Topics in Algebra, Chapter 1 Solutions

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1.1 - Set Theory

(1a) If $A \subset B$ and $B \subset C$, then every element $a \in A$ is also $a \in B$. Likewise, $B \subset C \Leftrightarrow (a \in B \Rightarrow a \in C)$. Thus, every element $a \in A$ is also in C. Therefore, $A \subset C$.

(1b) If we presuppose that $B \subset A$, this means that $x \in B \Rightarrow x \in A$. In set-builder notation, we know that

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

Because every element of B is also in A, the set builder is reduced to

$$A \cup B = \{x \mid x \in A\} = A$$

Conversely, if we start by supposing that $A \cup B = A$, this implies, by logical necessity, that

$$(x \in A) \Leftrightarrow (x \in A \vee x \in B)$$

As alluded to above, this is logically transformable into the statement that $x \in B \Rightarrow x \in A$, and thus $B \subset A$.

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(1c) In the set $B \cup C$, every element satisfies $x \in B \vee x \in C$. Likewise, in $A \cup C$, every element satisfies $x \in A \vee x \in C$. In the case that $x \in C$, the right side of the disjunction is satisfied in both sets. If $x \in B$, then $B \subset A$ implies that $x \in A$, and so both sets are satisfied again. This means that every element in $B \cup C$ is in $A \cup C$, and so $B \cup C \subset A \cup C$.

Similarly, every element of $B \cap C$ satisfies $x \in B \wedge x \in C$ and every element of $A \cap C$ satisfies $x \in A \wedge x \in C$. Because, again, due to $B \subset A$, we have that $x \in B \Rightarrow x \in A$ and thus every element in $B \cap C$ must also be in $A \cap C$ and so $B \cap C \subset A \cap C$.

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(2a) The commutativity of set intersection ($\cap$) and set union ($\cup$) follow from the analogous commutativity properties of logical operators $\wedge$ and $\vee$, respectively. Because $A \cap B$ consists of the elements which satisfy

$$x \in A \wedge x \in B \quad \equiv \quad x \in B \wedge x \in A$$

and thus it follows that $A \cap B = B \cap A$. The same argument follows for $A \cup B = B \cup A$.

(2b) As with the last problem, the associativity of $\cap$ follows from the associativity of logical $\wedge$. That is, the set $(A \cap B) \cap C$ consists of elements which satisfy,

$$(x \in A \wedge x \in B) \wedge x \in C \quad \equiv \quad x \in A \wedge (x \in B \wedge x \in C)$$

And thus, exactly those same elements satisfy the necessary condition for being elements of $A \cap (B \cap C)$, giving the equality $(A \cap B) \cap C = A \cap (B \cap C)$.

(3) Logically speaking, the elements of $A \cup (B \cap C)$ are those which satisfy $x \in A \vee (x \in B \wedge x \in C)$. By the distributivity of $\vee$, it follows that this logical condition is equivalent to the condition,

$$x \in A \vee (x \in B \wedge x \in C) \quad \equiv \quad (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$$

And thus, the elements in the former set must be exactly those elements in the latter set, giving the equality $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(4a) The elements of $(A \cap B)'$ are those which satisfy $\neg(x \in A \wedge x \in B)$. Applying De Morgan's logical negation rules, we get that this is logically equivalent to $x \notin A \vee x \notin B$. The set whose elements all satisfy this condition is identically $A' \cup B'$.

(4b) As in the last problem, we translate the set $(A \cup B)'$ into the membership condition $\neg(x \in A \vee x \in B)$. De Morgan transforms this into the membership condition $x \notin A \wedge x \notin B$. This membership condition corresponds to the identical set, $A' \cap B'$.

(5) There are two cases: (i) A is disjoint from B ($A \cap B = \emptyset$), or (ii) A is not disjoint from B ($A \cap B \neq \emptyset$).

In case (i), we trivially have that $o(A \cap B) = o(\emptyset) = 0$. So, it is trivially the case that every element of A is represented in $A \cup B$ and likewise for B, and all of these elements are distinct. Therefore, $o(A \cup B) = o(A) + o(B)$, which works for our hypothesis given that $o(A \cap B) = 0$.

In case (ii), let's suppose that there are k elements in common between A and B, such that $o(A \cap B) = k$. Because $A \cup B$ will only contain one copy of each of the common elements, we must subtract $o(A \cap B)$ from $o(A) + o(B)$ to get the number of unique elements in $A \cup B$.

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(6) First, we construct on A a bijective index $i : A \to [n]$. Every subset $S \subset A$ is uniquely identified by an n-tuple of binary variables $(b_1, \ldots, b_n) \in B$, with $b_i \in \{0, 1\}$. It's trivial to see that the set of all binary n-tuples B has 2^n elements. From this set, we have a bijection which generates (and indexes) the set of all subsets of A (hereafter called the *power set* of A, $\mathcal{P}(A)$). That is, we define $\sigma : B \to \mathcal{P}(A)$ via the map

$$\sigma : b \mapsto \bigcup_{b_j = 1} i^{-1}(j)$$

In other words, if a particular subset had $b_j = 1$, this means the element $a \in A$ with $i(a) = j$ is included in the subset. Because there exists a bijective map between the finite sets B and $\mathcal{P}(A)$, we have that they are the same size. Therefore, $\mathcal{P}(A)$ has 2^n elements.

