On Kuroda Formula

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Abstract

We give some characterizations of the class of intuitionistic predicate Kripke frames validating the Kuroda formula. We, first, prove that the Kuroda formula does not define a class of frames definable by a classical first-order sentence and, second, establish a criterion for countable frames to validate this formula.

1 Introduction

One of the first papers on completeness for intermediate predicate logics was Dov Gabbay's [1]. Theorem 2 from that paper proves that the intuitionistic predicate logic enriched with the Kuroda formula is complete with respect to a class of trees enjoying the McKinsey property: every point sees a maximal point. Later on, Dov Gabbay gave another prove showing that this logic is complete with respect to the class of all Kripke frames (not necessarily trees) with the McKinsey property [2, Chapter 3, Section 4]. The McKinsey property is elementary (i.e., first-order definable), but the class $\mathscr{KF}$ of all predicate Kripke frames validating the Kuroda formula is much larger.

In the present paper, we give some characterizations of the class $\mathscr{KF}$. First, we prove that $\mathscr{KF}$ is not elementary. Second, we give an explicit criterion for countable members of $\mathscr{KF}$.

With regret, we share the sad news that Dmitrij Skvortsov passed away when the work on this paper was nearly complete.

2 Preliminaries

2.1 Formulas and logics

We consider logics in a pure predicate language $\mathcal{L}$ containing the following symbols: countably many individual variables; for every $n \geqslant 0$, countably many n-ary predicate letters (nullary letters are also called proposition letters); the propositional constant $\bot$; the binary connectives $\wedge$, $\vee$, and $\to$; the quantifier symbols $\exists$ and $\forall$ (note that $\mathcal{L}$ does not contain individual constants, function symbols, or equality).

The definition of $\mathcal{L}$-formulas (or, simply, formulas) is standard. We use the standard abbreviations $\neg A := A \to \bot$ and $A \leftrightarrow B := (A \to B) \wedge (B \to A)$, and adopt the usual conventions on omitting parentheses. In what follows, the language $\mathcal{L}$ is identified with the set of its formulas. A formula is propositional if it contains no individual variables, and hence no predicate letters other than nullary and no quantifier symbols. A formula A is negative if $A = \neg B$, for some B.

The free variables of a formula are its parameters. The universal closure of a formula A, which may be assumed to be unique up to the enumeration of its parameters, is denoted by $\bar{\forall} A$. A formula without parameters is called a sentence. A theory is a set of sentences.

A superintuitionistic predicate logic, or simply logic, is a set of formulas that includes the intuitionistic predicate logic **QH** and is closed under Predicate Substitution, Modus Ponens, and Generalisation; thus, **QH** is the smallest superintuitionistic predicate logic. If $\mathbf{L}$ is a logic and A a formula, then $\mathbf{L} \vdash A$ means the same as $A \in \mathbf{L}$. The smallest logic that includes a logic $\mathbf{L}$ and a set Γ of formulas is denoted by $\mathbf{L} + \Gamma$; if A is a formula, we write $\mathbf{L} + A$ instead of $\mathbf{L} + \{A\}$. If $\mathbf{L}$ is a logic, and A and B are formulas, then we say that

- B is derivable from A in $\mathbf{L}$, and write $A \vdash_{\mathbf{L}} B$, if $\mathbf{L} + A \vdash B$;
- A and B are deductively equivalent in $\mathbf{L}$ if they are mutually derivable, i.e., if $A \vdash_{\mathbf{L}} B$ and $B \vdash_{\mathbf{L}} A$;
- B is simply derivable from A, and write $A \vdash B$, if B is derivable from A in **QH**;
- A and B are simply deductively equivalent if they are mutually derivable, i.e., if $A \vdash B$ and $B \vdash A$.

It should be clear that, if A and B are deductively equivalent, then $A \in \mathbf{L}$ if and only if $B \in \mathbf{L}$, for every logic $\mathbf{L}$.

It is well known that the set of all logics forms a lattice with respect to the set-theoretic inclusion; we denote this lattice by $\mathfrak{L}$. Algebraically, the meet of $\mathbf{L}_1, \mathbf{L}_2 \in \mathfrak{L}$

is the logic $\mathbf{L}_1 \cap \mathbf{L}_2$, and the join of $\mathbf{L}_1, \mathbf{L}_2 \in \mathfrak{L}$ is the logic $\mathbf{L}_1 + \mathbf{L}_2$. The least element of $\mathfrak{L}$ is the logic $\mathbf{QH}$; the greatest, the inconsistent logic $\mathcal{L}$ (the set of all formulas). The lattice $\mathfrak{L}$ is complete since it is closed under intersection.

In what follows, we will be referring to the following formulas and logics:

$$\begin{array}{lcll} CD & := & \forall x(P(x) \vee q) \to \forall x P(x) \vee q & \text{(the constant domain principle);} \\ EM & := & p \vee \neg p & \text{(the law of excluded middle);} \\ Z & := & (p \to q) \vee (q \to p) & \text{(Dummett's formula);} \\ \mathbf{QLC} & := & \mathbf{QH} + Z & \text{(Dummett's logic).} \end{array}$$

Definition 2.1. Let $n \geqslant 1$, let C be a propositional formula in proposition letters $p_1, \ldots, p_k$, let $P_1, \ldots, P_k$ be a list of distinct n-ary predicate letters, and let $\boldsymbol{x}$ be a fixed list of n distinct individual variables. An n-shift of C, denoted by C^n, is the formula obtained by replacing in C, for every $i \in \{1, \ldots, k\}$, each occurrence of p_i with the atomic formula $P_i(\boldsymbol{x})$.

2.2 Some properties of posets

We denote the cardinality of a set X by $|X|$.

