D. Sherman – Why Operator Spaces? 

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$C^*$-algebra: (concrete) closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$
(abstract) Banach $*$-algebra where $\|x^*x\| = \|x\|^2$

These two definitions are equivalent, via Gelfand-Naimark theorem of 1943.

Operator Spaces: (idea) closed subspace of $\mathcal{B}(\mathcal{H})$ (or any $C^*$-algebra)

Fact: Any Banach space embeds isometrically in a commutative $C^*$-algebra.

let $X$ be a Banach space

$$X \hookrightarrow C(X_1^*)$$ isometric via Hahn-Banach Theorem.

dual unit ball.

$$x \mapsto \hat{x}(\cdot)$$

so $\hat{x}(\varphi) = \varphi(x)$

where the weak-$*$ topology makes this compact

any $S \subseteq X_1^*$ : $\|\hat{x}|_S\| = \|\hat{x}\| = \|x\|$, $\forall x \in X$
which is norm attaining

If $X$ is separable, $S$ could be countable.

$$X \hookrightarrow \ell^\infty(S) \subseteq \mathcal{B}(\ell^2(S))$$

– these are even diagonal..

Operator spaces: (concrete) closed subspace $V \subseteq \mathcal{B}(\mathcal{H})$ with norms on $M_n(V)$ inherited from $M_n(\mathcal{B}(\mathcal{H}))$.

(abstract) a linear space $V$ with norms on $M_n(V)$, $n \geq 1$, satisfying

- $\left\|\begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}\right\| = \max(\|v\|, \|w\|)$, $\forall v \in M_m(V)$, $\forall w \in M_n(V)$
- $\|\alpha v \beta\| \leq \|\alpha\| \|v\| \|\beta\|$ $v \in M_m(V)$, $\alpha \in M_{n,m}(\mathbb{C})$, $\beta \in M_{m,n}(\mathbb{C})$.

3 choices of starting point for operator algebras.

These 2 definitions of operator spaces are equivalent, via Ruan, 1988.

History: Arveson 1969–1972 "Subalgebras of $C^*$-algebras", I & II.
Stinespring, 1955 comp. positive maps.

A Banach space can be turned into an operator space (almost always) in multiple ways. ("quantizations")

<u>Example 1</u>: $\ell_2^2 = V$ $\quad e_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

2-dim'l Hilbert space $\quad e_2 \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$\left\| \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \right\| = \sqrt{|\alpha|^2 + |\beta|^2}$$

$$\left\| \begin{pmatrix} e_1 & e_2 \\ 0 & 0 \end{pmatrix} \right\|_{M_2(V)} = \sqrt{2}$$

"row Hilbert space"

<u>Example 2</u>: $\ell_2^2 = V$ $\quad e_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$e_2 \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$$\left\| \begin{pmatrix} e_1 & e_2 \\ 0 & 0 \end{pmatrix} \right\|_{M_2(V)} = 1.$$

"column Hilbert space"

not same quantization in Example 1 & 2. (compare norms)

<u>Theorem</u> (Arveson, 1972): Let $S, T$ be compact, irreducible operators on $\ell^2$. Consider span$\{I, S\}$, span$\{I, T\}$, 2-dimensional operator spaces. The map $\alpha + \beta S \to \alpha + \beta T$ induces a complete isometry (i.e. same norm at every matrix level) <u>iff</u> $S$ and $T$ are unitarily equivalent.

<u>Back to K-theory</u>:

$A$ $C^*$-algebra

$\mathcal{P}_\infty(A) = \bigcup_{n\geq 1} \mathcal{P}(M_n(A))$

$\uparrow$ projections in $M_n(A)$.

For $p \in M_m(A)$, $q \in M_n(A)$,

$p \sim_0 q$ means there exists $v \in M_{m,n}(A)$, with $vv^* = p$, $v^*v = q$.

More natural: $\mathcal{P}_\infty(A) = \mathcal{P}(\bigcup_{n\geq 1} M_n(A))$

nested union
$*$-algebra
not a $C^*$-algebra in general.

$x \in M_n(A)$

$x = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$

$p \sim_0 q$ is Murray-von Neumann equivalence in $\bigcup_{n\geq 1} M_n(A)$ or unitary equivalence or homotopy equivalence.

$\mathcal{P}_\infty(A)/\sim_0$ has a $+$ operation:

$$[p] + [q] = [p \oplus q] \left( = \left[ \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \right] \right)$$

This operation is well-defined, commutative, associative, has $[0]$ as an additive identity

$\Rightarrow (\mathcal{P}_\infty(A)/\sim_0, +)$ is an abelian semi-group. (actually, a monoid)

denoted $D(A)$, the dimension semigroup of $A$.

<u>Note</u>:
If $p, q \in M_n(A)$, $p \perp q$, then $[p] + [q] = [p+q]$.

<u>proof</u>: $\begin{pmatrix} p \\ q \end{pmatrix}$ is a partial isometry.

// Chapter 2

A unital

$K_0(A)$ is the group generated by $D(A)$.

Examples

- $A = \mathbb{C}$

  $P_\infty(A) / \text{same rank}$    no $\infty$ rank projections

  identify $(D(A), +)$ with $(\underbrace{\mathbb{N} \cup \{0\}}_{\mathbb{N}_0}, +)$ : $[p] \to$ rank $p$.

- $A = \mathcal{B}(\ell^2)$

  same idea: $D(\mathcal{B}(\ell^2)) = (\mathbb{N}_0 \cup \{\infty\}, +)$

the "Grothendieck Construction" produces a group from an abelian semigroup

ex semigroups: $(\mathbb{Z}, \times)$, $(\mathbb{Z} \setminus \{0\}, \times)$
$(\mathbb{N}, +)$, $(\mathbb{N}_0, +)$

$S$ an abelian semigroup

$G(S)$ is constructed as

$\{(s_1, s_2) \mid s_1, s_2 \in S\}$ : $(s_1, s_2) = (s_3, s_4)$

when: (naïve try) $s_1 + s_4 = s_2 + s_3$ (not necessarily transitive)

(correct) $\exists s_5$ with $s_1 + s_4 + s_5 = s_2 + s_3 + s_5$

get a group, where operation is component-wise addition

$(x,y)^{-1} = (y,x)$, zero element $= (x,x)$

$(\text{or } = (0,0))$

Ex.:

$K_0(\mathbb{C}) = G(\mathbb{N}_0, +) = (\mathbb{Z}, +)$

$K_0(M_n(\mathbb{C})) = \mathbb{Z}$

$K_0(\mathcal{B}(\ell^2))$ : $(x,y) = (+\infty, +\infty)$, $\forall x, y$

$\quad = \{0\}$.