Factorization of analytic functions and operator inequalities

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Abstract. If A and B are contraction operators on a Hilbert space $\mathcal{H}$ that commute with a shift operator S, it is shown that $A = BC$ for some contraction operator C on $\mathcal{H}$ that commutes with S if and only if $AA^* \leq BB^*$.

Mathematics Subject Classification (2010). Primary 47A56; Secondary 47A45, 47A63. 47A68.

Keywords. Shift operator, analytic operator, contraction, factorization, Leech's theorem.

An isometry S on Hilbert space such that $S^{*n} \to 0$ strongly as $n \to \infty$ will be called a **shift operator**. We fix a Hilbert space $\mathcal{H}$ and a shift operator S on $\mathcal{H}$. We write $\mathcal{C} = \ker S^*$ and we let P_n denote the projection mapping $\mathcal{H}$ onto $S^n\mathcal{C}$ for each nonnegative integer n. A contraction T on $\mathcal{H}$ will be called S**-analytic** if $TS = ST$.

Theorem. *Let A and B be S-analytic contractions on $\mathcal{H}$. A necessary and sufficient condition that $A = BC$ for some S-analytic contraction C is that $AA^* \leq BB^*$.*

A special case of this theorem [3, Proposition V.5.3] is used by Sz.-Nagy and Foias to show the existence of "nontrivial" factorizations. This special case is presented in another form by de Branges and Rovnyak [1, Theorem 3]. The proof of the theorem has been translated from the context of the de Branges-Rovnyak model and makes extensive use of the following lemma, which can be found in [2].

Lemma. *Let A and B be bounded operators with final space $\mathcal{H}$. A necessary and sufficient condition that $A = BC$ for some contraction C is that $AA^* \leq BB^*$.*

Proof of Theorem. Since $BB^* - AA^* \leq BB^*$, we have

$$BB^* - AA^* = BTT^*B^*$$

For the history of the present paper see the next paper is this issue.

for some contraction T. Also,

$$\begin{aligned} BB^* - AA^* &\leq BB^* - AA^* + AP_0A^* \\ &= BB^* - ASS^*A^* \\ &= BP_0B^* + S(BB^* - AA^*)S^* \\ &= B(P_0 + STT^*S^*)B^* \\ &= B(P_0 + STS^*)(P_0 + ST^*S^*)B^* \end{aligned}$$

and, therefore,

$$BT = B(P_0 + STS^*)U$$

for some contraction U. We define a sequence of operators $\{V_n\}$ by $V_0 = T$, $V_{n+1} = (P_0 + SV_nS^*)U$. It can be verified inductively that for each n, $BT = BV_n$, $V_n^*V_n \leq I$, and

$$V_n = \sum_{k=0}^{n-1} S^kP_0(US^*)^kU + S^nT(S^*U)^n.$$

Since $P_NV_n = P_NV_{N+1}$ for $n \geq N+1$, the sequence $\{V_n\}$ converges weakly to an operator V, $|V| \leq 1$, $BV = BT$, and $V = (P_0 + SVS^*)U$. Therefore,

$$\begin{aligned} I - VV^* &= I - (P_0 + SVS^*)UU^*(P_0 + SV^*S^*) \\ &\geq I - (P_0 + SVS^*)(P_0 + SV^*S^*) \\ &= S(I - VV^*)S^*. \end{aligned}$$

Since

$$\begin{aligned} AP_0A^* &= (BB^* - ASS^*A^*) - (BB^* - AA^*) \\ &= B(P_0 + SVV^*S^* - VV^*)B^* \\ &= B(I - VV^* - S(I - VV^*)S^*)B^*, \end{aligned}$$

we have $AP_0 = BRP_0$, where

$$R = (I - VV^* - S(I - VV^*)S^*)^{1/2}Q$$

for some contraction $Q \in \mathcal{B}(\mathcal{C}, \mathcal{H})$. From the inequality

$$\begin{aligned} \sum_M^N S^nRR^*S^{*n} &\leq \sum_M^N S^n(I - VV^* - S(I - VV^*)S^*)S^{*n} \\ &\leq S^M(I - VV^*)S^{*M}, \end{aligned}$$

which holds for $0 \leq M \leq N$, it follows that the expression

$$\sum_0^\infty S^nRP_0S^{*n}$$

converges strongly to a contraction C. Clearly $CS = SC$, and the relation

$$AP_0 = BRP_0 = BC$$

implies that $A = BC$. $\square$

References

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