Linear combinations of projections in operator algebras.

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Joint work with

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Outline

1. Questions
2. Linear combinations of projections
3. Positive linear combinations of projections (PCP)
4. Infinite sums of projections
5. Finite sums of projections & open questions

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- For which C*-algebras are all elements linear combinations of projections? If not all, which are?

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  not necessarily mutually orthogonal.

The questions that started us on this project were:

- Which positive elements are infinite sums of projections? (Converging in the strict topology in a multiplier algebra or in the strong topology in a von Neumann algebra.) This question originated in frame theory (Dykema, Larson & all, 2004)
  It is now solved in $B(H)$.
- Which positive elements are finite sums of projections?
  Still open in $B(H)$.

Which W*-algebras are the span of their projections?

- $B(H)$:
  - Fillmore, 1967 (n=257),
    then progress via link to commutators:
  - Pearcy & Topping, 1967 (n=16),
  - Matsumoto, 1984 (n=10)

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  - Fillmore, 1967 (n=257),
    then progress via link to commutators:
  - Pearcy & Topping, 1967 (n=16),
  - Matsumoto, 1984 (n=10)
- All W*-algebras with the exception of finite type I algebras with infinite dimensional center (e.g., $\bigoplus_1^\infty M_n(\mathbb{C})$ or algebras having such a direct summand. (Pearcy &Topping (1967), Fack & de La Harpe (1980), Goldstein & Paszkiewicz (1992))

Universal constants

Definition

A C*-algebra $\mathcal{A}$ has universal constants V, N if for every $b = b^* \in \mathcal{A}$,

$$b = \sum_1^N \alpha_j p_j \quad \text{and} \quad \sum_1^N |\alpha_j| \leq V\|b\|$$

with $\alpha_j \in \mathbb{R}$ and $p_j \in \mathcal{A}$ projections.

Universal constants for a W*-algebra M

- M properly infinite, $N = 6,\ V = 8$
- M type II_1, $N = 12,\ V = 14$
- M direct sum of m matrix algebras: $N = m + 4,\ V = m + 4$.

These universal constants are quite useful. E.g., Fong&Murphy's proof that positive invertible in $B(H)$ are positive combinations of projections.

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- Unital simple AF-algebras, AT-algebras, or AH-algebras (with bounded dimension growth) with real rank zero and with $|\,\partial_e(\mathrm{T}(\mathcal{A}))\,| < \infty$.: The collection of the tracial states $\mathrm{T}(\mathcal{A})$ is convex and compact (in the w*-topology), $\partial_e(\mathrm{T}(\mathcal{A}))$ is the collection of the extreme points of $\mathrm{T}(\mathcal{A})$.

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More C*-algebras that are the span of their projections

The following includes all the finite algebras studied by Marcoux:

Theorem (2014)

A unital simple separable finite C-algebra $\mathcal{A}$ with real rank zero, stable rank one, strict comparison of projections is the linear span of its projections if and only if $|\,\partial_{\mathrm{e}}(\mathrm{T}(\mathcal{A}))\,| < \infty$.*

The key steps are

- elements in the kernel of all traces are sums of commutators (this holds also when $|\,\partial_{\mathrm{e}}(\mathrm{T}(\mathcal{A}))\,| = \infty$.)
- commutators are linear combinations of projections (by Marcoux this holds under very general conditions)
- every element can be decomposed as a linear combination of projections plus an element in the kernel of all traces (this requires $|\,\partial_{\mathrm{e}}(\mathrm{T}(\mathcal{A}))\,| < \infty$.)

C*-algebras that are not the span of their projections

Already mentioned:

- W*-algebras with finite type I direct summand with infinite dimensional center
- C*-algebras as in previous slide but with $|\,\partial_{\mathrm{e}}(\mathrm{T}(\mathcal{A}))\,| = \infty$.

C*-algebras that are not the span of their projections

Already mentioned:

- W*-algebras with finite type I direct summand with infinite dimensional center
- C*-algebras as in previous slide but with $|\,\partial_e(\mathrm{T}(\mathcal{A}))\,| = \infty$.

