SIGNAL PROCESSING

PERIODICITY TRANSFORMS

TEAM MEMBERS

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ABSTRACT

We can observe patterns and periodicities in signals, which computers fail to identify. DFT is represented by sinusoidal basis elements and the wavelet transform is represented as a sum of basis elements that are defined by a family of scaling functions, but none of them search for the underlying periodicities directly . The goal is to identify a suitable periodic basis for the signal by projecting onto non-orthogonal periodic subspaces and studying it's strengths and weaknesses. We build a self-adjusting basis, linear in period. It has various applications like rhythmic parsing of a musical score, the separation of waveforms, the finding of a harmonic template, and a search for patterns in astronomical data.

The goal is to identify a suitable periodic basis for the signal by projecting onto non-orthogonal periodic subspaces and studying it's strengths and weaknesses.

IMPORTANT CONCEPTS

Periodicity Transform theory uses basic concepts of Linear Algebra like Basis vectors, Vector Spaces, Subspaces, Orthogonality etc.

SUBSPACES

For a Vector space V over a field F , a non-empty set W is a subspace of V if it satisfies Closed Vector Addition and Scalar Multiplication over field F .

BASIS VECTORS

Basis set B of a Vector space V is a set of linearly independent vectors of V, which spans the entire vector space V .

ORTHOGONALITY

If the inner product of two vectors is 0 , then the vectors are said to be orthogonal vectors.

INNER PRODUCT

Let x , y, z be vectors of a vector space, the inner product of x and y is denoted as $\langle x, y \rangle$. Inner product must satisfy following properties.

Properties :

1. Commutativity : $\langle x, y\rangle = \langle y, x\rangle$
2. Additivity : $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
3. Scalar Multiplication : $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$
4. Positivity : $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ iff $x = 0$.

PROPOSED METHOD

Fourier series and wavelet series decompose the signals in terms of frequency or scale. The ***Periodicity Transform (PT)*** decomposes a sequence into periodic sequences by projecting it onto periodic subspaces, leaving residuals whose periodicities have been removed. PT doesn't need frequency or scale, it decomposes solely using periodicities. Thus PT is Linear -in -period.

Periodicity transform finds its own set of basis elements and it's subspaces are not orthogonal.

PERIODIC SUBSPACES

Definitions :

- P_p is set of all $p-$ periodic sequences
- P is set of all periodic sequences.

Consider an N-length real valued sequence x . It is clear that $x_N \in P_N \subset P$. PT decomposes x by finding smaller periodicities with in x , and projects x_N onto subspace P_p (where $p < N$).

$x_p \in P_p$ is closer to the original sequence x_N in P_p which is used while decomposing x .

> *P and P_p are both linear vector spaces and $P = \cup\, P_p$*

Basis Elements :

For every period p and time-shift s , $\delta_p^s(j)$ is a sequence for all integers (j) given by,

$$\delta_p^s(j) = \begin{cases} 1 & \text{if (j-s)mod p} = 0 \\ 0 & \text{otherwise} \end{cases}$$

For $s = 0, 1, \ldots p-1$, δ_p^s forms basis vectors for P_p . δ_p^s is periodic with p.

Inner Product :

Inner product of two vectors $x \in P_{p_1}$ and $y \in P_{p_2}$ is defined as,

$$\langle x, y\rangle = \frac{1}{p_1 p_2}\sum_{i=0}^{p_1 p_2 - 1} x(i)y(i)$$

(The sequence $x(i)y(i) \in P_{p_1p_2}$ and inner product is average over a single period p_1p_2)

Also , $||x|| = \langle x, x \rangle$.

Orthogonal signals :

Two signals x and y in P are said to be orthogonal if $\langle x, y \rangle = 0$. Similarly, signal x is orthogonal to subspace P_p if $\langle x, x_p \rangle = 0 \ \forall x_p \in P_p$.

> *The periodic subspaces (P_p) are not orthogonal to each other.*

Theorem 1

$P_{np} \cap P_{mp} = P_p$ when n and m are mutually prime.

PROJECTION ONTO PERIODIC SUBSPACES

Let $x \in P$ and x_p^* be a p - periodic minimizing vector in P_p (closes vector to x) . Then x_p satisfies,

$||x - x_p^*|| \leq ||x - x_p||$, $\forall x_p \in P_p$.

The Projection Theorem

For $x_p^* \in P_p$, to be the minimizing vector , the error $x - x_p^*$ should be orthogonal to P_p .

(necessary and sufficient condition).

As P_p is finite dimensional subspace , x_p^* exists and it can be expressed as the linear combination of basis vectors of P_p .

