DIFFERENTIAL GEOMETRY OF IMPLICIT SURFACES

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This technical report presents a brief introduction to the differential geometry of implicit surfaces. We follow the definitions of the important works [2, 3, 4, 8, 9].

1. Tubular Neighborhood

This section discusses the bridge of regular surfaces in $\mathbb{R}^3$ and implicit surfaces. We first recall one direction of the bridge. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function having zero as a *regular value*, i.e. $\nabla f \neq 0$ in $f^{-1}(0)$. The *inverse function theorem* implies that the zero-level set $S = f^{-1}(0)$ is a regular surface. In particular, S is *oriented* since it admits a smooth field of *normal vectors* $N = \frac{\nabla f}{|\nabla f|}$.

For the other bridge direction, consider S being a compact oriented surface in $\mathbb{R}^3$, then there is an implicit function $f : \mathbb{R}^3 \to \mathbb{R}$ having S as its zero-level set. To find a function f satisfying these properties, we use the *tubular neighborhood* of S. The construction of this set depends on the existence of a continuous normal field N on S, which exists since S is oriented. Then, define a normal line $\alpha(t) = p + tN(p)$ passing through each point $p \in S$ towards its normal direction $N(p)$. Let I_p be an open interval, with length 2ϵ, in the neighborhood of p along the normal line, i.e. $I_p = \alpha(-\epsilon, \epsilon)$. The union of these intervals $\bigcup_{p\in S} I_p$ is a *tubular neighborhood* of S iff for each pair $p \neq q \in S$, we have $I_p \cap I_q = \emptyset$. The existence of ϵ can be proved, again, using the inverse function theorem (see Prop. 1 in [3, Section 2.7]).

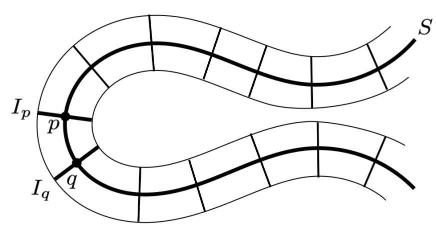

Figure 1. Slice of a tubular neighborhood. Illustration inspired by [3].

With the tubular neighborhood V of S in hands, we can define the function f restricted to V. By its construction, each point $q \in V$ belongs to a unique interval I_p passing through a point $p \in S$ towards $N(p)$. Thus, define $f(q) = t$, where t is the parameter satisfying $q = p + tN(p)$. It can be proved that f is smooth, has zero as a regular value, and $f^{-1}(0) = S$ (Prop. 2 in [3, Section 2.7]).

Without going into the details, we can extend the domain of f to $\mathbb{R}^3$ considering it to be constant negative inside S e constant positive outside S. The remaining of these notes will be dedicated to the differential geometry of level sets.

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2. The shape operator

Let S be a regular surface given by the zero-level set of an implicit function $f : \mathbb{R}^3 \to \mathbb{R}$. The differential dN_p of the normal map N at $p \in S$ is a linear map on T_pS, it is called the *shape operator* of S at p. Let v be a vector tangent to S at p, we compute the directional derivative of N along v using $\frac{\partial N}{\partial v}(p) = dN_p(v)$. Calculations give us the following shape operator formula [6].

$$dN = (I - NN^\top)\frac{\mathbf{H}f}{|\nabla f|}. \tag{2.1}$$

Where the matrix $\mathbf{H}f$ denotes the *Hessian* of the function f and I is the 3×3 identity matrix. Thus, the shape operator is the product of the Hessian of f scaled by the gradient norm $|\nabla f|$ and a linear projection along the normal field N.

