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Groebner-basis solution of the three-dimensional resection problem (P4P)

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Abstract. The three-dimensional (3-D) resection problem is usually solved by first obtaining the distances connecting the unknown point $P\{X, Y, Z\}$ to the known points $P_i\{X_i, Y_i, Z_i\} \mid i = 1, 2, 3$ through the solution of the three nonlinear Grunert equations and then using the obtained distances to determine the position $\{X, Y, Z\}$ and the 3-D orientation parameters $\{\Lambda_\Gamma, \Phi_\Gamma, \Sigma_\Gamma\}$. Starting from the work of the German J. A. Grunert (1841), the Grunert equations have been solved in several substitutional steps and the desire as evidenced by several publications has been to reduce these number of steps. Similarly, the 3-D ranging step for position determination which follows the distance determination step involves the solution of three nonlinear ranging ('Bogenschnitt') equations solved in several substitution steps. It is illustrated how the algebraic technique of Groebner basis solves explicitly the nonlinear Grunert distance equations and the nonlinear 3-D ranging ('Bogenschnitt') equations in a single step once the equations have been converted into algebraic (polynomial) form. In particular, the algebraic tool of the Groebner basis provides symbolic solutions to the problem of 3-D resection. The various forward and backward substitution steps inherent in the classical closed-form solutions of the problem are avoided. Similar to the Gauss elimination technique in linear systems of equations, the Groebner basis eliminates several variables in a multivariate system of nonlinear equations in such a manner that the end product normally consists of a univariate polynomial whose roots can be determined by existing programs e.g. by using the *roots* command in Matlab.

Keywords: Groebner basis – Three-dimensional resection – Grunert equations

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1 Introduction

The search towards the solution of the three-dimensional (3-D) resection problem originates in the work of a German mathematician, J.A. Grunert (1841). Grunert solved the 3-D resection problem – what was then known as the 'Pothenot' problem – in a closed form by solving an algebraic equation of degree four. The problem is normally solved by iterative means mainly in photogrammetry and computer vision. Procedures developed later for solving the 3-D resection problem revolved around improvements to the approach of Grunert (1841) with the aim of searching for the optimal means of distance determination. Whereas Grunert (1841) solves the problem by a substitution approach in three steps, the more recent desire has been to solve the distance equations in fewer steps, as exemplified in the works of Finsterwalder and Scheufele (1937), Merritt (1949), Fischler and Bolles (1981), Linnainmaa et al. (1988), Grafarend et al. (1989) and Lohse (1990).

Whereas the approaches listed above involved forward and backward steps with substitution of variables, a more direct procedure is offer by algebraic techniques such as the Groebner basis and the multipolynomial resultant. The advantage of algebraic techniques is that they directly (explicitly) solve the nonlinear system of equations inherent in the 3-D problem once they have been converted into algebraic form. As an appetizer to the solution of the 3-D resection problem using the Groebner basis approach, Awange (2002b) solved the planar resection problem in a closed form using the Groebner basis. In Awange and Grafarend (in press b), the current problem is solved by using the alternative algebraic technique of multipolynomial resultants. The overdetermined version of the problem is solved by the present authors in Awange and Grafarend (2003).

The authors have further used the algebraic technique of the Groebner basis to solve the geodetic problems of seven-parameter datum transformation $C_7(3)$ in Awange and Grafarend (2002a, in press a, submitted). In Awange and Grafarend (2002b, c), the algebraic techniques of the Groebner basis and the multipolynomial resultant are used to solve GPS pseudoranging problems in both closed and overdetermined form.

As a contribution towards the ongoing search for direct procedures for solving 3-D resection problems, we present a solution of the 3-D resection problem by solving the Grunert distances equations using the Groebner basis technique. The resulting fourth-order univariate polynomial is solved for the unknown distance and the admissible solution substituted in other elements of the Groebner basis to determine the remaining two distances. Once we have the spatial distances, the position is computed by solving the 3-D ranging problem ('Bogenschnitt') using the Groebner basis approach. For the orientation step which concludes the solution of the 3-D resection problem, we refer to the works of Awange (2003) and Grafarend and Awange (2000) who solved the 3-D orientation problem by using the simple Procrustes algorithm.

Groebner basis has become a household name in algebraic manipulations and finds application in fields such as statistics and engineering. It has found use as a tool for discovering and proving theorems and solving systems of polynomial equations as elaborated in publications by Buchberger and Winkler (1998). Groebner bases also give a solution to the 'ideal' membership problem. By reducing a given polynomial f with respect to the Groebner basis G, f is said to be a member of the ideal if the zero remainder is obtained. Thus let $G = \{g_1, \ldots, g_s\}$ be a Groebner basis of an ideal $I \subset k[x_1, \ldots, x_n]$ and let $f \in k[x_1, \ldots, x_n]$ be a polynomial, then $f \in I$ if and only if the remainder on division of f by G is zero. The Groebner basis can also be used to show the equivalence of polynomial equations. Two sets of polynomial equations will generate the same ideal if and only if their Groebner bases are equal with respect to any term ordering. This implies that a system of polynomial equations $f_1(x_1, \ldots, x_n) = 0, \ldots, f_s(x_1, \ldots, x_n) = 0$ will have the same solution as a system arising from any Groebner basis of $f_1, \ldots, f_s$ with respect to any term ordering. This is the main property of the Groebner basis that is used to solve a system of polynomial equations in the present work.

For a complete reference to the theory of the Groebner basis, the reader is referred to the work of Awange (2002b). This work presents the solution of planar 2-D resection using the Groebner basis approach, which provides a soft landing for the 3-D resection problem. The remainder of the study is organized as follows: in Sect. 2 we present the application of the Groebner basis approach to the solution of the 3-D resection problem, and in Sect. 3 we consider an example based on the the test network 'Stuttgart Central'.

2 Solution of 3-D resection problem

The closed-form 3-D resection procedure is carried out in three steps, namely: the distance-derivation step (solution of the Grunert equations), the position-derivation step and the orientation-derivation step. It may be argued that the distance derivation step is irrelevant in light of modern distance-measuring equipment such as electromagnetic distance measuring (EDM) equipment. Distance measurements in forest areas may, however, fail due to tree cover that may block the EDM signals, while on the other hand we may be able to measure the angles. In this case, the distances can be derived indirectly using directions/angles as already proposed by Grunert (1841). In photogrammetry, the distances have to be derived from the image coordinates in order to obtain the perspective center coordinates and the orientation parameters, i.e. the elements of exterior orientation. These two examples illustrate the necessity of still having procedures for deriving spatial distances despite the existence of EDM equipment.

In the closed-form 3-D resection problem, we are interested in determining the position and orientation of a point P connected by angular observations of type horizontal directions T_i and vertical directions B_i to three other known points P_1, P_2, P_3 as shown in Fig. 1. From the angular measurements, the distances are derived in the distance-derivation step by solving the Grunert equations. Once the distances have been established, the unknown position P is determined in the position-derivation step. The closed-form 3-D resection problem is completed in the orientation-derivation step by solving the orientation parameters that relate the global reference system $\mathbb{F}^{\bullet}$ to the local-level reference system of type $\mathbb{F}^{*}$.

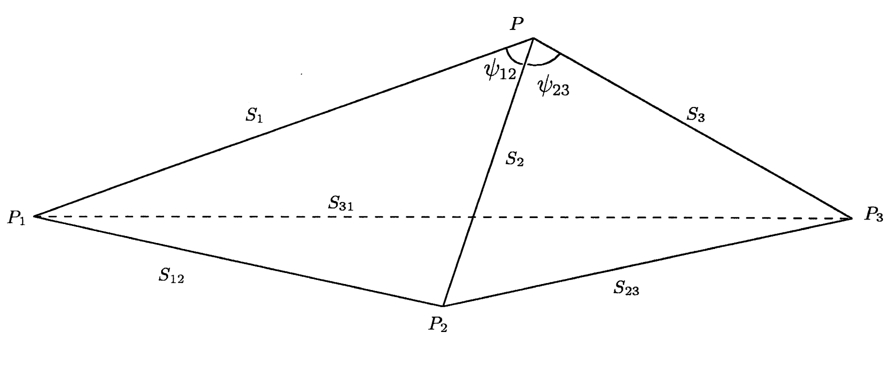

Fig. 1. Tetrahedron for closed-form 3-D resection

2.1 The distance-derivation step (solution of the Grunert equations)

We begin in Box 1 by presenting the derivation of the distance equations (also known as the Grunert equations) relating the known distances $S_{ij}, i,j = 1,2,3|i \neq j$, between the known GPS stations in the global reference frame $\mathbb{F}^{\bullet}$, the unknown distances $S_i, i = 1,2,3$, between the unknown station $P \in \mathbb{E}^3$ and the known stations $P_i \in \mathbb{E}^3$ and the spatial angles $\psi_{ij}, i,j = 1,2,3|i \neq j$. The spatial angles $\psi_{ij}, i,j = 1,2,3|i \neq j$, are obtained from observations of type horizontal directions T_i and vertical directions B_i as shown in Box 1 in the local level reference frame $\mathbb{F}^*$. The relationship between observations in the local-level reference frame and the global reference frame $\mathbb{F}^{\bullet}$ in Box 1 is presented in Eq. (3), where $X_i, Y_i, Z_i \mid i \in \{1,2,3\}$ are GPS Cartesian coordinates of known points $P_i \in \mathbb{E}^3 \mid i \in \{1,2,3\}$, $S_i, T_i, B_i \mid i \in \{1,2,3\}$ are spherical coordinates of types spatial distances, horizontal and vertical directions respectively, linking the old and new GPS points, while X, Y, Z are the required GPS coordinates of the unknown point $P \in \mathbb{E}^3$ (Fig. 1). $\mathbf{R}$ is the rotation matrix containing the 3-D orientation parameters. Multiplying Eq. (3) by Eq. (5), leading to Eq. (6), the relationship between the spherical coordinates of types horizontal directions T_i, vertical directions B_i and the space angles ψ_{ij} can be derived. After manipulations of Eqs. (7), (8) and (9), the spatial angle ψ_{ij} can be written in terms of the spherical coordinates $\{T_i, B_i\}, \{T_j, B_j\}$ of points P_i and P_j with respect to a theodolite orthogonal Euclidean frame $\mathbb{F}^*$ as in Eq. (10). The Grunert equations for the three unknown distances S_1, S_2, S_3 can now be written in terms of the known distances S_{12}, S_{23}, S_{31} and the space angles $\psi_{12}, \psi_{23}, \psi_{31}$ illustrated in Fig. 1, as in Eqs. (12). In photogrammetry, the relationship between the space angles and the measured image coordinates with respect to an orthogonal Euclidean frame centered at the perspective center with f being the focal length is given by Grafarend and Shan [1997, Eq. (1.1), p. 218] as