$x_p^* = \alpha_0 \delta_p^0 + \alpha_1 \delta_p^1 + \ldots + \alpha_{p-1} \delta_p^{p-1}$

According to the projection theorem , $x - x_p^*$ is orthogonal to all δ_p^s . ie , $\langle x - x_p^*, \delta_p^s \rangle = 0$. By solving it , we get $\alpha_s = p \langle x, \delta_p^s \rangle$.

Therefore , $\alpha_s = \frac{1}{N} \sum_{n=0}^{N-1} x(s + np)$. α_s can be written in two ways :

1. N/p is an integer : $\alpha_s = \frac{1}{N/p} \sum_{n=0}^{N/p-1} x(s + np)$
2. N/p is not an integer : $\alpha_s = \frac{1}{\lfloor N/p \rfloor} \sum_{n=0}^{\lfloor N/p \rfloor - 1} x_{\overline{N}}(s + np)$

where , $x_{\overline{N}}$ is a $\overline{N}$ periodic sequence constructed from the first $\overline{N} = p \lfloor N/p \rfloor$ elements of x .

Representation : $\pi(x, P_p)$ represents the projection of vector x on P_p .

When x is projected onto P_{np} , all np - periodic components in residual , $r = x - P_{np}$ are removed. (which also removes p - periodic elements).

Theorem 2

For any n , Let r be the residual after projecting x onto P_{np} , then $r = x \; - \pi(x, P_{np})$ and $\pi(r, P_{np}) \; = \; 0$.

Theorem 3

Let $r_p = x \; - \pi(x, P_p)$ be the residual after projecting x onto P_p and let
$r_{np} = x \; - \pi(x, P_{np})$

be the residual after projecting x onto P_{np} , then, $r_{np} = r_p - \; \pi(r_p, P_{np})$

NonUniqueness

The periodic subspaces of PT are non-orthogonal unlike Fourier Transforms and others. The order of projection doesn't matter in other transforms but it matters in PT . As the projection of x onto one subspace depends on others, it is not independent, the order of projections affects the decomposition and does not give a unique representation for x. The PT can provide several different decompositions into periodic basis elements, depending on how the order of projections.

> *The orthogonality of basis elements imply independence of projections onto subspaces.*

Algorithms For Periodic Decompositions

The goal is to find the closest p - periodic vector to x , given by $x_p \; = \; \pi(x, P_p)$. The residual vector given by $r_p \; = \; x \; - \; x_p$ is stripped of all p - periodicities . But both x_p and r_p may contain other periodicities .They may be decomposed into other q - periodic components by projection onto P_q.

1. Small To Large Algorithm

The ' Small To Large Algorithm ' makes use of theorem (1). As projecting x onto P_{np} gives us no new information as $x_p \; \epsilon \; P_{np}$ .The algorithm checks for periodicities only until $p \; = \; \frac{N}{2}$. It uses a **threshold** value to decide the basis elements . Each chosen basis element removes at least a factor of the power from the signal. (0.01 < T < 0.1) is considered an ideal value for T.

Advantages

- It is simple and calculates periodicities only until $p \; = \; \frac{N}{2}$. As it calculates basis elements x_p from $p \; = \; 2$ to $p \; = \; \frac{N}{2}$, there is no need to further decompose and check for q - periodicities ($q \; < \; p$).
- It favors small periodicities and gives a compact representation.

Disadvantages

- The value of $\mathbf{T}$ must be selected carefully. A "very small" threshold could just output the first linearly independent set from among the p-periodic basis vectors. A "very large" threshold will give very few basis elements.

2. M-Best Algorithm

The 'M-Best Algorithm' maintains a list of the M best periodicities and the corresponding basis elements. There are two steps involved. The first step is to prepare the list of M-best periodicities and their basis elements. The M-best periodicities are chosen such that it removes maximum energy from the sequence. The periodicities will generally be large. In the second step these M-Best basis are further decomposed into their constituent periodic elements to see if these smaller (sub)periodicities removes more energy from the signal than another currently on the list. If so, then the new one replaces the old. We only check for the factors of these M-Best periodicities.

Advantages

- It is sensible and does not need a Threshold value unlike the 'Small To Large' algorithm.

Disadvantages

- The M - Best algorithm is sometimes fooled into returning multiples of the basic periodicities.

3. Best Correlation

In the ' Best Correlation ' algorithm searches for p with the largest correlation , by projecting the signal x onto all the periodic basis elements δ_p^s for all p and s . A good p will tend to have good correlation with at least one of the p -periodic basis vectors. It measures the correlation between x and the individual periodic basis elements.

