The Forced Simple Pendulum and Chaos

Dominika Kubanek 32636156

April 2024

Abstract

In this paper we attempt to answer what chaos is and how it affects nonlinear systems by first looking at the forced simple pendulum. We go through the forced pendulum in harmonic oscillation, steady periodicity, and finally in chaos, at which point we allow room for statistical interpretations to better understand how chaos affects the system. We then lean into a different nonlinear system known as the Lorenz Attractor to compare chaos in the two systems and see what patterns can be extracted from them.

Contents

1 Introduction

Chaos, at its simplest, is mathematical randomness. With it we can start noticing patterns in otherwise random physical phenomena and develop our understanding of previously unknown to us systems.

The forced simple pendulum (FSP) is a simple approach to seeing how quickly a nonlinear system can fall into chaos. By slightly adjusting parameters we notice large shifts in the motion of the system. For example, provided we keep the damping force constant, if we slowly increase the driving force, we will notice period doubling which eventually would lead to chaos. This is known as bifurcation.[1] While the FSP system does exhibit chaotic behaviour, it still remains rooted in periodic oscillations that are easy to interpret. We will later see how bifurcation affects the period of the oscillations as well as the system in regular simple harmonic oscillations.

Another application of nonlinear systems in chaos comes from the Lorenz Attractor system. This system arises from three coupled nonlinear differential equations, which when solved, are often connected to chaos. We look at chaos within the Lorenz system and solve to test how the parameters affect the chaotic behaviour and compare our findings with the FSP to see if chaos acts the same throughout all nonlinear systems.

To fully appreciate chaos it isn't enough to observe it. Doing so has no practical interpretations as there are no discernible patterns to follow. With this in mind, we take a statistical approach to the problem. To do so, we first solve the second order differential equation of motion for the FSP system.

2 Method

2.1 The Dimensionless Equation

Starting from the equation of motion for the FSP

$$mL^2\ddot{\theta} + k\dot{\theta} + mgLsin(\theta) = FLcos(\Omega t) \tag{1}$$

we divided through by mgL to obtain

$$\frac{L}{g}\ddot{\theta} + \frac{k}{mgL}\dot{\theta} + sin(\theta) = \frac{F}{mg}cos(\Omega t) \tag{2}$$

We then used $t = \tau\sqrt{\frac{L}{g}}$ to re-express the equation as a second order differential equation in terms of τ, the dimensionless time parameter.

$$\frac{dt}{d\tau} = \sqrt{\frac{L}{g}} \tag{3}$$

Let $\frac{d\theta}{d\tau} = \bar{\theta}$,

$$\bar{\theta} = \dot{\theta}\frac{dt}{d\tau} = \sqrt{\frac{L}{g}}\dot{\theta} \tag{4}$$

Now we have the relation between our differential with respect to t and differential with respect to τ,

$$\dot{\theta} = \sqrt{\frac{g}{L}}\bar{\theta} \tag{5}$$

so we can finally complete our dimensionless equation.

$$\sqrt{\frac{k}{m}}\frac{d\theta}{dt} \rightarrow \frac{k}{m}\sqrt{\frac{L}{g}} \tag{6}$$

$$\frac{k}{mgL}\sqrt{\frac{g}{L}} = \frac{k}{mL^{\frac{3}{2}}g^{\frac{1}{2}}} \tag{7}$$

$$\ddot{\theta} = \frac{g}{L}\bar{\bar{\theta}} \tag{8}$$

and finally using $\Omega = (1-\eta)\sqrt{\frac{g}{L}}$ we get

$$\begin{aligned}\bar{\bar{\theta}} &+ \frac{k}{mL\sqrt{gL}}\bar{\theta} + sin(\theta) \\ &= \frac{F}{mg}cos((1-\eta)\tau)\end{aligned} \tag{9}$$

To simplify the equation some more we make $K = \frac{k}{mL\sqrt{gL}}$ and $F_f = \frac{F}{mg}$ where K is the damping coefficient and F_f[1] is the driving coefficient. The equation then becomes $\bar{\bar{\theta}} + K\bar{\theta} + sin(\theta) = F_f cos((1-\eta)\tau)$

The equation of motion for the FSP is now dimensionless and we proceed to turn it into a system of first order differential equations by letting $\bar{\theta} = \gamma$. The resulting system of first order differential equations is hence

$$\begin{bmatrix}\theta \\ \gamma\end{bmatrix} = \begin{bmatrix}\gamma \\ K\gamma - sin(\theta) + F_f cos((1-\eta)\tau)\end{bmatrix} \tag{10}$$

2.2 Solving the Initial Value Problem

Now that we have our system of first order differential equations we define a function for the system that returns the right hand side in equation (8). We solve the problem using the Runge-Kutta 4 (RK4) method, which is pre-installed into the scipy.integrate package under the name solve_ivp. Solve_ivp defaults to the RK4 method so we don't specify which method to use within the function when calling the IVP solver.

In this code we have defined our initial values as $[\theta, \gamma] = [\frac{\pi}{6}, 0]$, which means the system starts with the velocity at zero but at an initial angle of $\frac{\pi}{6}$. We could change the initial angle if we want, but for the sake of

[1] In the code we use F instead of F_f.

consistency, the initial angle throughout this paper remains as $\frac{\pi}{6}$. We call the initial values of the angles (θ_0) within the solve_ivp function, as well as our dimensionless time span and the dimensionless equation of motion function. From there we use standard procedure to solve the differential equations (DE).

In order for our figures to remain readable even in large F_f cases, we constrained the position plot to loop within the range of $-\pi < \theta < \pi$.[2] Using numpy's arctan2 function worked for us in this paper. Without the looping angles, the plots become harder to interpret. From there we plot the velocity of the pendulum against the looped position to see the phase plot, and our looped position against time to see how the FSP system oscillates through time.

2.3 Successive flips

In order to find within the chaotic regime the time between complete circuits of the simple pendulum, we define a function that checks consecutive angles of the system and whether the new angle has become negative. This uses the Poincaré method for time-dependent ordinary differential equations.[3] The Poincaré map essentially acts as a plane across the angular boundary which records when the angle changes direction.[4] This allows us to iterate between consecutive steps of the DE. Within the function we test to see whether two consecutive angles (implementing the iteration scheme as just stated) produce a negative and if the new angle is bigger than zero. If both are true, our function has gone through a complete circuit.

We solve the system of first-order DE again using scipy, this time with an additional dependency within the solve_ivp function for the event of complete circuits. Our output should provide us with a list of each time the pendulum goes over the top, this in turn we loop over the length of the list and subtract consecutive elements to determine the time between each complete circuit. This is, as expected, our time between circuits which we plot as a histogram using seaborn.[2]

3 Results

Our results look at the FSP and its motion both through time and as a phase plot of velocity against position. Three main cases can be seen when collecting the results, and we categorise those as "steady harmonic motion", "complete oscillations", and the "chaotic regime". In order to see any of these cases within the pendulum we note once again that the angles must be within the range of $-\pi < \theta < \pi$.