The shape operator $dN_p : T_pS \to T_pS$ at $p \in S$ is symmetric. Indeed, let u and v be tangent vectors in T_pS, using that the symmetric matrix $I - NN^\top$ is a linear projection towards the normal direction N and that the Hessian matrix $\mathbf{H}f$ is symmetric, we obtain the symmetry of the shape operator:

$$\begin{aligned}
\langle v, dN(u)\rangle &= \left\langle v, (I - NN^\top)\frac{\mathbf{H}f}{|\nabla f|}u \right\rangle = \left\langle (I - NN^\top)v, \frac{\mathbf{H}f}{|\nabla f|}u \right\rangle \\
&= \left\langle v, \frac{\mathbf{H}f}{|\nabla f|}u \right\rangle = \left\langle \frac{\mathbf{H}f}{|\nabla f|}v, u \right\rangle = \left\langle \frac{\mathbf{H}f}{|\nabla f|}v, (I - NN^\top)u \right\rangle \\
&= \left\langle (I - NN^\top)\frac{\mathbf{H}f}{|\nabla f|}v, u \right\rangle = \langle dN(v), u\rangle .
\end{aligned}$$

Then, the spectral theorem states that there is an orthogonal basis $\{e_1, e_2\}$ of T_pS called the *principal directions*, where the shape operator can be expressed as a diagonal 2×2 matrix. The two elements of this diagonal are the *principal curvatures* k_1 and k_2. These curvatures are obtained using the equations $dN(e_i) = -k_ie_i$, for $i = 1, 2$. We now provide a geometrical interpretation of the shape operator dN.

The *second fundamental form* of the implicit surface S is a map that attributes to each point $p \in S$ the quadratic form on the tangent space T_pS

$$\mathbf{II}_p(v) = \langle -dN_p(v), v\rangle . \tag{2.2}$$

Let α be a curve passing through p with unit tangent direction v. The number $k_n(p) = \mathbf{II}_p(v)$ is the *normal curvature* of α at p. We provide a geometrical interpretation of $k_n(p)$. Let α be the *normal section* of S at p along v, i.e. the local intersection of S and the plane spanned by v and N. In this setting, $k_n(p)$ coincides with the curvature k of α. Indeed, consider α to be parameterized by arc length and that $\alpha(0) = p$ and $\alpha'(0) = v$. Remember that $\alpha'' = kn$, where n is the normal of α, which in this case (normal section) is aligned to N. Then, taking the derivative of $\langle N(s), \alpha'(s)\rangle = 0$ implies in $k_n(0) = -\langle N'(0), \alpha'(0)\rangle = \langle N(0), \alpha''(0)\rangle = k(0)$.

Restricted to the unit circle centered in the origin of T_pS, $\mathbf{II}_p$ reaches a maximum value and a minimum value, and these coincide with the principal curvatures k_1 and k_2, respectively. See [3, Section 3.2] for the details.

The principal curvatures measure the maximum and minimum bending of a surface at each point. For an illustrative example, consider the *double-torus* surface given by the zero-level set $f^{-1}(0)$ of the function

$$f(x, y, z) = 2y(y^2 - 3x^2)(1 - z^2) + (x^2 + y^2)^2 - (9z^2 - 1)(1 - z^2). \tag{2.3}$$

Figure 2(left) shows the surface of the double-torus with a shading indicating its minimum curvature. Specifically, a transfer function is used to map lower values of curvature to red, higher values to blue, and intermediary values to white. Analogously, Figure 2(right) illustrates the maximum curvature function.

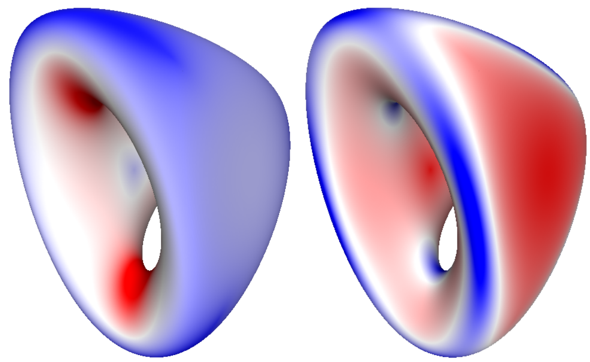

FIGURE 2. Minimum and maximum curvatures of the double-torus.