Advantages

- This method tends to pick out periodicities with large regular spikes over those that are more uniform.

Disadvantages

- Too many computations , as it calculates the correlation for every value of p.

4. Best Frequency

We can determine the best periodicity by Fourier methods and then project p onto P_p .

Advantages

- It is well understood and Fourier methods have good resolution at high frequencies (small periodicities) while the PT has better resolution at large periodicities (low frequencies).

Disadvantages

- It is not very suitable as all the periodicities are not guaranteed.

APPLICATIONS

Periodicity transforms is used in many fields. Some of the major applications are grouping of rhythm motifs in a musical score, signal separation and identifying periodicities in astronomical data.

Musical Score

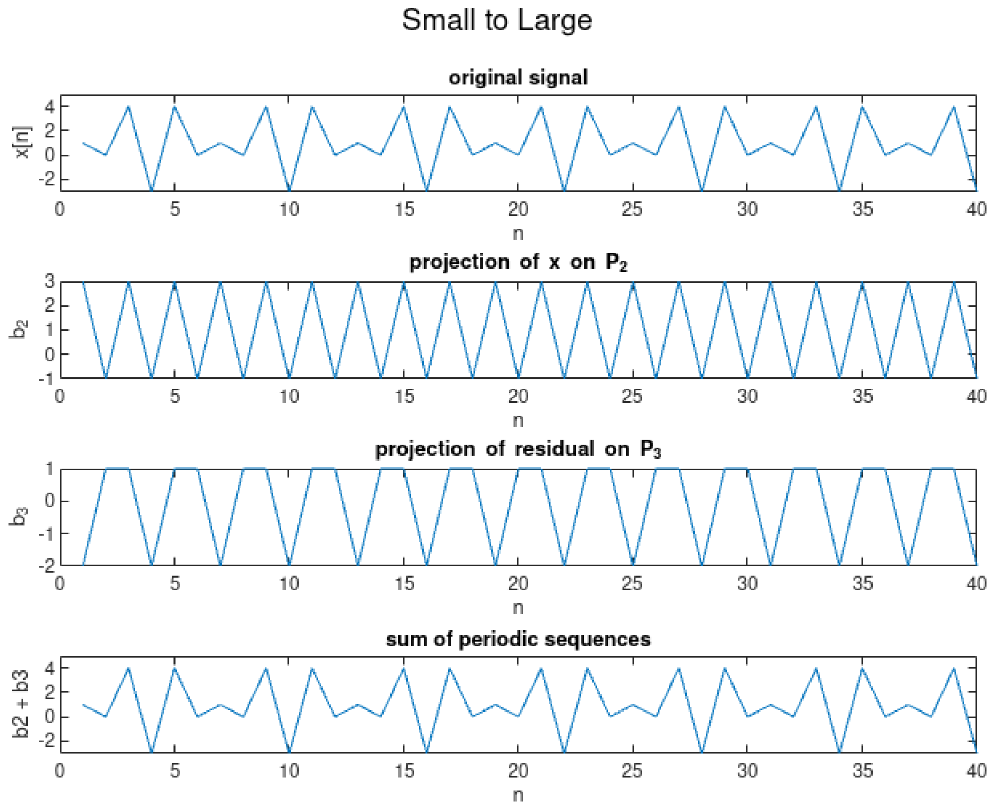
The grouping of musical rhythms is majorly related to periodicities. There are many musical elements like Quarter notes (pulses) , phrases etc. All these groupings and clusters are hard to be recognised by computers. Many scientists worked on this and many theories were proposed. Rosenthal's rhythm parsing program, "Fa" searches for regularly placed onset times in Musical Instrument Digital Interface (MIDI). The song "La Marseillaise" was coded into a binary periodic sequence and was decomposed by various periodic algorithms. Small to Large algorithm with threshold 0.1 , decomposed into four periodicities, 4 (quarter notes) , 16 (bar lines), 32(measured phrases), 64 (others) which removed 28% , 13% , 20% and 39% of powers. This is well agreed with Rosenthal's program. Best Correlation algorithm (with ratio 0.1) found the three periodicities
4, 16, and 32. M- Best algorithm detected 8, 16, 32, and 64, whereas the M-Best γ algorithm returned 2, 4, 16, and 32. Best Frequency Algorithm returned 4,16,5. In the magnitude spectrum analysis of Fourier Transform, with sampling rate 8 Hz and 0.5 s duration of quarter note, the largest peak at 2 Hz represents quarter note and peak at 0.5 Hz corresponds to measure bar. It is still unclear on how to interpret the data.

