MODEL THEORY OF OPERATOR ALGEBRAS I: STABILITY

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Abstract. Several authors have considered whether the ultrapower and the relative commutant of a C*-algebra or II_1 factor depend on the choice of the ultrafilter. We settle each of these questions, extending results of Ge–Hadwin and the first author.

1. Introduction

Suppose that A is a separable object of some kind: a C*-algebra, II_1-factor or a metric group. In various places, it is asked whether all ultrapowers of A associated with nonprincipal ultrafilters on $\mathbb{N}$, or all relative commutants of A in an ultrapower, are isomorphic. For instance, McDuff ([23, Question (i) on p. 460]) asked whether all relative commutants of a fixed separable II_1 factor in its ultrapowers associated with nonprincipal ultrafilters on $\mathbb{N}$ are isomorphic. An analogous question for relative commutants of separable C*-algebras was asked by Kirchberg and answered in [12] for certain C*-algebras. (The argument in [12] does not cover all cases of real rank zero as stated.) As a partial answer to both questions, Ge and Hadwin ([19]) proved that the Continuum Hypothesis implies a positive answer. They also proved that if the Continuum Hypothesis fails then some C*-algebras have nonisomorphic ultrapowers associated with nonprincipal ultrafilters on $\mathbb{N}$. We give complete answers to all these questions in Theorems 3.1, 4.8 and 5.1.

During the December 2008 Canadian Mathematical Society meeting in Ottawa, Sorin Popa asked the first author whether one can find uncountably many non-isomorphic tracial ultraproducts of finite dimensional matrix algebras $M_i(\mathbb{C})$, for $i \in \mathbb{N}$. In Proposition 3.3 we show the if the Continuum Hypothesis fails then there are nonisomorphic ultraproducts and if the continuum is sufficiently large then Popa's question has a positive answer.

These results can be contrasted with the fact that all ultrapowers of a separable Hilbert space (or even a Hilbert space of character density $\leq \mathfrak{c} = 2^{\aleph_0}$) associated with nonprincipal ultrafilters on $\mathbb{N}$ are isomorphic to $\ell^2(\mathfrak{c})$. We show that the separable tracial von Neumann algebras whose ultrapowers are all isomorphic even when the Continuum Hypothesis fails are exactly those of type I (Theorem 4.7).

We now introduce some terminology for operator algebraic ultrapowers that we will use throughout the paper. **Unless we say otherwise, all ultrafilters we use in this paper are non-principal ultrafilters on $\mathbb{N}$.**

A von Neumann algebra M is *tracial* if it is equipped with a faithful normal tracial state τ. A finite factor has a unique tracial state which is automatically normal. The metric induced by the ℓ^2-norm, $\|a\|_2 = \sqrt{\tau(a^*a)}$, is not complete on M, but it is complete on the unit ball (in the operator norm). The completion of M with respect to this metric is isomorphic to a Hilbert space (see e.g., [6] or [21]).

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The algebra of all sequences in M bounded in the operator norm is denoted by $\ell^\infty(M)$. If $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ then

$$c_{\mathcal{U}} = \{\vec{a} \in \ell^\infty(M) : \lim_{i\to\mathcal{U}} \|a_i\|_2 = 0\}$$

is a norm-closed two-sided ideal in $\ell^\infty(M)$, and the *tracial ultrapower* $M^{\mathcal{U}}$ (also denoted by $\prod_{\mathcal{U}} M$) is defined to be the quotient $\ell^\infty(M)/c_{\mathcal{U}}$. It is well-known that $M^{\mathcal{U}}$ is tracial, and a factor if and only if M is—see e.g., [6] or [30]. In the sequel to this paper ([14]) we shall demonstrate that this follows from axiomatizability in first order continuous logic of tracial von Neumann algebras and the Fundamental Theorem of Ultraproducts.

The elements of $M^{\mathcal{U}}$ will be denoted by boldface Roman letters such as $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$,... and their representing sequences in $\ell^\infty(M)$ will be denoted by $a(i)$, $b(i)$, $c(i)$, ..., for $i \in \mathbb{N}$, respectively. Identifying a tracial von Neumann algebra M with its diagonal image in $M^{\mathcal{U}}$, we will also work with the *relative commutant* of M in its ultrapower,

$$M' \cap M^{\mathcal{U}} = \{\mathbf{b} : (\forall a \in M) a\mathbf{b} = \mathbf{b}a\}.$$

A brief history of tracial ultrapowers of II_1 factors can be found in the introduction to [28].

We will use several variations of the ultraproduct construction. For C*-algebras, $c_{\mathcal{U}}$ consists of sequences which go to zero in the operator norm. For groups with bi-invariant metric, $c_{\mathcal{U}}$ is the normal subgroup of sequences whose ultralimit along $\mathcal{U}$ is the identity ([25]). One may also form the *ultraproduct* of a sequence of distinct algebras or groups; in fact all of these are special cases of the ultraproduct construction from the model theory of metric structures (see [14], [4], or [19]).

The methods used in the present paper make no explicit use of logic, and a reader can understand all proofs while completely ignoring all references to logic. However, the intuition coming from model theory was indispensable in discovering our results. With future applications in mind, we shall outline the model-theoretic framework for the study of operator algebras in [14].

2. The order property

Our results are stated and proved for tracial von Neumann algebras. Later on we shall point out that the arguments work in more general contexts, including those of C*-algebras and unitary groups.

Let M be a tracial von Neumann algebra and let $M_{\leq 1}$ denote its unit ball with respect to the operator norm. For $n \geq 1$ and a *-polynomial $P(x_1, \dots, x_n, y_1, \dots, y_n)$ in $2n$ variables consider the function

$$g(\vec{x}, \vec{y}) = \|P(\vec{x}, \vec{y})\|_2$$

on the $2n$-th power of $M_{\leq 1}$. In this section all functions of this kind range over the unit ball of M. Note that with this convention we have the following:

Properties 2.1. (G1) g defines a uniformly continuous function on the $2n$-th power of the unit ball of any tracial von Neumann algebra. The uniform continuity does not depend on the particular algebra; that is, for every ϵ there is a δ independent of the choice of algebra;

(G2) For every ultrafilter $\mathcal{U}$, the function g can be canonically extended to the $2n$-th power of the unit ball of the ultrapower

The discussion provided below applies verbatim to any g satisfying Properties 2.1. Readers familiar with model theory will notice that if g is an interpretation in continuous logic of a $2n$-ary formula, then Properties 2.1 are satisfied (see [14] for definition of a formula and properties of an interpretation of a formula). We shall furthermore suppress the mention of n whenever it is irrelevant.