We start off with testing to see if the program works as intended, as seen in Figure 1, the system under both no force and no damping, produces a simple harmonic oscillation. It is as expected: a pendulum with no driving force and no damping force is just a simple harmonic oscillator.

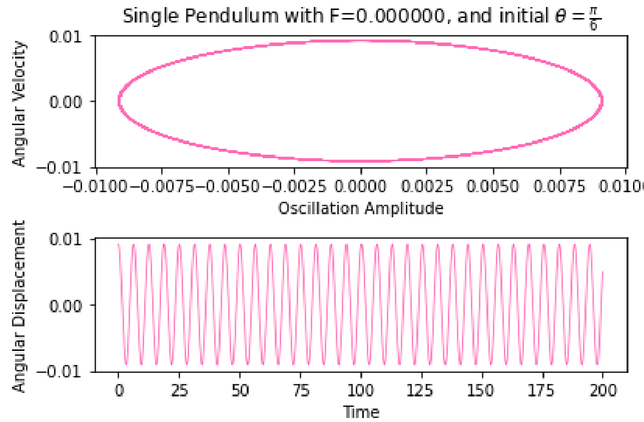

Figure 1: The single pendulum with both driving force and damping coefficient equal to zero (F_f=0, K=0). We see it shows simple harmonic motion, as expected.

3.1 Steady Harmonic Motion

After seeing the correct solution for a pendulum of no force or damping, we set the damping coefficient to remain constant until stated otherwise at K=0.5. We slowly increase the driving force and observe how the motion of the system changes.

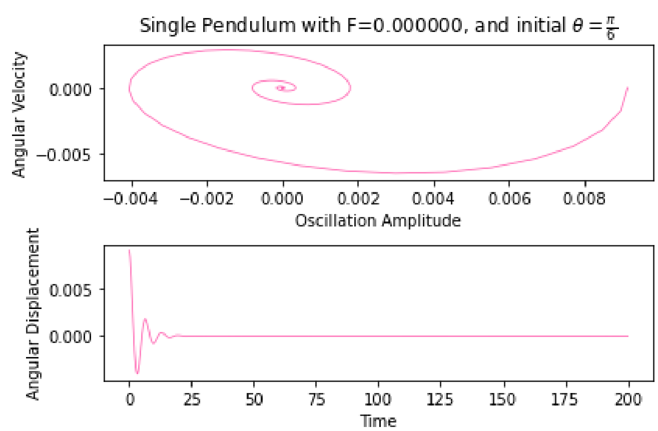

Figure 2: The FSP with driving force F_f=0. Our damping coefficient is K=0.5. We see a steady periodic motion.

We have two sets of plots with Figure 2 having a driving force of $F_f = 0$ and Figure 3 $F_f = 0.5$.

[2]Can be done in MATPLOTLIB. We use seaborn as it is presented neater.

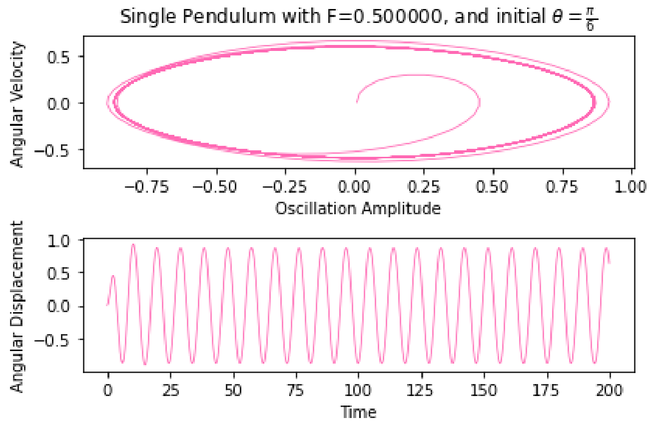

Figure 3: The FSP with F_f=0.5 and K=0.5. The motion still depicts steady harmonic oscillations despite first 25 seconds on the plot. The motion settles and becomes steady from t=25.

In both Figure 2 and Figure 3 we see how the pendulum's motion follows that of a steady harmonic oscillation. In Figure 2 we see the damping of the pendulum with no driving force allows for a short periodic oscillation before its velocity eventually goes to zero, meaning the pendulum has completely stopped; whereas in Figure 3 we observe the oscillations start at a higher amplitude due to the driving force, before it settles into a steady, stable periodic oscillation thanks to both the driving force and damping force balancing the system.

While the system is steadily periodic even at $F_f = 1.0$, the plots look remarkably similar to that of $F_f = 0.50$ and hence we don't include them in the text.

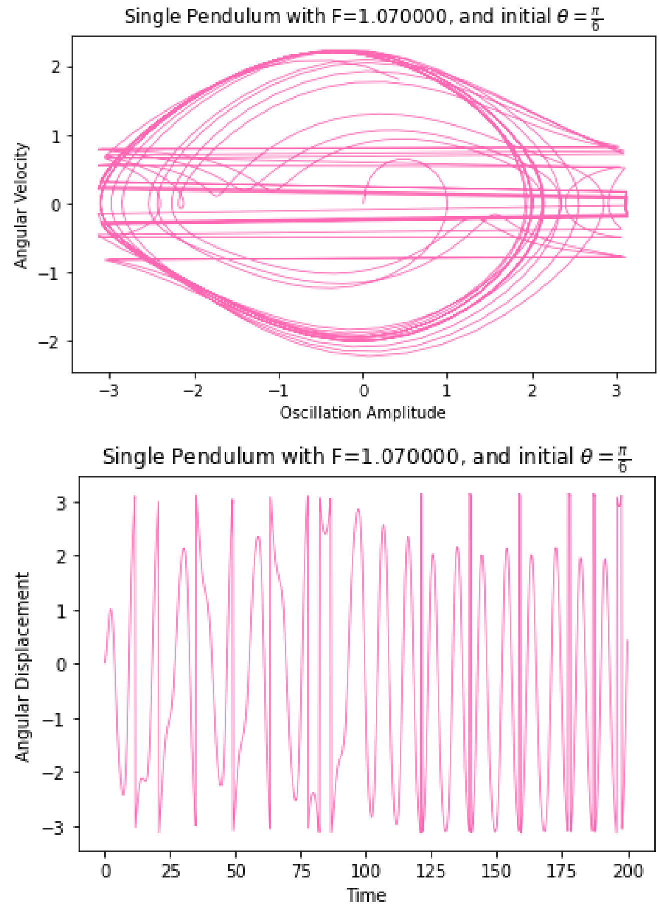

Figure 4: We can see that the simple pendulum at F_f=1.07 is chaotic. From the time plot alone we note the inconsistency throughout its propagation. In the top plot we see it completes oscillations as all other examples thus far, but at a much more sporadic rate.