$$\cos\psi_{ij} = \frac{x_i x_j + y_i y_j + f^2}{\sqrt{x_i^2 + y_i^2 + f^2}\sqrt{x_j^2 + y_j^2 + f^2}} \tag{1}$$

In order to understand the usefulness of the Groebner basis in the solution of the 3-D resection problem, we present first the hand computation of the Grunert distances for a regular tetrahedron using the Groebner basis before considering the general case of the Grunert equations for distances in the general 3-D resection problem whose Groebner bases are computed using Mathematica software.

2.1.1 Groebner basis solution of the Grunert equations for a regular tetrahedron

We begin by expressing the Grunert distance equations, Eqs. (12), whose geometrical behaviour has been studied by Grafarend et al. (1989), in the form

Box 1. Derivation of the Grunert equations

GPS and LPS coordinate systems

$$\begin{bmatrix} x_i - x \\ y_i - z \\ z_i - z \end{bmatrix}_* = \mathbf{R} \begin{bmatrix} X_i - X \\ Y_i - Y \\ Z_i - Z \end{bmatrix}_{\bullet} \tag{2}$$

$$S_i \begin{bmatrix} \cos T_i \cos B_i \\ \sin T_i \cos B_i \\ \sin B_i \end{bmatrix} = \mathbf{R} \begin{bmatrix} X_i - X \\ Y_i - Y \\ Z_i - Z \end{bmatrix} \tag{3}$$

$$\mathbf{R} \in SO(3) := \{\mathbf{X} \in \mathbb{R}^{3\times 3} \mid \mathbf{X}^T\mathbf{X} = \mathbf{I}_3, |\mathbf{X}| = +1\} \tag{4}$$

$$(-2)\left[\cos T_j \cos B_j, \sin T_j \cos B_j, \sin B_j\right] S_j \tag{5}$$

$$\begin{aligned} &(-2)[\cos T_j \cos B_j, \sin T_j \cos B_j, \sin B_j] S_i S_j \begin{bmatrix} \cos T_i \cos B_i \\ \sin T_i \cos B_i \\ \sin B_i \end{bmatrix} \\ &\quad = (-2)[(X_j - X), (Y_j - Y), (Z_j - Z)] \begin{bmatrix} X_i - X \\ Y_i - Y \\ Z_i - Z \end{bmatrix} \end{aligned} \tag{6}$$

$$\begin{aligned} &\left.\begin{aligned} &(X_j - X)(X_i - X) = X_j X_i - X_j X - X_i X + X^2 \\ &(X_i - X_j)(X_i - X_j) = X_i^2 - 2X_i X_j + X_j^2 \\ &(X_i - X)(X_i - X) = X_i^2 - 2X_i X + X^2 \\ &(X_j - X)(X_j - X) = X_j^2 - 2X_j X + X^2 \end{aligned}\right] \\ &\quad \Rightarrow (X_i - X_j)^2 - (X_i - X)^2 - (X_j - X)^2 = -2(X_j - X)(X_i - X) \end{aligned} \tag{7}$$

$$\begin{aligned} &(-2)[\cos T_j \cos B_j, \sin T_j \cos B_j, \sin B_j] S_i S_j \begin{bmatrix} \cos T_i \cos B_i \\ \sin T_i \cos B_i \\ \sin B_i \end{bmatrix} \\ &\quad = (X_i - X_j)^2 + (Y_i - Y_j)^2 + (Z_i - Z_j)^2 - (X_i - X)^2 - (Y_i - Y)^2 \\ &\qquad - (Z_i - Z)^2 - (X_j - X)^2 - (Y_j - Y)^2 - (Z_j - Z)^2 \end{aligned} \tag{8}$$

$$\begin{aligned} &-2\{\sin B_j \sin B_i + \cos B_j \cos B_i \cos(T_j - T_i)\} S_i S_j \\ &\quad = (X_i - X_j)^2 + (Y_i - Y_j)^2 + (Z_i - Z_j)^2 - (X_i - X)^2 - (Y_i - Y)^2 \\ &\qquad - (Z_i - Z)^2 - (X_j - X)^2 - (Y_j - Y)^2 - (Z_j - Z)^2 \end{aligned} \tag{9}$$

$$\cos\psi_{ij} = \cos B_i \cos B_j \cos(T_j - T_i) + \sin B_i \sin B_j \tag{10}$$

$$\begin{aligned} &-2\cos\psi_{ij} S_i S_j = S_{ij}^2 - S_i^2 - S_j^2 \\ &S_{ij}^2 = S_i^2 + S_j^2 - 2S_i S_j \cos\psi_{ij} \end{aligned} \tag{11}$$

$$\begin{aligned} S_{12}^2 &= S_1^2 + S_2^2 - 2S_1 S_2 \cos\psi_{12} \\ S_{23}^2 &= S_2^2 + S_3^2 - 2S_2 S_3 \cos\psi_{23} \\ S_{31}^2 &= S_3^2 + S_1^2 - 2S_3 S_1 \cos\psi_{31} \end{aligned} \tag{12}$$

$$\begin{aligned} a_0 &= x_1^2 + x_2^2 - 2a_{12} x_1 x_2 \\ b_0 &= x_2^2 + x_3^2 - 2b_{23} x_2 x_3 \\ c_0 &= x_3^2 + x_1^2 - 2c_{31} x_3 x_1 \end{aligned} \tag{13}$$

where

$$\begin{aligned} &S_1 = x_1 \in \mathbb{R}^+, \quad S_2 = x_2 \in \mathbb{R}^+, \quad S_3 = x_3 \in \mathbb{R}^+ \\ &\cos\psi_{12} = a_{12}, \quad \cos\psi_{23} = b_{23}, \quad \cos\psi_{31} = c_{31} \\ &S_{12}^2 = a_0, \quad S_{23}^2 = b_0, \quad S_{31}^2 = c_0 \end{aligned} \tag{14}$$

Grafarend et al. (1989) demonstrate that for each of the quadratic equations of Eqs. (13), there exists an elliptical cylinder in the planes $\{x_1,x_2\}$, $\{x_2,x_3\}$, and $\{x_3,x_1\}$ for the first, second, and third equations respectively.

These cylinders are constrained to their first quadrant since the distances are positive, thus $\{x_1 \in \mathbb{R}^+\}$, $\{x_2 \in \mathbb{R}^+\}$, and $\{x_3 \in \mathbb{R}^+\}$. In Box 2 we apply the Groebner basis technique to solve for the distances $x_i, i = 1,2,3$, between the unknown station $P \in \mathbb{E}^3$ and the known stations $P_i \in \mathbb{E}^3$. For a regular tetrahedron, the distances $x_1 = x_2 = x_3$ joining the unknown point $P \in \mathbb{E}^3$ to the known points $P_i \in \mathbb{E}^3$ are equal to the distances $S_{12} = S_{23} = S_{31}$ between the known stations. Let us consider these distances to be equal to $+\sqrt{d}$. The spatial angles are also equal (i.e. $\psi_{12} = \psi_{23} = \psi_{31} = 60°$). The task now is to compute by hand the Groebner bases of Eqs. (13) and use them to find the Grunert distances for the regular tetrahedron (i.e. to show that the desired solutions for $\{x_1,x_2,x_3\} \in \mathbb{R}^+$ are $x_1 = x_2 = x_3 = +\sqrt{d}$).

Box 2. Hand computation of Groebner basis of the Grunert equations for a regular tetrahedron

Upon lexicographic ordering $x_1 > x_2 > x_3$ and subtracting the left-hand side of Eq. (13) from the right-hand side, we have

$$\begin{aligned} &x_1^2 - 2a_{12}x_1x_2 + x_2^2 - a_0 = 0 \\ &x_2^2 - 2b_{23}x_2x_3 + x_3^2 - b_0 = 0 \\ &x_1^2 - 2c_{31}x_1x_3 + x_3^2 - c_0 = 0 \end{aligned} \tag{15}$$

as polynomials in $\mathbb{R}[x_1,x_2,x_3]$.