A partially ordered set (a poset, for short) is a pair $(U, \leqslant)$ where U is a set and $\leqslant$ is a partial order on U. In this paper, all posets are non-empty (i.e., $U \neq \varnothing$). If $\leqslant$ is understood from the context, we may simply say that U is a poset. If $(U, \leqslant)$ is a poset and $x, y \in U$, then, as usual, we write $x < y$ to mean that $x \leqslant y$ and $x \neq y$. The set of maximal points of a poset U is denoted by $max[U]$. If U is a poset and $u \in U$, then the cone of u in U is the set $U \uparrow u := \{w \in U \mid u \leqslant w\}$; if U is understood from the context, we write $u\!\uparrow$ instead of $U \uparrow u$. A subset X of a poset U is upward-closed if it includes the cones of all its points, i.e., if $(\forall u \in X)\, u\!\uparrow \subseteq X$. We define

$$\begin{array}{rcl} X\!\uparrow & := & \{u \in U \mid (\exists x \in X)\, x \leqslant u\}; \\ X\!\downarrow & := & \{u \in U \mid (\exists x \in X)\, u \leqslant x\}. \end{array}$$

Note that X is upward-closed iff $X\!\uparrow = X$.

Definition 2.2. Let $(U, \leqslant)$ be a poset and $X, Y \subseteq U$. We say that X is cofinal for Y if $Y \subseteq X\!\downarrow$. If X is cofinal for U itself, i.e., if $U \subseteq X\!\downarrow$, then we say that X is a cofinal subset of U or that X is cofinal in U. The cofinality $cf[U]$ of a poset U is the minimal cardinality of a cofinal subset of U.

Note that, since U itself is cofinal in U, surely $cf[U] \leqslant |U|$.

Lemma 2.3. Let $(U, \leqslant)$ be a poset and $X, Y \subseteq U$.

(1) If X is cofinal in U and $U \neq \varnothing$, then $X \neq \varnothing$.

(2) If Y is upward-closed, then X is cofinal for Y iff $X \cap Y$ is a cofinal subset of Y.

Remark 2.4. If Y is not upward-closed, then Lemma 2.3 (2) in general does not hold.

Lemma 2.5. Let $(U, \leqslant)$ be a linearly ordered set, X an upward-closed subset of U, and Y a non-empty subset of U. Then, X is cofinal for Y iff $X \neq \varnothing$.

Definition 2.6. A poset $(U, \leqslant)$ satisfies the McKinsey property if the set $max[U]$ of its maximal points is cofinal in U, i.e., if $U \subseteq max[U]{\downarrow}$.

Lemma 2.7. Let $(U, \leqslant)$ be a poset and $X \subseteq U$.

(1) If X is a cofinal in U, then $max[U] \subseteq X$.

(2) If U satisfies the McKinsey property, then X is cofinal in U iff $max[U] \subseteq X$.

Lemma 2.8. Let U be a poset.

(1) $cf[U] \geqslant |max[U]|$.

(2) If U satisfies the McKinsey property, then $cf[U] = |max[U]|$.

Remark 2.9. The converse of Lemma 2.8 (2) in general does not hold.

Lemma 2.10. The intersection of a finite number of upward-closed cofinal subsets of a poset U is itself cofinal (and similarly for sets that are upward-closed and cofinal for some subset Y of U).

2.3 Constant domain Kripke semantics

Recall that a predicate Kripke frame with a constant domain is a triple $(U, \leqslant, D)$ where $(U, \leqslant)$ is a non-empty poset n(elements of U are called possible words) and D is a non-empty set (called the domain of individuals); the frame $(U, \leqslant, D)$ is called a frame over $(U, \leqslant)$ with the domain D.

For a non-empty set D, a D-sentence is an expression obtained from a predicate formula by replacing all its parameters with elements of D.[1]

[1] More precisely, with constants naming elements of D.

If F is a frame over $(U, \leqslant)$ with a domain D, then a valuation over F, or an F-valuation, is a map V sending each D-sentence A to upward-closed subsets of $(U, \leqslant)$ and satisfying the following conditions:[2]

$$\begin{array}{lcl} V(\bot) & = & \varnothing, \\ V(A \wedge B) & = & V(A) \cap V(B), \\ V(A \vee B) & = & V(A) \cup V(B), \\ V(A \to B) & = & \{u \in U \mid u{\uparrow} \cap V(A) \subseteq V(B)\}, \\ V(\forall x A(x)) & = & \bigcap\{V(A(d)) \mid d \in D\}, \\ V(\exists x A(x)) & = & \bigcup\{V(A(d)) \mid d \in D\}. \end{array}$$

These clauses imply that $V(\neg A) = \{u \in U \mid u{\uparrow} \cap V(A) = \varnothing\}$.

We can rewrite $u \in V(A)$ as $u \models A$ ('the truth of A at u') to see the usual inductive clauses for the intuitionistic constant domain Kripke semantics [3, Chapter 3]:

$$\begin{array}{lcl} u \not\models \bot; & & \\ u \models A \wedge B & \iff & v \models A \text{ and } v \models B; \\ u \models A \vee B & \iff & v \models A \text{ or } v \models B; \\ u \models A \to B & \iff & (\forall v \geqslant u)\,(v \models A \Rightarrow v \models B); \\ u \models \forall x A(x) & \iff & (\forall d \in D)\, u \models A(d); \\ u \models \exists x A(x) & \iff & (\exists d \in D)\, u \models A(d). \end{array}$$

These clauses imply that $u \models \neg A$ iff $\forall v \geqslant u\, (v \not\models A)$.