3.2 Chaotic Behaviour

Chaos is best described as a mathematical approach to unpredictable behaviours seen in systems.[5] Although it is random, the value of chaos is that there is often an underlying pattern the randomness seems to abide by which can tell us a lot about the system's nature.

In our forced pendulum system we notice chaos when the driving force is around $F_f = 1.07$ and $F_f = 1.5$.

While both $F_f = 1.07$ and $F_f = 1.5$ are chaotic regimes of the FSP system, had we allowed for the simulation to continue for much longer, we would begin to notice that for the case in Figure 4 the motion eventually settles into a periodic regime. While this would normally pose problems, as we are restricting ourselves to measure only the first 200 steps, it is not. Within this time period the system behaves as a chaotic one, and therefore we view it as such.

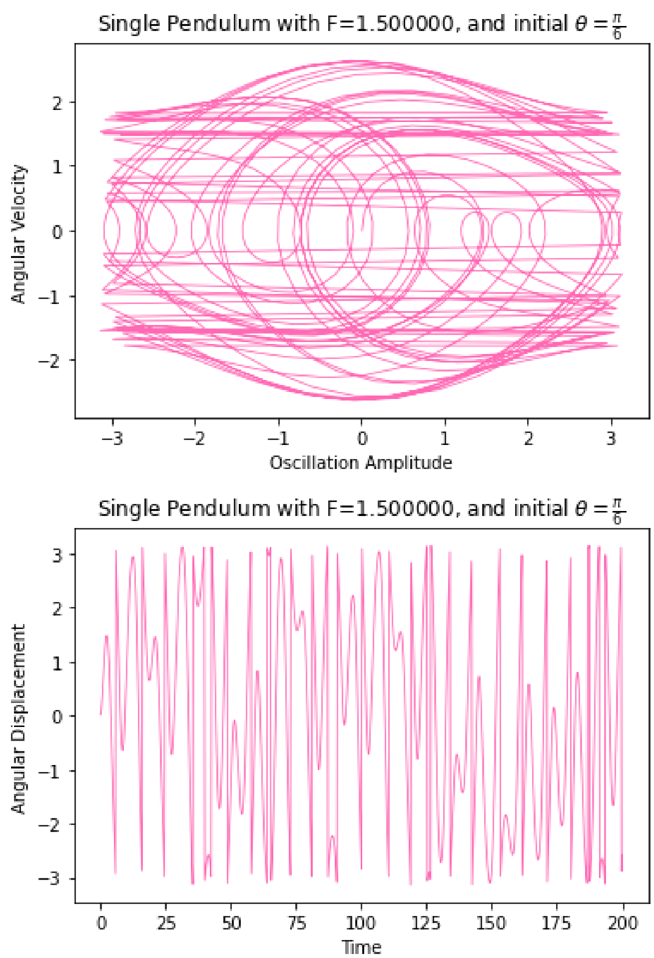

Figure 5: As in Figure 4, we observe the chaotic regime of the simple pendulum, this time with the driving force, F_f=1.5. Much like with Figure 4, we notice complete circuits but at infrequent intervals and with no obvious pattern.

3.3 Complete Oscillations

We've covered both steady harmonic motion and the chaotic behaviour of a FSP, but there is a case where the pendulum follows neither path. While the pendulum does complete full circuits, it isn't harmonic like in the first case, yet it is too stable to be chaotic.

Our first look at this case is when $F_f = 1.1$ as in Figure 6. We see the plot of position against time in the bottom of Figure 6 that the pendulum completes full oscillations despite it clearly not being steady. We note the damping force affecting the system more at this driving force. Through the phase diagram we confirm that full oscillations are made.

When $F_f = 1.37$ (Figure 7) we observe a similar phenomenon to Figure 6. While the system does complete full oscillations it is far from a steady harmonic pattern. When we look at the phase plot for $F_f = 1.37$ we definitely observe full circuits, as well as oscillations from a smaller amplitude. The phase plot compared with the time plot confirms that the oscillation of the pendulum grows and shrinks in amplitude as it varies in time.

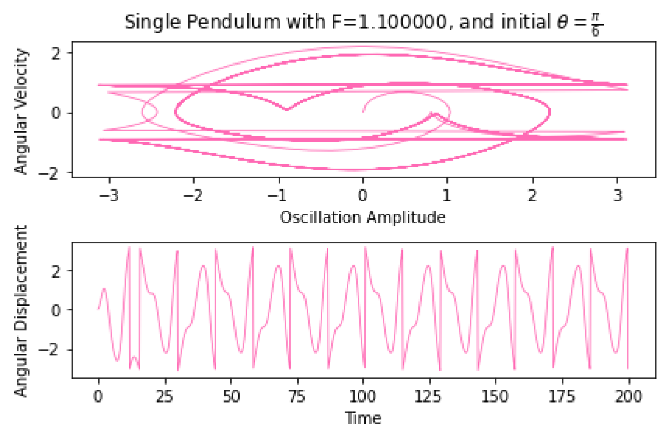

Figure 6: We see the simple pendulum has broken the steady periodic motion, both from the time graph as well as from the phase diagram itself. The pendulum still undergoes complete oscillations at equal intervals.

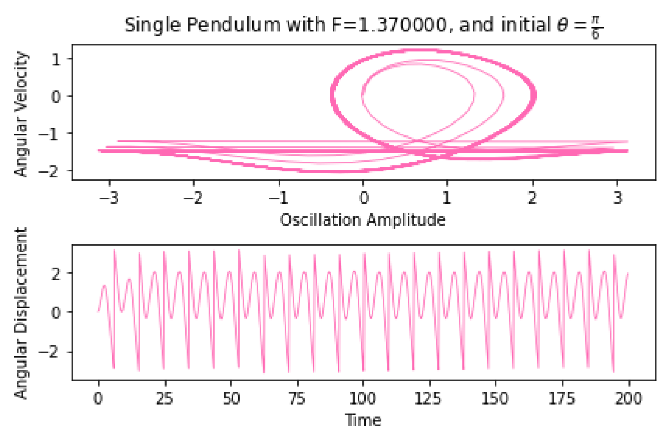

Figure 7: The simple pendulum is not a harmonic oscillation while also not being chaotic, but it completes full oscillations regardless. We can see from the time graph that the pendulum is in fact stable as it oscillates.

3.4 Statistical Behaviour of the Chaotic Regime

An interesting focal point is how the chaotic regime completes full circuits. Throughout this section we have kept $F_f = 1.5$. We could test both chaotic regimes we have outlined in section 3.2 but we choose not to as it would be trivial to do so. We tested to see how the time between successive circuits depends on length of the pendulum, mass of the pendulum, and the initial conditions.