Convention. In the remainder of this section we refer to g and n that satisfy Properties 2.1 as 'a $2n$-ary formula.'

Each g used in our applications will be of the form $\|P(\vec{x},\vec{y})\|_2$ for some *-polynomial P.

For $0 \leq \varepsilon < 1/2$ define the relation $\prec_{g,\varepsilon}$ on $(M_{\leq 1})^n$ by

$$\vec{x}_1 \prec_{g,\varepsilon} \vec{x}_2 \text{ if } g(\vec{x}_1,\vec{x}_2) \leq \varepsilon \text{ and } g(\vec{x}_2,\vec{x}_1) \geq 1-\varepsilon$$

Note that we do not require that $\prec_{g,\varepsilon}$ defines an ordering on its domain. However, if $\vec{x}_i$, for $1 \leq i \leq k$, are such that $\vec{x}_i \prec_{g,\varepsilon} \vec{x}_j$ for $i < j$ then these n-tuples form a linearly ordered chain of length k. We call such a configuration a *g-ε-chain of length k in M*.

We write $\prec_g$ for $\prec_{g,0}$. The following is a special case of Łos' theorem for ultraproducts in the logic of metric structures (see [4, Theorem 5.4] or [14, Proposition 4.3)]).

Lemma 2.2. *For a formula g, an ultrafilter $\mathcal{U}$, and $\mathbf{a}$ and $\mathbf{b}$ in $\prod_i (M_i)_{\leq 1}$ the following are equivalent.*

(1) *For every $\varepsilon > 0$ we have $\{i : a(i) \prec_{g,\varepsilon} b(i)\} \in \mathcal{U}$,*
(2) $\mathbf{a} \prec_g \mathbf{b}$.

The following special case of the abstract definition of the order property in a model (see [5, Section 7] or [14, Definition 5.2]) is modeled on the order property in stability theory ([27]).

Definition 2.3. A tracial von Neumann algebra M has the *order property* if there exists a formula g such that for every $\varepsilon > 0$, M has arbitrarily long finite g-ε-chains. If we wish to make the g explicit, we say that M has the order property with respect to g.

The following terminology is non-standard but convenient for what follows: A sequence M_i, for $i \in \mathbb{N}$, of tracial von Neumann algebras has the *order property with respect to g* if for every $\varepsilon > 0$ and every $k \in \mathbb{N}$ all but finitely many of the M_i, for $i \in \mathbb{N}$, have a g-ε-chain of length k. We say that the sequence of M_i's has the order property if it has the order property with respect to some g.

The analysis of gaps in quotient structures is behind a number of applications of set theory to functional analysis (see e.g., [7], [13]). Let λ be a regular cardinal. An *$(\aleph_0,\lambda)$-g-pregap* in M is a pair consisting of a $\prec_g$-increasing family $\mathbf{a}_m$, for $m \in \mathbb{N}$, and a $\prec_g$-decreasing family $\mathbf{b}_\gamma$, for $\gamma < \lambda$, such that $\mathbf{a}_m \prec_g \mathbf{b}_\gamma$ for all m and γ. A $\mathbf{c}$ such that $\mathbf{a}_m \prec_g \mathbf{c}$ for all m and $\mathbf{c} \prec_g \mathbf{b}_\gamma$ for all γ is said to *fill* (or *separate*) the pregap. An $(\aleph_0,\lambda)$-pregap that is not separated is an *$(\aleph_0,\lambda)$-gap*.

Assume M_i, $i \in \mathbb{N}$, are tracial von Neumann algebras. Assume $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$ such that for every $m \geq 1$ the set of all i such that M_i has a g-$1/m$-chain of length m belongs to $\mathcal{U}$. Then we can find sets $Y_m \in \mathcal{U}$ such that $Y_m \supseteq Y_{m+1}$, $\bigcap_m Y_m = \emptyset$ and for every $i \in Y_m \setminus Y_{m+1}$ there exists a g-$1/m$-chain $\vec{a}_0(i), \vec{a}_1(i), \ldots, \vec{a}_{m-1}(i)$ in M_i. Letting $Y_0 = \mathbb{N}$ and using $\bigcap_m Y_m = \emptyset$ for each $i \in \mathbb{N}$ we define $m(i)$ as the unique m such that $i \in Y_m \setminus Y_{m+1}$.

For $h \in \mathbb{N}^{\mathbb{N}}$ define $\vec{a}_h$ in $\prod_{i\in\mathbb{N}} M_i^n$ by $\vec{a}_h(i) = \vec{a}_{h(i)}(i)$ if $h(i) \leq m(i)-1$ and $\vec{a}_h(i) = \vec{a}_{m(i)-1}(i)$ otherwise. Then $\mathbf{a_h}$ denotes the element of $\prod_{\mathcal{U}} M_i^n$ with the representing sequence $\vec{a}_h$. Write $\bar{m}$ for the constant function $\bar{m}(i) = m$. Let $\mathbf{a}_m$ denote $\mathbf{a}_{\bar{m}}$. By $\mathbb{N}^{\nearrow\mathbb{N}}$ we denote the set of all nondecreasing functions $f\colon \mathbb{N} \to \mathbb{N}$ such that $\lim_n f(n) = \infty$, ordered pointwise.

Lemma 2.4. *With the notation as in the above paragraph, assume $\mathbf{b} \in \prod_{\mathcal{U}} M_i^n$ is such that $\mathbf{a}_m \prec_g \mathbf{b}$ for all m. Then there is $h \in \mathbb{N}^{\nearrow\mathbb{N}}$ such that $\mathbf{a}_h \prec_g \mathbf{b}$ and $\mathbf{a}_m \prec_g \mathbf{a}_h$ for all m.*

Proof. For $m \in \mathbb{N}$ let

$$X_m = \{i \in Y_m : (\forall k \leq m) g(a_k(i), b(i)) \leq 1/m \text{ and } g(b(i), a_k(i)) \geq 1 - 1/m\}.$$

Clearly $X_m \in \mathcal{U}$ and $\bigcap_m X_m = \emptyset$. For $i \in X_m \setminus X_{m+1}$ let $h(i) = m$ and let $h(i) = 0$ for $i \notin X_0$. Then for $m < m'$ and $i \in X_{m'}$ we have $h(i) \geq m' \geq m$ hence $g(a_m(i), a_{h(i)}(i)) \leq 1/m'$ and $g(\mathbf{a}_m, \mathbf{a}_h) = 0$. Similarly $g(\mathbf{a}_h, \mathbf{a}_m) = 1$ and therefore $\lim_{i\to\mathcal{U}} h(i) = \infty$.