Let F be a predicate Kripke frame with a constant domain. An F-model is a pair $M = (F, V)$, where V is an F-valuation. We say that a formula A is true in a model M, or M-true, and write $M \models A$, if $V(\bar{\forall} A) = U$, i.e., if $V(A(\boldsymbol{d})) = U$, for every list $\boldsymbol{d}$ of constants from D corresponding to the parameters of A. We say that a formula A is valid on F, or F-valid, and write $F \models A$, if it is M-true in every F-model M. The set of all F-valid formulas is called the logic of a frame F and is denoted by $\mathbf{L}F$. It is well known that, for every frame F, the set $\mathbf{L}F$ is, indeed, a logic and that, moreover,

$$\bigcap\{\mathbf{L}F \mid F \text{ is a predicate frame with a constant domain}\} \quad = \quad \mathbf{QH} + CD.$$

The following simple lemma plays an important role in our considerations:

[2] These are the usual inductive clauses for truth in an algebraic structure over the Heyting algebra $H(U)$ of upward-closed subsets of a poset $(U, \leqslant)$ with a domain D; see [3, Chapter 4].

Lemma 2.11 (Double negation lemma). Let F be a frame over $(U, \leqslant\ )$ with the domain D, let V be an F-valuation, and A a D-sentence. Then,

(1) $V(\neg\neg A) = \{u \in U \mid V(A) \text{ is cofinal for } u{\uparrow}\}$.

(2) $V(\neg\neg A) = U$ iff $V(A)$ is cofinal in U.

Lemma 2.12. Let F be a frame with the domain D, let $M = (F, V)$ be an F-model, and let A be a formula. Then,

(1) $M \models \neg\neg A$ iff, for all lists $\boldsymbol{d}$ of elements of D, the sets $V(A(\boldsymbol{d}))$ are cofinal in U.

(2) $F \models \neg\neg A$ iff, for all F-valuations V and all lists $\boldsymbol{d}$ of elements of D, the sets $V(A(\boldsymbol{d}))$ are cofinal in U.

2.4 Elementarity

In what follows, $\Vdash$ denotes the classical validity, understood in the usual way.

To describe Kripke frames with constant domains classically, we fix the classical predicate language with two sorts of variables (one for worlds, denoted by $u, v, u_1, \ldots$, and one for individuals, denoted by $x, y, x_1, \ldots$), with equality applicable to pairs of variables of the same sort, and with a binary predicate symbol R restricted to variables for worlds. Every predicate Kripke frame F over $(U, \leqslant)$ with the domain D gives rise to the classical structure $\bar{F}$ with two domains, U and D, and with the binary predicate R on U defined so that

$$\bar{F} \Vdash R(u, v) \quad \text{iff} \quad u \leqslant v.$$

It should be clear that the classical theory T_{po} of the class of all structures of this type contains (and is axiomatized by) the usual axioms for partial orders together with equality axioms for both sorts of variables.[3]

An (intuitionistic predicate) formula A is called elementary (or first-order definable) w.r.t. to a class of frames $\mathscr{F}$ (within a fixed semantics) if there exists a classical theory $T \supseteq T_{po}$ such that, for every $F \in \mathscr{F}$,

$$F \models A \quad \Longleftrightarrow \quad \bar{F} \Vdash T,$$

in which case we say that the classical theory T corresponds to the formula A. If $\mathbf{L}$ a logic, we say that a formula A is elementary w.r.t. $\mathbf{L}$ if A is elementary w.r.t. the class $\{F \in \mathscr{F} \mid F \models \mathbf{L}\}$.

[3]Equality of worlds can be defined by $u = v := R(u, v) \wedge R(v, u)$, and is, therefore, strictly not needed.

3 Kuroda formula and Kuroda logic

3.1 Different axiomatizations of Kuroda logic

We will be using two deductively equivalent versions of the Kuroda formula:

$$\begin{array}{lcll} K & := & \neg\neg\forall x(P(x) \vee \neg P(x)) & (= \neg\neg\forall x EM^1), \\ DNS & := & \forall x \neg\neg P(x) \to \neg\neg\forall x P(x) & \text{(the double negation shift).} \end{array}$$

Since **QH** proves the converse of DNS, the formula DNS is **QH**-equivalent to the formula

$$DNS_{\leftrightarrow} \quad := \quad \forall x \neg\neg P(x) \leftrightarrow \neg\neg\forall x P(x).$$

Observation.

(1) Formulas K and DNS are deductively equivalent.

(2) For every $n \geqslant 1$, the formula DNS is deductively equivalent to

$$DNS_n \quad = \quad \forall x_1 \ldots \forall x_n \neg\neg P^{(n)}(x_1, \ldots, x_n) \to \neg\neg\forall x_1 \ldots \forall x_n P^{(n)}(x_1, \ldots, x_n)$$

(and, hence, to the formula obtained from DNS_n by replacing $\to$ with $\leftrightarrow$).

(3) For every $n \geqslant 1$, the formula K is deductively equivalent to

$$K_n \quad = \quad \neg\neg\forall x_1 \ldots \forall x_n EM^n.$$

We call the logic

$$\mathbf{QHK} \quad := \quad \mathbf{QH} + K$$

the Kuroda logic.

The next proposition describes an extensive family of axiomatizations of **QHK**:

Proposition 3.0. Let C be a propositional formula axiomatizing the classical propositional logic **CL**, i.e., such that $\mathbf{H} + C = \mathbf{CL}$. Then, for every $n \geqslant 1$, the formula

$$K_n^C := \neg\neg\forall x_1 \ldots \forall x_n \, C^n(x_1, \ldots, x_n)$$

is deductively equivalent to K.

3.2 Extremal properties of Kuroda logic

The Kuroda formula K had been discovered as a historically first example of a classically, but not intuitionistically, valid formula of the form $\neg A$, i.e, as a counterexample to the purported predicate counterpart of the celebrated Glivenko theorem. In fact, K is the strongest formula with the said property:

Claim 3.1. Let A be a formula such that $\neg A \in \mathbf{QCL} \setminus \mathbf{QH}$. Then, $K \vdash_{\mathbf{QH}} \neg A$.