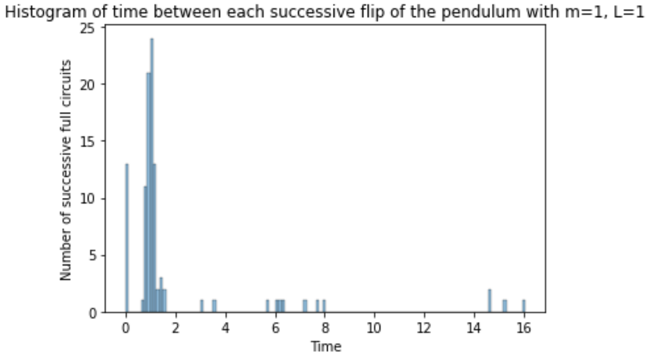

Figure 8: The damping coefficient is at K=0.5. Here we see how the time between successive flips is distributed without making any changes to the parameters. The driving force F_f=1.5 (it is in the chaotic regime). We see the time between successive flips on the x-axis, and the number of successive flips in the y-axis.

Our first point of investigation was seeing what the successive circuits looked like if we kept everything the exact same. In Figure 8 we see that majority of the complete circuits are separated by approximately $\tau = 1$ but the separation in time extends to as far as $\tau = 16$.

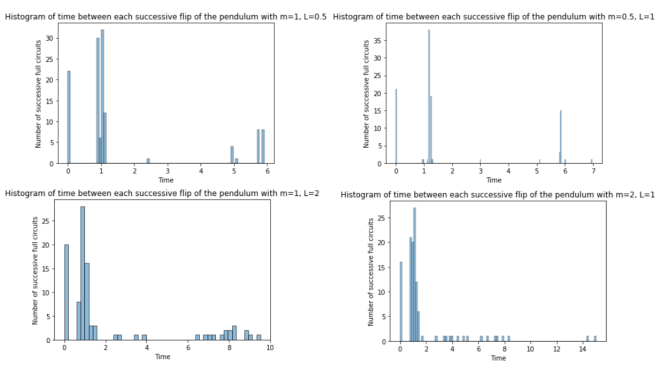

Figure 9: On the left hand side we test to see the dependence of the successive circuits on the length, L, while keeping mass, m, constant. On the right hand side we change the mass, m, to test how mass affects the time between complete circuits while keeping length constant.

By changing the length and mass separately we keep the investigation fair. The first thing we notice when changing the parameters is that changing the mass causes a wider distribution, with more 'bins' than what we see when we change the length. When we half the length and mass respectively, we see for the length that the distribution is centered around $\tau \approx 1$, while for mass it moves farther out to be around $\tau \approx 1.2$, with a far higher number of successive full circuits being at that point.

When we double the length and mass respectively, we see a similar situation occur as when we half the values, but with a larger range in the time (going as high as $\tau = 15$ for the mass change). Still, we observe the length change has a much smaller distribution of times than that for the mass change.

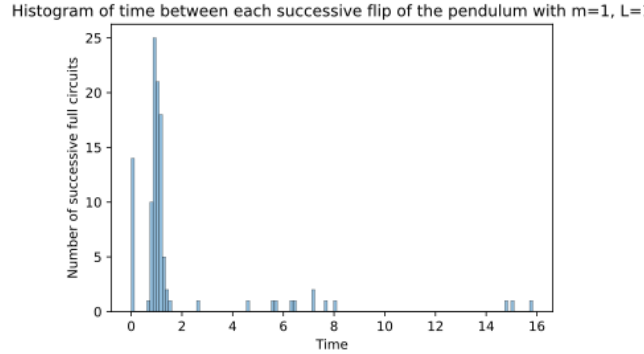

Figure 10: As chaos is heavily dependent on initial conditions we also changed the initial value of the velocity to $\frac{\pi}{10}$ to see how it affects the systems completed circuits.

We now change the initial conditions to see how the chaos is affected. We don't include any plots where we change the initial position value as the distribution remains the same, albeit with a larger number of successive full circuits in each bin (statistically speaking, the value hence stays the same).

Setting the initial velocity to $\frac{\pi}{10}$ shows an almost identical distribution to that from Figure 8, but we do notice a couple differences, namely a more even spread of data centered around $\tau = 1$. The points beyond $\tau = 2$ also seem to be more even.

4 Discussion

4.1 Simple Pendulum and its behaviour

As stated in sections 3.1 and 3.3, we see complete circuits throughout the system, a lot of them following a steady periodic pattern. In Figure 2 we clearly see that the damping force causes the pendulum to stop moving when it experiences no driving force, which, despite being intuitive, shows us that the pendulum is physically realistic. The effect of the damping force is again seen in Figure 6 and 7 when the motion is no longer harmonic. From the time plot we can clearly see the damping force's effect on the system through the bumps as the system attempts to complete a full oscillation. The driving force, in turn, continues to push the system forward and hence the system completes the oscillation.

Another thing to note is that while in sections 3.1 and 3.3 we see periodic motion, plots in 3.1 are harmonic while the plots in 3.3 are not. While both seem to be steady in their periodicity, we do note that this only appears to be as such due to the wrapping of the angles.

Overall, this is as we expect from a simple pendulum with added forces. In all four cases we present in sections 3.1 and 3.3 the system doesn't act out of the ordinary. That is however changed when we look at the cases from section 3.2.

4.2 Chaos in the Simple Pendulum

As stated in section 1, bifurcation causes for the period to double with an increase in F_f amplitude and eventually will lead to chaos in a nonlinear system.

Evidently from Figure 9, the forced simple pendulum in chaos is sensitive to changes in both the initial conditions as well as the pendulum parameters. We notice that when we vary the length of the pendulum, the chaotic system becomes a lot less spread out whereas when we vary the mass, the system seems to fall deeper into chaos, with the distribution being much wider. In both cases, the system seemed to average across $\tau \approx 1$. We also note that if we were to extend the range of τ, we would notice the spread change only marginally, with a higher density of circuits being focused on the same average point and therefore we are content with the time range we have set.

But still we see some strange behaviour from the chaos. When we keep everything the same in Figure 8, there are 14 successive circuits where the time is 0 between each circuit. This would mean that the system changes direction right after changing direction in the previous event. This could explain why the system in chaos has such inconsistent spikes throughout its allowed cycle however assuming this instantaneous switch in direction isn't an error, the velocity should also switch instantaneously. Fortunately from the phase graph in Figures 4 and 5 we see that this is the case.

As we know, chaos can seem to follow specific patterns despite its randomness. If we look at Figure 5, we see a somewhat sinusoidal pattern for the peaks and troughs in the time plot. We emphasise that the shape is not exactly sinusoidal as we only see a snapshot of the full cycle. What we can infer from this is that whilst the system varies randomly in time, the randomness seems to repeat, though not quite periodically.