Since dN is symmetric, the principal directions associated with the principal curvatures $\{e_1, e_2\}$ form an orthogonal frame at each point. Again, an important geometrical property is that they are parallel to the directions in which the surface curves more or less. Figure 3 shows the principal direction of the double-torus.

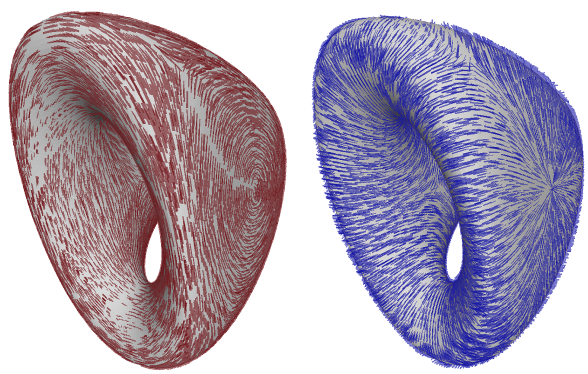

FIGURE 3. Minimum and maximum directions of the double-torus.

In the frame $\{e_1, e_2\}$, the second fundamental form $\mathbf{II}_p$ can be written in the standard quadratic form. Specifically, let $v = x_1e_1 + x_2e_2$ be a tangent vector at a point $p \in S$ expressed in the basis $\{e_1, e_2\}$. After simple calculations, we obtain $\mathbf{II}_p(v) = x_1^2k_1 + x_2^2k_2$. This classifies the points in S: *Elliptic* if $k_1k_2 > 0$, *hyperbolic* if $k_1k_2 < 0$, *parabolic* if only one k_i is zero, and *planar* if $k_1 = k_2 = 0$.

Each elliptic point $p \in S$ admits a neighborhood that belongs to the same side of its tangent plane T_pS. On the other hand, each neighborhood of a hyperbolic point has points on both sides of the tangent plane. No such statement can be made for the parabolic and planar points of S [3, Section 3.3].

3. Gaussian and mean curvatures

The above classification is related to the *Gaussian curvature* $K = k_1 k_2$ of S. Elliptic points have positive Gaussian curvature. In these points, the surface is similar to a dome. Hyperbolic points have negative Gaussian curvature. At such points, the surface is saddle-shaped. Parabolic and planar points have null curvature.

Gaussian curvature has relations with Euclidean and Non-Euclidean geometries. Let S be a complete surface in $\mathbb{R}^3$. If S has a constant zero Gaussian curvature, then it is either a cylinder or a plane (Theorem in [3, Section 5.8]), thus S has the *Euclidean* geometry. If S has a constant positive Gaussian curvature it must be a sphere (Theorem 1 in [3, Section 5.2]) and its geometry is *spherical.* There is no complete surface in $\mathbb{R}^3$ with a constant negative Gaussian curvature (Theorem in [3, Section 5.11]), however, allowing S to have a boundary, we can consider the *pseudosphere* (see Exercise 6 in [3, Page 171]) which has the *hyperbolic* geometry.

The *mean curvature* $H = \frac{k_1+k_2}{2}$, is an extrinsic measure that locally describes the curvature of the embedded surface S in $\mathbb{R}^3$. Note that by its definition, H is written in terms of the shape operator *trace* which does not depend on the choice of basis. Therefore, $2H = \operatorname{trace}(dN)$. To obtain a formula of H in terms of the derivatives of f we expand the trace of $dN = (I - NN^\top)\frac{\mathbf{H}_f}{|\nabla f|}$:

$$\begin{aligned}|\nabla f|^3 \operatorname{trace}(dN) = {} & f_{xx}(f_y^2 + f_z^2) + f_{yy}(f_x^2 + f_z^2) + f_{zz}(f_x^2 + f_y^2) \\ & - 2f_x f_y f_{xy} - 2f_x f_z f_{xz} - 2f_y f_z f_{yz}.\end{aligned}$$

On the other hand, computing $\operatorname{div}\frac{\nabla f}{|\nabla f|}$ and using the above formula of $\operatorname{trace}(dN)$, we get $2\operatorname{div}\frac{\nabla f}{|\nabla f|} = \operatorname{trace}(dN)$. Thus, the mean curvature of S is expressed as $2H = \operatorname{div}\frac{\nabla f}{|\nabla f|}$. Then, when $f : \mathbb{R}^3 \to \mathbb{R}$ is a signed distance function, the mean curvature is given by the *Laplacian* Δf.