This is an immediate consequence of the following:

Claim 3.2. Let A be a formula. Then, $\mathbf{QHK} \vdash \neg A$ if and only if $\mathbf{QCL} \vdash \neg A$.

We do not know whether the set $\mathbf{QCL} \setminus \mathbf{QH}$ (or its subset containing only formulas of the form $\neg A$) has the weakest formula. On the other hand, due to the disjunctive property of $\mathbf{QH}$, this set cannot contain more than one minimal formula.

Claim 3.2 implies that $\mathbf{QHK}$ is the least (weakest) logic with the property we are considering:

Proposition 3.3. Let $\mathbf{L}$ is a predicate logic. Then, the following conditions are equivalent:

(i) $\mathbf{L} \vdash \neg A$ iff $\mathbf{QCL} \vdash \neg A$, for every formula A;

(ii) $\mathbf{QHK} \subseteq \mathbf{L} \subseteq \mathbf{QCL}$.

Proof. $(ii) \Rightarrow (i)$: In view of Claim 3.2, 'if' and 'only if' follow, respectively, from the first and the second inclusions of (ii).

$(i) \Rightarrow (ii)$: Since K is classically valid, it follows, by (i), that $K \in \mathbf{L}$; hence, $\mathbf{QHK} \subseteq \mathbf{L}$. Next, if the second inclusion does not hold, then there exists $A \in \mathbf{L} \setminus \mathbf{QCL}$. Hence, $\neg\neg A \in \mathbf{L} \setminus \mathbf{QCL}$, in contradiction with (i). □

3.3 Connections with the system of negative slices

The negative fragment $\mathbf{L}^{\neg}$ of a logic $\mathbf{L}$ is the set

$$\mathbf{L}^{\neg} := \{\neg A \mid \mathbf{L} \vdash \neg A\}.$$

The double negative fragment $\mathbf{L}^{\neg\neg}$ of a logic $\mathbf{L}$ is the set

$$\mathbf{L}^{\neg\neg} := \{\neg\neg A \mid \mathbf{L} \vdash \neg\neg A\}.$$

The fragments[4] $\mathbf{L}^{\neg}$ and $\mathbf{L}^{\neg\neg}$ determine each other, in the following sense:

$$\mathbf{L}^{\neg\neg} = \{\neg\neg A \in \mathcal{L} \mid \neg\neg A \in \mathbf{L}^{\neg}\} \quad \text{and} \quad \mathbf{L}^{\neg} = \{\neg A \in \mathcal{L} \mid \neg\neg\neg A \in \mathbf{L}^{\neg\neg}\}.$$

We say that logics $\mathbf{L}_1$ and $\mathbf{L}_2$ are negatively equivalent, and write $\mathbf{L}_1 \equiv_{Neg} \mathbf{L}_2$, if their negative (equivalently, double negative) fragments coincide:

$$\mathbf{L}_1 \equiv_{Neg} \mathbf{L}_2 \quad := \quad \mathbf{L}_1^{\neg} = \mathbf{L}_2^{\neg} \quad (\iff \mathbf{L}_1^{\neg\neg} = \mathbf{L}_2^{\neg\neg}).$$

It should be clear that the relation $\equiv_{Neg}$ is an equivalence on the structure $\mathfrak{L}$ of predicate logics. These equivalence classes form a system of slices in $\mathfrak{L}$, as defined in [7, Section 2] and [5, Section 3.2]. We call them negative slices. This system of negative slices was introduced and briefly discussed in [7, Section 3]; the reader can compare it to the system of positive slices introduced and studied in [5].

We denote the negative slice of a logic $\mathbf{L}$ by $[\mathbf{L}]_{Neg}$. It turns out that all negative slices are segments $\{\mathbf{L} : \mathbf{L}_1 \subseteq \mathbf{L} \subseteq \mathbf{L}_2\}$. On the one hand, the least logic of the slice $[\mathbf{L}]_{Neg}$ is $\mathbf{QH} + \mathbf{L}^{\neg}$, since the negative fragments of all logics from a given negative slice coincide. On the other hand, the greatest logic of a slice is the sum of all its logics. In general, we do not know a more satisfactory description of the greatest logic of an arbitrary negative slice.

Coming back to the Kuroda logic **QHK**, we see that Proposition 3.3 means that the negative slice $[\mathbf{QCL}]_{Neg}$ of the classical logic **QCL** is just the segment between **QHK** and **QCL**, i.e., the segment of all logics with the classical negative fragment. In other words, the Kuroda logic is the least logic with the classical negative fragment (the crucial characteristic property of the Kuroda logic). The negative fragments of some superclassical logics, i.e., extensions of **QCL**, were briefly described in [7, Section 2]; a more detailed description will be presented in the series of papers announced in [6].

3.4 Non-elementarity of the Kuroda formula

Proposition 3.4. The formula K is non-elementary w.r.t. the class of all Kripke frames with constant domains; moreover, it is non-elementary, in this semantics, w.r.t. the logic **QLC**, i.e., w.r.t. the class of Kripke frames with a constant domain over a linearly ordered set of possible worlds.

Proof. Suppose that K, and hence DNS, corresponds to a classical theory $T \supseteq T_{po}$. Let Θ_n (here, $n \geqslant 1$) be the theory obtained by adding to T the following formulas:

[4] These fragments were studied in [7], where they were denoted by, respectively, $Neg[\mathbf{L}]$ and $\mathbf{L}^{\neg\neg}$.

- the axiom of linearity $Lin := \forall u \forall v\, (R(u,v) \vee R(v,u))$;
- the axiom of seriality $Ser := \forall u \exists v\, (R(u,v) \wedge (u \neq v))$;
- the axiom postulating that $|D| \geqslant n$, i.e.,

$$E_n \quad := \quad \exists x_1 \ldots \exists x_n \bigwedge_{i \neq j} (x_i \neq x_j).$$

Also, let $\Theta := \bigcup\limits_{n \in \omega} \Theta_n$. First, we prove the following:

(I) For every $n \in \omega$, the theory Θ_n has a model over the set ω of natural numbers.

