Truncated Toeplitz Operators

Note Title 1/16/2008

$\boxed{\text{I}}$ Toeplitz Operators

H^2, L^2, $P : L^2 \to H^2$ (Riesz proj)

$$(Pf)(z) = \int_{\mathbb{T}} \frac{f(\xi)}{1 - \bar{\xi} z} \, dm(\xi)$$

For $\varphi \in L^2$, define $T_\varphi : H^2 \to \mathrm{Hol}(\mathbb{D})$

$$(T_\varphi f)(z) = \int_{\mathbb{T}} \frac{\varphi(\xi) f(\xi)}{1 - \bar{\xi} z} \, dm(\xi)$$

<u>Facts</u>. 1) $T_\varphi : H^2 \to H^2$ is bd $\Leftrightarrow \varphi \in L^\infty$

2) $\|T_\varphi\| = \|\varphi\|_\infty$

3) $T_\varphi \equiv 0 \Leftrightarrow \varphi = 0$ a.e.

4) $T_\varphi \simeq \begin{pmatrix} a_0 & a_{-1} & & \\ a_1 & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots \end{pmatrix}$

matrix w.r.t. $\longrightarrow$
$1, z, z^2, \ldots$

$a_n = \hat{\varphi}(n)$

5) T_φ compact $\Leftrightarrow T_\varphi \equiv 0$

Q When is $T \in \mathcal{B}(H^2)$ a Toep. op.?

Let S = unilateral shift on H^2.

$$\langle S^* T_\varphi S f, g \rangle = \langle T_\varphi S f, S g \rangle$$
$$= \langle M_\varphi S f, S g \rangle$$
$$= \langle M_\varphi f, g \rangle$$
$$= \langle T_\varphi f, g \rangle$$

So $S^* T_\varphi S = T_\varphi$

Thm (Brown-Halmos) $T \in \mathcal{B}(H^2)$ is a Toep op $\iff S^* T S = T$

$\boxed{\text{II}}$ <u>Truncated</u> <u>Toeplitz</u> <u>operators</u>

Θ = inner function

$K_\Theta = H^2 \cap (\Theta H^2)^\perp$

model space

Note $S^* K_\Theta \subseteq K_\Theta$

<u>Thus</u> (Douglas–Shapiro–Shields)

For $f \in H^2$, TFAE.

1) $f \in K_\Theta$

2) $f|_{\mathbb{T}} \in H^2(\mathbb{T}) \cap \Theta \overline{H_0^2(\mathbb{T})}$

3) f/Θ has a P.C. $F_\Theta \in H_0^2(\mathbb{D}_e)$

Reproducing kernel for K_Θ

$$k_\lambda(z) = \frac{1 - \overline{\Theta(\lambda)}\,\Theta(z)}{1 - \bar{\lambda} z}$$

$P_\Theta : L^2 \to K_\Theta$ ortho. proj.

$$(P_\Theta f)(\lambda) = \langle f, k_\lambda \rangle$$

For $\boxed{\varphi \in L^2}$

$$A_\varphi : K_\Theta \to Hol(\mathbb{D})$$

$$(A_\varphi f)(\lambda) = \int_{\mathbb{T}} \varphi(\xi) f(\xi) \overline{k_\lambda(\xi)}\, dm(\xi)$$

A_φ is a "truncated Toeplitz operator"

Remarks:

1. When $\varphi \in L^\infty$,

$$A_\varphi = P_\Theta T_\varphi | K_\Theta$$

is bounded and is the compression of the Toeplitz operator T_φ

2. Unlike T_φ where $T_\varphi \equiv 0 \Leftrightarrow \varphi \equiv 0$ it is possible to have $A_\varphi \equiv 0$ but $\varphi \not\equiv 0$.

3. If A_φ is bd can we find a $\varphi_1 \in L^\infty$ so that $A_\varphi = A_{\varphi_1}$?

Ex: If $\varphi \in H^2$ and $A_{\bar{\varphi}}$ is bd the NF dilation says that $A_{\bar{\varphi}} = A_{\bar{\psi}}$ for some $\psi \in H^\infty$. Moreover $\| A_{\bar{\varphi}} \| = \| \psi \|_\infty$.

$\boxed{\text{III}}$ <u>Uniqueness</u>

Recall $T_\varphi \equiv 0 \iff \varphi = 0$ a.e.

when is $A_\varphi \equiv 0$?

<u>Thm</u> (Sarason)

$A_\varphi \equiv 0 \iff \varphi \in \underbrace{\Theta H^2 + \overline{\Theta H^2}}_{\text{(think on } \mathbb{T}\text{)}}$

Pf: ($\Longleftarrow$)

Suppose $\varphi = \Theta h_1 + \overline{\Theta}\,\overline{h_2}$

$(h_1, h_2 \in H^2)$

$$\varphi f = \underbrace{\Theta h_1 f}_{\mathrm{I}} + \underbrace{\overline{\Theta}\,\overline{h_2}\, f}_{\mathrm{II}}$$

$(f \in K_\Theta \cap H^\infty)$
(dense!)

