum_{k=1}^{n} e_{kk} \right),$$

where X_n is $\{\omega \mid \mathrm{Tr}(q(\omega)) = n\}$.

By direct calculation, we see that $C^p(\mathcal{A})$ is the Bochner space $L^p(\ell^2, X, \mu)$, and

$$(p_n \otimes \chi_{X_n})C^p \simeq L^p(\ell^2_n, X_n, \mu),$$

where we still use μ to denote the restricted measure on X_n. This last is a constant measurable field of Hilbert spaces with norm

$$\left(\int_{X_n} \|f(\omega)\|^p d\mu(\omega)\right)^{1/p}.$$

The full module qC^p is a central/ℓ^p sum of these,

$$qC^p = (\sum p_n \otimes \chi_{X_n})C^p = \bigoplus_{\ell^p} L^p(\ell^2_n, X_n, \mu),$$

which is exactly a p-direct integral of the measurable field of Hilbert spaces which has dimension n over X_n. It is clear that any such p-direct integral can be obtained in this way, so we are done. □

We summarize the results in

Theorem 6.9. *Let $\mathcal{M}$ and $\mathcal{N}$ be σ-finite algebras, let $1 < p < \infty$, $p \neq 2$, and let $\mathfrak{X}$ be an $\mathcal{M}-\mathcal{N}$ L^p-bimodule which is countably generated and faithful for each action.*

The centers of $\mathcal{M}$ and $\mathcal{N}$ are isomorphic and act identically on $\mathfrak{X}$. Let z be the largest central projection which is abelian for both $\mathcal{M}$ and $\mathcal{N}$. Then the left action of $z^\perp\mathcal{M}$ and the right action of $z^\perp\mathcal{N}$ are commutants on $z^\perp\mathfrak{X}$. On the other hand, $z\mathfrak{X}$ is isomorphic to a p-direct integral of a measurable field of Hilbert spaces over (X, μ), where $z\mathcal{M} = z\mathcal{N} \simeq L^\infty(X, \mu)$.

This has an appealing consequence.

Theorem 6.10. *Under the same assumptions as above, there exists an $\mathcal{M}-\mathcal{N}$ L^p-bimodule if and only if $\mathcal{M}$ and $\mathcal{N}$ are Morita equivalent.*

Proof. $\mathcal{M}$ and $\mathcal{N}$ are Morita equivalent exactly when $\mathcal{N} \simeq qM_\infty(\mathcal{M})q$ for some projection q with central support one. (This fact, and many other fundamental ideas, may be found in Rieffel's discussions of Morita equivalence [Ri1], [Ri2].) When this happens, one may take $\mathfrak{X} \simeq R^p(\mathcal{M})q$ and notice $\mathcal{N} \simeq (\mathcal{M}')^{\mathrm{op}}$.

If there is an $\mathcal{M}-\mathcal{N}$ L^p-bimodule $\mathfrak{X}$, then $\mathfrak{X} \simeq R^p(\mathcal{M})q$ and $\mathcal{N} \simeq \pi(\mathcal{N}) \subset qM_\infty(\mathcal{M})q$. Let z be as in Theorem 6.9. We have that $\pi(z^\perp\mathcal{N}) = z^\perp qM_\infty(\mathcal{M})q$, but

$$\pi(z\mathcal{N}) = z\mathcal{M} = ze_{11}M_\infty(\mathcal{M})e_{11} \subset zqM_\infty(\mathcal{M})q.$$

Then

$$\mathcal{N} \simeq (z^\perp q + ze_{11})M_\infty(\mathcal{M})(z^\perp q + ze_{11}),$$

so $\mathcal{M}$ and $\mathcal{N}$ are Morita equivalent. □

In fact, an $\mathcal{M}-\mathcal{N}$ L^p-bimodule $\mathfrak{X}$ which does not degenerate on its abelian component (so zq is abelian, and $L(\mathcal{M})$ and $R(\mathcal{N})$ are commutants) implements an equivalence of representation categories just as in the Hilbert C*-module case. Here the densities are nonzero, and one makes use of the generalized relative tensor product, with functorial equivalence given by

$$Left\ L^qMod(\mathcal{N}) \to Left\ L^rMod(\mathcal{M}) : {}_{\mathcal{N}}\mathfrak{Y} \longmapsto ({}_{\mathcal{M}}\mathfrak{X}_{\mathcal{N}}) \otimes_{\mathcal{N},p,q,r} ({}_{\mathcal{N}}\mathfrak{Y}).$$

To see that this is an isomorphism, we let ${}_{\mathcal{N}}\bar{\mathfrak{X}}_{\mathcal{M}}$ be the contragredient and note that "$({}_{\mathcal{N}}\bar{\mathfrak{X}}_{\mathcal{M}})\otimes_{\mathcal{M},p,r,q}$" is the inverse map. By associativity of the relative tensor product, it suffices to show that

$$\mathfrak{X} \otimes_{\mathcal{N},p,q,r} \bar{\mathfrak{X}} \simeq L^p(\mathcal{M}); \qquad \bar{\mathfrak{X}} \otimes_{\mathcal{M},p,r,q} \mathfrak{X} \simeq L^p(\mathcal{N}).$$

We verify the first, using Proposition 5.6:

$$\mathfrak{X} \otimes_{\mathcal{N},p,q,r} \bar{\mathfrak{X}} \simeq qC^p(\mathcal{N}) \otimes_{\mathcal{N},p,q,r} R^p(\mathcal{N})q \simeq qL^p(M_\infty(N))q \simeq L^p(\mathcal{M}).$$

The second follows by symmetry.

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References

[B] E. Behrends et al., *L^p-structure in real Banach spaces*, Springer-Verlag, Berlin-New York, 1977.