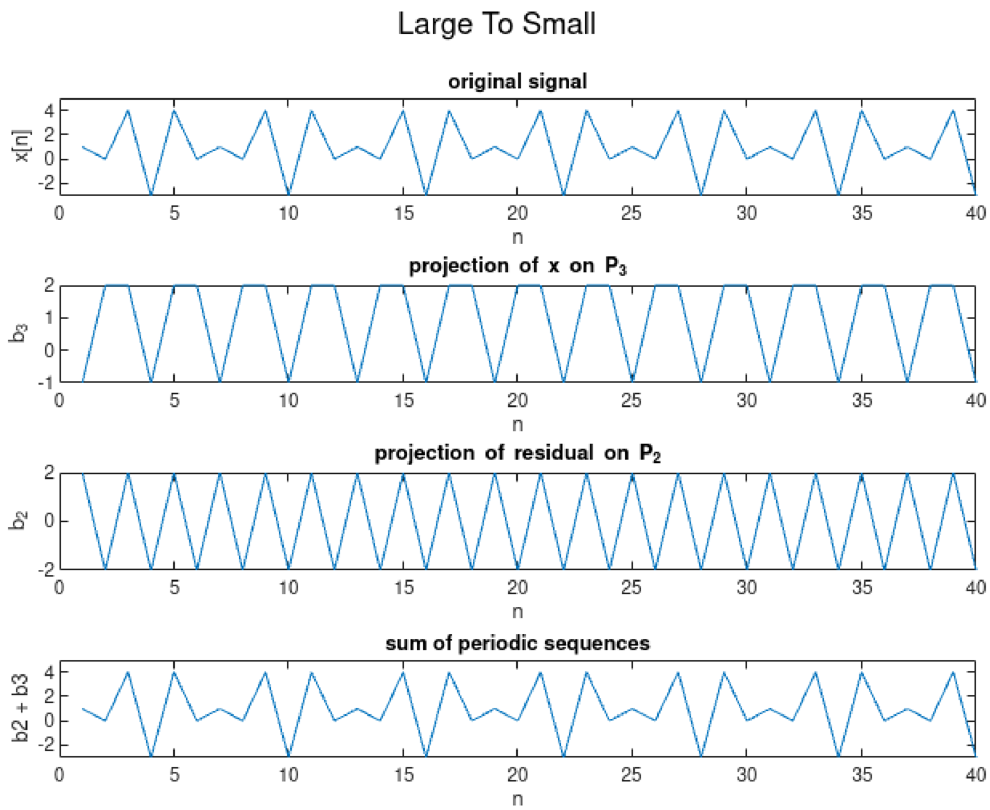
Signal Separation

If the underlying signals are periodic in nature, then the PT can be used to recover signals from
their sum. Consider sequences x and y which are 13 ad 19 periodic sequences respectively. By adding x and y to give z = x + y , we end up getting a complex spectrum for z. When we apply PT on z to give periodic components, it detects periods 13 and 19. Suppose it gives signals x_{13} and y_{19} , then , $x_{13} = x + c_1$ and $y_{19} = y + c_2$ ie, both signals are recovered up to a constant. Even when there is noise in z , PT still locates periodicities and recovers the signals x and y , although they are noisy. Generally, Fourier transforms and others work well but in some cases it splits into many frequencies making it difficult to analyse every component. Where as PT works accurately and finds periodicites in the signal.