Figure 4(left) illustrates the Gaussian curvature of the double-torus. The blue color indicates the (elliptic) points with positive Gaussian curvature. The white color shows the (parabolic and planar) points with zero Gaussian curvature. Finally, the red color illustrates the (hyperbolic) points with negative Gaussian curvature. Figure 4(middle) shows the mean curvature of the double-torus. Observe that the mean curvature highlights more expressive geometrical features of the surface.

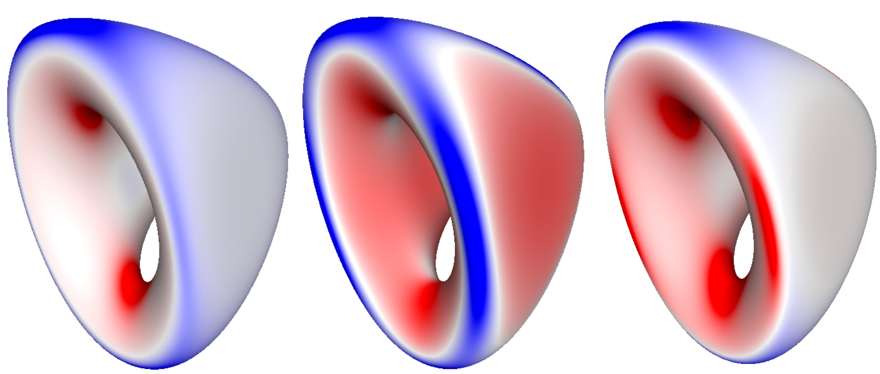

Figure 4. Gaussian (left) and mean (middle) curvatures of the double-torus, and the corresponding Harris function (right).

4. Harris corner detector

The Gaussian and mean curvatures can be used to decompose S into regions with different geometries, e.g. in elliptic, hyperbolic, parabolic, and planar regions.

In the context of image segmentation, the *Harris corner detector* [5] decomposes a given image in the corner, edges, and planar regions. Such decomposition is reached using the *Harris response function* of the surface given by the 3D graph of a gray-scale image.

$$R = k_1 k_2 - \tau(k_1 + k_2)^2 = K - 4\tau H^2. \tag{4.1}$$

Where k_i are the principal curvatures of the graph given by the image function, and τ is an empirical constant commonly taken in the interval $[0.04, 0.06]$.

The Harris response function R can be easily extended to the implicit surface S providing a decomposition of S in *corner regions* $(R > 0)$, *edge regions* $(R < 0)$, and *planar regions*$(R \approx 0)$. Figure 4(right) illustrate the Harris response function of the double-torus. Blue/red/white colors indicate the corner/edge/planar regions.

5. Umbilical points

A point $p \in S$ is called *umbilical* if its principal curvatures are equal, i.e. $k_1 = k_2$. Note that planar points are umbilical. There is the interesting fact that a region of S containing only umbilical points must coincide with a piece of a plane $(k_1 = k_2 = 0)$ or a piece of a sphere $(k_1 = k_2 \neq 0)$. Umbilical points are singularities of the principal directions. Figure 3 gives an illustrative example.

Umbilical points can be connected by some integral lines (separatrices) of the principal directions. The resulting graph is called the *topological graph* and decomposes the surface in regions containing no umbilical points.

6. Computing the principal curvatures

This section presents the explicit formulas of the curvatures of the surface S given by the zero-level set of a function $f : \mathbb{R}^3 \to \mathbb{R}$.