With all this in mind, we can interpret that the FSP, as a nonlinear system of motion, is highly sensitive to external forces and conditions. By small changes to the system we notice doubling in the period, and even occasional chaos. As with any nonlinear system[6] we also see the system return to non-chaotic behaviour when the conditions are right after it has already been observed in chaos.

But what does this mean for chaos? If we have a system that can fall into chaotic behaviour, while not being chaotic itself in nature, how do we know that chaos is a real observable and not just an error? Luckily,[7] we have observed chaos in numerous nonlinear system, all concluding similar findings regarding the reality of chaos and so we are content with the FSP and its chaotic regime.

5 More Chaos in the form of the Lorenz Attractor

While the chaotic behaviour of the FSP showed interesting results, for a deeper look into chaos theory we now solve the Lorenz Attractor, sometimes referred to as the "butterfly effect".

The Lorenz system is one known markedly for its chaotic behaviour. It can be used to describe most commonly the weather, but we will use it to aid our understanding of chaos theory itself. Due to the random nature of the system, by studying the way it evolves through changing parameters, and how that leads to its variation in time, we can begin to spot patterns and common behaviours in chaotic systems.

5.1 Lorenz Attractor function

As with the single pendulum, we solve the system's differential equations:[8]

$$\dot{x} = \sigma(y - x) \tag{11}$$

$$\dot{y} = x(\rho - z)y \tag{12}$$

$$\dot{z} = xy - \beta z \tag{13}$$

for which we define a function that takes the input parameters of time and a three-dimensional vector which holds our x, y, z coordinates. We then return the three-dimensional array and run that through the IVP solver within scipy to obtain our solutions. As with the simple pendulum problem, we fix our initial conditions for the three coordinates, in our case we went for $[x_0, y_0, z_0] = [0, 1, 1]$. The initial values are not too important, provided we don't have all set to 0. In this case the problem doesn't exist as it has no starting point.

Once our function has been solved and we have our three coordinates associated with the correct element in the vector, we plot the three coordinates against each other in a three dimensional plot (Figure 11). We also plot each coordinate against time to see how the functions evolve through time.

5.2 Lorenz Attractor Results

At $\sigma = 10, \rho = 28, \beta = \frac{8}{3}$ the system displays blatant chaos and so we will use those values[9] as our basis for the problem.

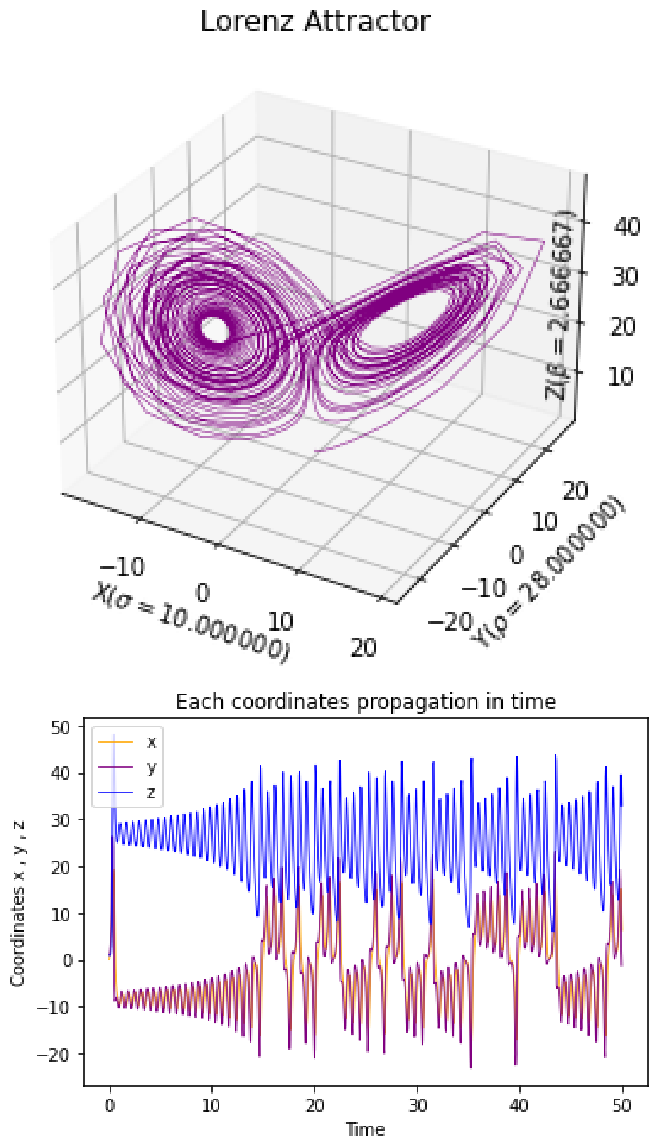

Figure 11: We set the parameters to be $\sigma = 10$, $\rho = 28$, $\beta = \frac{8}{3}$. We can see how the motion looks in the top plot, while the bottom plot shows us how each coordinate varies in time.

As our goal is to test how the parameters affect the system, we change them individually. In Figure 12 we have changed σ by halving the value. This causes the system to move in the x-z plane after its initial motion through space. Once it settles in the x-z plane it moves to an infinitesimally small point centered on approximately x=-5 and z=30.

In the coordinate against time plot we see that all the coordinate axes seem to propagate in a similar fashion, with x being negative, y being a smaller amplitude but also negative, and z being the same amplitude as x but in the positive.

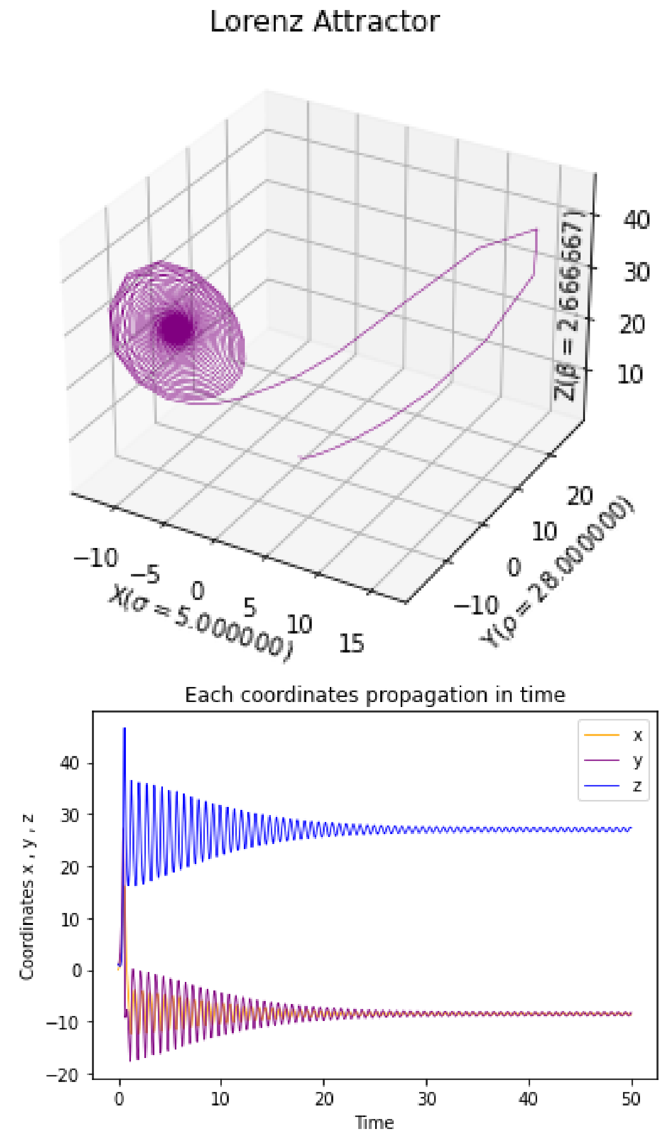

Figure 12: We have changed σ and set it to be $\sigma = 5$ to test how that affects the motion. We kept both ρ and β as in Figure 11.