SIMULATIONS AND RESULTS

1. ***Non-Uniqueness of Projections***



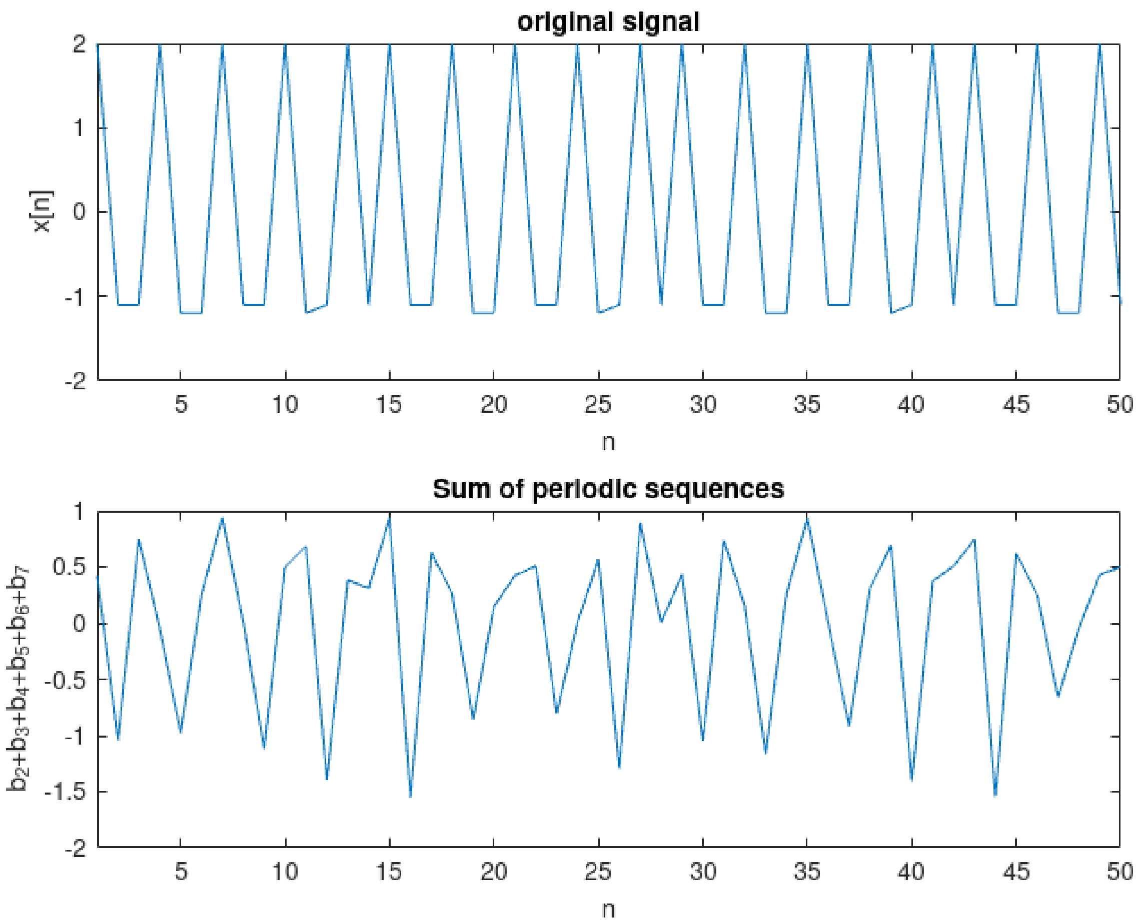


The basis elements of PT are non- orthogonal, hence the order of periodicity affects the decomposition. In Small to Large algorithm projections start from $P_2, P_3, \ldots P_{N/2}$. Where as in Large to small , it occurs in reverse order. It is based on the fact that, np periodic sequence contains p - periodic elements also. When projection occurs on np periodic subspace, p-periodic components are also stripped away along with np - periodic elements. So , the order of projection matters and it results in non-uniqueness of decomposed signals.