These equations form [following Definition 2-5 in Awange (2002a, pp. 14–15)] the ideal I as

$$I = \langle x_1^2 - 2a_{12}x_1x_2 + x_2^2 - a_0, x_2^2 - 2b_{23}x_2x_3 + x_3^2 - b_0, x_1^2 - 2c_{31}x_1x_3 + x_3^2 - c_0 \rangle \subset \mathbb{R}[x_1,x_2,x_3] \tag{16}$$

For a regular tetrahedron, where $\psi_{ij} = 60°$, $2\cos(60°) = 1$ and $a_0 = b_0 = c_0 = d$. Equation (15) is then written as

$$\begin{aligned} &x_1^2 - x_1x_2 + x_2^2 - d = 0 \\ &x_2^2 - x_2x_3 + x_3^2 - d = 0 \\ &x_1^2 - x_1x_3 + x_3^2 - d = 0 \end{aligned} \tag{17}$$

giving rise from Eq. (16) to the ideal I as

$$I = \langle x_1^2 - x_1x_2 + x_2^2 - d, x_2^2 - x_2x_3 + x_3^2 - d, x_1^2 - x_1x_3 + x_3^2 - d \rangle \subset \mathbb{R}[x_1,x_2,x_3] \tag{18}$$

whose generators G are written as

$$\begin{aligned} &g_1 = x_1^2 - x_1x_2 + x_2^2 - d \\ &g_2 = x_1^2 - x_1x_3 + x_3^2 - d \\ &g_3 = x_2^2 - x_2x_3 + x_3^2 - d \end{aligned} \tag{19}$$

Now we require the Groebner basis for the generators of Eqs. (19) of the ideal I in Eq. (16). One proceeds to compute the S-polynomial pairs $(g_1,g_2),(g_1,g_3),(g_2,g_3)$ from the generators of Eqs. (19) of Eq. (16). From Buchberger's third criterion (Buchberger 1979), we notice that $\mathbf{LM}(g_2) = x_1^2$ divides the $\mathbf{LCM}(g_1,g_3) = x_1^2x_2^2$. It suffices therefore to suppress the consideration of (g_1,g_3) and instead consider only $(g_1,g_2),(g_2,g_3)$. $S(g_1,g_2)$ gives

$$S(g_1,g_2) = -x_1x_2 + x_1x_3 + x_2^2 - x_3^2 \tag{20}$$

which is reduced with respect to G by subtracting g_3 to obtain

$$-x_1x_2 + x_1x_3 - 2x_3^2 + x_2x_3 + d \tag{21}$$

which does not reduce to zero and is added in the original list G of the generating set of the ideal I as g_4. The S-polynomial pairs to be considered next are $S(g_2,g_3), S(g_2,g_4), S(g_3,g_4)$ from the new generating set $G = \{g_2,g_3,g_4\}$. Since $LM(g_2)$ and $LM(g_3)$ are relatively prime, $S(g_2,g_3)$ reduces to zero modulo G ($S(g_2,g_3) \rightarrow_G 0$). The S-polynomial pairs remaining for consideration are (g_2,g_4) and (g_3,g_4). $S(g_2,g_4)$ gives

$$S(g_2,g_4) = x_1^2x_3 + x_1d - 2x_1x_3^2 + x_2x_3^2 - x_2d \tag{22}$$

which is reduced with respect to G by subtracting x_3g_2 to give

$$x_1d - x_1x_3^2 + x_2x_3^2 - x_2d - x_3^3 + x_3d \tag{23}$$

which does not reduce to zero and is added to the list G of the generating set of the ideal I as g_5. The S-polynomial set to be considered next is $S(g_3,g_4)$ from the new generating set $G = \{g_2,g_3,g_4,g_5\}$. $S(g_3,g_4)$ gives

$$S(g_3,g_4) = -x_1x_3^2 + x_1d + 2x_2x_3^2 - x_2^2x_3 - x_2d \tag{24}$$

which is reduced with respect to G by subtracting g_5 and adding x_3g_3 to give

$$2x_3^3 - 2x_3d \tag{25}$$

which is a univariate polynomial and completes the set of the reduced Groebner basis of the set G summarized as follows:

$$G := \begin{bmatrix} g_2 = x_1^2 - x_1x_3 + x_3^2 - d \\ g_3 = x_2^2 - x_2x_3 + x_3^2 - d \\ g_4 = -x_1x_2 + x_1x_3 - 2x_3^2 + x_2x_3 + d \\ g_5 = x_1d - x_1x_3^2 + x_2x_3^2 - x_2d - x_3^3 + x_3d \\ g_6 = 2x_3^3 - 2x_3d \end{bmatrix} \tag{26}$$

From the computed reduced Groebner basis in Eq. (26) we note that the element $g_6 = 2x_3^3 - 2x_3 d$ is a univariate polynomial in x_3 and readily gives the values of $x_3 = \{0, \pm\sqrt{d}\}$. We then proceed to derive the solution to the Grunert distance equations, Eqs. (15), as follows. Since $S_3 = x_3 \in \mathbb{R}^+$, the value of $S_3 = +\sqrt{d}$. This is substituted back into $g_3 = x_2^2 - x_2x_3 + x_3^2 - d$ and $g_2 = x_1^2 - x_1x_3 + x_3^2 - d$ to give $x_2 = \{0, +\sqrt{d}\}$ and $x_1 = \{0, +\sqrt{d}\}$ respectively, thus completing the solution of the Grunert equations, Eqs. (15), for the unknown distances $x_1 = x_2 = x_3 = +\sqrt{d}$, as we had initially assumed.

2.1.2 Groebner basis solution of the Grunert equations for the general 3-D resection problem

We next present the application of Groebner basis technique to the solution of the Grunert equations of Eqs. (12) expressed as follows:

$$\begin{aligned} &x_1^2 + x_2^2 + 2a_{12}x_1x_2 + a_0 = 0 \\ &x_2^2 + x_3^2 + 2b_{23}x_2x_3 + b_0 = 0 \\ &x_3^2 + x_1^2 + 2c_{31}x_3x_1 + c_0 = 0 \end{aligned} \tag{27}$$

where

$$\begin{aligned} &S_1 = x_1 \in \mathbb{R}^+,\ S_2 = x_2 \in \mathbb{R}^+,\ S_3 = x_3 \in \mathbb{R}^+ \\ &-2\cos\psi_{12} = a_{12},\ -2\cos\psi_{23} = b_{23}, -2\cos\psi_{31} = c_{31} \\ &-S_{12}^2 = a_0,\ -S_{23}^2 = b_0,\ -S_{31}^2 = c_0 \end{aligned} \tag{28}$$

We then have the ideal formed from Eqs. (27) as

$$\begin{aligned} \text{ideal } I = \langle &x_1^2 + x_2^2 + 2a_{12}x_1x_2 + a_0, x_2^2 + x_3^2 + 2b_{23}x_2x_3 + b_0, \\ &x_3^2 + x_1^2 + 2c_{31}x_3x_1 + c_0\rangle \end{aligned} \tag{29}$$

whose Groebner bases are computed following the advice of Buchberger (pers. commun. 1999) using Mathematica 3.0 after Lexicographic ordering $(x_1 > x_2 > x_3)$ by the command Groebner Basis as

$$\begin{aligned} &\text{Groebner Basis } [\{x_1^2 + x_2^2 + 2a_{12}x_1x_2 + a_0, \\ &x_2^2 + x_3^2 + 2b_{23}x_2x_3 + b_0, \\ &x_1^2 + x_3^2 + 2c_{31}x_1x_3 + c_0\}, \\ &\{x_1, x_2, x_3\}] \end{aligned} \tag{30}$$

The execution of the Mathematica 3.0 command above gives the computed Groebner basis of the ideal, Eq. (29), as expressed in Boxes 3a and 3b.

From the computed Groebner basis of the ideal $I \subset \mathbb{R}[x_1, x_2, x_3]$ above, we note that the element g_1 in Box 3a is a univariate polynomial in x_3. With the coefficients of g_1 known, the univariate polynomial is then solved for $x_3 \in \mathbb{R}^+$ and the admissible values inserted in g_{11} in Box 3b to obtain $x_1 \in \mathbb{R}^+$. The obtained values of $x_3 \in \mathbb{R}^+$ and $x_1 \in \mathbb{R}^+$ are now inserted in any of the remaining elements of the Groebner basis $g_2, \ldots, g_{10}$ in Box 3b to obtain the remaining variable $x_2 \in \mathbb{R}^+$. The correct distances are finally deduced with the help of prior information (e.g. from an existing map).

Fischler and Bolles (1981, pp. 386–387, Fig. 5) have demonstrated that because every term in Eqs. (12) is either a constant or of degree 2, for every real positive solution there exists a geometrically isomorphic negative solution. Thus there are at most four positive solution to Eq. (12). This is because Eq. (12) has eight solutions according to Chrystal (1964, p. 415), who states that for n independent polynomial equations in n unknowns, there can be no more solutions than the product of their respective degrees. Since each equation of Eqs. (12) is of degree 2 there can only be up to eight solutions.