Restricted to the tangent plane T_pS at $p \in S$, the characteristic polynomial $\det[dN_p - \lambda I] = 0$ of the shape operator can be written as $\lambda^2 - 2H\lambda + K = 0$, where $H = \frac{k_1+k_2}{2}$ is the mean curvature and $K = k_1k_2$ is the Gaussian curvature. Thus the principal curvatures are given by:

$$k_1 = H - \sqrt{H^2 - K} \text{ and } k_2 = H + \sqrt{H^2 - K}. \tag{6.1}$$

As we saw in Section 3, the mean curvature H can be computed using the divergence of the normal field, i.e. $2H = \operatorname{div} \frac{\nabla f}{|\nabla f|}$.

The Gaussian curvature K of S can be calculated using the following elegant formula. We refer to the work of Goldman [4] for the deduction of this formula.

$$K = -\frac{1}{|\nabla f|^4} \det \left[\begin{array}{c|c} \mathbf{H}f & \nabla f \\ \hline \nabla f^\top & 0 \end{array} \right]. \tag{6.2}$$

Therefore, the curvatures of S can be calculated analytically considering only the coefficients of the gradient and Hessian of the implicit function f. Now we focus on computing the principal directions of S.

7. Computing the principal directions

Let $v = (v_x, v_y, v_z)$ be a tangent direction at a point $p \in S$. By definition, v is a principal direction of S if and only if $dN(v) = \lambda v$. In other words, $dN(v)$ must belong to the line spanned by v which is equivalent to $\langle v, N \wedge dN(v)\rangle = 0$. Then, using the formula of the shape operator dN, given in Equation 2.1, we obtain

$$\langle v, \nabla f \wedge \mathbf{H}f(v)\rangle = 0. \tag{7.1}$$

Where $\mathbf{H}f(v)$ is the Hessian applied to v, i.e. $\mathbf{H}f(v) = \big(\langle v, \nabla f_x\rangle, \langle v, \nabla f_y\rangle, \langle v, \nabla f_z\rangle\big)$ with $\nabla f_x = (f_{xx}, f_{xy}, f_{xz})$ being the gradient of f_x, analogous for ∇f_y and ∇f_z. Thus, Equation 7.1 can be written in the determinant form

$$\det \begin{bmatrix} v_x & v_y & v_z \\ f_x & f_y & f_z \\ \langle v, \nabla f_x\rangle & \langle v, \nabla f_y\rangle & \langle v, \nabla f_z\rangle \end{bmatrix} = 0. \tag{7.2}$$

Therefore, satisfying Equation 7.2 is a necessary and sufficient condition for v to be a principal direction of S. To solve Equation 7.2, we express it in a tensor form

$$\begin{bmatrix} v_x & v_y & v_z \end{bmatrix} \begin{bmatrix} A & B & C \\ B & D & E \\ C & E & F \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = 0. \tag{7.3}$$

Where the coefficients of this symmetric matrix can be expressed in terms of the gradient and Hessian of f [2]:

$$\begin{aligned} A &= f_y f_{zx} - f_z f_{yx},\ D = f_z f_{xy} - f_x f_{zy},\ F = f_x f_{yz} - f_y f_{xz}, \\ B &= (f_z f_{xx} - f_x f_{zx} + f_y f_{zy} - f_z f_{yy})/2, \\ C &= (f_y f_{zz} - f_z f_{yz} + f_x f_{yx} - f_y f_{xx})/2, \\ E &= (f_x f_{yy} - f_y f_{xy} + f_z f_{xz} - f_x f_{zz})/2. \end{aligned}$$

