Title: When is an operator S an analytic function of an operator T ?

Abstract: Suppose X is a Banach space, $S, T : X \to X$, $W : \ell^2 \to X$ are (bounded, linear) operators. We define the linear transformation $W^{-1} : ranW \to (\ker W)^{\perp}$ to be the linear inverse of the restriction of W to $(\ker W)^{\perp}$. The closed graph theorem easily shows $T(ranW) \subseteq ranW \Leftrightarrow W^{-1}TW$ is defined $\Leftrightarrow W^{-1}TW$ is bounded. This is joint work with John Conway.

THEOREM If T is not algebraic and $\|T\| < R$, then $S = \varphi(T)$ for some analytic function φ on $D(0, R)$ if and only if, for each $\|T\| < r < R$, there is a number α_r such that

$$\left\|W^{-1}SW\right\| \leq \alpha_r$$

for every compact operator $W : \ell^2 \to X$ with $\left\|W^{-1}TW\right\| \leq r$.

When X is a Hilbert space, we have a similar statement considering invertible operators $W \in B(X)$. We also prove an analogue of this theorem in a type I von Neumann algebra acting on a separable Hilbert space, where the analytic function φ is center-valued rather than scalar-valued.