2.2 The position-derivation step (3-D ranging or '3-D Bogenschnitt')

This step is commonly referred to in German literature as the 'Bogenschnitt' problem and in English literature as the 'ranging problem' or 'Arc section' (Kahmen and Faig 1988, p. 215), and is the problem of establishing the position of a point given the distances from the unknown point $P \in \mathbb{E}^3$ to three other known stations $P_i \in \mathbb{E}^3 \mid i = 1, 2, 3$. In general the 3-D 'Bogenschnitt' problem can be formulated as follows: Given distances as observations or pseudo-observations from an unknown point $P \in \mathbb{E}^3$ to a minimum of three known points $P_i \in \mathbb{E}^3 \mid i = 1, 2, 3$, determine the position $\{X, Y, Z\}$ of the unknown point $P \in \mathbb{E}^3$. When only three known stations are used to determine the position of the unknown station in three dimensions, the problem reduces to that of a 3-D closed-form solution. We present below the Groebner basis approach for solving the '3-D Bogenschnitt' problem in a closed form.

Starting from three nonlinear 3-D Pythagorus distance observation equations, Eqs. (31) in Box 4, relating to the three unknowns $\{X, Y, Z\}$, two equations with three unknowns are derived. Equation (31) is expanded in the form given by Eq. (32) and differenced in Eq. (33) to eliminate the quadratic terms $\{X^2, Y^2, Z^2\}$. Collecting all the known terms of Eq. (33) on the right-hand side and those relating to the unknowns on the left-hand side leads to Eq. (34) with the terms $\{a, b\}$ given by Eq. (35). The solution of the unknown terms $\{X, Y, Z\}$ now involves solving equation Eqs. (34), which consist of two equations with three unknowns. To circumvent the problem of having more unknowns than equations, two of the unknowns are sought in terms of the third unknown [e.g. $X = g(Z), Y = g(Z)$].

2.2.1 Groebner basis approach

We express Eqs. (34) in the form of Eqs. (36) in Box 5 with the coefficients as given in Eqs. (37). The Groebner basis is then obtained using the Groebner Basis command in Mathematica 3.0, as illustrated by Eq. (38), giving the computed Groebner basis as in Eqs. (39). The first equation of Eqs. (39) is solved for $Y = g(Z)$ and is as presented in Eq. (40). This value is substituted in the second of Eqs. (39) to give $X = g(Z)$, presented in the first of Eqs. (41). The obtained values of Y and X are substituted in the first of Eqs. (31) to give a quadratic equation in Z. Once this quadratic has been solved for Z,

Box 3a. Computed Groebner basis for the Grunert distance equations – univariate term

$$
\begin{aligned}
g_1 = & \left\{ \begin{array}{l} \{16 - 8a_{12}^2 + a_{12}^4 - 8b_{23}^2 + 2a_{12}^2 b_{23}^2 + b_{23}^4 - 8a_{12}b_{23}c_{31} + 2a_{12}^3 b_{23}c_{31} + 2a_{12}b_{23}^3 c_{31} - 8c_{31}^2 \\ + 2a_{12}^2 c_{31}^2 + 2b_{23}^2 c_{31}^2 + a_{12}^2 b_{23}^2 c_{31}^2 + 2\mathrm{a}_{12}b_{23}c_{31}^3 + c_{31}^4 \end{array} \right\} x_3^8 \\
& + \left\{ \begin{array}{l} -32a_0 + 8a_0 a_{12}^2 + 32b_0 - 16a_{12}^2 b_0 + 2a_{12}^4 b_0 + 16a_0 b_{23}^2 - 2a_0 a_{12}^2 b_{23}^2 - 8b_0 b_{23}^2 + 2a_{12}^2 b_0 b_{23}^2 \\ - 2a_0 b_{23}^4 + 32c_0 - 16a_{12}^2 c_0 + 2a_{12}^4 c_0 - 16b_{23}^2 c_0 + 4a_{12}^2 b_{23}^2 c_0 + 2b_{23}^4 c_0 + 4a_0 a_{12} b_{23} c31 + a_0 a_{12}^3 b_{23} c_{31} \\ - 12a_{12}b_0 b_{23}c_{31} + 3a_{12}^3 b_0 b_{23} c_{31} - a_0 a_{12} b_{23}^3 c_{31} + a_{12} b_0 b_{23}^3 c_{31} - 12a_{12}b_{23}c_0 c_{31} \\ +3a_{12}^3 b_{23} c_0 c_{31} + 3a_{12} b_{23}^3 c_0 c_{31} + 16a_0 c_{31}^2 - 2a_0 a_{12}^2 c_{31}^2 - 16b_0 c_{31}^2 + 4a_{12}^2 b_0 c_{31}^2 - 8a_0 b_{23}^2 c_{31}^2 \\ + 2a_0 a_{12}^2 b_{23}^2 c_{31}^2 + 2b_0 b_{23}^2 c_{31}^2 + a_{12}^2 b_0 b_{23}^2 c_{31}^2 + a_0 b_{23}^4 c_{31}^2 - 8c_0 c_{31}^2 + 2a_{12}^2 c_0 c_{31}^2 \\ + 2b_{23}^2 c_0 c_{31}^2 + a_{12}^2 b_{23}^2 c_0 c_{31}^2 - a_0 a_{12} b_{23} c_{31}^3 + 3a_{12} b_0 b_{23} c_{31}^3 + a_0 a_{12} b_{23}^3 c_{31}^3 + a_{12} b_{23} c_0 c_{31}^3 \\ - 2a_0 c_{31}^4 + 2b_0 c_{31}^4 + a_0 b_{23}^2 c_{31}^4 \end{array} \right\} x_3^6 \\
& + \left\{ \begin{array}{l} 24a_0^2 - 2a_0^2 a_{12}^2 - 48a_0 b_0 + 12a_0 a_{12}^2 b_0 + 24b_0^2 - 10a_{12}^2 b_0^2 + a_{12}^4 b_0^2 - 10a_0^2 b_{23}^2 + a_0^2 a_{12}^2 b_{23}^2 \\ + 12a_0 b_0 b_{23}^2 - 2b_0^2 b_{23}^2 + a_{12}^2 b_0^2 b_{23}^2 + a_0^2 b_{23}^4 - 48a_0 c_0 + 12a_0 a_{12}^2 c_0 + 48b_0 c_0 - 28a_{12}^2 b_0 c_0 + 4a_{12}^4 b_0 c_0 \\ + 20a_0 b_{23}^2 c_0 - 4a_0 a_{12}^2 b_{23}^2 c_0 - 12b_0 b_{23}^2 c_0 + 2a_{12}^2 b_0 b_{23}^2 c_0 - 2a_0 b_{23}^4 c_0 + 24c_0^2 - 10a_{12}^2 c_0^2 + a_{12}^4 c_0^2 \\ - 10b_{23}^2 c_0^2 + 3a_{12}^2 b_{23}^2 c_0^2 + b_{23}^4 c_0^2 - 2a_0^2 a_{12} b_{23} c_{31} + 4a_0 a_{12} b_0 b_{23} c_{31} + a_0 a_{12}^3 b_0 b_{23} c_{31} \\ - 2a_{12} b_0^2 b_{23} c_{31} + a_{12}^3 b_0^2 b_{23} c_{31} + a_0^2 a_{12} b_{23}^3 c_{31} + a_0 a_{12} b_0 b_{23}^3 c_{31} + 4a_0 a_{12} b_{23} c_0 c_{31} \\ + a_0 a_{12}^3 b_{23} c_0 c_{31} - 20a_{12} b_0 b_{23} c_0 c_{31} + 4a_{12}^3 b_0 b_{23} c_0 c_{31} - 2a_0 a_{12} b_{23}^3 c_0 c_{31} + a_{12} b_0 b_{23}^3 c_0 c_{31} \\ - 2a_{12} b_{23} c_0^2 c_{31} + a_{12}^3 b_{23} c_0^2 c_{31} + a_{12} b_{23}^3 c_0^2 c_{31} - 10a_0^2 c_{31}^2 + a_0^2 a_{12}^2 c_{31}^2 + 20a_0 b_0 c_{31}^2 \\ - 4a_0 a_{12}^2 b_0 c_{31}^2 - 10b_0^2 c_{31}^2 + 3a_{12}^2 b_0^2 c_{31}^2 + 3a_0^2 b_{23}^2 c_{31}^2 - 4a_0 b_0 b_{23}^2 c_{31}^2 + a_0 a_{12}^2 b_0 b_{23}^2 c_{31}^2 \\ + b_0^2 b_{23}^2 c_{31}^2 + 12a_0 c_0 c_{31}^2 - 12b_0 c_0 c_{31}^2 + 2a_{12}^2 b_0 c_0 c_{31}^2 - 4a_0 b_{23}^2 c_0 c_{31}^2 + a_0 a_{12}^2 b_{23}^2 c_0 c_{31}^2 \\ + a_{12}^2 b_0 b_{23}^2 c_0 c_{31}^2 - 2c_0^2 c_{31}^2 + a_{12}^2 c_0^2 c_{31}^2 + b_{23}^2 c_0^2 c_{31}^2 + a_0^2 a_{12} b_{23} c_{31}^3 - 2a_0 a_{12} b_0 b_{23} c_{31}^3 \\ + a_{12} b_0^2 b_{23} c_{31}^3 + a_0 a_{12} b_{23} c_0 c_{31}^3 + a_{12} b_0 b_{23} c_0 c_{31}^3 + a_0^2 c_{31}^4 - 2a_0 b_0 c_{31}^4 + b_0^2 c_{31}^4 \end{array} \right\} x_3^4 \\
& + \left\{ \begin{array}{l} -8a_0^3 + 24a_0^2 b_0 - 2a_0^2 a_{12}^2 b_0 - 24a_0 b_0^2 + 4a_0 a_{12}^2 b_0^2 + 8b_0^3 - 2a_{12}^2 b_0^3 + 2a_0^3 b_{23}^2 - 4a_0^2 b_0 b_{23}^2 \\ + 2a_0 b_0^2 b_{23}^2 + 24a_0^2 c_0 - 2a_0^2 a_{12}^2 c_0 - 48a_0 b_0 c_0 + 16a_0 a_{12}^2 b_0 c_0 + 24b_0^2 c_0 - 14a_{12}^2 b_0^2 c_0 + 2a_{12}^4 b_0^2 c_0 \\ - 6a_0^2 b_{23}^2 c_0 + a_0^2 a_{12}^2 b_{23}^2 c_0 + 8a_0 b_0 b_{23}^2 c_0 - 2b_0^2 b_{23}^2 c_0 + a_{12}^2 b_0^2 b_{23}^2 c_0 - 24a_0 c_0^2 + 4a_0 a_{12}^2 c_0^2 \\ + 24b_0 c_0^2 - 14a_{12}^2 b_0 c_0^2 + 2a_{12}^4 b_0 c_0^2 + 6a_0 b_{23}^2 c_0^2 - 2a_0 a_{12}^2 b_{23}^2 c_0^2 - 4b_0 b_{23}^2 c_0^2 + 8c_0^3 - 2a_{12}^2 c_0^3 - 2b_{23}^2 c_0^3 \\ + a_{12}^2 b_{23}^2 c_0^3 + a_0^3 a_{12} b_{23} c_{31} - a_0^2 a_{12} b_0 b_{23} c_{31} - a_0 a_{12} b_0^2 b_{23} c_{31} + a_{12} b_0^3 b_{23} c_{31} \\ - a_0^2 a_{12} b_{23} c_0 c_{31} + 6a_0 a_{12} b_0 b_{23} c_0 c_{31} + a_0 a_{12}^3 b_0 b_{23} c_0 c_{31} - 5a_{12} b_0^2 b_{23} c_0 c_{31} + a_{12}^3 b_0^2 b_{23} c_0 c_{31} \\ - a_0 a_{12} b_{23} c_0^2 c_{31} - 5a_{12} b_0 b_{23} c_0^2 c_{31} + a_{12}^3 b_0 b_{23} c_0^2 c_{31} + a_{12} b_{23} c_0^3 c_{31} + 2a_0^3 c_{31}^2 - 6a_0^2 b_0 c_{31}^2 \\ + a_0^2 a_{12}^2 b_0 c_{31}^2 + 6a_0 b_0^2 c_{31}^2 - 2a_0 a_{12}^2 b_0^2 c_{31}^2 - 2b_0^3 c_{31}^2 + a_{12}^2 b_0^3 c_{31}^2 - 4a_0^2 c_0 c_{31}^2 \\ + 8a_0 b_0 c_0 c_{31}^2 - 4b_0^2 c_0 c_{31}^2 + 2a_0 c_0^2 c_{31}^2 - 2b_0 c_0^2 c_{31}^2 + a_{12}^2 b_0 c_0^2 c_{31}^2 \end{array} \right\} x_3^2 \\
& + a_0^4 - 4a_0^3 b_0 + 6a_0^2 b_0^2 - 4a_0 b_0^3 + b_0^4 - 4a_0^3 c_0 + 12a_0^2 b_0 c_0 - 2a_0^2 a_{12}^2 b_0 c_0 - 12a_0 b_0^2 c_0 + 4a_0 a_{12}^2 b_0^2 c_0 + 4b_0^3 c_0 \\
& - 2a_{12}^2 b_0^3 c_0 + 6a_0^2 c_0^2 - 12a_0 b_0 c_0^2 + 4a_0 a_{12}^2 b_0 c_0^2 + 6b_0^2 c_0^2 - 4a_{12}^2 b_0^2 c_0^2 + a_{12}^4 b_0^2 c_0^2 - 4a_0 c_0^3 + 4b_0 c_0^3 - 2a_{12}^2 b_0 c_0^3 + c_0^4
\end{aligned}
$$

