$\Rightarrow$ a commutes with $|z|^2$ $\Rightarrow$ a commutes with any element of $C^*(1, |z|^2)$.
In particular, $a|z|^{-1} = |z|^{-1}a$, by an "easy approximation argument."

So $$ua = z|z|^{-1}a$$
$$= z\,a\,|z|^{-1}$$
$$= b\,z\,|z|^{-1}$$
$$= bu.$$

$\Rightarrow$ $b = uau^*$. ∎

<u>Proposition 2.2.6</u>: Let $p, q \in \mathcal{P}(A)$. Then

$$p \sim_h q \text{ in } \mathcal{P}(A) \iff q = upu^* \text{ where } u \in \mathcal{U}_0(\tilde{A}).$$

10/16/07

<u>proof</u>:
<u>Easy way</u>: Write $1 = 1_{\tilde{A}}$.
Suppose $q = upu^*$, $u \in \mathcal{U}_0(\tilde{A})$
Let $t \mapsto u_t$ in $\mathcal{U}(\tilde{A})$ connecting u to 1.
u_t^* defines a path from u^* to 1.
projection $= u_t p u_t^*$
$=$ path from p to $upu^* = q$

<u>Converse</u>:
Assume first that $\|p - q\| < \frac{1}{2}$
Let $z = pq + (1-p)(1-q)$

$$\|z - 1\| = \|pq + (1-p)(1-q) - 1\|$$

$$= \|p(q-p) + (1-p)(1-q) - (1-p)\|$$

$$= \|p(q-p) + (1-p)(1-q-(1-p))\| \quad (\text{since } (1-p)^2 = (1-p))$$

$$= \|(p - (1-p))(q-p)\| \leq 2\underbrace{\|q-p\|}_{<1/2} < 1$$

Prop. 2.1.11

So z^{-1} exists and $z \sim_h 1$ in $GL(\tilde{A})$
[Recall. If a^{-1} exists and $\|b-a\| < \frac{1}{\|a^{-1}\|}$, then b^{-1} exists and $b \sim_h a$ in $GL(A)$, A unital C^*-algebra]
[we use ↑ with $a=1$].

$$pz = p[pq + (1-p)(1-q)] = pq \qquad (p\cdot(1-p)=0)$$

$$zq = [pq + (1-p)(1-q)]q = pq$$

$\Rightarrow$ $pz = zq$.

So $q = z^{-1}pz$.

Let $z = u|z|$, u unitary

$\Rightarrow$ $q = u^* p u$ (Proposition 2.2.5)

Then $u \sim_h z$ in $GL(\tilde{A})$

Since $z \sim_h 1$ in $GL(\tilde{A})$, then $u \sim_h 1$ in $\mathcal{U}(\tilde{A})$ by Proposition 2.1.8 (iii).

Then $q = u^* p u$ since $u \in \mathcal{U}_0(\tilde{A})$.
Correct conclusion, wrong hypothesis, i.e. $\|p-q\| < 1/2$.
Correct hypothesis: $p \sim_h q$ in $\mathcal{P}(A)$ = projections in A.

Let $t \mapsto p_t$ be a curve in $\mathcal{P}(A)$ with $p_0 = p$, $p_1 = q$

$0 = t_0 \quad t_1 \quad t_2 \qquad 1 = t_n$

There exists $0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$ so that
$\|p_{t_{i-1}} - p_{t_i}\| < 1/2$, $i = 1, \ldots, n$.
(by uniform continuity). By the above, there exists $u_i \in \mathcal{U}_0(\tilde{A})$ with $p_{t_{i-1}} = u_i^* p_{t_i} u_i$, $p = p_0 = \underbrace{u_1 u_2 \cdots u_n}_{=u} p_1 \underbrace{u_n^* \cdots u_2^* u_1^*}_{=u^*}$
(since $p = uqu^*$)
$= uqu^*$. ∎

Summary (Proposition 2.2.7):
Let $p, q \in A$.
(i) $p \sim_h q \Rightarrow p \sim_u q$

(ii) $p \sim_u q \Rightarrow p \sim q$

proof.
(i) previous proposition
(ii) If $q = upu^*$, $u \in \mathcal{U}(\tilde{A})$,
let $v = up \in A$.
$v^*v = pu^*up = p$
$vv^* = uppu^* = q$
$\Rightarrow p \sim q$ ∎

Note: neither reverse implication holds in general.

