FINITE DIFFERENCES:
Taylor Series, Higher Order Accuracy Cont'd

- Using $$f(x_{i+1}) = f(x_i) + \Delta x\, f'(x_i) + \frac{\Delta x^2}{2!} f''(x_i) + \frac{\Delta x^3}{3!} f'''(x_i) + \ldots + \frac{\Delta x^n}{n!} f^n(x_i) + R_n$$

$$R_n = \frac{\Delta x^{n+1}}{n+1!} f^{(n+1)}(\xi)$$

- Estimate the second-derivative with forward finite-differences at first-order accuracy:

$$\left.\begin{aligned} f(x_{i+1}) &= f(x_i) + \Delta x\, f'(x_i) + \frac{\Delta x^2}{2!} f''(x_i) + O(\Delta x^3) \\ f(x_{i+2}) &= f(x_i) + 2\Delta x\, f'(x_i) + \frac{4\Delta x^2}{2!} f''(x_i) + O(\Delta x^3) \end{aligned}\right\} \left.\begin{aligned} &*(-2) \\ &*(1) \end{aligned}\right\} \Rightarrow f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{\Delta x^2} + O(\Delta x)$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{\Delta x} - \frac{\Delta x}{2!} f''(x_i) + O(\Delta x^2)$$

$$\Rightarrow \boxed{f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{\Delta x} - \frac{\Delta x}{2!} \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{\Delta x^2} + O(\Delta x^2) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2\Delta x} + O(\Delta x^2)}$$