We then change ρ by also halving it in Figure 13. The plot once again ends at an infinitesimally small point in space but not centered on any one plane. Each coordinate seems to give an equal contribution to the motion, with only the z coordinate being of a larger amplitude.

Finally, in Figure 14 we change β by halving it and notice the familiar butterfly wing shape, with less dense spirals through each loop. The change in β provides us with a far less drastic change to the system than the previous two changes.

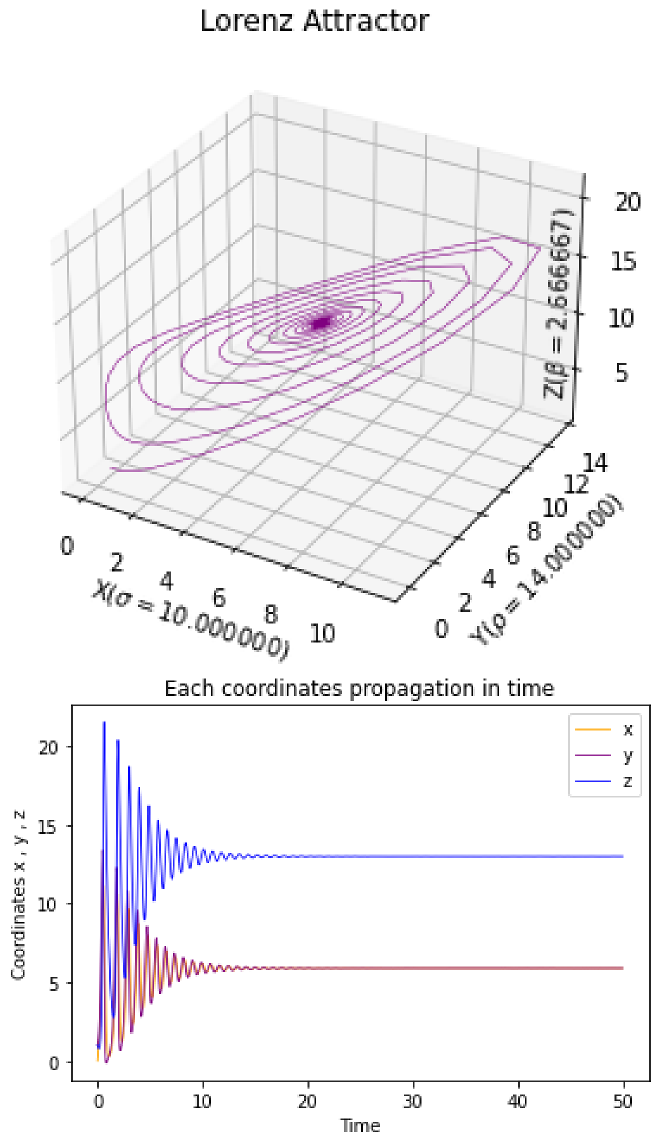

Figure 13: We have changed ρ and set it to be $\rho = 14$ to test how that affects the motion. We have kept both σ and β as in Figure 11.

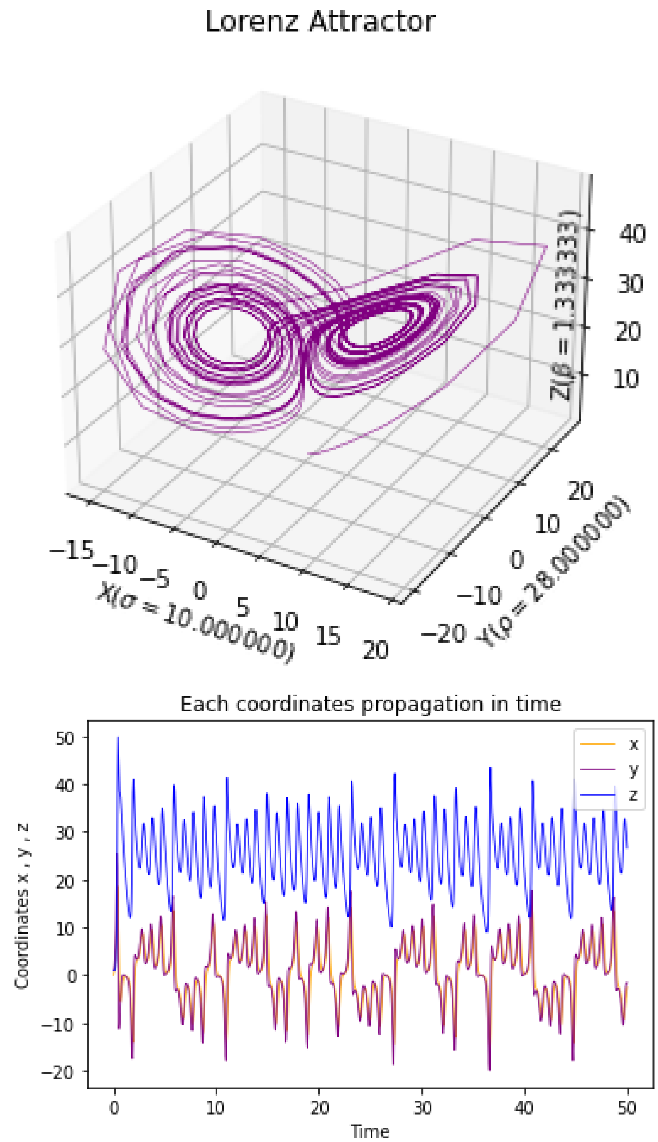

Figure 14: We have changed β and set it to be $\beta = \frac{4}{3}$ to test how that affects the motion. We have kept both σ and ρ as in Figure 11.