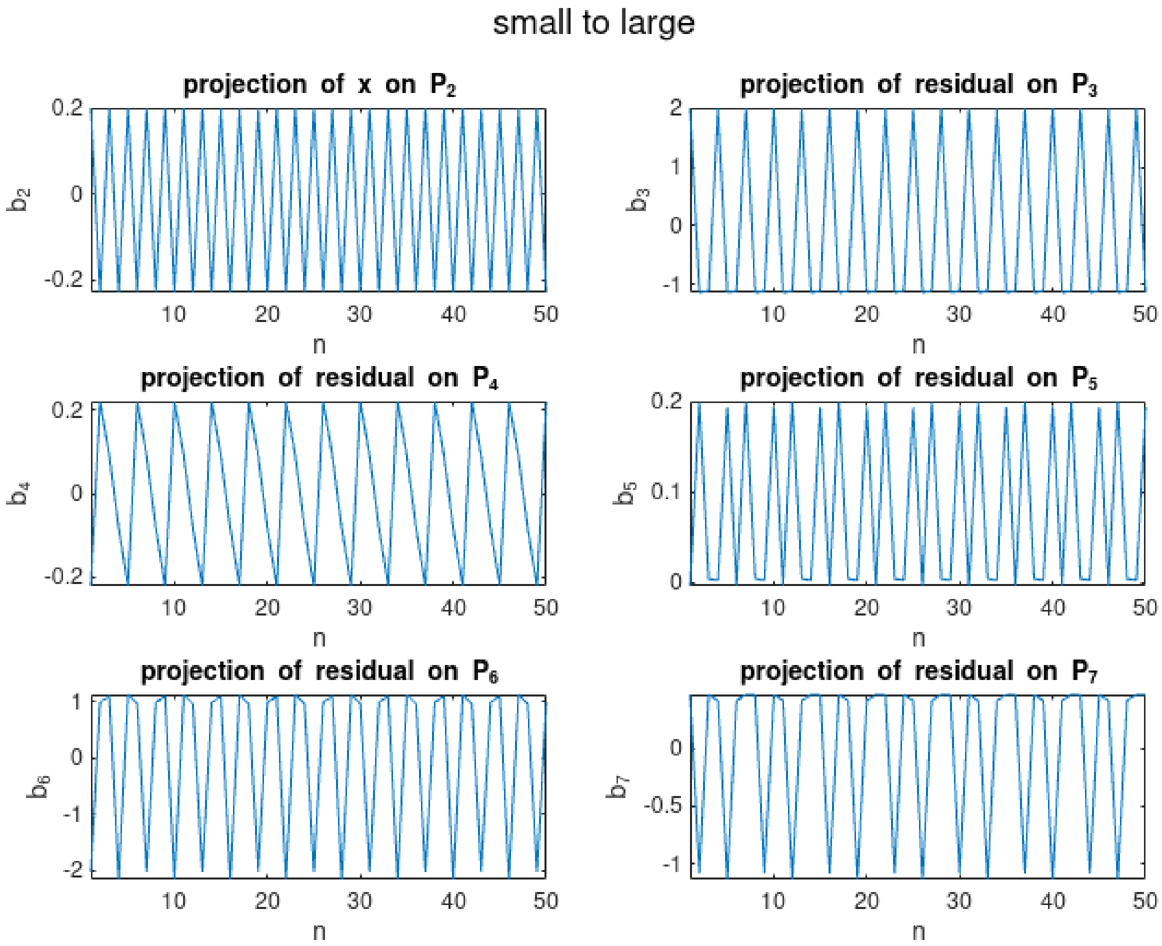
2. ***Comparison of Algorithms***

For the sequence x = [2,-1.1,-1.1,2,-1.2,-1.2,2,-1.1,-1.1,2,-1.2,-1.1,2,-1.1] , Small to Large Algorithm detected 2,3,4,5,6,7 as periods and Best Correlation Algorithm detected 7,14 as periods. Best Correlation Algorithm gave exact decomposition but Small to Large Algorithm failed to do so. (Exact decomposition means the detected periodic sequences sum up to give the original signal).

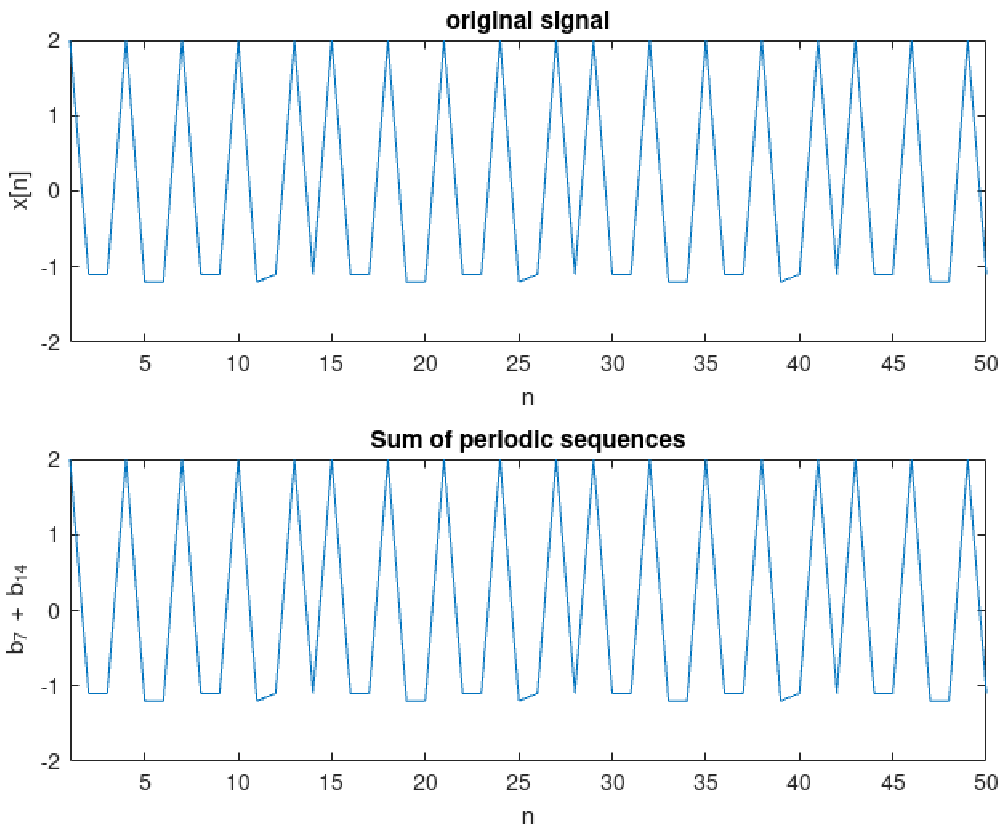
Small To Large Algorithm :