What else?

- Some nonunital algebras, e.g. $\mathcal{K}$ ($\mathcal{K} = K(H)$): linear combinations of projections must have finite rank.

Theorem (in press)

Let $\mathcal{A}$ finite as previous slide and with $|\,\partial_e(\mathrm{T}(\mathcal{A}))\,| < \infty$. Then $b \in \mathcal{A} \otimes \mathcal{K}$ is a linear combination of projections in $\mathcal{A} \otimes \mathcal{K}$ if and only if $\tau(R_b) < \infty \ \forall \tau \in \mathrm{T}(\mathcal{A})$ (R_b range projection of b.)

Positive combinations of projections in $B(H)$

- If $b \in B(H)_+ \setminus \mathcal{K}$ then b is PCP (Fillmore, 1969).
- $\mathcal{K}_+ \ni b = \sum_1^n \lambda_j p_j$ with $\lambda_j > 0 \Rightarrow b$ has finite rank.
  $b \geq 0$ has finite rank $\Rightarrow b$ is PCP. (Both are obvious)
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Yet, knowing about PCP is useful, e.g., it is the first step to obtain decomposition into sum of projections.

Positive combination of projections - the "easy" cases

Theorem (with H. Halpern, 2013)

The following positive elements in a W-algebra are PCP:*

- *Simple ($M_n(\mathbb{C})$, type II_1 or type III σ-finite factors): all.*
- *Type II_∞ factors: iff either b is not in the Breuer ideal of compact operators or the range projection R_b is finite.*
- *"Small center" (finite direct sums of factors): same as above.*
- *"Large center": obstruction in terms of the central essential spectrum. (E.g., $\bigoplus \frac{1}{n} 1_n \in \bigoplus B(H_n)$ is not PCP.)*

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Theorem (2011)

Let $\mathcal{A}$ be a purely infinite simple σ-unital C-algebra. Then every positive element of $\mathcal{A}$ and of the multiplier algebra $\mathcal{M}(\mathcal{A})$ is a PCP.*

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More on multiplier algebras

By Marcoux, $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ is the span of its projections. But more:

Theorem (in press)

Let $\mathcal{A}$ be unital simple separable C-algebra with real rank zero, stable rank one, strict comparison of projections and with $\partial_e(\mathrm{T}(\mathcal{A}))$ finite.*

- *If $P \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ is a projection and $B \in P\mathcal{M}(\mathcal{A} \otimes \mathcal{K})P$, then B is a linear combination of projections in $P\mathcal{M}(\mathcal{A} \otimes \mathcal{K})P$.*
- *If $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ has real rank zero, then $B \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})_+$ is a PCP if and only if and only if $\tau(R_B) < \infty$ for every $\tau \in \mathrm{T}(\mathcal{A})$ for which $B \in I_\tau := \overline{\mathit{span}}\{X \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})_+ \mid \tau(X) < \infty\}$.*

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Real rank zero needed to use Brown's interpolation theorem. We recently removed the condition that $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ has real rank zero by proving that $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$ has strict comparison of positive elements.

Infinite sums of projections in $B(H)$

Theorem (Choi & Wu, 1994-2014)

If $b \in B(H)_+$ and $\|b\|_{ess} > 1$ then b is a sum of finitely many projections.

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Theorem (2009)

Let $b \in B(H)_+$, then b is a sum of projections iff either
$\operatorname{Tr}\left((b - R_b)_+\right) = \infty$ *or*
$\operatorname{Tr}\left((b - R_b)_+\right) - \operatorname{Tr}\left((b - R_b)_-\right) \in \mathbb{N} \cup \{0\}$.

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Where does that integer come from?