Let F_n be a Kripke frame over $(\omega, \leqslant)$ with the n-element domain $\{d_1, \ldots, d_n\}$. We show that $F_n \models \mathit{DNS}$, i.e., that $V(\forall x \neg\neg P(x)) \subseteq V(\neg\neg \forall x P(x))$, for every F_n-valuation V. Suppose that $u \in V(\forall x \neg\neg P(x)) (= \bigcap\limits_{i=1}^{n} V(\neg\neg P(d_i)))$. Then, by Lemma 2.11 (1), for every $i \in \{1, \ldots, n\}$, the set $V(P(d_i))$ is cofinal for $u{\uparrow}$. Since $\varnothing$ is not cofinal for any non-empty subset of U, it follows that $V(P(d_i)) \neq \varnothing$, for every $i \in \{1, \ldots, n\}$. Choose $u_i \in V(P(d_i))$ and put $u_0 := \max\{u_1, \ldots, u_n\}$. Then, by upward closure, $u_0 \in \bigcap\limits_{i=1}^{n} V(P(d_i)) (= V(\forall x P(x)))$. Since $V(\forall x P(x))$ is non-empty and ω is linear, it follows, by Lemma 2.5, that $V(\forall x P(x))$ is cofinal in U. Hence, by Lemma 2.11 (2), $V(\neg\neg \forall x P(x)) = U$. Thus, $F_n \models \mathit{DNS}$. Therefore, the classical structure $\bar{F}_n$ corresponding to F_n validates T, and hence Θ_n.

(II) It follows from (I), by Compactness theorem, that the theory Θ has a model; hence, by the Löwenheim–Skolem theorem, it has a countable model, say $\bar{F}$. To obtain a contradiction, we will show that $\bar{F} \not\Vdash T$. To that end, it suffices to prove the following:

(III) The Kripke frame F corresponding to $\bar{F}$ does not validate DNS.

First, note that the set of worlds U of the frame F is a denumerable chain; this chain cannot be finite due to seriality of R. Also note that, since $\bar{F} \Vdash E_n$ whenever $n \in \omega$, the domain D of F is also denumerable. Fix a bijection $\varphi \colon D \to U$ and define an F-valuation V as follows: for every $d \in D$,

$$V(P(d)) \quad := \quad \varphi(d){\uparrow} \setminus \{\varphi(d)\}. \tag{$*$}$$

Due to seriality, for every $d \in D$, the set $V(P(d))$ is non-empty, and so, by Lemma 2.5, cofinal in U. Thus, by Lemma 2.11 (2), $V(\neg\neg P(d)) = U$, for every $d \in D$. Hence,

$$V(\forall x \, \neg\neg P(x)) \quad = \quad \bigcap_{d \in D} V(\neg\neg P(d)) \quad = \quad U.$$

On the other hand, the set $V(\forall x P(x))(= \bigcap_{d\in D} V(P(d)))$ is empty since, for every $u \in U$, by $(*)$, $u \notin V(P(\varphi^{-1}(u)))$.
Hence, $V(\neg\forall x P(x)) = U$, and so $V(\neg\neg\forall x P(x)) = \varnothing$. Thus, $V(DNS) = \varnothing$, and so $F \not\models DNS$. $\square$

We next observe that, in the statement of Proposition 3.4, the logic **QLC** cannot be replaced with logics of finite heights.

Recall that the height of a poset U, denoted by $h[U]$, is the supremum of cardinalities of the chains in U; if F is a predicate Kripke frame over a poset U, then we define $h[F] := h[U]$. Thus, over frames of a finite height, the formula K is valid, and so corresponds to the empty theory (hence, it is, trivially, elementary w.r.t. logics of finite heights).

We also recall the formulas P_h and P_h^+ corresponding to frames of finite height h; these formulas are defined by recursion (here, q_h and Q_h, for each $h \in \omega$, are distinct, respectively, nullary and unary predicate letters):

$$\begin{array}{lllllll} P_0 & := & \bot, & P_{h+1} & := & q_h \vee (q_h \to P_h); \\ P_0^+ & := & \bot, & P_{h+1}^+ & := & \forall x\,(Q_h(x) \vee (Q_h(x) \to P_h^+)). \end{array}$$

It is well known that, for every predicate Kripke frame F with constant or expanding domains,[5] for every $h \in \omega$,

$$F \models P_h \iff F \models P_h^+ \iff h[F] \leqslant h.$$

The formula K is valid on every Kripke frame F with expanding domains over a poset of a finite height. It is known (see [3, Theorem 6.3.8]) that every logic $\mathbf{QH} + P_h^+$ is complete in the semantics with expanding domains, i.e., for every $h \in \omega$,

$$\mathbf{QH} + P_h^+ \quad = \quad \bigcap (\mathbf{L}F : h[F] \leqslant h).$$

However, if $h \geqslant 2$, then the logic $\mathbf{QH} + P_h$ is Kripke incomplete, i.e., it is a proper sublogic of the logic $\mathbf{QH} + P_h^+$; this follows from the fact, proved by H. Ono [4], that $\mathbf{QH}+P_2 \nvdash K$ and so, since $\mathbf{QH}+P_h \subseteq \mathbf{QH}+P_2$ whenever $h \geqslant 2$, also $\mathbf{QH}+P_h \nvdash K$.[6]

Since K is valid on every Kripke frame of a finite height, it is elementary w.r.t. the logic $\mathbf{QH} + P_h^+$, for every $h \in \omega$. Since every frame validating P_h validates P_h^+, an analogous claim holds for weaker logics $\mathbf{QH} + P_h$. It should also be clear that K is elementary w.r.t. the logic $\mathbf{L}_\infty^+ := \bigcap_{h\in\omega} (\mathbf{QH} + P_h^+)$ of all frames of a finite height.