Also, if $i \in X_m$ then $g(a_{h(i)}, b(i)) \leq 1/m$ and therefore $g(\mathbf{a}_h, \mathbf{b}) = 0$. Similarly $g(\mathbf{b}, \mathbf{a}_h) = 1$ and the conclusion follows. □

Following [10] (see also [12]), for an ultrafilter $\mathcal{U}$ we write $\kappa(\mathcal{U})$ for the *coinitiality* of $\mathbb{N}^{\nearrow\mathbb{N}}/\mathcal{U}$, i.e., the minimal cardinality of $X \subseteq \mathbb{N}^{\nearrow\mathbb{N}}$ such that for every $g \in \mathbb{N}^{\nearrow\mathbb{N}}$ there is $f \in X$ such that $\{n\colon f(n) \leq g(n)\} \in \mathcal{U}$. (It is not difficult to see that this is equal to $\kappa(\mathcal{U})$ as defined in [10, Definition 1.3].)

Lemma 2.5. *Assume M_i, $i \in \mathbb{N}$, is a sequence of tracial von Neumann algebras with the order property with respect to g and $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$. Then $\kappa(\mathcal{U})$ is equal to the minimal cardinal λ such that $\prod_{\mathcal{U}} M_i$ contains an $(\aleph_0, \lambda)$-g-gap.*

Proof. The proof is similar to the last paragraph of the proof of [12, Proposition 6]. Let Y_m and $\vec{a}_0(i), \vec{a}_1(i), \dots, \vec{a}_{m-1}(i)$ be as in the paragraph before Lemma 2.4 and we shall use the notation $\mathbf{a}_h$ and $\mathbf{a}_m$ as introduced there.

Fix functions $h(\gamma)$, $\gamma < \kappa(\mathcal{U})$, which together with the constant functions $\bar{m}$, for $m \in \mathbb{N}$ form a gap. We claim that $\mathbf{a}_{\bar{m}}$, for $m \in \mathbb{N}$, form a gap with $\mathbf{b}_\gamma = \mathbf{a}_{h(\gamma)}$, for $\gamma < \kappa(\mathcal{U})$. It is clear that these elements form a $(\aleph_0, \kappa(\mathcal{U}))$-$g$-pregap and by Lemma 2.4 this pregap is not separated.

Now assume λ is the minimal cardinal such that $\prod_{\mathcal{U}} M_i$ contains an $(\aleph_0, \lambda)$-g-gap, and let $\mathbf{a}_m$, for $m \in \mathbb{N}$, $\mathbf{b}_\gamma$, for $\gamma < \lambda$, be an $(\aleph_0, \lambda)$-g-gap in $\prod_{\mathcal{U}} M_i$. Fix a representing sequence $a_m(i)$, for $i \in \mathbb{N}$, of $\mathbf{a}_m$. For each m the set

$$X_m = \{i \in Y_m : i \geq m \text{ and } a_0(i), a_1(i), \dots, a_{m-1}(i) \text{ form a } g\text{-}1/m\text{-chain}\}$$

belongs to $\mathcal{U}$ and $X_m \supseteq X_{m+1}$ for all m. As before, for $h \in \mathbb{N}$ define $\mathbf{a}_h$ via its representing sequence. Let $a_h(i) = a_{h(i)}(i)$ if $i \in X_{h(i)}$ and $a_h(i) = a_m(i)$ if $i \notin X_{h(i)}$ and m is the maximal such that $i \in X_m$. For $i \notin X_0$ define $a_h(i)$ arbitrarily. Note that $\lim_{i\to\mathcal{U}} h(i) = \infty$ if and only if $\mathbf{a}_m \prec_g \mathbf{a}_h$.

By Lemma 2.4 for every $\gamma < \lambda$ we can find $h(\gamma)$ such that $\mathbf{b}'_\gamma = \mathbf{a}_{h(\gamma)}$ is $\prec_g \mathbf{b}_\gamma$ and $\mathbf{a}_m \prec_g \mathbf{b}'_\gamma$ for all m. In addition we choose $h(\gamma)$ so that $h(\gamma) \leq^* h(\gamma')$ for all $\gamma' < \gamma$. This is possible by the minimality of λ. The functions $h(\gamma)$, for $\gamma < \lambda$, together with the constant functions form an $(\aleph_0, \lambda)$-gap in $\mathbb{N}^{\mathbb{N}}/\mathcal{U}$. □

Proposition 2.6. *Assume the Continuum Hypothesis fails. Assume M_i, for $i \in \mathbb{N}$, is a sequence of tracial von Neumann algebras with the order property. Then there exist ultrafilters $\mathcal{U}$ and $\mathcal{V}$ such that the ultraproducts $\prod_i M_i/c_{\mathcal{U}}$ and $\prod_i M_i/c_{\mathcal{V}}$ are not isomorphic.*

Proof. Fix n and a $2n$-ary formula g such that for every $m \in \mathbb{N}$ the set of all i such that there is a g-$1/m$-chain of length $\geq m$ in M_i^n is cofinite.

Let $X \subseteq \mathbb{N}$ be an infinite set such that for all $m \in \mathbb{N}$ there is a g-$1/m$-chain of length m in the unit ball of M_i^n for all but finitely many i in X. By [10, Theorem 2.2] (also proved by Shelah, [27]) there are $\mathcal{U}$ and $\mathcal{V}$ so that $\kappa(\mathcal{U}) = \aleph_1$ and $\kappa(\mathcal{V}) = \aleph_2$ (here $\aleph_1$ and $\aleph_2$ are the least two uncountable cardinals; all that matters for us is that they are both $\leq 2^{\aleph_0}$ and different).

By Lemma 2.5 the ultraproduct associated with $\mathcal{U}$ has $(\aleph_0, \aleph_1)$-g-gaps and the ultraproduct associated with $\mathcal{V}$ has $(\aleph_0, \aleph_2)$-g-gaps but no $(\aleph_0, \aleph_1)$-g-gaps. Therefore these ultraproducts are not isomorphic. $\square$

Our definition of the order property for relative commutants is a bit more restrictive than that of the order property for tracial von Neumann algebras. We say that a and b in a tracial von Neumann algebra $1/m$-*commute* if $\|[a, b]\|_2 < 1/m$.

