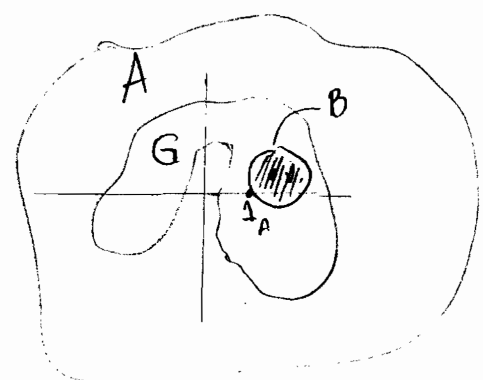
number in Rørdam

Proposition 2.1.6: Let A be a unital C^*-algebra.
(i) $\mathcal{U}_0(A)$ is a normal subgroup in $\mathcal{U}(A)$
(ii) $\mathcal{U}_0(A)$ is open and closed in $\mathcal{U}(A)$.
(iii) $u \in A$ is in $\mathcal{U}_0(A) \Leftrightarrow u = e^{ih_1}e^{ih_2}\cdots e^{ih_n}$
and each h_j is self-adjoint.

09/18/07

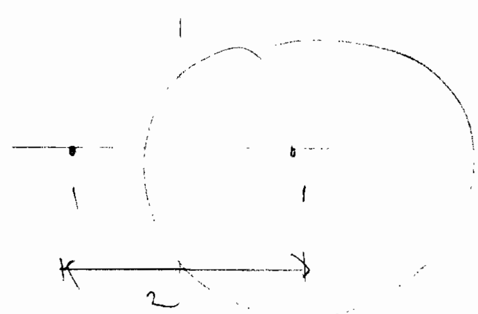
proof: (i)
clear: $\mathcal{U}_0(A)$ is closed under multiplication
If $t \mapsto w_t$ is a path in $\mathcal{U}(A)$ from 1 to u, then
$t \mapsto w_t^*$ is a path in $\mathcal{U}(A)$ from 1 to u^{-1}.
$\Rightarrow \mathcal{U}_0(A)$ is a group
Normality: If v is unitary, then $v w_t v^{-1}$ is a path from 1 to vuv^{-1}
So $vuv^{-1} \in \mathcal{U}_0(A) \Rightarrow$ normal.

(ii)+(iii): Let G = set of all $e^{ih_1} \cdot e^{ih_2} \cdots e^{ih_n}$, $n \in \mathbb{N}$
where each h_j is self-adjoint.
G is a subgroup of $\mathcal{U}_0(A)$.



Let $v \in G$.
Consider u in the set
$B = \{u \in \mathcal{U}(A) : \|u - v\| < 2\}$
Then
$$\|1 - uv^*\| = \|(v-u)v^*\|$$
$$= \|v - u\| < 2$$
(using that v is unitary)

So $-1 \notin \sigma(uv^*)$



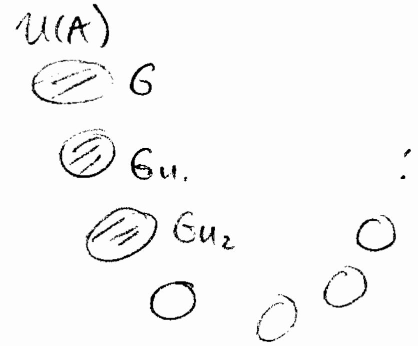
$\Rightarrow \sigma(uv^*) \subsetneq \mathbb{T}$

$\Rightarrow uv^* = e^{ih}$ for some self-adjoint h

$\Rightarrow u = \underbrace{e^{ih} v}_{\text{both in } G} \in G$

So $B \subset G$. Thus G is open in $\mathcal{U}(A)$.

Also G is closed in $\mathcal{U}(A)$.
To see this, consider the cosets
$$\{Gu : u \in \mathcal{U}(A)\}.$$
A familiar argument implies if $u, v \in \mathcal{U}(A)$
either $Gu = Gv$ or $Gu \cap Gv = \phi$.
$\mathcal{U}(A) = \bigcup_{u \in \mathcal{U}(A)} Gu$, disjoint union of open sets in $\mathcal{U}(A)$.