[5]The semantics of expanding domains will be sketched out in Section 3.5.2; more details can be found in [3, Chapter 3].

[6]Note that $\mathbf{QH}+P_0^+ = \mathbf{QH}+P_0 = \mathcal{L}$ (the inconsistent logic) and $\mathbf{QH}+P_1^+ = \mathbf{QH}+P_1 = \mathbf{QCL}$ (the classical predicate logic); these logics are, clearly, Kripke complete.

3.5 On the validity of the Kuroda formula in Kripke semantics

3.5.1 Validity of K in the constant domain Kripke semantics

Definition 3.5. Let $(U, \leqslant)$ be a poset and $\mathcal{X}$ a family of subsets of U. We say that $\mathcal{X}$ is K-critical if there exists $u \in U$ such that

- every member of $\mathcal{X}$ is an upward-closed cofinal subset of $u\!\uparrow$;
- $\bigcap \mathcal{X}$ is not cofinal in $u\!\uparrow$.

If, additionally, $\bigcap \mathcal{X} = \varnothing$, then $\mathcal{X}$ is $K^{\varnothing}$-critical. If K-criticality of $\mathcal{X}$ is witnessed by $u \in U$, we say that $\mathcal{X}$ is K-critical for u. We say that a poset $(U, \leqslant)$ is K-critical if there exists a K-critical family in $(U, \leqslant)$.

Definition 3.6. Let $(U, \leqslant)$ be a poset. The K-cofinality of U, denoted by $cf_K[U]$, is defined as follows:

- if U is K-critical, then $cf_K[U]$ is the minimal cardinality of a K-critical family in $(U, \leqslant)$;
- otherwise, $cf_K[U] := \infty$ (we assume that ∞ is greater than any cardinal).

Lemma 3.7. Let $(U, \leqslant)$ be a poset. Then, $cf_K[U]$ equals the minimum cardinality of a $K^{\varnothing}$-critical family in $(U, \leqslant)$.

Lemma 3.8. A poset is not K-critical if and only if it does not contain $K^{\varnothing}$-critical families.

Theorem 3.9. Let F be a constant domain Kripke frame over a poset $(U, \leqslant)$ with the domain D. Then, F validates K if and only if $|D| < cf_K[U]$.

Proof. Suppose, first, that $F \not\models K$, and so $F \not\models DNS$. Then, there exist $u \in U$ and an F-valuation V such that $u \in V(\forall x \neg\neg P(x)) \setminus V(\neg\neg\forall x P(x))$.

Then, on the one hand, $u \in V(\neg\neg P(d))$, for every $d \in D$, i.e., by Lemma 2.11 (1) and Lemma 2.3(2), for every $d \in D$, the upward-closed set $X_d := V(P(d)) \cap u\!\uparrow$ is cofinal in $u\!\uparrow$. On the other hand, again by Lemma 2.11 (1), the set $V(\forall x P(x))$ $(= \bigcap\limits_{d \in D} V(P(d)))$ is not cofinal for $u\!\uparrow$ in U, and so $\bigcap\limits_{d \in D} X_d$ is not cofinal in $u\!\uparrow$. Hence, the family $\mathcal{X} := \{X_d \mid d \in D\}$ is K-critical (for u) in $(U, \leqslant)$, and so $|D| (= |\mathcal{X}|) \geqslant cf_K[U]$.

Suppose, next, that $|D| \geqslant |\mathcal{X}| \geqslant cf_K[U]$. Then, $(U, \leqslant)$ contains a K-critical family $\mathcal{X}$ (say, for $u \in U$) with $|\mathcal{X}| \leqslant |D|$. Fix a surjection $\varphi : D \to \mathcal{X}$ and define an F-valuation V so that $V(P(d)) = \varphi(d)$, for every $d \in D$. Then, for every $d \in D$, the set $V(P(d))$ is a cofinal subset of $u\uparrow$, and so, by Lemma 2.11 (1), $(\forall d \in D)\, u \in V(\neg\neg P(d))$, i.e., $u \in \bigcap_{d \in D} V(\neg\neg P(d))(= V(\forall x \neg\neg P(x)))$. On the other hand, since $\bigcap \mathcal{X}(= V(\forall x P(x)))$ is not cofinal in $u\uparrow$, if follows, by Lemma 2.11 (1), that $u \notin V(\neg\neg\forall x P(x))$. Hence, $u \in V(\forall x \neg\neg P(x)) \setminus V(\neg\neg\forall x P(x))$, and so $F \not\models DNS$. Hence, $F \not\models K$. □

Corollary 3.10. Let $(U, \leqslant)$ be a poset. Then, the following conditions are equivalent:

(i) Every Kripke frame with constant domains over $(U, \leqslant)$ validates K;

(ii) $cf_K[U] = \infty$ (i.e., if U is not K-critical).

Lemma 3.11. Every poset with the McKinsey property is not K-critical.

Lemma 3.12. The formula K is valid on every Kripke frame with constant domains over posets satisfying the McKinsey property.

In fact, Lemma 3.12 extends to the Kripke semantics with expanding domains—see Proposition 3.22.

Conjecture 3.13. A poset has the McKinsey property if and only if it is not K-critical.

Remark 3.14. Due to Lemma 2.10, $cf_K[U] \geqslant \aleph_0$ for every poset U.

Lemma 3.15. Let U be a linearly ordered poser without the greatest point. Then, $cf_K[U] = cf[U]$.