Definition 2.7. If M is a tracial von Neumann algebra then we say that the *relative commutant type of M has the order property with respect to g* if there are n and a *-polynomial $P(x_1, \dots, x_n, y_1, \dots, y_n)$ in $2n$ variables such that with $g(\vec{x}, \vec{y}) = \|P(\vec{x}, \vec{y})\|_2$, for every finite $F \subseteq M$, and every $m \in \mathbb{N}$, there is a g-$1/m$-chain of length m in $M^n_{\leq 1}$ all of whose elements $1/m$-commute with all elements of F.

Another remark for model-theorists is in order. In this definition we could have allowed g to be an arbitrary atomic formula, but we don't have an application for the more general definition.

Lemma 2.8. *Assume M is a separable tracial von Neumann algebra and $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$. If the relative commutant type of M has the order property with respect to g, then for every uncountable regular cardinal λ the following are equivalent.*

(1) $\kappa(\mathcal{U}) = \lambda$.
(2) *The relative commutant $M' \cap M^{\mathcal{U}}$ contains an $(\aleph_0, \lambda)$-g-gap.*

Note that g and $\prec_g$ are isomorphism invariants for relative commutants.

Proof. The proof is very similar to the proof of Lemma 2.5 and we add just a few clarifying remarks. Returning to the paragraphs before Lemma 2.4, let us define the sets Y_m in the context of the relative commutant. Fix a countable dense subset F of our separable M and write F as an increasing sequence of finite sets F_n. Now define a decreasing sequence of sets $Y_m \in \mathcal{U}$ such that $\bigcap Y_m = \emptyset$ and such that for all $i \in Y_m$, there is a g-$1/m$-chain $\vec{a}_0(i), \vec{a}_1(i), \dots, \vec{a}_{m-1}(i)$ made up of elements of M which $1/m$-commute with all the elements of F_m. The point of the exercise is that if we now define $\vec{a}_h$ as before, the resulting element of the ultraproduct will be in the relative commutant. Lemma 2.4 can now be reformulated so that if $\mathbf{b}$ is in the relative commutant then so is the constructed $\mathbf{a}_h$. Looking at the proof of Lemma 2.5, the proof may be copied verbatim simply replacing each mention of the ultraproduct with the relative commutant and noting that the various elements of the ultraproduct constructed in the proof are now provably in the relative commutant.

The fact that g and $\prec_g$ are isomorphism invariants for relative commutants follows from the fact that the evaluation of g depends only on the elements of the relative commutant (i.e., g is a quantifier-free formula). $\square$

Using Lemma 2.8 instead of Lemma 2.5 the proof of Proposition 2.6 gives the following.

Proposition 2.9. *Assume the Continuum Hypothesis fails. If M is a separable tracial von Neumann algebra whose relative commutant type has the order property then there are nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$ such that the relative commutants $M' \cap M^{\mathcal{U}}$ and $M' \cap M^{\mathcal{V}}$ are not isomorphic.* □

The preceding proofs used very little of the assumption that we were working with tracial von Neumann algebras and they apply to the context of C*-algebras, as well as unitary groups of tracial von Neumann algebras or C*-algebras.

The following proposition can be proved by copying the proofs of Proposition 2.6 and Proposition 2.9 verbatim.

Proposition 2.10. *Assume the Continuum Hypothesis fails, and let A be a separable C*-algebra or metric group. If A has the order property then there are nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$ such that $A^{\mathcal{U}}$ and $A^{\mathcal{V}}$ are not isomorphic. If the relative commutant type of A has the order property then there are nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$ such that the relative commutants $A' \cap A^{\mathcal{U}}$ and $A' \cap A^{\mathcal{V}}$ are not isomorphic.* □

3. Type II_1 factors

The main result of this section is Theorem 3.1. The case (3) implies (1) was proved in [19].

For a II_1 factor M let $U(M)$ denote its unitary group. It is a complete metric group with respect to the ℓ^2-metric. Hence the ultrapower of $U(M)$ is defined in the usual manner (see [25] and [4]).

Theorem 3.1. *For a II_1 factor M of cardinality $\mathfrak{c}$ the following are equivalent.*

(1) *For all nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ the ultrapowers $M^{\mathcal{U}}$ and $M^{\mathcal{V}}$ are isomorphic.*
(2) *For all nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ the ultrapowers of the unitary groups $U(M)^{\mathcal{U}}$ and $U(M)^{\mathcal{V}}$ are isomorphic.*
(3) *The Continuum Hypothesis holds.*

In fact the implications (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) are true for an arbitrary II_1 factor.

The proof of this theorem will be given after a sequence of lemmas.

On a II_1 factor M define $g\colon (M_{\leq 1})^4 \to \mathbb{R}$ by

$$g(a_1, b_1, a_2, b_2) = \|[a_1, b_2]\|_2.$$

In the following we consider $M_{2^n}(\mathbb{C})$ with respect to its ℓ_2-metric. The analogous statements for the operator metric are true, and easier to prove (see [12]).

Lemma 3.2. (1) *The sequence $M_{2^n}(\mathbb{C})$, for $n \in \mathbb{N}$, has the order property.*
(2) *The sequence $M_n(\mathbb{C})$, for $n \in \mathbb{N}$, has the order property.*
(3) *Every II_1 factor (and every sequence of II_1 factors) has the order property.*

Proof. (1) We prove that for every n in $M_{2^n}(\mathbb{C})$ there is a g-chain of length $n-1$. Identify $M_{2^n}(\mathbb{C})$ with $\bigotimes_{i=0}^{n-1} M_2(\mathbb{C})$. Let $x = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}$ and let $y = x^* = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}$. Then we have

$\|x\|_2 = 1 = \|y\|_2$. Also $[x, y] = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ and $\|[x, y]\|_2 = 2$. For $1 \leq i \leq n-1$ let

$$a_i = \bigotimes_{j=0}^{i} x \otimes \bigotimes_{j=i+1}^{n-1} 1 \text{ and } b_i = \bigotimes_{j=0}^{i} 1 \otimes y \otimes \bigotimes_{j=i+2}^{n-1} 1.$$

We have $\|a_i\|_2 = \prod_{j=0}^{i} \|x_j\|_2 = 1 = \|b_i\|_2$ for all i. Clearly $\|[a_i, b_j]\|_2 = 0$ if $i \leq j$ and $\|[a_i, b_j]\|_2 = \|[x, y]\|_2 = 2$ if $i > j$ and therefore pairs (a_i, b_i), for $1 \leq i \leq n-1$, form a g-chain.