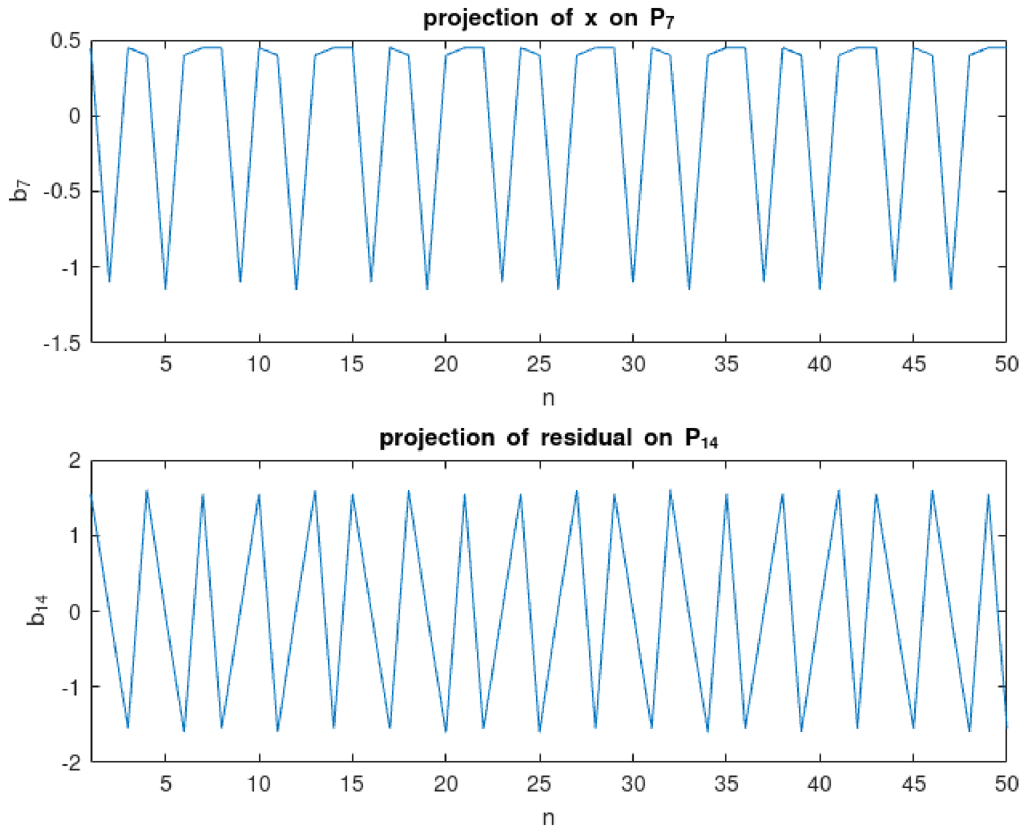
Detected Periodic components :



Best Correlation Algorithm :