The values of Y and X can be obtained from Eqs. (40) and (41) respectively. We mention here that the direct solution of $X = g(Z)$ as presented in the second of Eqs. (41) could be obtained by computing theˇreduced Groebner basis (Awange 2002b), rather than solving for $Y = g(Z)$ and substituting in the second of Eqs. (39) to give $X = g(Z)$ presented in first of Eqs. (41). Similarly we could obtain $Y = g(Z)$ alone by replacing Y with X in the option section of the reduced Groebner basis.

A pair of solutions (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2) are obtained. The correct solution from this pair is obtained with the help of prior information, e.g. from an existing map. Of importance is the problem of bifurcation, that is, identifying the point where the quadratic equation has only one solution, i.e. bifurcates. Bancroft (1985), Abel and Chaffee (1991), Chaffee and Abel (1994), and Grafarend and Shan (1997) have already treated this problem.

In Box 6 the critical configuration of the 3-D ranging problem is presented. First the derivatives of the ranging equations, Eqs. (42), are computed as in Eqs. (43). The determinants of the matrices formed by the derivatives are obtained by a triple scalar product to give Eq. (45). The computed determinants and Eqs. (47) and (48) indicate the critical configuration to be the case that the points $P(X, Y, Z)$, $P_1(X_1, Y_1, Z_1)$, $P_2(X_2, Y_2, Z_2)$, and $P_3(X_3, Y_3, Z_3)$ lie on a plane. A complete treatment of the ranging problem is presented in Awange et al. (in press)

3 Test network 'Stuttgart Central'

We consider in this section the closed-form solution of the 3-D resection problem using the Groebner basis approach (Awange 2002b). The test network 'Stuttgart Central' in Fig. 2 is selected for study. First, we consider the observations of the test network 'Stuttgart Central' of types GPS coordinates, horizontal directions T_i and vertical directions B_i that will be used in the experiment.

3.1 Observations

The following experiment was performed at the centre of Stuttgart on one of the pillars of the University buildings

Box 3b. Computed Groebner basis for the Grunert distance equations – multivariate terms

$$
\begin{aligned}
g_2 = &(a_0a_{12}b_{23} + a_{12}b_0b_{23} - a_{12}b_{23}c_0 + 2a_0c_{31} - 2b_0c_{31} - 2c_0c_{31})x_1 \\
&+ (2b_0b_{23} - 2a_0b_{23} + 2b_{23}c_0 - a_{12}^2b_{23}c_0 - 2a_{12}c_0c_{31})x_2 \\
&+ (b_{23}^2c_0 - a_0b_{23}^2 - b_0b_{23}^2 + a_0a_{12}b_{23}c_{31} - a_{12}b_0b_{23}c_{31} \\
&+ 2a_0c_{31}^2 - 2b_0c_{31}^2)x_3 + (b_{23}^2c_{31} - 4c_{31})x_1x_3^2 \\
&+ (4b_{23} - a_{12}^2b_{23} - b_{23}^3 - 2a_{12}c_{31} - a_{12}b_{23}^2c_{31} - 2b_{23}c_{31}^2)x_2x_3^2 \\
&- (a_{12}b_{23}c_{31} + 2c_{31}^2)x_3^3
\end{aligned}
$$

$$
\begin{aligned}
g_3 = &(a_{12}b_0 - a_0a_{12} + a_{12}c_0)x_1 + a_{12}^2c_0x_2 + (b_0b_{23} - a_0b_{23} + b_{23}c_0 \\
&- a_0a_{12}c_{31} + a_{12}b_0c_{31})x_3 + (2a_{12} + b_{23}c_{31})x_1x_3^2 \\
&+ (a_{12}^2 + b_{23}^2 + a_{12}b_{23}c_{31})x_2x_3^2 + (2b_{23} + a_{12}c_{31})x_3^3
\end{aligned}
$$

$$
\begin{aligned}
g_4 = &(2a_0b_0 - a_0^2 + -b_0^2 + 2a_0c_0 - 2b_0c_0 - c_0^2)x_1 \\
&+ (a_0a_{12}c_0 - a_{12}b_0c_0 - a_{12}c_0^2)x_2 + (2a_0b_0c_{31} - a_{12}b_0b_{23}c_0 \\
&- a_0^2c_{31} - b_0^2c_{31} + a_0c_0c_{31} - b_0c_0c_{31})x_3 + (4a_0 - 4b_0 - a_0b_{23}^2 \\
&- 4c_0 + b_{23}^2c_0)x_1x_3^2 + (a_0a_{12} - a_{12}b_0 - 3a_{12}c_0 - b_{23}c_0c_{31})x_2x_3^2 \\
&+ (-a_{12}b_0b_{23} - a_{12}b_{23}c_0 + 3a_0c_{31} - 3b_0c_{31} - a_0b_{23}^2c_{31} \\
&- c_0c_{31})x_3^3 + (-4 + b_{23}^2)x_1x_3^4 + (-2a_{12} - b_{23}c_{31})x_2x_3^4 \\
&+ (-a_{12}b_{23} - 2c_{31})x_3^5
\end{aligned}
$$