(2) We prove that for all $\varepsilon > 0$, $n \in \mathbb{N}$ and a large enough m there is a g, ε-chain of length n in $M_m(\mathbb{C})$. Assume m is much larger than 2^n so that $m = k \cdot 2^n + r$ with r/m sufficiently small. Then pick a projection p in $M_m(\mathbb{C})$ with $\tau(p) = k \cdot 2^n$, identify the corner $pM_m(\mathbb{C})p$ with $M_{2^n}(\mathbb{C}) \otimes M_k(\mathbb{C})$ and apply (1).

(3) Since every II_1 factor M has a unital copy of $M_{2^n}(\mathbb{C})$ for every n this follows immediately from (1). □

The question whether there are non-isomorphic non-atomic tracial ultraproducts of full matrix algebras $M_n(\mathbb{C})$ was raised by S. Popa (personal communication). This question first appeared in [22], after Theorem 3.2.

Proposition 3.3. *Assume the Continuum Hypothesis fails. Then there are nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$ such that the II_1 factors $\prod_{\mathcal{U}} M_n(\mathbb{C})$ and $\prod_{\mathcal{V}} M_n(\mathbb{C})$ are nonisomorphic. Moreover, there are at least as many nonisomorphic ultraproducts as there are uncountable cardinals below $\mathfrak{c}$.*

Proof. Again this follows by Proposition 2.6 and Lemma 3.2. □

We don't know whether the Continuum Hypothesis implies that all ultraproducts of $M_n(\mathbb{C})$, for $n \in \mathbb{N}$, associated with nonprincipal ultrafilters on $\mathbb{N}$ are isomorphic. This is equivalent to asking whether the continuous first order theories of matrix algebras $M_n(\mathbb{C})$ converge as $n \to \infty$ (see [14]).

Proof of Theorem 3.1. Assume the Continuum Hypothesis holds. Clause (2) is an immediate consequence of clause (1), which was proved by Ge and Hadwin in [19] (note their proof in [19, Section 3] only uses the fact that the algebra has cardinality $\mathfrak{c}$).

Now assume the Continuum Hypothesis fails. Then (1) follows by Proposition 2.6 and Lemma 3.2.

It remains to show (2). This can be proved directly by taking advantage of the commutators (cf. the proof of Lemma 3.2) but instead we show that it already follows from what was proved so far. It is well-known that $U(M^{\mathcal{U}}) = U(M)^{\mathcal{U}}$. In model-theoretic terms (see [4]) this equality states that the unitary group is definable in a II_1 factor. The conclusion then follows from Dye's result ([11]) that two von Neumann algebras are isomorphic if and only if their unitary groups are isomorphic (even as discrete groups). □

4. Tracial von Neumann algebras

We remind the reader that any tracial von Neumann algebra can be written as $M_{\mathrm{II}_1} \oplus M_{\mathrm{I}_1} \oplus M_{\mathrm{I}_2} \dots$, with subscripts the types of the summands. The goal of this section is to prove that a tracial von Neumann algebra does not have the order property if and only if it is type I (Theorem 4.7).

Since most of the literature on tracial ultrapowers has focused on factors, we start by establishing that this operation commutes with forming weighted direct sums (Lemma 4.1) and taking the center (Corollary 4.3). Corollary 4.3 follows from Lemma A.4.2 in [29] but for the reader's convenience we give a short proof of our special case that contains some extra information in Lemma 4.2. Let (M,τ) be a tracial von Neumann algebra, with $\mathcal{Z}(M)$ its center. Fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$.

Lemma 4.1. *If $\{z_j\} \subset M$ are nonzero central projections summing to 1, then*

$$(M,\tau)^{\mathcal{U}} \simeq \sum\nolimits^{\oplus} (z_j M, \tfrac{1}{\tau(z_j)}\tau|_{z_j M})^{\mathcal{U}}.$$

Proof. We claim that the map

$$M^{\mathcal{U}} \ni (x_i) \mapsto ((z_j x_i)_i)_j \in \sum\nolimits_j^{\oplus} (z_j M)^{\mathcal{U}}$$

is a well-defined *-isomorphism.

To see that it is well-defined, suppose $(x_i) = (x_i')$ in $M^{\mathcal{U}}$. Then as $i \to \mathcal{U}$, $\|x_i - x_i'\|_2 \to 0$. Thus for any j, $\|z_j(x_i - x_i')\|_2 \to 0$. By rescaling the trace this implies that $(z_j x_i)_i = (z_j x_i')_i$ in $(z_j M)^{\mathcal{U}}$.

It is then clearly a *-homomorphism.

We show next that it is injective. Suppose that (x_i) belongs to the kernel. We may assume that $\sup \|x_i\| \leq 1$. We have that for all j, $\|z_j x_i\|_2 \to 0$ as $i \to \mathcal{U}$. Given $\varepsilon > 0$, let N be such that $\sum_{j>N} \tau(z_j) < \varepsilon$. Then

$$\|x_i\|_2^2 = \sum\nolimits_j \|z_j x_i\|_2^2 < \sum\nolimits_{j<N} \|z_j x_i\|_2^2 + \varepsilon,$$

which is $< 2\varepsilon$ for i near $\mathcal{U}$. Since ε was arbitrary, $x_i \to 0$ in L^2.

Finally, for surjectivity let $((y_i^j)_i)_j \in \sum^{\oplus}(z_j M)^{\mathcal{U}}$. We may assume that $1 = \|((y_i^j)_i)_j\|_\infty = \sup_j \|(y_i^j)_i\|_\infty$, and then we may also assume that the representing sequences have been chosen so that $\|y_i^j\|_\infty \leq 1$ for each i and j. Identifying each y_i^j with its image in the inclusion $z_j M \subseteq M$, set $x_i = \sum_j y_i^j$. Since each set $\{y_i^j\}_j$ consists of centrally orthogonal elements, $\|x_i\|_\infty = \sup_j \|y_i^j\|_\infty \leq 1$. Thus (x_i) defines an element of $M^{\mathcal{U}}$, and by construction (x_i) is mapped to the element initially chosen. □

The following lemma may be in the literature, but we do not know a reference.

