On the preceding paper by R. B. Leech

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Abstract. The history of the paper by R. B. Leech, *Factorization of analytic functions and operator inequalities*, is described.

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Keywords. Leech's theorem, corona problem, commutant lifting.

The preceding paper by R. B. Leech was written in 1971-72 when the author was at the University of Wisconsin–Milwaukee. This note explains why the paper is being published only now, more than four decades after it was written.

At the time of its writing, the paper was submitted to a journal and "received a terse rejection," according to the author, who at that time had one published mathematical paper (Proc. Amer. Math. Soc. 1969). It did not help matters that on another occasion a senior operator theorist, sought out by Leech at an AMS meeting, made the critical remark that the paper lacked an application. After his term at the University of Wisconsin–Milwaukee, Leech did not pursue a mathematical research career.

Unknown to Leech, his typescript was circulated privately among operator theorists (by J. W. Helton and others) and soon attracted interest. Rosenblum [15] gave a proof of the main result and a generalization using the commutant lifting theorem, and at the same time he gave an application to the corona problem. The connection with the commutant lifting theorem was also known to Foias at an early stage (Ball [5, p. 175]). Leech's theorem appears in the account of corona problems and analytic Toeplitz operators in Foias and Frazho [10, Section VIII.6]. In the course of the years Leech's theorem has been adapted to a variety of settings, including, for example, situations requiring less regularity of the functions involved and indefinite inner product spaces. Nowadays there is a special interest in rational matrix solutions. The theorem also plays a role in an interdisciplinary setting, which is a bit surprising given the fact that "lack of applications" was one of the early critical comments on the paper.

In reference to this episode and the plan to publish Leech's original paper, J. W. Helton observed: "The story is interesting and gives a bit of perspective on how science works. Usually math history is told from the view of successful people. It might be valuable to give this view of the business."

Another reason to publish the original paper is that Leech's own method of proof, as it appears in his paper, is now published here for the first time, at least as far as we know. Leech's approach does not use lifting techniques nor any of the other methods to deal with norm constrained interpolation problems that are known presently. It is not excluded that something more could be made of Leech's original method.

Concerning Leech's proof, we remark that the series defining C at the end of the proof converges weakly, but it is not clear that it converges strongly. Work instead with the adjoint $C^* = \sum_{n=0}^{\infty} S^n P_0 R^* S^{*n}$, which is given by a strongly convergent series. By construction $AP_0 = BRP_0$. Thus $AS^n P_0 S^{*n} = BS^n RP_0 S^{*n}$ for all n, yielding $A = BC$.

The bibliography below lists a sample of works in which theorems of Leech type play a role.

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