The integer in the N&S condition

Let $b = \sum_1^\infty p_j$ with $p_j = \xi_j \otimes \xi_j$ rank-one projections.
$((\xi \otimes \eta)(\zeta) := (\zeta, \eta)\xi$ rank one operator, $\xi, \eta, \zeta \in H)$.
Let $\{\chi_j\}$ be an o.n. basis,

$\theta = \sum_j^\infty \chi_j \otimes \xi_j$	frame transform, aka analysis operator
$\theta = wb^{1/2}$	polar decomposition, so $w^*w = R_b$
$\theta\theta^*$	Gram matrix of $\{\xi_j\}$

Now, an easy computation:
$E(wbw^*) = E(\theta\theta^*) = 1$ and $b = (b - R_b)_+ - (b - R_b)_- + R_b$

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$\mathrm{Tr}\,\big((b - R_b)_+\big) - \mathrm{Tr}\,\big((b - R_b)_-\big) = \mathrm{Tr}(I - ww^*) = -ind(w)$,

that is, the integer is the Fredholm index of the partial isometry w.

Infinite sums of projections in W*-factors

Theorem (2009)

Let M be a W-factor with separable predual (e.g., H is separable), let $b \in M_+$*

- *If M is type III, then b is a sum of projections iff either $\|b\| > 1$ or b is a projection.*
- *If M is type II and b is a sum of projections, then*

$$\tau((b - R_b)_+) \geq \tau((b - R_b)_-).$$

The condition is also sufficient if $\tau((b - R_b)_+) = \infty$ or if b is diagonalizable (i.e., $b = \oplus_1^\infty \gamma_n p_n$.)

Finite sums of projections in W*-algebras: Suff condition

Theorem (2013)

Let M be a σ-finite factor. A sufficient condition for $b \in M_+$ to be a finite sum of projections is

- *If M is of type I_∞: $\|b\|_{ess} > 1$ (usual essential norm)*
- *If M is of type II_∞: $\|b\|_{ess} > 1$ (essential norm relative to the Breuer ideal)*
- *If M is of type III: $\|b\| > 1$ (operator norm)*
- *If M is of type II_1 and b is diagonalizable:* $\tau((b - R_b)_+) > \tau((b - R_b)_-)$.

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- *If M is of type III: $\|b\| > 1$ (operator norm)*
- *If M is of type II_1 and b is diagonalizable: $\tau((b - R_b)_+) > \tau((b - R_b)_-)$.*

But what if $\|b\|_{ess} = 1$?

Finite sums of projections in $B(H)$: Nec condition

Choi&Wu's test case, 2014

$b = 1 - k_1 \oplus 1 + k_2$ with $\mathrm{Tr}(k_1) = \mathrm{Tr}(k_2) < \infty$
They showed b can fail to be a finite sum of projections.

Why?

Theorem (with Halpern, 2013)

If b is a finite sum of projections, and $(I-b)_+ R_b \in K$ then $(I-b)_- R_b \in K$ and $(I-b)_+ R_b$ and $(I-b)_- R_b$ generate the same (non-closed) ideal of $B(H)$.
The same holds in W-algebras.*

Question

Is the condition that k_1 and k_2 as above generate the same ideals sufficient for b to be a finite sum of projections?

Finite sums of proj in C*-algebras

Theorem (2012)

Let $\mathcal{A}$ be purely infinite simple C-algebra whose K_o is a torsion group (e.g., $\mathcal{O}_n$ with $n < \infty$) and let $b \in \mathcal{A}_+$. Then b is a finite sum of projections if and only if either $\|b\| > 1$ or b is a projection.*
Question: Is the torsion of K_o necessary? What about $\mathcal{O}_\infty$?

Theorem (2011&2012)

Let $\mathcal{A}$ be a purely infinite simple unital C-algebra and $b \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})_+ \setminus \mathcal{A} \otimes \mathcal{K}$ not a projection.*
If b is a finite or infinite sum of proj then $\|b\|_{ess} \geq 1$ and $\|b\| > 1$.
If $\|b\|_{ess} > 1$ then b is a finite sum of projections.
If $\|b\|_{ess} = 1$ and $\|b\| > 1$ then b is an infinite sum of projections.
Question: can it be a finite sum of projections?

THANK YOU