Lemma 4.2. *Let M be a finite von Neumann algebra and T its unique center-valued trace. For any $x \in M$, we have*

$$\|x - T(x)\|_2 \leq \sup_{y \in M_{\leq 1}} \|[x,y]\|_2 \leq 2\|x - T(x)\|_2.$$

Proof. We will use the Dixmier averaging theorem ([8]), which says in this situation that for any $\varepsilon > 0$ there is a finite set of unitaries $\{u_j\} \subset M$ and positive constants λ_j adding to 1 such that $\|T(x) - \sum \lambda_j u_j x u_j^*\|_\infty < \varepsilon$. We only need the ℓ^2 estimate to compute

$$\begin{aligned}\|T(x) - x\|_2 &\leq \varepsilon + \left\| \sum \lambda_j u_j x u_j^* - x \right\|_2 \\ &= \varepsilon + \left\| \sum \lambda_j [u_j, x] \right\|_2 \\ &\leq \varepsilon + \sum \lambda_j \|[u_j, x]\|_2 \\ &\leq \varepsilon + \sup_{y \in M_{\leq 1}} \|[x, y]\|_2.\end{aligned}$$

Since ε is arbitrary, the first inequality of the lemma follows.

The second inequality is more routine:

$$\sup_{y \in M_{\leq 1}} \|[x, y]\|_2 = \sup_{y \in M_{\leq 1}} \|[x - T(x), y]\|_2 \leq 2\|x - T(x)\|_2. \quad \square$$

Corollary 4.3. *For M a tracial von Neumann algebra, $\mathcal{Z}(M^{\mathcal{U}}) \simeq \mathcal{Z}(M)^{\mathcal{U}}$.*

Proof. The inclusion $\supseteq$ is obvious. For the other inclusion, suppose $(x_i) \in \mathcal{Z}(M^{\mathcal{U}})$. For each i, let $y_i \in M_{\leq 1}$ be such that $\|[x_i, y_i]\|_2 \geq \frac{1}{2} \sup_{y \in M_{\leq 1}} \|[x_i, y]\|_2$. By Lemma 4.2, $\|x_i - T(x_i)\|_2 \leq \sup_{y \in M_{\leq 1}} \|[x_i, y]\|_2 \leq 2\|[x_i, y_i]\|_2$. Centrality of (x_i) means that the last term goes to 0 as $i \to \mathcal{U}$, so the first term does as well, and $(x_i) = (T(x_i))$ is central in $M^{\mathcal{U}}$. $\square$

Lemma 4.4. *If M is type I_n, then $M^{\mathcal{U}}$ is type I_n. More specifically, $M \simeq \mathbb{M}_n \otimes \mathcal{Z}(M)$ and $M^{\mathcal{U}} \simeq \mathbb{M}_n \otimes \mathcal{Z}(M)^{\mathcal{U}}$.*

Proof. A direct calculation shows that an abelian projection in M is carried under the diagonal embedding $M \hookrightarrow M^{\mathcal{U}}$ to an abelian projection in $M^{\mathcal{U}}$.

An algebra is type I_n if and only if it contains n equivalent abelian projections summing to 1. If M has such projections, their images in $M^{\mathcal{U}}$ have the same properties.

The second sentence of the lemma follows from Corollary 4.3, or just the fact that M is a finite-dimensional module over its center. $\square$

Lemma 4.5. *The categories of abelian tracial von Neumann algebras and probability measure algebras are equivalent.*

Proof. Given a tracial von Neumann algebra (M, τ) define the probability measure algebra $\mathcal{B}(M, \tau)$ to be the complete Boolean algebra of projections of M, with $\mu(p) = \tau(p)$ giving a probability measure on $\mathcal{B}$. Since τ is normal, μ is σ-additive.

Given a probability measure algebra $(\mathcal{B}, \mu)$ define the tracial von Neumann algebra $M(\mathcal{B}, \mu)$ to be the associated L^∞ algebra, with τ being integration against μ.

It is clear that $\mathcal{B}(M(\mathcal{B}, \mu))$ is isomorphic to $(\mathcal{B}, \mu)$ and that $M(\mathcal{B}(M, \tau))$ is isomorphic to (M, τ). It is also clear that these two operations agree with morphisms, so we have an equivalence of categories. $\square$

The following is a consequence of Maharam's theorem and we shall need only the case when $\mathcal{B}$ is countably generated.

Proposition 4.6. *If $(\mathcal{B}, \nu)$ is a probability measure algebra then all of its ultrapowers associated with nonprincipal ultrafilters on $\mathbb{N}$ are isomorphic.*

Proof. Recall that the *Maharam character* of a measure algebra is the minimal cardinality of a set that generates the algebra and that a measure algebra is *Maharam homogeneous* if for every nonzero element b the Maharam character of the algebra restricted to b is equal to the Maharam character of the whole algebra. Maharam's theorem ([17, 331L]) implies that a Maharam-homogeneous probability measure algebra of Maharam character κ is isomorphic to the Haar measure algebra on 2^κ.

Let us first give a proof in the case when $\mathcal{B}$ is Maharam homogeneous. By Maharam's theorem it will suffice to show $\mathcal{B}^\mathcal{U}$ is Maharam homogeneous of character $\kappa^{\aleph_0}$ for every nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. By $\kappa^{<\mathbb{N}}$ we denote the set of all finite sequences of ordinals less than κ. Since $\mathcal{B}$ is isomorphic to the Haar measure algebra on 2^κ and the cardinality of $\kappa^{<\mathbb{N}}$ is equal to κ, we can fix a stochastically independent sequence x_s, for $s \in \kappa^{<\mathbb{N}}$. Hence $\nu(x_s) = 1/2$ for all s and $\nu(x_s \cap x_t) = 1/4$ whenever $s \neq t$.

For $f\colon \mathbb{N} \to \kappa$ define $\mathbf{x}_f$ by its representing sequence $x_{f\restriction n}$, for $n \in \mathbb{N}$. For $f \neq g$ and for all large enough n we have $\nu(x_{f\restriction n} \cap x_{g\restriction n}) = 1/4$. Therefore $\mathbf{x}_f$, for $f\colon \mathbb{N} \to \kappa$, is a stochastically independent family of size $\kappa^{\aleph_0}$. Since the cardinality of $\mathcal{B}^\mathcal{U}$ is $\kappa^{\aleph_0}$ we conclude it is a Maharam-homogeneous algebra of Maharam character $\kappa^{\aleph_0}$. Using Maharam's theorem again we conclude all such ultrapowers are isomorphic.