Gu is open: the map $v \mapsto vu$ is a homeomorphism of $\mathcal{U}(A)$.

$$\mathcal{U}(A) \setminus G = \bigcup_{\substack{u \in \mathcal{U}(A) \\ Gu \neq G}} Gu, \quad \text{union of open sets}$$
$\Rightarrow G$ is closed.

So G is the component of $\mathcal{U}(A)$ containing 1.
So $G = \mathcal{U}_0(A)$. (So (ii) + (iii) hold.) $\boxtimes$

If $\varphi : A \to B$, is a $*$-homomorphism write $\varphi_n : M_n(A) \to M_n(B)$
by
$$\varphi_n((a_{ij})) = (\varphi(a_{ij})).$$
If φ is onto, then $\varphi(1_A) = \varphi(1_B)$ if A, B are unital.
Let $b \in B$. $\varphi(1_A)b$.
But $b = \varphi(a)$, so
$$\varphi(1_A)b = \varphi(1_A)\varphi(a) = \varphi(1_A a) \cdot \varphi(a) = b.$$

If φ is onto, so is φ_n.

Also, if $u \in \mathcal{U}(A)$, $\varphi(u) \in \mathcal{U}(B)$ since
$\varphi(u)\varphi(u)^* = \varphi(u)\varphi(u^*) = \varphi(uu^*) = \varphi(1_A) = 1_B$.

So if φ is onto, $\varphi(\mathcal{U}(A)) \subset \mathcal{U}(B)$.

Lemma (2.1.7): A, B are unital, φ ($*$-homomorphism) is onto $(\varphi: A \to B)$
(i) $\varphi(\mathcal{U}_0(A)) = \mathcal{U}_0(B)$
(ii) If $u \in \mathcal{U}(B)$, $\exists\, v \in \mathcal{U}_0(M_2(A))$ so that
$\varphi_2(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$.

(iii) If $u \in \mathcal{U}(B)$ and $\exists\, v \in \mathcal{U}(A)$ with $\varphi(v) \sim_h u$ in $\mathcal{U}(B)$
then $\exists\, w \in \mathcal{U}(A)$ with $\varphi(w) = u$.

proof: know $\varphi(\mathcal{U}(A)) \subset \mathcal{U}(B)$.
Suppose $u \in \mathcal{U}_0(A)$, $u \sim_h 1_A$
Then $\varphi(u) \sim_h \varphi(1_A) = 1_B$. (since φ(unitary) = unitary).
So
$$\varphi(\mathcal{U}_0(A)) \subset \mathcal{U}_0(B).$$
For $\supset$:
Let $u = e^{ih_1}e^{ih_2}\cdots e^{ih_n}$, $n \in \mathbb{N}$, where each h_j is self-adjoint in B.
(note: $u \in \mathcal{U}_0(B)$)
φ is onto: $\exists\, x_i \in A$ with $\varphi(x_i) = h_i$, $i = 1, \dots, n$.
If x_i is not self-adjoint, replace x_i with $k_i = \frac{x_i + x_i^*}{2}$.
Then k_i is self-adjoint, $\varphi(k_i) = h_i$, for all i.

So $\varphi(\underbrace{e^{ik_1} \cdots e^{ik_n}}_{v}) = e^{ih_1} \cdots e^{ih_n} = u$.

so $\varphi: \mathcal{U}_0(A) \xrightarrow{\text{onto}} \mathcal{U}_0(B)$.
$\Rightarrow \varphi(\mathcal{U}_0(A)) = \mathcal{U}_0(B)$. //

Let $u \in \mathcal{U}_0(B)$. $\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \in \mathcal{U}(M_2(B))$.