To solve Equation 7.3, we use that the gradient of f is perpendicular to the tangent direction v, i.e. $\langle \nabla f, v\rangle = v_x f_x + v_y f_y + v_z f_z = 0$. As S is a regular surface ($\nabla f \neq 0$), we can consider, "without loss of generality", $f_z \neq 0$. This leads us to $v_z = (v_x f_x + v_y f_y)/f_z$. Replacing this expression in Equation 7.3 provides the following quadratic equation in terms of v_x and v_y

$$U v_x^2 + 2V v_x v_y + W v_y^2 = 0 \tag{7.4}$$

Where its coefficients are given by

$$\begin{aligned} U &= A f_z^2 - 2C f_x f_z + F f_x^2 \\ V &= 2(B f_z^2 - C f_y f_z - E f_x f_z + F f_x f_y) \\ W &= D f_z^2 - 2E f_y f_z + F f_y^2 \end{aligned}$$

Equation 7.4 can be solved using the Bhaskara formula. If $\Delta = V^2 - 4UW \neq 0$, the principal direction are given by

$$e_1 = \Big(X_1 f_z,\ 2U f_z,\ -X_1 f_x - 2U f_y\Big) \text{ and } e_2 = \Big(X_2 f_z,\ 2U f_z,\ -X_2 f_x - 2U f_y\Big)$$

where $X_1 = -V + \mathrm{sgn}(f_z)\sqrt{\Delta}$ and $X_2 = -V - \mathrm{sgn}(f_z)\sqrt{\Delta}$ with $\mathrm{sgn}(f_z)$ being the sign of the z-coordinate f_z of ∇f. There is no solution for Equation 7.4 in the points of S satisfying $\Delta = 0$. These points are umbilicals.

We give a brief explanation of the term $\operatorname{sgn}(f_z)$ in the principal direction formulas. We considered a region of S satisfying $f_z \neq 0$ to parameterize the tangent planes using the x, y-coordinates. Then, changing the sign of f_z induces a change of orientation on the tangent planes which permutes the principal directions. Therefore, $\operatorname{sgn}(f_z)$ is used to maintain the formulas coherent within the considered region.

8. Silhouettes, Valleys, and Ridges

Silhouettes are common objects in non-photorealistic rendering, they highlight the transitions between the front surface and the back-surface [6]. A point p belongs to the silhouette regions of the surface S, associated to an observer points q, if it satisfies $|\langle v, N(p)\rangle| < \epsilon$; v is the view direction of the ray connecting q to p, and $\epsilon > 0$ is a threshold radius of the region. Clearly, the silhouettes regions are view dependent. Figure 5 illustrates the silhouettes of the Armadillo model in black.

Figure 5. The silhouettes are in black.

The extreme points of the principal curvatures along the principal directions compose the *ridge* and *ravines* of S [1, 7]. These are lines encoding the local information of how S is bending. Specifically, a point p in S is a *ridge* (*ravine*) if k_1 (k_2) attains a maximum (minimum) along e_1 (e_2). In other words, when the directional derivative of k_i along e_i vanishes $\frac{\partial k_i}{\partial e_i} = \langle \nabla k_i, e_i\rangle = 0$, then p is a ridge if $i = 1$ and $\frac{\partial^2 k_1}{\partial e_1^2} < 0$ or p is a ravine if $i = 2$ and $\frac{\partial^2 k_2}{\partial e_2^2} > 0$. Observe that reversing the orientation of S, that is, considering the normal field $-N$ instead of N, ridges and ravines are permuted. Figure 6 illustrates a ridge curve.

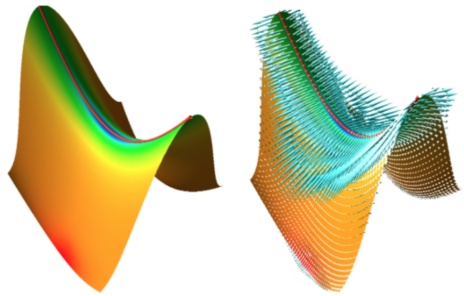

FIGURE 6. The color map illustrates the maximum curvature. The red line is a ridge. The maximum directions scaled by the corresponding principal curvature are given on the right. Image from [7].

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