Remark 3.16. If U is a linear poset with the greatest point, then $cf[U] = 1$, by Lemma 2.5, while, due to Lemma 3.11, $cf_K[U] = \infty$.

Thus, Theorem 3.9 gives us the following:

Corollary 3.17. A Kripke frame F with constant domain over a linearly ordered U without the greatest point validates K if and only if $|D| < cf[U]$.

This claim gives us many linear frames with a constant domain validating K; e.g.,

Example 3.18. Let F be a frame over $U = \aleph_1$ with a denumerable domain D. Then $cf[U] = \aleph_1$. Therefore, F validates K.

Lemma 3.19. The formula K is valid on every Kripke frame with a finite constant domain.

Proof. Let F be a constant domain frame with the domain $\{d_1, \ldots, d_m\}$ and let V be an F-valuation. Then,

$$V(\forall x \neg\neg P(x)) = \bigcap_{d=1}^{n} V(\neg\neg P(d_i)) = V(\neg\neg P(d_1)) \cap \ldots \cap V(\neg\neg P(d_n)),$$

and, likewise,

$$V(\neg\neg\forall x P(x)) = V(\neg\neg(P(d_1)) \cap \ldots \cap P(d_n)).$$

Since $\mathbf{H} \vdash \neg\neg p_1 \wedge \ldots \wedge \neg\neg p_n \leftrightarrow \neg\neg(p_1 \wedge \ldots \wedge p_n)$, it follows that $V(\forall x \neg\neg P(x)) = V(\neg\neg\forall x P(x))$, and so F validates DNS, and hence K. □

3.5.2 Expanding domains Kripke semantics

We assume that the reader is familiar with the expanding domains Kripke semantics (see, e.g., [3, Chapter 3]). However, to fix terminology and notation in a way consistent with our treatment of the constant domain semantics, we briefly recall the main differences between the two semantics.

In the expanding domains semantics, domains of worlds may differ. Hence, a predicate Kripke frame F with expanding domains, in addition to a non-empty poset $(U, \leqslant)$ and a non-empty domain D, has a system $\bar{D} := \{D_u \mid u \in U\}$ of domains satisfying the following conditions: first, for every $u \in U$, the set D_u is non-empty; second, $D_u \subseteq D_v \subseteq D$ whenever $u \leqslant v$. The set D_u is called the individual domain of a world $u \in U$, or a local domain of u. The global domain of a frame F is the set $D[F] := \bigcup\{D_u \mid u \in U\}$. Clearly, $D[F] \subseteq D$. Sometimes, it is assumed that $D[F] = D$; this restriction has no effect on the semantics since it does not affect truth or validity of formulas.

Let D be the global domain of a Kripke frame and let $d \in D$. The measure of existence of d is defined by $E(d) := \{u \in U \mid d \in D_u\}$. The measure of existence of a D-sentence A is defined by $E(A) := E(d_1) \cap \ldots \cap E(d_k)$, where $\{d_1, \ldots, d_k\}$ is the set of constants from D occurring in A; thus,

$$\begin{array}{rcl} E(A) & = & \{u \in U \mid \text{ all constants occurring in } A \text{ belong to } D_u\} \\ & = & \{u \in U \mid A \text{ is a } D_u\text{-sentence}\}. \end{array}$$

If A and B are D-sentences, then

$$\begin{array}{rcll} E(\neg A) & = & E(A); & \\ E(A \circ B) & = & E(A) \cap E(B) & \text{if } \circ \in \{\wedge, \vee, \rightarrow\}; \\ E(A(d)) & = & E(\kappa x A(x)) \cap E(d) & \text{if } \kappa \in \{\forall, \exists\}. \end{array}$$

It should be clear that, if A is a sentence without constants from D, then $E(A) = U$.

If F is a frame with expanding domains, then an F-valuation is a map V sending each D-sentence A to an upward-closed subset of $E(A)$ and satisfying the following conditions:

$$\begin{array}{lcl} V(\bot) & = & \varnothing; \\ V(A \wedge B) & = & V(A) \cap V(B); \\ V(A \vee B) & = & V(A) \cup V(B); \\ V(A \to B) & = & \{u \in E(A \to B) \mid (u\!\uparrow \cap V(A)) \subseteq V(B)\}; \\ V(\forall x A(x)) & = & \{u \in E(\forall x A(x)) \mid u\!\uparrow \subseteq \bigcap\{V(A(d)) \mid d \in D\}\}; \\ V(\exists x A(x)) & = & E(\exists x A(x)) \cap \bigcup\{V(A(d)) \mid d \in D\}. \end{array}$$

We can write $u \in V(A)$ as $u \models A$ to see the usual inductive clauses for the intuitionistic Kripke semantics with expanding domains (see, e.g., [3, Section 3.2]) such as

$$\begin{array}{lcl} u \models A \to B & \iff & (\forall v \geqslant u)\,(v \models A \Rightarrow v \models B); \\ u \models \forall x A(x) & \iff & (\forall v \geqslant u)\,(\forall d \in D_v)\, v \models A(d); \\ u \models \exists x A(x) & \iff & (\exists d \in D_u)\, u \models A(d). \end{array}$$

An F-model is a pair $M = (F, V)$, where F is a frame and V is an F-valuation. A formula A is true in a model M over frame F if $V(\bar{\forall} A) = U$, i.e., if, for all lists $\boldsymbol{d} = (d_1, \ldots, d_k)$ of constants from D corresponding to parameters of A,

$$V(A(\boldsymbol{d})) = E(A(\boldsymbol{d})) = E(d_1) \cap \ldots \cap E(d_k).$$

We say that a formula A is valid on a Kripke frame with expanding domains F (or, simply, F-valid), and write $F \models A$, if A is true in every model over F. The set of all F-valid formulas is called the logic of the frame F and is denoted by $\mathbf{L}F$. It is well known that $\mathbf{L}F$ is, indeed, a predicate logic; moreover,

$$\bigcap(\mathbf{L}F \mid F \text{ is a frame with expanding domains}) \quad = \quad \mathbf{QH}.$$

Lemma 3.20. Let F be a Kripke frame (with expanding domains) over a poset U with the domain D, let V be an F-valuation, and let A be a D-sentence. Then,

(1) $V(\neg\neg A) = \{u \in E(A) \mid V(A) \text{ is cofinal for } u\!\uparrow \text{ (in } U)\}$.