I: $P_\Theta(\Theta h_1 f) = 0$ (since $\Theta h_1 f \in \Theta H^2$)

II: DSS $\Rightarrow f \in \Theta\, \overline{H_0^\infty}$

$\Rightarrow \overline{\Theta} f \in \overline{H_0^\infty}$

$\Rightarrow \overline{h_2}(\overline{\Theta} f) \in \overline{H_0^2} = (H^2)^{\perp_{L^2}}$

$\Rightarrow P_\Theta(\overline{h_2}\,\overline{\Theta} f) = 0$ ✓

($\Longrightarrow$) Kind of technical

$\boxed{IV}$ Membership.

Recall : $T \in B(H^2)$ is a Toep. op

$\Longleftrightarrow \quad S^* T S = T$

$T_\theta = \{ A_\varphi : \varphi \in L^2, A_\varphi : K_\theta \to K_\theta$ is bounded $\}$

(vector space of bd T.T.O)

Thm (Sarason) $A \in B(K_\theta)$ belongs to $T_\theta \Longleftrightarrow \exists\, g_1, g_2 \in K_\theta$ so that

$$A = A_z^* A A_z + g_1 \otimes S^*\theta + S^*\theta \otimes g_2$$

(nice condition but difficult to apply)

Cor T_θ is W.O.T. closed.

$\boxed{V}$ Truncated Toeplitz and Clark

Recall OT Seminars (2/1, 2/8, 2/15 in 2005)

WLOG $\Theta(0) = 0$

For $\alpha \in \mathbb{T}$

$$U_\alpha = A_z + \alpha\left(1 \otimes \frac{\Theta}{z}\right)$$

- U_α are the Clark unitary operators
- U_α is cyclic and unitary
- $\{U_\alpha : \alpha \in \mathbb{T}\}$ are all the cyclic unitary rk 1 pert of A_z

Thm (Clark) There exists $\mu_\alpha \in M_+(\mathbb{T})$ and a unitary $V_\alpha : K_\Theta \to L^2(\mu_\alpha)$ such that

$$V_\alpha^* (M_\xi, L^2(\mu_\alpha)) V_\alpha = U_\alpha$$

Cor For say a bd-Borel function φ

$$V_\alpha^* (M_\varphi, L^2(\mu_\alpha)) V_\alpha = \varphi(U_\alpha)$$

Thm (Aleksandrov) $m = \int \mu_\alpha \, dm(\alpha)$

$$\int f \, dm = \int \left(\int f \, d\mu_\alpha \right) dm(\alpha)$$

Thm (Poltoratski)

For $f \in K_\Theta$

$$\lim_{\angle z \to \xi} f(z) = (V_\alpha f)(\xi)$$

μ_α — a.e.

For a bd. Borel φ and $f, g \in K_\theta$

$$\langle A_\varphi f, g \rangle = \int \varphi f \bar{g} \, dm$$

(Alek.) $\Longrightarrow$ $$= \int \left(\int \varphi f \bar{g} \, d\mu_\alpha \right) dm(\alpha)$$

(Polt) $\Longrightarrow$ $$= \int \left(\int \varphi V_\alpha f \, \overline{V_\alpha g} \, d\mu_\alpha \right) dm(\alpha)$$

$$= \int \langle M_\varphi V_\alpha f, V_\alpha g \rangle_{L^2(\mu_\alpha)} dm(\alpha)$$

(Clark) $$\begin{cases} = \int \langle V_\alpha^* M_\varphi V_\alpha f, V_\alpha^* V_\alpha g \rangle_{K_\theta} dm(\alpha) \\ = \int \langle \varphi(U_\alpha) f, g \rangle_{K_\theta} dm(\alpha) \end{cases}$$

Thus $$A_\varphi = \int \varphi(U_\alpha) \, dm(\alpha)$$

(Cool formula!)

$\boxed{\text{VI}}$ Compact A_φ.

Recall that T_φ is compact iff $\varphi = 0$.

It is possible to have compact A_φ.

Thm: (Sarason) For $\lambda \in \mathbb{D}$, let

$$k_\lambda(z) = \frac{1 - \overline{\Theta(\lambda)}\,\Theta(z)}{1 - \bar{\lambda} z}$$

$$\tilde{k}_\lambda(z) = \frac{\Theta(z) - \Theta(\lambda)}{z - \lambda}$$

Then $k_\lambda, \tilde{k}_\lambda \in K_\Theta$ and

$$k_\lambda \otimes \tilde{k}_\lambda = A_{\overline{\left(\frac{\Theta}{z-\lambda}\right)}}$$

is a rank-one truncated Toeplitz operator.