Whitehead's Lemma $\Rightarrow$ $\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
But φ_2 is onto, so
$\exists\, v \in \mathcal{U}_0(M_2(A))$ with $\varphi_2(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$. //

Suppose $u \in \mathcal{U}(B)$ and $\exists\, v \in \mathcal{U}(A)$ with
$\varphi(v) \sim_h u$ in $\mathcal{U}(B)$
Then

$$\underbrace{\varphi(v)\underbrace{\varphi(v)^*}_{\text{fixed unitary}}}_{= 1_B} \sim_h u\varphi(v^*) \quad \text{in } \mathcal{U}(B).$$

$\Rightarrow$ $u\varphi(v^*) \in \mathcal{U}_0(B)$.
So $\exists\, w \in \mathcal{U}_0(A)$ with $\varphi(w) = u\varphi(v^*)$.

So $u = \varphi(w)\varphi(v) = \varphi(wv)$.
We are done since $wv \in \mathcal{U}(A)$. ∎

A is unital
$GL(A) = \{a \in A : a^{-1} \text{ exists}\}$ = general linear group of A.
$GL_0(A) = \{a \in GL(A) : a \sim_h 1_A \text{ in } GL(A)\}$

If $a \in A$, $|a| = \sqrt{a^*a}$ (via continuous functional calculus)
We want to look at the polar decomposition of a
when a^{-1} exists. In this case set $\omega(a) = a|a|^{-1}$
$\omega(a) \in A$.

<u>Lemma 1.2.5:</u> K is compact $\subset \mathbb{R}$. Let $f \in C(K)$. Let
A be a unital C^*-algebra. $\Omega_K = \{a = a^* \in A : \sigma(A) \subset K\}$.
The map $a \mapsto f(a)$, for $a \in \Omega_K$ (non-linear) is continuous
from Ω_K to A.

proof: Let $n \in \mathbb{N}$.
The map $a \mapsto a^n$ is continuous on A.
If p is a polynomial, $a \mapsto p(a)$ is continuous on A.
Let $a \in \Omega_K$, $f \in C(K)$.
By Stone-Weierstrass Theorem, $\exists$ polynomial p
with $\|p - f\|_K = \sup_t |f(t) - p(t)| < \varepsilon$

$\exists \delta > 0$ so that if $b \in A$, $\|b - a\| < \delta$, then $\|p(a) - p(b)\| < \varepsilon$.
So for such b lying in Ω_K,

$$\|f(a) - f(b)\| \leq \underbrace{\|f(a) - p(a)\|}_{\|f-p\|_{\sigma(a)}} + \|p(a) - p(b)\| + \underbrace{\|p(b) - f(b)\|}_{\|f-p\|_{\sigma(b)}}$$

$$< 3\varepsilon. \qquad \blacksquare$$

Proposition 2.1.8: Let A be a unital C^*-algebra
(i) If $a \in GL(A)$, so is $|a|$ and $\omega(a) \equiv a|a|^{-1} \in \mathcal{U}(A)$.
We get polar decomposition: $a = \omega(a)|a|$. (done already)
(ii) $\omega: GL(A) \to \mathcal{U}(A)$ is continuous, {clear
$\omega(u) = u$ for $u \in \mathcal{U}(A)$, and
$\omega(a) \sim_h a$ in $GL(A)$.

proof: Let $a \in GL(A)$. Let $a_t = \omega(a)(t|a| + (1-t)1_A)$.
each $t|a| + (1-t)1_A$ is invertible:
$|a|^{-1}$ exists, so $|a| \geq \lambda 1_A$, some $\lambda > 0$.

$$t|a| + (1-t)1_A \geq (t\lambda + (1-t))1_A$$

$$= [1 - t(1-\lambda)]\, 1_A$$

$$\geq \lambda 1_A \quad \Rightarrow \text{invertible.}$$

(λ needs to be < 1, so let $t = 1$ to get this)

$t \mapsto a_t$ is continuous. $a_0 = \omega(a)$, $a_1 = a$. $\blacksquare$
$\dot{a}(t)$