In the general case, we can write $\mathcal{B}$ as a direct sum of denumerably many Maharam-homogeneous algebras. We assume this sequence is infinite since the finite case is slightly easier. Write $\mathcal{B}$ as a direct sum $\bigoplus_{i=0}^\infty \mathcal{B}_i$ where each $\mathcal{B}_i$ is Maharam homogeneous. It is clear from Lemma 4.1 that $\mathcal{B}^\mathcal{U}$ is isomorphic to $\bigoplus_{i=0}^\infty \mathcal{B}_i^\mathcal{U}$ for any ultrafilter $\mathcal{U}$ and the conclusion follows by the first part of the proof. □

As a side remark, we note that by a result from [14] the above implies that the theory of probability measure algebras does not have the order property, and thus gives a different proof of the same result from [4] (see also [3]).

Theorem 4.7. *Assume the Continuum Hypothesis fails. A separable tracial von Neumann algebra has the property that all of its ultrapowers associated with nonprincipal ultrafilters on* $\mathbb{N}$ *are isomorphic if and only if it is type I.*

Proof. It follows from Lemmas 4.1 and 4.4 and Proposition 4.6 that tracial type I algebras do not have the order property.

Note that a tracial II_1 algebra (even a non-factor) unitally contains the sequence $\mathbb{M}_{2^n}$, so has the order property, and therefore has non-isomorphic ultrapowers. A tracial von Neumann algebra which is not of type I must have a type II_1 summand, so it has non-isomorphic ultrapowers by Lemma 4.1 and Proposition 2.6. □

We are now in the position to handle the relative commutants of separable II_1 factors. Recall that a separable II_1 factor is said to be *McDuff* if it is isomorphic to its tensor product with the hyperfinite II_1 factor. By McDuff's theorem from [23], being McDuff is equivalent to having non-abelian relative commutant. The following Theorem says that a II_1 factor is McDuff iff its relative commutant has the order property.

Theorem 4.8. *Suppose that* $\mathcal{A}$ *is a separable* II_1 *factor. If the Continuum Hypothesis fails then the following are equivalent:*

(1) *there are nonprincipal ultrafilters* $\mathcal{U}$ *and* $\mathcal{V}$ *on* $\mathbb{N}$ *such that the relative commutants* $A' \cap A^\mathcal{U}$ *and* $A' \cap A^\mathcal{V}$ *are not isomorphic.*

(2) *there are nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$ such that the unitary groups of the relative commutants $A' \cap A^{\mathcal{U}}$ and $A' \cap A^{\mathcal{V}}$ are not isomorphic.*
(3) *some (all) relative commutant(s) of $\mathcal{A}$ are type II_1.*

If the Continuum Hypothesis holds then all the relative commutants of $\mathcal{A}$ and all their unitary groups are isomorphic.

Proof. As in the proof of Theorem 3.1, the equivalence of the first two clauses of the first paragraph follows from Dye's result ([11]) that two von Neumann algebras are isomorphic if and only if their unitary groups are isomorphic. The last sentence in the Theorem follows from [19]. McDuff showed ([23]) that for nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$, the relative commutants $A' \cap A^{\mathcal{U}}$ and $A' \cap A^{\mathcal{V}}$ are always either

(1) $\mathbb{C}$,
(2) abelian, non-atomic and of density character $\mathfrak{c}$, or
(3) of type II_1.

In the first case, clearly all relative commutants are isomorphic. In the second case, as in the proof of Theorem 4.6, one can show that the relative commutant is in fact isomorphic to $L^\infty(\mathcal{B})$ for a Maharam-homogeneous measure algebra $\mathcal{B}$ of character density $\mathfrak{c}$. It follows then by Maharam's theorem that all relative commutants in this case are isomorphic. We are left then with the possibility that a relative commutant of $\mathcal{A}$ is type II_1. By the previous theorem, this means that that relative commutant has the order property. The proof then follows from Proposition 2.9. □

5. C*-algebras

In this subsection we consider norm ultrapowers of C*-algebras and their unitary groups. Here $U(A)$ denotes the unitary group of a unital C*-algebra A. The following is the analogue (although simpler) of Theorems 3.1 and 4.8 in the C*-algebra case.

Theorem 5.1. *For an infinite-dimensional separable unital C*-algebra A the following are equivalent.*

(1) *For all nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ the ultrapowers $A^{\mathcal{U}}$ and $A^{\mathcal{V}}$ are isomorphic.*
(2) *For all nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ the relative commutants of A in the ultrapowers $A^{\mathcal{U}}$ and $A^{\mathcal{V}}$ are isomorphic.*
(3) *For all nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ the ultrapowers $U(A)^{\mathcal{U}}$ and $U(A)^{\mathcal{V}}$ are isomorphic.*
(4) *For all nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ the relative commutants of $U(A)$ in the ultrapowers $U(A)^{\mathcal{U}}$ and $U(A)^{\mathcal{V}}$ are isomorphic.*
(5) *The Continuum Hypothesis holds.*

If A is separable but not unital then (1)*,* (2) *and* (5) *are equivalent.*

Some instances of Theorem 5.1 were proved in [19] and [12]. If the separability of A is not assumed then (1) and (2) in Theorem 5.1 need not be equivalent (see [15]).

For a unital C*-algebra A let $U(A)$ denote its unitary group equipped with the norm metric. The analogue of Dye's rigidity result used in the proofs of Theorems 3.1 and 4.8 for unitary groups of C*-algebras used above is false (Pestov, see [2] also [18]) although this is true for simple AF C*-algebras by [2].

A C*-algebra A has the *order property with respect to g* if for every $\varepsilon > 0$ there are arbitrarily long finite $\prec_{g,\varepsilon}$-chains (see §2). We say that the *relative commutant type of A has the order property with respect to g* if there are n, a *-polynomial $P(x_1, \ldots, x_n, y_1, \ldots, y_n)$ in $2n$ variables such that with $g(\vec{x}, \vec{y}) = \|P(\vec{x}, \vec{y})\|$ for every finite $F \subseteq A$, and every $m \in \mathbb{N}$, there is a g-$1/m$-chain of length m in $A^n_{\leq 1}$ all of whose elements $1/m$-commute with all elements of F.