(2) $V(\neg\neg A) = E(A)$ iff $V(A)$ is cofinal in $E(A)$.

Lemma 3.21. A sentence $\neg\neg A$ is valid on a Kripke frame (with expanding domains) F if, for every F-valuation V, the set $V(A)$ is cofinal in $E(A)$.

3.5.3 Criterion of validity of K on Kripke frames with expanding domains

It should be clear that the sentence $\forall x EM^1(x)$ is classically valid; therefore, it is valid at every maximal point of every Kripke frame with expanding domains. Hence, due to Lemma 3.21, we obtain the following well-known result (this result is tacitly assumed by Gabbay in [2, Chapter 3, Section 4]):

Proposition 3.22. The formula K is valid on every Kripke frame with expanding domains satisfying the McKinsey property.

Since every frame of a finite height satisfies the McKinsey property, we obtain the following:

Corollary 3.23. The formula K is valid on every Kripke frame of a finite height, and in particular every Kripke frame over a finite poset.

If $F = (U, \leqslant, \bar{D})$ is a frame with expanding domains and $u \in U$, then the cone of F at u is the frame $F \uparrow u$ obtained by restricting both $\leqslant$ and $\bar{D}$ to $u\uparrow$. Next, we define the set of cones of F that contain only worlds with a finite constant domain:

$$fcd[F] \quad := \quad \{u \in U \mid F \uparrow u \text{ is a frame with a finite constant domain}\}.$$

It should be clear that $fcd[F]$ is an upward-closed subset of U. By Lemma 3.19, for every $u \in fcd[F]$, the frame $F \uparrow u$ validates K.

Definition 3.24. A Kripke frame with expanding domains F over a poset U is an fcd-frame if the set $max[U] \cup fcd[F]$ is cofinal in U.

Due to Lemma 3.21 and Lemma 3.19, we obtain the following:

Proposition 3.25. Every *fcd*-frame validates K.

We now establish a criterion for the validity of K on Kripke frames with expanding domains over countable posets:

Theorem 3.26. Let F is a Kripke frame with expanding domains over a countable poset U. Then, F validates K if and only if it is an *fcd*-frame.

Proof. The 'if' part is given by Proposition 3.25, so it remains to prove the 'only if' part.

Let F be a frame with expanding domains over U that is not an *fcd*-frame; this means that the set $max[U] \cup fcd[F]$ is not cofinal in U. Hence, there exists $u_0 \in U$ such that $u_0\uparrow \cap (max[U] \cup fcd[F]) = \varnothing$. We now replace F with the frame $F \uparrow u_0$ and prove the following lemma about it; this will conclude the proof of the theorem. $\square$

Lemma 3.27. Let F be a frame with the root u_0 over a denumerable poset U such that

$$max[U] \cup fcd[F] = \varnothing.$$

Then, there exists an F-valuation V falsifying K; moreover, $V(\forall x EM^1(x)) = \varnothing$ (and hence $V(\neg\forall x EM^1(x)) = U$ and $V(K) = \varnothing$).

Proof. We enumerate the set of worlds so that $U = \{u_i \mid i \in \omega\}$ (recall that u_0 is the root of U). For every $i \in \omega$ we choose, inductively, a world $u'_i \geqslant u_i$ and a fresh individual $d_i \in D_{u'_i}$ (i.e., $d_i \neq d_j$ whenever $j < i$). If, for some $u'_i \geqslant u_i$, the domain $D_{u'_i}$ is infinite, then we choose this u'_i and a fresh $d_i \in D_{u'_i}$. Second, if, for every $v \geqslant u$, the domain D_v is finite, then we choose an increasing chain $u_i < v_0 < \ldots < v_i$ so that $D_{v_0} \subset D_{v_1} \subset \ldots \subset D_{v_i}$ (since $fcd[F] = \varnothing$). Then, $|D_{v_i}| > i$, and we can put $u'_i := v_i$ and choose d_i in $D_{u'_i} = D_{v_i}$ that is distinct from $d_0, \ldots, d_{i-1}$.

Finally, we put $V(P(d_i) := u'_i{\uparrow} \setminus \{u'_i\}$, for every $i \in \omega$; the values $V(P(d))$ for all other $d \in D$ are immaterial, so we can make all of them empty. Clearly, all $V(P(d_i)$ are non-empty, since $max[U] = \varnothing$.

Now, since $V(P(d_i)$ are non-empty subsets of $u'_i{\uparrow}$, it follows that $u'_i \notin V(P(d_i))$ and $u'_i \notin V(\neg P(d_i))$. Hence, $u'_i \notin V(P(d_i) \vee \neg P(d_i))$, for every $i \in \omega$. Thus, by monotonicity, for every $i \in \omega$,

$$u_i \notin V(P(d_i) \vee \neg P(d_i)) = V(EM^1(d_i)).$$

Thus, $u_i \notin V(\forall x EM^1(x))$, for every $u_i \in U$, which means that $V(\forall x EM^1(x)) = \varnothing$. □

Remark 3.28. Theorem 3.26 does not extend to the uncountable case: for a counterexample, consider the frame from Example 3.18, where both $max[U]$ and $fcd[F]$ are empty (and so non-cofinal).

It appears doubtful whether there exists a nice criterion characterizing the whole class $\mathscr{K}\mathscr{F}$.

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