$$
\begin{aligned}
g_5 = &\, a_0 - b_0 - c_0 + a_{12}x_1x_2 - c_{31}x_1x_3 - b_{23}x_2x_3 - 2x_3^2 \\
g_6 = &\, a_{12}b_0c_0c_{31} + (a_0c_{31} - b_0c_{31} - c_0c_{31})x_1x_2 \\
&+ (a_0a_{12} + a_{12}b_0 - a_{12}c_0 + a_0b_{23}c_{31} - b_{23}c_0c_{31})x_1x_3 \\
&+ (-2a_0 + 2b_0 + 2c_0 - a_{12}^2c_0 + a_0c_{31}^2 - b_0c_{31}^2)x_2x_3 \\
&+ (-a_0b_{23} - b_0b_{23} + b_{23}c_0 + a_0a_{12}c_{31} + a_{12}c_0c_{31} \\
&+ a_0b_{23}c_{31}^2)x_3^2 + (4 - a_{12}^2 - b_{23}^2 - a_{12}b_{23}c_{31} - c_{31}^2)x_2x_3^3 \\
g_7 = &\, a_{12}b_0b_{23}c_0 + (a_0b_{23} - b_0b_{23} - b_{23}c_0)x_1x_2 + (-2a_0 + 2b_0 \\
&+ a_0b_{23}^2 + 2c_0 - b_{23}^2c_0)x_1x_3 + (2a_{12}c_0 + a_0b_{23}c_{31} \\
&- b_0b_{23}c_{31})x_2x_3 + (a_{12}b_0b_{23} + a_{12}b_{23}c_0 - 2a_0c_{31} \\
&+ 2b_0c_{31} + a_0b_{23}^2c_{31})x_3^2 + (4 - b_{23}^2)x_1x_3^3 + (2a_{12} + b_{23}c_{31})x_2x_3^3 \\
&+ (a_{12}b_{23} + 2c_{31})x_3^4
\end{aligned}
$$

$$
\begin{aligned}
g_8 = &\, (a_0a_{12} + a_{12}b_0 - a_{12}c_0)x_1 + (-2a_0 + 2b_0 + 2c_0 - a_{12}^2c_0)x_2 \\
&+ (-a_0b_{23} - b_0b_{23} + b_{23}c_0 + a_0a_{12}c_{31} - a_{12}b_0c_{31})x_3 \\
&+ 2c_{31}x_1x_2x_3 + b_{23}c_{31}x_1x_3^2 + (4 - a_{12}^2 - b_{23}^2 - a_{12}b_{23}c_{31})x_2x_3^2 \\
&- a_{12}c_{31}x_3^3 \\
g_9 = &\, (-a_0 + b_0 + c_0)x_1 + a_{12}c_0x_2 + (-a_0c_{31} + b_0c_{31})x_3 + b_{23}x_1x_2x_3 \\
&+ 2x_1x_3^2 + (a_{12} + b_{23}c_{31})x_2x_3^2 + c_{31}x_3^3 \\
g_{10} = &\, -a_{12}b_0c_0 + (-a_0 + b_0 + c_0)x_1x_2 + (-a_0b_{23} + b_{23}c_0)x_1x_3 \\
&+ (-a_0c_{31} + b_0c_{31})x_2x_3 + (-a_{12}b_0 - a_{12}c_0 - a_0b_{23}c_{31})x_3^2 \\
&+ 2x_1x_2x_3^2 + (b_{23} + c_{31})x_2x_3^3 - a_{12}x_3^4 \\
g_{11} = &\, c_0 + x_1^2 + c_{31}x_1x_3 + x_3^2
\end{aligned}
$$

along Kepler Strasse 11, as depicted by Fig. 2. The test network 'Stuttgart Central' consisted of eight GPS points, three of which are here used are listed in Table 1. A theodolite was stationed at pillar K1 whose astronomical longitude Λ_Γ as well as astronomic latitude Φ_Γ were known from previous astrogeodetic observations made by the Department of Geodesy and GeoInformatics, Stuttgart University. Since theodolite observations of the types horizontal directions T_i as well as vertical directions B_i from the pillar K1 to the target points $i, i = 1, 2, \ldots, 3$, were only partially available, we decided to simulate the horizontal and vertical directions from the given values of $\{\Lambda_\Gamma, \Phi_\Gamma\}$ as well as the Cartesian coordinates of the station point (X, Y, Z) and target points (X_i, Y_i, Z_i). In detail, the directional parameters $\{\Lambda_\Gamma, \Phi_\Gamma\}$ of the local gravity vector were adopted from the astrogeodetic observations reported by Kurz (1996, p. 46) with a root-mean-square (RMS) error $\sigma_\Lambda = \sigma_\Phi = 10''$. Table 1 contains the (X, Y, Z) coordinates obtained from a GPS survey of the test-network 'Stuttgart Central', with RMS errors

Box 4. Differencing of the nonlinear distance equations

$$
\begin{aligned}
S_1^2 &= (X_1 - X)^2 + (Y_1 - Y)^2 + (Z_1 - Z)^2 \\
S_2^2 &= (X_2 - X)^2 + (Y_2 - Y)^2 + (Z_2 - Z)^2 \\
S_3^2 &= (X_3 - X)^2 + (Y_3 - Y)^2 + (Z_3 - Z)^2
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
S_1^2 &= X_1^2 + Y_1^2 + Z_1^2 + X^2 + Y^2 + Z^2 - 2X_1X - 2Y_1Y - 2Z_1Z \\
S_2^2 &= X_2^2 + Y_2^2 + Z_2^2 + X^2 + Y^2 + Z^2 - 2X_2X - 2Y_2Y - 2Z_2Z \\
S_3^2 &= X_3^2 + Y_3^2 + Z_3^2 + X^2 + Y^2 + Z^2 - 2X_3X - 2Y_3Y - 2Z_3Z
\end{aligned} \tag{32}
$$

differencing above

$$
\begin{aligned}
S_1^2 - S_2^2 &= X_1^2 - X_2^2 + Y_1^2 - Y_2^2 + Z_1^2 - Z_2^2 + 2X(X_2 - X_1) \\
&\quad + 2Y(Y_2 - Y_1) + 2Z(Z_2 - Z_1) \\
S_2^2 - S_3^2 &= X_2^2 - X_3^2 + Y_2^2 - Y_3^2 + Z_2^2 - Z_3^2 + 2X(X_3 - X_2) \\
&\quad + 2Y(Y_3 - Y_2) + 2Z(Z_3 - Z_2)
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
2X(X_2 - X_1) + 2Y(Y_2 - Y_1) + 2Z(Z_2 - Z_1) &= a \\
2X(X_3 - X_2) + 2Y(Y_3 - Y_2) + 2Z(Z_3 - Z_2) &= b
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
a &= S_1^2 - S_2^2 - X_1^2 + X_2^2 - Y_1^2 + Y_2^2 - Z_1^2 + Z_2^2 \\
b &= S_2^2 - S_3^2 - X_2^2 + X_3^2 - Y_2^2 + Y_3^2 - Z_2^2 + Z_3^2
\end{aligned} \tag{35}
$$

Box 5. Groebner basis approach

$$
\begin{aligned}
a_{02}X + b_{02}Y + c_{02}Z + f_{02} &= 0 \\
a_{12}X + b_{12}Y + c_{12}Z + f_{12} &= 0
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
&a_{02} = 2(X_1 - X_2), b_{02} = 2(Y_1 - Y_2), c_{02} = 2(Z_1 - Z_2) \\
&a_{12} = 2(X_2 - X_3), b_{12} = 2(Y_2 - Y_3), c_{12} = 2(Z_2 - Z_3) \\
&f_{02} = (S_1^2 - X_1^2 - Y_1^2 - Z_1^2) - (S_2^2 - X_2^2 - Y_2^2 - Z_2^2) \\
&f_{12} = (S_2^2 - X_2^2 - Y_2^2 - Z_2^2) - (S_3^2 - X_3^2 - Y_3^2 - Z_3^2)
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
\text{Groebner Basis } [\{&a_{02}X + b_{02}Y + c_{02}Z + f_{02}, \\
&a_{12}X + b_{12}Y + c_{12}Z + f_{12}\}, \{X, Y\}]
\end{aligned} \tag{38}
$$

$$
\begin{aligned}
g_1 &= a_{02}b_{12}Y - a_{12}b_{02}Y - a_{12}c_{02}Z + a_{02}c_{12}Z + a_{02}f_{12} - a_{12}f_{02} \\
g_2 &= a_{12}X + b_{12}Y + c_{12}Z + f_{12} \\
g_3 &= a_{02}X + b_{02}Y + c_{02}Z + f_{02}
\end{aligned} \tag{39}
$$

$$
Y = \frac{\{(a_{12}c_{02} - a_{02}c_{12})Z + a_{12}f_{02} - a_{02}f_{12}\}}{(a_{02}b_{12} - a_{12}b_{02})} \tag{40}
$$

$$
\begin{aligned}
X &= \frac{-(b_{12}Y + c_{12}Z + f_{12})}{a_{12}} \text{ or} \\
X &= \frac{\{(b_{02}c_{12} - b_{12}c_{02})Z + b_{02}f_{12} - b_{12}f_{02}\}}{(a_{02}b_{12} - a_{12}b_{02})}
\end{aligned} \tag{41}
$$