Lemma 5.2. *If A is an infinite-dimensional separable unital C*-algebra and $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$ then $A' \cap A^{\mathcal{U}}$ is infinite-dimensional and in fact, non-separable.*

Proof. We divide this proof into cases depending on whether or not the C^*-algebra is continuous trace. Recall that a C^*-algebra A is said to have continuous trace if its spectrum T is Hausdorff and is locally Morita equivalent to $C_0(T)$. We will show that an infinite-dimensional unital continuous trace C^*-algebra A must have infinite-dimensional center. We could not find this explicitly stated in the literature, so we give a short proof. Suppose A is continuous trace; it is type I and its spectrum $\hat{A}$ is Hausdorff ([6, IV.1.4.16]). By the Dauns-Hofmann theorem, $\mathcal{Z}(A) \simeq C(\hat{A})$. If $\mathcal{Z}(A)$ were finite-dimensional, then $\hat{A}$ would be finite and discrete; by [9, X.10.10.6a] A would be a direct sum of simple unital type I algebras. This would force A to be finite-dimensional, a contradiction. Always $\mathcal{Z}(A) \subseteq A' \cap A^{\mathcal{U}}$, so if A is continuous trace, the lemma follows.

In the remainder of the proof we suppose A is not continuous trace. By [1, Theorem 2.4], A has a nontrivial central sequence $a(n)$ – but note that these authors work with limits at infinity, not an ultrafilter. Passing to a subsequence if necessary, we may assume that there is $c > 0$ such that for all n,

$$\inf_{z \in \mathcal{Z}(A)} \|a(n) - z\| > c.$$

Now moving to the ultrapower, it is clear that $a(n)$ belongs to $A' \cap A^{\mathcal{U}}$ but is not equivalent to a constant sequence.

We have $\mathbf{a} = a(n) \in (A' \cap A^{\mathcal{U}}) \setminus A$. Let $\epsilon = d(\mathbf{a}, A)/2$; $\epsilon > 0$ since A is complete. We may assume $\epsilon \leq c$. Note that for every finite subset G of the sequence $\{a(n)\}$ the set $\{j : \|a - a(j)\| > \epsilon \text{ for all } a \in G\}$ is in U. Let F_n, for $n \in N$, be an increasing sequence of finite subsets of A with dense union. For every m and every $\delta > 0$ the set $\{j : \|[b, a(j)]\| < \delta$ for all $b \in F_m\}$ is in U. We can therefore recursively find disjoint sets G_n for n in N such that for all n

(1) $\|a(j) - a(k)\| \geq \epsilon$ for all distinct j and k in G_n
(2) For every $m \leq n$ and every $b \in F_m$ and $j \in G_n$, $\|[a(j), b]\| < 1/m$.
(3) $|G_n| = 2^n$ and G_n is enumerated as $a(s)$ for $s \in s^n$.

If $x \in 2^{\mathbb{N}}$ then (with $x|n$ denoting the first n digits of x) the sequence $a(x|n)$ for $n \in N$ is central. Let $\mathbf{a}(x)$ denote the element of A^U corresponding to it. By (2) it is in $(A' \cap A^U) \setminus A$. If $x \neq y$ are in $2^{\mathbb{N}}$ then $x|n \neq y|n$ for a large n and therefore (1) implies $\|\mathbf{a}(x) - \mathbf{a}(y)\| \geq \epsilon$.

We have therefore proved that the relative commutant of A in A^U is nonseparable. □

Lemma 5.3. *Assume A is an infinite-dimensional separable unital C*-algebra. Then both A and its relative commutant type have the order property.*

Proof. Both A and any of its relative commutants are infinite-dimensional, by the previous lemma, so their maximal abelian *-subalgebras are also infinite-dimensional ([24]). In order to prove the lemma then, it suffices to consider the case when A is abelian. By the Gelfand transform A is isomorphic to $C_0(X)$ for an infinite locally compact (possibly compact) Hausdorff space X. In X find a sequence of distinct x_n that converges to some x (with x possibly in the compactification of X). For each n find a positive $a_n \in C_0(X)$ of norm 1 such that $a_n(x_i) = 1$ for $i \leq n$ and $a_n(x_i) = 0$ for $i > n$. By replacing a_n with the maximum of a_j for $j \leq n$ we may assume this is an increasing sequence. Moreover, we may assume the support K_n of each a_n is compact and a_m is identically equal to 1 on K_n for $n < m$. Hence $a_m a_n = a_n$ if $n \leq m$.

Now consider the formula $\varphi(x, y) = \|xy - y\|$. Then $\varphi(a_m, a_n) = 0$ if $m > n$ and $\varphi(a_m, a_n) = 1$ if $m < n$, and the order property for A follows. □

In connection with the first paragraph of the proof of Lemma 5.3 it is worth mentioning that there is a nonseparable C*-algebra all of whose masas are separable ([26]).

The idea for the following is well-known. Compare for example with [20, Example 8.2].

Lemma 5.4. *For every infinite-dimensional unital C*-algebra both its unitary group and the relative commutant type of its unitary group have the order property.*

Proof. Let A be an infinite-dimensional unital C*-algebra. Since it contains an infinite-dimensional abelian C*-algebra we may assume $A = C(X)$ for an infinite compact Hausdorff space X. Therefore $U(A)$ is isomorphic to $C(X, \mathbb{T})$, where $\mathbb{T}$ denotes the unit circle. Let x_n, for $n \in \mathbb{N}$, be a nontrivial convergent sequence in X and let $U_n \ni x_n$ be disjoint open sets. Find $0 \leq f_n \leq 1/4$ in $C(X)$ such that $f_n(x_n) = 1/4$ and $\operatorname{supp}(f_n) \subseteq U_n$. Let $a_n = \exp(2\pi i \sum_{j \leq n} f_j)$ and $b_n = \exp(2\pi i f_n)$. Then

$$g_0(a, b, a', b') = \|1 - ab'\|$$

is such that $g_0(a_m, b_m, a_n, b_n)$ is equal to $|1 - i|$ if $m < n$ and to 2 if $m \geq n$. It is now easy to modify g_0 to g as required.

The result for the relative commutant type follows as in Lemma 5.3. □

Proof of Theorem 5.1. The implications from (5) were proved in [19] and the converse implications follow by Lemma 5.3, Lemma 5.4, and Proposition 2.10. □

Added June, 2010: Our results give only as many nonisomorphic ultrapowers as there are uncountable cardinals $\leq \mathfrak{c}$ (i.e., at least two). This inspired [16], where it was proved that every separable metric structure A either has all of its ultrapowers by ultrafilters on $\mathbb{N}$ isomorphic or it has $2^{\mathfrak{c}}$ nonisomorphic such ultrapowers (see [16, Corollary 4]). The analogous result for relative commutants of C*-algebras and II_1 factors is given in [16, Proposition 8.4]. The answer to Popa's question, mentioned before, can be found in [16, Proposition 8.3].

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