5.3 Lorenz Attractor Discussion

From an initial glance we can tell that the x and y component play equal parts in contributing to the systems chaos. When we change them, the system falls into a non-chaotic motion (chaos never ends at a single point in space or time). While the z component still plays a part in describing the system's chaotic motion, it does so in a much different way to the other two. While we don't show how the initial conditions of [x, y, z] changing affect the system, we do note that in doing do the system remains chaotic, but the circular motion is distributed in different densities throughout. The initial conditions are really only useful in starting the motion, as one would expect, but can later be fixed as we desire due to the trivial differences they make on the system.

When discussing the implications of chaos on a system we want to focus on the stability of the system, its predictability and how controllable the motion of the system is through chaos. For the Lorenz system we can see that the chaos makes the system much more predictable. Despite following randomness, it continues to create an infinity shaped loop, and the longer we run the system, the more tight the winding of the loops will be. This is something we can see just by observing the systems chaotic regime. The predictability of chaos in this system makes it desirable, as it allows us to control how the system is used. When we remove the chaotic behaviour of the system as in Figure 12 and 13 we lose the predictability aspect of the system and hence the system becomes less useful to us.

While this is the case for the Lorenz system, we want to know how this ties in with our FSP. Where chaos made the system easier to understand for the Lorenz Attractor, the FSP was much harder to visualise through chaos. There were no glaring patterns through it, and only when we solved it from a statistical point of view did the interpretation come to light. As previously stated, chaos can make a system easier to control, or on the contrary, completely incomprehensible. While the forced simple pendulum was far from predictable, it also wasn't incomprehensible.

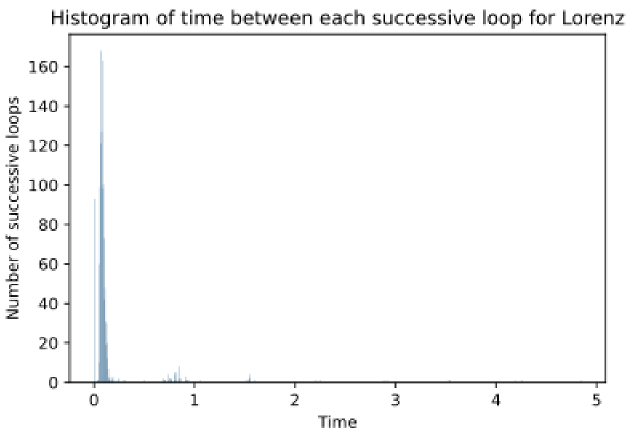

Figure 15: A plot of the time between each successive loop the Lorenz system makes. We see a much higher distribution between t=0 and t=1 than that in the forced pendulum.

When we compare Figure 8 to Figure 15 we immediately notice that they both seem to be situated on one particular average value, but where in Figure 8 we see the value to be approximately 1, Figure 15 has the average around ≈ 0.1. Also, we immediately notice the spread difference in the two histograms. Whilst the pendulum is spread up to a value of 15 and has obvious points throughout, the Lorenz Attractor stops at 1.5, with almost all of its points being settled between 0 and 1. While this could mean a number of things, the most obvious interpretation is that chaos does not always mean disarray in all systems. While there are obvious patterns in both, Figure 15 follows the patterns more strictly, or perhaps the patterns are more well-defined.

We can only assume this is due to the pendulum not being an inherently chaotic system. Had we done the same comparison with a double pendulum, we may have seen more similar patterns to those in Figure 15.

6 Conclusion

By looking through different cases of the forced simple pendulum and how the system changes with the driving force we found that the forced simple pendulum remains harmonic until $F_f \approx 1.0$ at which point the system remains steadily periodic while completing full oscillations. We saw that at $F_f \approx 1.2$ the systems steady periodicity starts falling apart while still allowing for chaos at $F_f \approx 1.07$ as seen in Figure 4. Within the chaotic regime of the forced pendulum system we also set out to find the statistical behaviour of chaos in plotting histograms of time between successive flips; we found that while the chaotic regime seems to average at around $\tau \approx 1$ regardless of how we change the parameters, it still displays a range of times going as high as $\tau = 15$ between successive flips. While we did investigate both parameters and initial conditions, we concluded that the initial conditions changing didn't cause enough of a difference in this system to be included in the analysis. We did however conclude that the mass affects chaos more significantly than the length of the pendulum.

We also briefly looked at the Lorenz system and how chaos changed its behaviour. We found, by also changing the parameters, that the system's dependence on chaos predominantly derived from its x and y components. The z component of the plot, while still important, changed the density of the systems chaos rather than pushing it into chaos like x and y did. When we compared the statistical behaviour of the Lorenz Attractor in chaos to that of the forced simple pendulum system we found that there were potential patterns, wherein both systems seemed to average around a single time between circuits, but the Lorenz Attractor had a much smaller range than he forced pendulum. This implied that the Lorenz Attractor in chaos was much more predictable and controlled than the forced pendulum in chaos.

In attempting to answer how chaos affects different nonlinear systems we discovered that predictability and patterns sometimes arise from chaos, but that doesn't strictly mean the chaos is useful. As with the case in Figure 8 we know the complete circuits are frequent, but that doesn't necessarily mean anything. In Figure 5 we see how chaos impacts the motion of the pendulum and note that in practice the simple pendulum may be ill-fit to derive further insights into chaos. Whereas going into the Lorenz Attractor showed that the system can be best understood through chaos, hence its applications to meteorology and more.

With all this we conclude that chaos follows patterns determined by external conditions and parameters and that sometimes its interpretation is flawed due to the nature of the problem.

References

[1] Anjali Sharma, Vinod Patidar, and G. Purohit. Bifurcation and chaos in periodically forced and nonlin-

early damped pendulum. *ijnsns*, 14(3–4):179–188, June 2013.

[2] Joan Jani. Simulation of chaotic operation f a damped driven pendulum using python. *WSEAS TRANSACTIONS ON ADVANCES in ENGINEERING EDUCATION*, 20:1–6, January 2023.

[3] Springer-Verlag, 2003.

[4] Divakar Viswanath. The lindstedt–poincaré technique as an algorithm for computing periodic orbits. *SIAM Review*, 43(3):478–495, January 2001.

[5] Adam Augustyn. chaos theory. encyclopedia britannica., 2024. Accessed on April 06, 2024.

[6] Stephen. H. Strogatz. *Nonlinear Dynamics and Chaos*. Taylor Francis, 2015.

[7] Kazuyuki Aihara. Chaos and its applications. *Procedia IUTAM*, 5:199–203, 2012.

[8] Masayasu Suzuki, Noboru Sakamoto, and Takashi Yasukochi. A butterfly-shaped localization set for the lorenz attractor. *Physics Letters A*, 372(15):2614–2617, April 2008.

[9] Robert L. Devaney Morris W. Hirsch, Stephen Smale. *DIFFERENTIAL EQUATIONS,DYNAMICAL SYSTEMS, AND AN INTRODUCTION TO CHAOS*. Elsevier Inc., 2013.

[10] A. Lesne. *Multiscale Approaches*, page 465–482. Elsevier, 2006.