Box 6. critical configuration of the 3-D ranging

$$f_1(X,Y,Z;X_1,Y_1,Z_1,S_1)=(X_1-X)^2+(Y_1-Y)^2+(Z_1-Z)^2-S_1^2$$
$$f_2(X,Y,Z;X_2,Y_2,Z_2,S_2)=(X_2-X)^2+(Y_2-Y)^2+(Z_2-Z)^2-S_2^2$$
$$f_3(X,Y,Z;X_3,Y_3,Z_3,S_3)=(X_3-X)^2+(Y_3-Y)^2+(Z_3-Z)^2-S_3^2 \quad (42)$$

$$\begin{aligned}
\frac{\partial f_1}{\partial X}&=-2(X_1-X), & \frac{\partial f_2}{\partial X}&=-2(X_2-X), & \frac{\partial f_3}{\partial X}&=-2(X_3-X)\\
\frac{\partial f_1}{\partial Y}&=-2(Y_1-Y), & \frac{\partial f_2}{\partial Y}&=-2(Y_2-Y), & \frac{\partial f_3}{\partial Y}&=-2(Y_3-Y)\\
\frac{\partial f_1}{\partial Z}&=-2(Z_1-Z), & \frac{\partial f_2}{\partial Z}&=-2(Z_2-Z), & \frac{\partial f_3}{\partial Zv}&=-2(Z_3-Z)
\end{aligned} \quad (43)$$

$$D=\left|\frac{\partial f_i}{\partial X_j}\right|=-8\begin{vmatrix}X_1-X & Y_1-Y & Z_1-Z\\ X_2-X & Y_2-Y & Z_1-Z\\ X_3-X & Y_3-Y & Z_1-Z\end{vmatrix}$$
$$D\Leftrightarrow\begin{vmatrix}X_1-X & Y_1-Y & Z_1-Z\\ X_2-X & Y_2-Y & Z_1-Z\\ X_3-X & Y_3-Y & Z_1-Z\end{vmatrix}=\begin{vmatrix}X & Y & Z & 1\\ X_1 & Y_1 & Z_1 & 1\\ X_2 & Y_2 & Z_2 & 1\\ X_3 & Y_3 & Z_3 & 1\end{vmatrix}=0 \quad (44)$$

$$\begin{aligned}
-\frac{1}{8}D=&\{-Z_1Y_3+Y_1Z_3-Y_2Z_3+Y_3Z_2--Y_1Z_2+Y_2Z_1\}X\\
&+\{-Z_1X_2-X_1Z_3+Z_1X_3+X_1Z_2-X_3Z_2+X_2Z_3\}Y\\
&+\{Y_1X_2-Y_1X_3+Y_3X_1-X_2Y_3-X_1Y_2+Y_2X_3\}Z\\
&+X_1Y_2Z_3-X_1Y_3Z_2-X_3Y_2Z_1+X_2Y_3Z_1-X_2Y_1Z_3+X_3Y_1Z_2
\end{aligned} \quad (45)$$

thus

$$\begin{vmatrix}X & Y & Z & 1\\ X_1 & Y_1 & Z_1 & 1\\ X_2 & Y_2 & Z_2 & 1\\ X_3 & Y_3 & Z_3 & 1\end{vmatrix} \quad (46)$$

describes six times the volume of the tetrahedron formed by the points $P(X,Y,Z)$, $P_1(X_1,Y_1,Z_1)$, $P_2(X_2,Y_2,Z_2)$, and $P_3(X_3,Y_3,Z_3)$. Therefore

$$D=\begin{vmatrix}X & Y & Z & 1\\ X_1 & Y_1 & Z_1 & 1\\ X_2 & Y_2 & Z_2 & 1\\ X_3 & Y_3 & Z_3 & 1\end{vmatrix}\begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix}=0 \quad (47)$$

results in a system of homogeneous equations

$$\begin{aligned}
&aX+bY+cZ+d=0\\
&aX_1+bY_1+cY_1+d=0\\
&aX_2+bY_2+cZ_2+d=0\\
&aX_3+bY_3+cZ_3+d
\end{aligned} \quad (48)$$

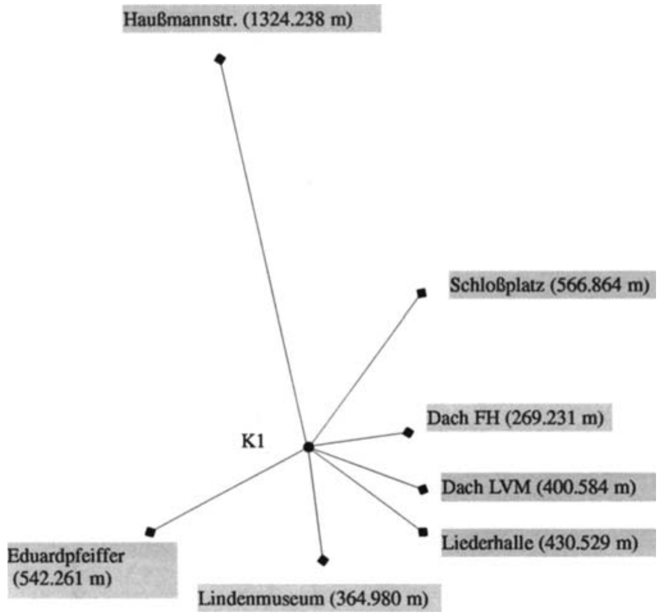

Fig. 2. Test network 'Stuttgart Central'

$(\sigma_X,\sigma_Y,\sigma_Z)$ neglecting the covariances $(\sigma_{XY},\sigma_{YZ},\sigma_{ZX})$. The spherical coordinates of the relative position vector, namely of the coordinate differences (x_i-x, y_i-y, z_i-z), are called horizontal directions T_i, vertical directions B_i and distance S_i, and are given in Table 2. The standard deviations/RMS errors were fixed to $\sigma_T=6'', \sigma_B=6''$. Such RMS errors can be obtained on the basis of a proper refraction model. Since the horizontal and vertical directions of Table 2 were simulated data, with zero noise level, we used a random generator randn in Matlab version 5.3 (Hanselman and Littlefield 1997, pp. 84, 144) to produce additional observational data sets within the framework of the given RMS errors. For each observable of types T_i and B_i, 30 randomly simulated data were obtained and the mean taken. Let us refer to the observational data sets $\{T_i,B_i\}, i=1,2,3,$ of Table 3 which were enriched by the RMS errors of the individual randomly generated observations as well as by the differences $\Delta T_i := T_i - T_i$ (generated), $\Delta B_i := B_i - B_i$(generated). Such differences $(\Delta T_i, \Delta B_i)$ indicate the difference between the ideal values of Table 2 and those randomly generated.

The observations are thus designed such that, by observing the other seven GPS stations, the orientation of the local-level reference frame $\mathbb{F}^*$, whose origin is station K1, with respect to the global reference frame $\mathbb{F}^\bullet$

Table 1. GPS coordinates in the global reference frame $\mathbb{F}^\bullet(X,Y,Z),(X_i,Y_i,Z_i), i=1,2,3$

Station name	X (m)	Y (m)	Z (m)	σ_X (m)	σ_Y (m)	σ_Z (m)
Haussmanstr.	4,156,749.5977	672,711.4554	4,774,981.5459	0.00177	0.00159	0.00161
Eduardpfeiffer	4,156,748.6829	671,171.9385	4,775,235.5483	0.00193	0.00184	0.00187
Liederhalle	4,157,266.6181	671,099.1577	4,774,689.8536	0.00129	0.00128	0.00134

Table 2. Ideal spherical coordinates of the relative position vector in the local horizontal reference frame $\mathbb{F}^*$: spatial distance, horizontal direction, vertical direction

Station observed from K1	Distance (m)	Horizontal direction (gon)	Vertical direction (gon)
Haussmanstr.	1324.2380	107.160333	0.271038
Eduardpfeiffer	542.2609	224.582723	4.036011
Liederhalle	430.5286	336.851237	−6.941728

is obtained. The direction of Schlossplatz is chosen as the zero direction of the theodolite and this leads to the determination of the third component Σ_Γ of the 3-D orientation parameters. The observations of types horizontal directions T_i and vertical directions B_i are measured to each of the GPS target points i. The spatial distances $S_i^2(\mathbf{X}, \mathbf{X}_i) = \|\mathbf{X}_i - \mathbf{X}\|$ are readily obtained from the observations of types horizontal directions T_i and vertical directions B_i using the algebraic computational technique of the Groebner basis (Awange 2002b). Once we have Euclidean distances S_i computed from the observations of types horizontal directions T_i and vertical directions B_i, a forward computation using the Groebner basis (to solve the 3-D ranging problem) is performed to compute the coordinates $\{X, Y, Z\}_{\text{GPS}}$, from which the direction parameters $(\Lambda_\Gamma, \Phi_\Gamma)$ of the local gravity vector at K1 and the 'orientation unknown' element Σ_Γ are finally computed as discussed in Awange (2003) and Grafarend and Awange (2000). The obtained values are then compared to the starting values. The following symbols are used: $\sigma_X, \sigma_Y, \sigma_Z$ are the standard errors of the GPS Cartesian coordinates; covariances $\sigma_{XY}, \sigma_{YZ}, \sigma_{ZX}$ are neglected; σ_T, σ_B are the standard deviation of horizontal and vertical directions respectively after an adjustment; and Δ_T, Δ_B are the magnitude of the noise in the horizontal and vertical directions, respectively.

3.2 Experiment

In order to position point K1 in the GPS network of 'Stuttgart Central' using local positioning system (LPS) observables of type horizontal directions T_i and vertical directions B_i, three known stations (Haussmanstr., Eduardpfeiffer and Liederhalle) of the test network 'Stuttgart Central' in Fig. 2 are used. We proceed in three steps: the first step considers the computation of the spatial distances, the second step is the computation of the coordinates of the unknown station, and the final step is the computation of the 3-D orientation parameters. In this section, the 3-D resection method is considered with the aim of providing the 3-D geocentric GPS coordinates in the global reference frame. The 3-D orientation parameters of types astronomical longitude, astronomical latitude, and the 'orientation unknown' in the horizontal plane and the deflection of the vertical can be obtained as in Awange (2003) and Grafarend and Awange (2000).