[C] A. Connes, *On the spatial theory of von Neumann algebras*, J. Funct. Anal. **35** (1980), 153-164.

[FT] T. Falcone and M. Takesaki, *The non-commutative flow of weights on a von Neumann algebra*, J. Funct. Anal. **182**, no.1 (2001), 170-206.

[H] U. Haagerup, *L^p-spaces associated with an arbitrary von Neumann algebra*, Algèbres d'opérateurs et leurs applications en physique mathématique, CNRS **15** (1979), 175-184.

[HRS] U. Haagerup, H. Rosenthal, and F. Sukochev, *Banach embedding properties of non-commutative L_p-spaces*, Mem. Amer. Math. Soc. **163** (2003), no. 776.

[HaS] H. Hanche-Olsen and E. Størmer, *Jordan operator algebras*, Pitman, Boston, 1984.

[Hi] M. Hilsum, *Les espaces L^p d'une algèbre de von Neumann*, J. Funct. Anal. **40** (1981), 151-169.

[J1] M. Junge, *Doob's inequality for non-commutative martingales*, J. Reine Angew. Math. **549** (2002), 149-190.

[J2] M. Junge, *A noncommutative Poisson process*, preprint.

[JNRX] M. Junge, N. Nielsen, Z.-J. Ruan, and Q. Xu, *$\mathcal{COL}_p$ spaces - the local structure of non-commutative L_p spaces*, preprint.

[JRX] M. Junge, Z.-J. Ruan, and Q. Xu, *$\mathcal{COL}_p$ structure for L_p spaces associated with hyperfinite type III von Neumann algebras*, in preparation.

[KR] R. Kadison and J. Ringrose, *Fundamentals of the theory of operator algebras II*, Graduate Studies in Mathematics **16**, AMS, Providence, 1997.

[K1] H. Kosaki, *Applications of the complex interpolation method to a von Neumann algebra: non-commutative L^p spaces*, J. Funct. Anal. **56** (1984), 29-78.

[K2] H. Kosaki, *On the continuity of the map $\varphi \to |\varphi|$ from the predual of a W^*-algebra*, J. Funct. Anal. **59** (1984), 123-131.

[K3] H. Kosaki, *Applications of uniform convexity of noncommutative L^p-spaces*, Trans. AMS **283** no. 1 (1984), 265-282.

[L] E. C. Lance, *Hilbert C*-modules: a toolkit for operator algebraists*, LMS Lecture Note Series **210**, Cambridge University Press, Cambridge, 1995.

[N] E. Nelson, *Notes on non-commutative integration*, J. Funct. Anal. **15** (1974), 103-116.

[Pa] W. Paschke, *Inner product modules over B^*-algebras*, Trans. AMS **182** (1973), 443-468.

[PX] G. Pisier and Q. Xu, *Noncommutative L^p spaces*, Handbook of the geometry of Banach spaces, Vol. II., North-Holland Publishing Co., Amsterdam, 2003.

[P] S. Popa, *Correspondences*, notes, 1986.

[R] Y. Raynaud, *On ultrapowers of non commutative L_p spaces*, J. Operator Theory **48** (2002), 41-68.

[RX] Y. Raynaud and Q. Xu, *On subspaces of non-commutative L_p-spaces*, J. Funct. Anal. **203** (2003), 149-196.

[Ri1] M. Rieffel, *Morita equivalence for C*-algebras and W*-algebras*, J. Pure and Appl. Algebra **5** (1974), 51-96.

[Ri2] M. Rieffel, *Morita equivalence for operator algebras*, in Operator Algebras and Applications **38**, no. 1, AMS Proc. of Symposia in Pure Math., 1982.

[Sa] J.-L. Sauvageot, *Sur le produit tensoriel relatif d'espaces de Hilbert*, J. Operator Theory **9** (1983), 237-252.

[Sc] L. Schmitt, *Order derivations on L^p-spaces of W*-algebras*, Math. Z. **196** (1987), 117-124.

[S1] D. Sherman, *The application of modular algebras to relative tensor products and non-commutative L^p modules*, Ph. D. dissertation, UCLA, 2001.

[S2] D. Sherman, *Applications of modular algebras*, in preparation.

[S3] D. Sherman, *Relative tensor products for modules over von Neumann algebras*, Contemp. Math. **328** (2003), 275-291.

[S4] D. Sherman, *Noncommutative L^p structure encodes exactly Jordan structure*, preprint.

[Su] F. Sukochev, *Non-isomorphism of L_p-spaces associated with finite and infinite von Neumann algebras*, Proc. Amer. Math. Soc. **124** no.5 (1996), 1517-1527.

[T] M. Takesaki, *Theory of Operator Algebras I*, Springer-Verlag, New York, 1979.

[Te] M. Terp, *L^p-spaces associated with von Neumann algebras*, notes, Copenhagen University, 1981.

[W1] K. Watanabe, *Dual of non-commutative L^p-spaces with $0 < p < 1$*, Math. Proc. Camb. Phil. Soc. **103** (1988), 503-509.

[W2] K. Watanabe, *An application of orthoisomorphisms to non-commutative L^p-isometries*, Publ. RIMS **32** (1996), 493-502.

[We] N. Weaver, *Mathematical quantization*, Chapman & Hall/CRC, Boca Raton, 2001.

[W-O] N. E. Wegge-Olsen, *K-theory and C*-algebras*, Oxford University Press, Oxford, 1993.

[Y] S. Yamagami, *Algebraic aspects in modular theory*, Publ. RIMS **28** (1992), 1075-1106.

[Ye] F. Yeadon, *Isometries of non-commutative L^p spaces*, Math. Proc. Camb. Phil. Soc. **90** (1981), 41-50.

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