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(7) Let S be the set of Americans. Let C be the set of Americans that like cheese, and A be the set of Americans that like apples. The proportions given suggest that $|C|/|S| = 0.63$ and that $|A|/|S| = 0.76$. Because both $C \subset S$ and $A \subset S$, we expect $C \cup A \subset S$. This necessitates that

$$|C \cup A| \leq |S|$$
$$|C| + |A| - |C \cap A| \leq |S|$$

Rearranging terms and dividing through by $|S|$, we get a bound on the proportion of Americans which like both cheese and apples:

$$|C \cap A| \geq \frac{|C|}{|S|} + \frac{|A|}{|S|} - 1.0$$
$$|C \cap A| \geq 0.63 + 0.76 - 1.0$$
$$|C \cap A| \geq 0.39$$

In English, no fewer than 39% of Americans like both cheese and apples.

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(8) Recall that set difference $A - B$ entails the membership condition $x \in A \land x \notin B$. This means that the *symmetric difference* $A * B$ entails the membership condition,

$$(x \in A \land x \notin B) \lor (x \in B \land x \notin A)$$

By double-distributing the disjunction, we get to a CNF representation, and then set-difference:

$$\begin{gathered}(x \in A \lor x \in B) \land (x \in A \lor x \notin A) \land \\ (x \notin B \lor x \in B) \land (x \notin B \lor x \notin A) \\ \Downarrow \\ (x \in A \lor x \in B) \land (x \notin B \lor x \notin A) \\ \Downarrow \\ (x \in A \lor x \in B) \land \neg(x \in A \land x \in B)\end{gathered}$$

This shows logical equivalence in membership between $A * B$ and $(A \cup B) - (A \cap B)$.

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(9a) For brevity, we argue non-symbolically. Every element of $(A+B)$ is either unique to A or unique to B. Taking the symmetric difference, then, $(A+B)+C$ must result in those elements which are either unique to A, unique to B, unique to C, or common to all three of them.

Likewise, the elements of $(B+C)$ are those elements unique to B and unique to C. Taking the symmetric difference, we see that $A+(B+C)$ is the set of elements which are unique to A, unique to B, unique to C, or common to all three. Therefore, the sets are equal and so $(A+B)+C = A+(B+C)$.

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(9b) Reducing the symmetric differences and distributing the leftmost conjunction,

$$A \cap ((B - C) \cup (C - B))$$
$$(A \cap (B - C)) \cup (A \cap (C - B))$$

We then use the fact that intersection distributes over set difference:

$$((A \cap B) - (A \cap C)) \cup ((A \cap C) - (A \cap B))$$
$$\Downarrow$$
$$A \cdot B + A \cdot C$$

(9c) It's trivially the case that $A \cdot A = A \cap A = A$.

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(9d) Every element of A is also (identically) an element of A, so there are no elements unique to either set. Therefore, $A + A = \emptyset$

(9e) Taking the left-symmetric-difference from A of both sides of the equation and applying the associativity from part (a) with the cancellation property of part (d), we get:

$$\begin{aligned} A + (A + B) &= A + (A + C) \\ (A + A) + B &= (A + A) + C \\ \emptyset + B &= \emptyset + C \end{aligned}$$

Trivially, the symmetric difference of any set with the empty set is simply the set itself, which gives us $B = C$.

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(10a) The relation of having a common ancestor is reflexive, symmetric and transitive. Therefore, it is a valid equivalence relation.

(10b) The relation of living within 100 miles of each other is reflexive and symmetric, but not transitive. Therefore, it is not a valid equivalence relation.

(10c) The relation of having the same father is reflexive, symmetric and transitive. Therefore, it is a valid equivalence relation.

(10d) The relation of having the same absolute value is reflexive, symmetric and transitive. Therefore, it is a valid equivalence relation.

(10e) The relation of being strictly greater and strictly lesser than one another is impossible to satisfy. Therefore, it is not a valid equivalence relation.

(10f) The relation of two lines having the same slope in the plane is reflexive, symmetric and transitive. Therefore, it is a valid equivalence relation.

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(11a) Using only symmetry and transitivity, we do not have a guarantee that there exists a b for a such that $a \sim b$. Thus, this argument does not account for cases that each equivalence class $[a]$ are each only individual elements.

(11b) If we include a property known as *seriality*, which necessitates that every a has a b such that $a \sim b$. Under this assumption, both symmetry and transitivity imply reflexivity.

(12) Clearly, $a \sim a$ because 0 is a multiple of n. Likewise, if $a \sim b$, then it means $a - b = pn$ and so $b - a = -pn$, which is also a multiple of n, implying $b \sim a$. Finally, $a \sim b$ and $b \sim c$ mean that $a - b = pn$ and $b - c = qn$. Adding the equations together gives us $a - c = (p + q)n$, so $a \sim c$. Therefore, differing by a multiple of n is a valid equivalence relation. Each of the equivalence classes are defined to be $\mathrm{cl}(i) = \{x \in \mathbb{Z} \mid x \equiv i \pmod n\}$. Because every integer must have a remainder in $[n]$ after division by n, we have that $\mathrm{cl} : [n] \to \mathbb{Z}$ is a surjection, and so there are at most n equivalence classes. Because each equivalence class $\mathrm{cl}(i)$ trivially contains i for $0 \leq i < n$, we have that there are at least n equivalence classes. Therefore, there are exactly n equivalence classes.

(13) It's clear that being in the same mutually disjoint subset A_α is a valid equivalence relation. This follows from the demonstrable reflexivity, symmetry and transitivity. Moreover, the equivalence classes must be the distinct A_α's because that is how we defined our equivalence.