The solution of Grunert equations is carried out, followed by the position-derivation step which involves computing the desired 3-D GPS Cartesian coordinates $\{X, Y, Z\}_{\text{GPS}}$ of the unknown point $P \in \mathbb{E}^3$ in the global reference frame using the algebraic computational technique of the Groebner basis.

Using the computed univariate polynomial (element of the Groebner basis of the ideal $I \subset \mathbb{R}[x_1, x_2, x_3]$) in Box 3a in Sect. 2, we determine the distances $S_i = x_i \in \mathbb{R}^+, i = \{1, 2, 3\} \in \mathbb{Z}^3_+$, between the unknown station $P \in \mathbb{E}^3$ and the known stations $P_i \in \mathbb{E}^3$ expressed in Eqs. (12) for the test network 'Stuttgart Central' in Fig. 2. The unknown point P in this case is the pillar K1 on top of the university building at Kepler Strasse 11. Points P_1, P_2, P_3 of the tetrahedron $\{PP_1P_2P_3\}$ in Fig. 1 correspond to the chosen known GPS stations Haussmannstr., Eduardpfeiffer, and Liederhalle. The distance from K1 to Haussmannstr. is designated $S_1 = x_1 \in \mathbb{R}^+$, K1 to Eduardpfeiffer $S_2 = x_2 \in \mathbb{R}^+$, while that of K1 to Liederhalle is designated $S_3 = x_3 \in \mathbb{R}^+$. The distances between the known stations $\{S_{12}, S_{23}, S_{31}\} \in \mathbb{R}^+$ are computed from their respective GPS coordinates as indicated in Box 7. Their corresponding space angles $\psi_{12}, \psi_{23}, \psi_{31}$ are computed from Eq. (10). In order to control the computations, the Cartesian GPS coordinates of point K1 are also known. Box 7 gives the complete solution of the unknowns $\{x_1, x_2, x_3\} \in \mathbb{R}^+$ from the computed Groebner basis of Boxes 3a and 3b in Sect. 2. The univariate polynomial in x_3 has eight roots, four of which are complex and four real. Of the four real roots, two are positives and two are negative. The desired distance $x_3 \in \mathbb{R}^+$ is thus chosen from the two positive roots with the help of prior information and substituted in g_{11} in Box 3b to give two solutions of x_1, one of which is positive. Finally the obtained values of $\{x_1, x_3\} \in \mathbb{R}^+$ are substituted in g_5 (Box 3b) to obtain the remaining indeterminate x_2. Using this procedure, we have in Box 7 that $S_3 = \{430.5286, 153.7112\}$. Since $S_3 = x_3 \in \mathbb{R}^+$, from a priori information we choose $S_3 = 430.5286$, leading to $S_1 = 1324.2381$ and

Table 3. Randomly generated spherical coordinates of the relative position vector: horizontal direction T_i and vertical direction $B_i, i = 1, 2, 3$ RMS errors of individual observations, differences $\Delta T_i := T_i - T_i(\text{generated})$, $\Delta B_i := B_i - B_i(\text{generated})$ with respect to (T_i, B_i) ideal data of Table 2

Station observed from K1	Horizontal direction (gon)	Vertical direction (gon)	σ_T (gon)	σ_B (gon)	Δ_T (gon)	Δ_B (gon)
Haussmanstr.	54.840818	0.271340	0.0027559	0.0028895	−0.000088	0.001659
Eduardpfeiffer	172.263416	4.035779	0.0023929	0.0032068	−0.000296	0.000232
Liederhalle	284.532013	−6.942079	0.0027289	0.0032386	−0.000379	0.000351

Box 7. Computation of distances for test network 'Stuttgart Central'

Using the entries of Table 1, the computed inter-station distances by pythagorus, $S_{ij} = \sqrt{(X_j - X_i)^2 + (Y_j - Y_i)^2 + (Z_j - Z_i)^2}$, and spatial angles from Eq. (10) are given as

$$\begin{array}{lcl} S_{12} = 1560.3302\,\mathrm{m} & & \psi_{12} = 1.843620 \\ S_{23} = 755.8681\,\mathrm{m} & \text{and} & \psi_{23} = 1.768989 \\ S_{31} = 1718.1090\,\mathrm{m} & & \psi_{31} = 2.664537 \end{array}$$

and substituted in Eqs. (28) to compute the terms $\{a_{12}, b_{23}, c_{31}, a_0, b_0, c_0\}$ which are needed to compute the coefficients of the Groebner basis element g_1 in Box 3a in Sect. 2. Expressing theunivariate polynomial g_1 in Box 3a in Sect. 2 as $A_8 x_3^8 + A_6 x_3^6 + A_4 x_3^4 + A_2 x_3^2 + A_0 = 0$, the computed coefficients are

$$A_0 = 4.833922266706213\mathrm{e} + 023$$
$$A_2 = -2.306847176510587\mathrm{e} + 019$$
$$A_4 = 1.104429253262719\mathrm{e} + 014$$
$$A_6 = -3.083017244255380\mathrm{e} + 005$$
$$A_8 = 4.323368172460818\mathrm{e} - 004$$

The solution to the univariate polynomial equation is then obtained from the Matlab command 'roots' (see e.g. Hanselman and Littlefield 1997, p. 146) as

$$c = [A_8\ A_7\ A_6\ A_5\ A_4\ A_3\ A_2\ A_1\ A_0]$$
$$x_3 = \mathrm{roots}(c)$$

the other coefficients being zero. The obtained values of x_3 are

$$x_3 = \left\{ \begin{array}{l} -20757.2530734872 + 8626.43262759353\mathrm{i} \\ -20757.2530734872 - 8626.43262759353\mathrm{i} \\ 20757.2530734872 + 8626.4326275935\mathrm{i} \\ 20757.2530734872 - 8626.4326275935\mathrm{i} \\ 430.528578109464 \\ -430.528578109464 \\ 153.711222705295 \\ -153.711222705295 \end{array} \right\}$$

where the chosen value 430.5286 of $x_3 \in \mathbb{R}^+$ using prior information is substituted in g_{11} in Box 3b to give

$$x_1 = \{-2089.15882397074, 1324.23808451951\}$$

and finally the values of $\{x_1, x_3\} \in \mathbb{R}^+$ are substituted in g_5 in Box 3b to give $x_2 = 542.260767703842$

$S_2 = 542.2608$. These values compare well with the real values depicted in Fig. 2.

The computation procedure using the Buchberger algorithm (Groebner basis) is summarized as follows.

1. Arrange the given polynomial equations using a chosen monomial order as in Eqs. (27).
2. Determine the polynomial ideal as in Eq. (29).
3. Compute the Groebner basis of this ideal using either Mathematica or Maple software.
4. From the computed Groebner basis of the ideal, solve the univariate polynomial for the desired roots.
5. Substitute the obtained chosen root of the univariate polynomial in the other Groebner basis elements in order to obtain the remaining variables.

For the observations in Table 3, distances are obtained using the Groebner basis technique as illustrated above. The results of the computed distances are presented in Table 4. The computed distances are used, with the help of the Groebner basis approach, to determine the position of K1.

4 Results

In Table 4 are presented the results of the computed closed-form 3-D resection distances from K1 to Haussmanstr., K1–Eduardpfeiffer, and K1–Liederhalle and their deviation ΔS obtained by subtracting the computed distance S_i from its ideal value S in Table 2. Observational set number 0* comprised the ideal values of Table 2 which are used as the control experiment. The position results for the set under study are presented in Table 5, with set 0* indicating the results of the theoretical set.

We have demonstrated the power of algebraic computational tools (Groebner basis) in solving the problem of 3-D resection. We have succeeded in demonstrating that by converting the nonlinear observation equations of the 3-D resection into algebraic (polynomial) form, the multivariate system of polynomial equations relating the unknown variables (indeterminate) to the known variables can be reduced to a system of polynomial equations consisting of a univariate polynomial. We have therefore managed to provide symbolic solutions to the 3-D resection problem by obtaining univariate polynomials that can readily be solved numerically once the observations are available. The deviations from the real values (Tables 1 and 2) of the computed distances $\Delta S_i \mid i = 1, \ldots, 3$ and position $\{\Delta X, \Delta Y, \Delta Z\}$ from the Groebner basis procedure were in the millimeter range, as depicted in Tables 4 and 5.

Table 4. Distances computed explicitly by Groebner basis algorithm

Observational set no.	S_1 (m)	S_2 (m)	S_3 (m)	ΔS_1 (m)	ΔS_2 (m)	ΔS_3 (m)
0* (ideal values)	1324.2380	542.2609	430.5286	0.0000	0.0000	0.0000
1	1324.2375	542.2594	430.5299	0.0005	0.0015	−0.0013

Table 5. Position of station K1 computed explicitly by Groebner basis or multipolynomial resultants algorithm

Set no.	X (m)	Y (m)	Z (m)	ΔX (m)	ΔY (m)	ΔZ (m)
0	4,157,066.1116	671,429.6655	4,774,879.3704	0	0	0
1	4,157,066.1107	671,429.6657	4,774,879.3721	0.0009	−0.0002	−0.0017

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