Example: S = shift in $B(\ell^2)$.

$\underbrace{S^*S}_{=P} = I$, $\underbrace{SS^*}_{\equiv Q}$ = Proj onto $\{(0, x_1, x_2, \ldots) \in \ell^2\}$

$P = I \sim Q$ but $P \not\sim_u Q$

unitization of $M_2(A)$

Proposition 2.2.8. Let $p, q \in A$.
(i) If $p \sim q$, then $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_u \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ in $M_2(A)^\sim$
(ii) If $p \sim_u q$, then $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_h \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ in $\mathcal{P}(M_2(A))$.

proof:
(i): let $p = v^*v$, $q = vv^*$.
Then $v = vp = qv = qvp$.
So $v^* = pv^* = v^*q$
Consider 2 matrices.
$u = \begin{pmatrix} v & 1-q \\ 1-p & v^* \end{pmatrix}$, $w = \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix}$

since $1 = 1_{\tilde{A}}$

Both are unitary in $M_2(\tilde{A})$.

$$u^*u = \begin{pmatrix} v^* & 1-p \\ 1-q & v \end{pmatrix}\begin{pmatrix} v & 1-q \\ 1-p & v^* \end{pmatrix}$$

$$= \begin{pmatrix} v^*v + (1-p) & v^*(1-q) + (1-p)v^* \\ (1-q)v + \underline{v}(1-p) & (1-q) + vv^* \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Similarly for the other computations.

$$u\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}u^* = \begin{pmatrix} v & 1-q \\ 1-p & v^* \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v^* & 1-p \\ 1-q & v \end{pmatrix}$$

$$= \begin{pmatrix} vp & 0 \\ 0 & 0 \end{pmatrix}u^* = \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v^* & 1-p \\ 1-q & v \end{pmatrix}$$

$$= \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}.$$

u won't work since $u \in \mathcal{U}(M_2(\tilde{A}))$.

Also check $\quad w\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}w^* = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}.$

$$wu\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}u^*w^* = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$$

and wu is unitary.

$$wu = \begin{pmatrix} v + (1-q)(1-p) & (1-q)v^* \\ q(1-p) & (1-q) + qv^* \end{pmatrix}$$

which is unitary in $\mathcal{U}(M_2(A)^\sim)$;

<u>note</u>:

$M_2(A)^\sim \subset M_2(\tilde{A})$.

note: diagonal entries of $wu \in A + \mathbb{C}1_{\tilde{A}}$.
off diagonal of $wu \in A$

So $M_2(A)^{\sim}$ $\begin{pmatrix} a+\lambda & b \\ c & d+\lambda \end{pmatrix}$

$M_2(\tilde{A})$ $\begin{pmatrix} a+\lambda & b+\mu \\ c+\nu & d+\gamma \end{pmatrix}$ (bigger!) ✓(i)

(ii): Suppose $u \in \mathcal{U}(\tilde{A})$, $q = upu^*$.

Whitehead's Lemma (2.1.5) for unital C^*-algebras A says

$$\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

in $\mathcal{U}(M_2(\tilde{A}))$.

Let $t \mapsto w_t$, $0 \le t \le 1$, be a path from $w_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ to $w_1 = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$.

Let $e_t = w_t \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} w_t^*$.

Note: $e_t \in \mathcal{P}(M_2(A))$, $e_0 = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$,

$e_1 = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_h \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}.$$

∎