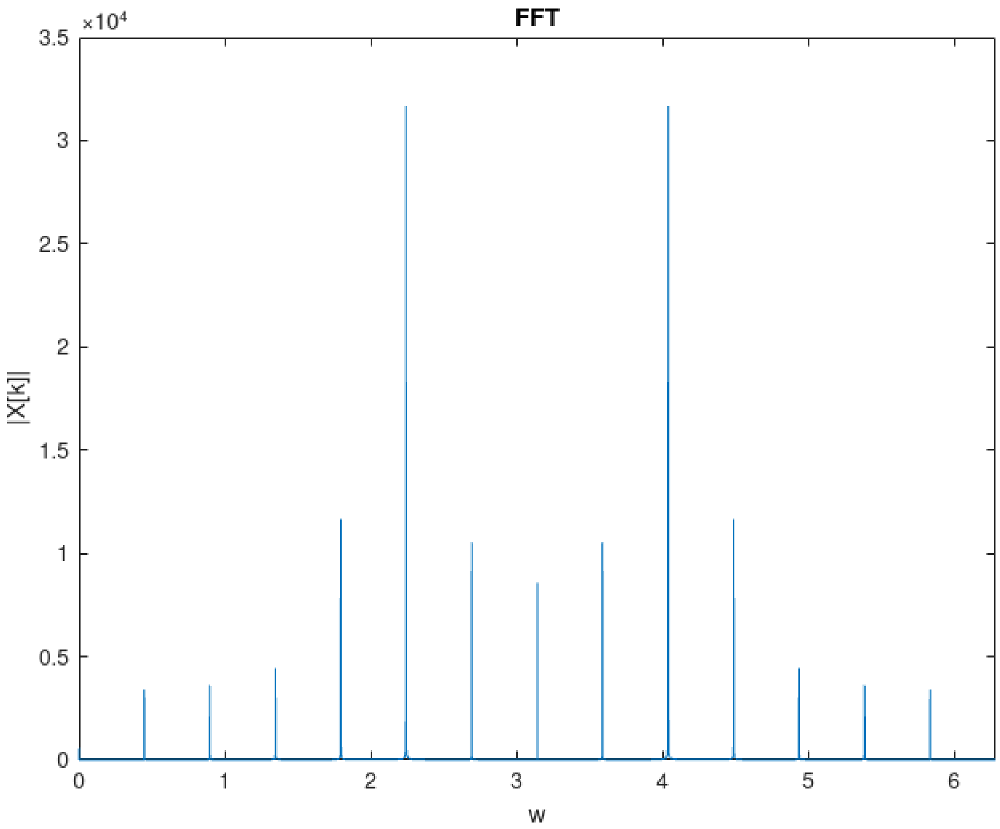
Detected Periodic Components :



- The Best Correlation algorithm works best when the periodic data is spiky.
- M -Best algorithm is sometimes gives multiples of periodicities instead of the actual periods .
- M -Best$_\gamma$ is overall the most reliable and noise resistant algorithm.
- The Best Frequency algorithm doesn't work well when the frequency with the largest magnitude does not closely correspond to an integer periodicity.

3. ***DFT vs PT***

   When we apply fourier transform for the above sequence (in 2) , we get the plot as follows. We notice that many frequencies contribute to the signal and it is difficult to analyse it. Where as with PT (Best Correlation), only 2 components are found which makes it easier to understand the signal nature and properties.



For signals which involve many periodicities PT outperforms DFT but for signals with clearer frequency relationships, FFT gives better results.

CONCLUSION

The Periodicity transform decomposes the signals into periodic sequences by projecting them onto the non-orthogonal periodic subspaces.

The periodicity transforms (PT) finds periodicities with in the data and creates its own basis elements by linearly combining p -periodic basis vectors ,

$$\delta_p^s(j) = \begin{cases} 1 & \text{if (j-s)mod p} = 0 \\ 0 & \text{otherwise} \end{cases}$$

where $s = 0, 1, \ldots p - 1$ and j ϵ integers , δ_p^s is p-periodic basis vector.

The inner product used here is defined by

$$< x, y > \; = \frac{1}{P_1 P_2} \sum_{i\,=\,0}^{P_1 P_2\,-\,1}$$

To construct a periodic vector $x_p^* \;\epsilon\; P_p$ which is arbitrarily close to the original signal x , it must follow the **Projection Theorem* , that is , $x \;-\; x_p^*$ should be orthogonal to P_p

$\pi(x, P_p)$ represents the projection of x onto P_p and is equal to
$\pi(x, P_p) = \; \sum_{s=0}^{p-1} \alpha_s \delta_p^s$

Periodicity transforms have non orthogonal subspaces and hence the order in which projections occurs affects the decomposition, and the PT does not in general provide a unique representation.

We looked at four different algorithms.

The PT searches for the best periodic characterization of the N length signal x projecting it onto some periodic subspace giving the closest p-periodic vector to x given by $x_p = \pi(x, P_p)$. This periodicity is then removed from x, leaving the residual r_p stripped of its -periodicities.

The four algorithms studied above are.

- Small To Large Algorithm
- Best Correlation Algorithm
- M - Best Algorithm
- Best Frequency Algorithm

There are some drawbacks in every algorithm and they work efficiently for different parameters and conditions.

PT can be used to separate signals and analysis of musical signals and astronomical data.

- Some of the drawbacks are
  - It is hard to reconstruct the signals with less probability of error using periodicity transform if they do not contain any underlying periodic signals.
  - DFT will provide a clearer result than PT in case of frequency incorporated signals.
  - The calculations required to project onto P_p overlap the calculations required to project onto P_{np} in a nontrivial way, and can be exploited in more efficient implementation.

REFERENCES

https://ieeexplore.ieee.org/abstract/document/796431