[11] T. N. Kato and F. Takahara. Application of random walk theory to the first-order fermi acceleration in shock waves. *Monthly Notices of the Royal Astronomical Society*, 321(4):642–660, March 2001.

[12] J Kempe. Quantum random walks: An introductory overview. *Contemporary Physics*, 44(4):307–327, 2003.

[13] Manfred Gilli, Dietmar Maringer, and Enrico Schumann. *A gentle introduction to financial simulation*, page 153–188. Elsevier, 2019.

[14] Edward A Codling, Michael J Plank, and Simon Benhamou. Random walk models in biology. *Journal of The Royal Society Interface*, 5(25):813–834, April 2008.

[15] C. M. Topakoglu and D. F. Bruley. *A Mathematical Model Applying the Random-Walk Method to the Environment of a Neuron*, page 181–187. Springer US, 1986.

[16] S.H. El-Essawy, M.I. Nouh, A.A. Soliman, H.I. Abdel Rahman, and G.A. Abd-Elmougod. Monte carlo simulation of lane–emden type equations arising in astrophysics. *Astronomy and Computing*, 42:100665, January 2023.

[17] Mauricio Moralles, Carla C. Guimaraes, Daniel A. B. Bonifacio, Emico Okuno, Helio M. Murata, Marcio Bottaro, Mario O. Menezes, Valdir Guimaraes, Valdir Guimaraes, Jose R. B. Oliveira, Kita C. D. Macario, and Frederico A. Genezini. Applications of the monte carlo method in nuclear physics using the geant4 toolkit. In *AIP Conference Proceedings*. AIP, 2009.

[18] H. M. Cuppen, L. J. Karssemeijer, and T. Lamberts. The kinetic monte carlo method as a way to solve the master equation for interstellar grain chemistry. *Chemical Reviews*, 113(12):8840–8871, November 2013.

[19] Open Risk Manual contributors. Monte carlo simulation of credit portfolios, 2022. Accessed on April 10, 2024.

[20] Julien Bert and David Sarrut. *Monte Carlo simulations for medical and biomedical applications*, page 23–53. Elsevier, 2022.

[21] Magdalena Kruza, David Shaw, Jacob Shaw, and Nicola Carslaw. Towards improved models for indoor air chemistry: A monte carlo simulation study. *Atmospheric Environment*, 262:118625, October 2021.

[22] Edward Nuhfer, Christopher Cogan, Steven Fleischer, Eric Gaze, and Karl Wirth. Random number simulations reveal how random noise affects the measurements and graphical portrayals of self-assessed competency. *Numeracy*, 9(1), January 2016.

[23] derose. The geiger counter and counting statistics., 1994. Accessed on April 10, 2024.

[24] Audrey Sederberg and Ilya Nemenman. Randomly connected networks generate emergent selectivity and predict decoding properties of large populations of neurons. *PLOS Computational Biology*, 16(5):e1007875, May 2020.

[25] Ulrich Scholten Peter Smirnoff. The use of random number generators in cryptography, 2019. Accessed on April 11, 2024.

[26] M. Andrecut. Logistic map as a random number generator. *International Journal of Modern Physics B*, 12(09):921–930, April 1998.

[27] Ben Moews. On random number generators and practical market efficiency. 2023.

7 Addendum

7.1 Random Walks

Random walks are often regarded as a simplification of Brownian motion, using a number of calculated, mathematical steps to get to a destination. It calculates the next step in the walk using a probability distribution.

7.1.1 In Academia

Random walks are widely used in simulating the diffusion of particles using its statistical behaviour to run through its probabilistic motion over a given time period.[10] As the random walks take into account the diffusion coefficients and other correlations of the particle in motion, it is a good fit for the motion of a diffusing particle.

In physics, random walks are used in Fermi estimations.[11] The probabilistic nature of random walks aligns with the Fermi estimations as we often have to make assumptions or approximations based off limited information. We do so through a probability distribution. That and the flexibility of random walks make Fermi estimations a good fit for a random walk model.

Perhaps a more intriguing application of random walks would be through quantum walks.[12] As with the previous two examples, quantum physics is subject to its probability distribution as to where its next step in time will be. Random walks provide a visual description of the quantum particle's probabilistic motion as it evolves through time. While quantum faces wave-particle duality and hence is not fit for classical solutions, we can approach it with random walks as the probability distribution of the quantum system is accounted for.

7.1.2 Outside Academia

Aside from the academic applications of random walk simulations, random walks are common in the financial industry. An application of random walks can be seen in stock price modelling. Stock prices follow a non-stationary process[13] but through detrending the system, one obtains a stationary equation which makes the model suitable for random walks. As prices grow, their average over time does the same and the model persists on a forward time basis but is not influenced by any past events.

We can yet again see random walks through a more biological approach through organism movement.[14] An organism's movement is dependent on its surrounding factors, be it temperature or resource scarcity, or even pest repellent while it varies spatially, but these factors affecting motion are assumed to be stationary. With the number of affecting factors, all in their own way random, random walks are ideal for modelling the movement of organisms.

Also from a biological perspective, we see random walks used to model neuron firing in the brain.[15] The process of the neurons getting fired in the first place is done through a probability distribution, making it fit to be modelled via random walks. This probability distribution is then extended to the motion of the neuron through the brain so the neuron's entire motion can be modelled as a random walk.

7.2 Monte Carlo

Monte Carlo is a well known method used throughout physics; much like random walks, it takes a large dataset modelled on a probability distribution and outputs a result based on those probabilities. Monte Carlo is used frequently all throughout both academia and industry alike due to its reliability to provide a progressively more accurate result with each added calculation.

7.2.1 In Academia

One application in academia for Monte Carlo simulations is through simulating galaxy formation.[16] As Monte Carlo is an iterative method, it updates a system based on probability and random samples on a step-by-step basis, the galaxy formation is hence possible to simulate despite its non-linearity. In addition, galaxy formations also involve random processes such as galaxy mergers, gas accretion, and so on. This means that Monte Carlo is well suited due to its random sampling.

We can also make use of Monte Carlo simulations in nuclear physics through simulating particles decaying.[17] Particle decay involves a parent particle decaying into a daughter particle and often becomes a complex chain of successive decays based on a set of probabilistic conditions. Monte Carlo simulates such a complex chain of events by iterating through the events based on the probability of the system. This in turn allows us to simulate decay chains that could last thousands of years and optimise the results.

In chemistry we can use Monte Carlo to model how solvents affect chemical reactions.[18] We can simulate the motion of the solvents around the reacting species. Monte Carlo simulations can provide insights into reaction mechanisms and solvent dynamics.

7.2.2 Outside Academia

Monte Carlo is well fit for finance, especially in credit risk modelling.[19] Monte Carlo uses the probability of credit losses and credit portfolio performances to simulate individual probabilities of credit risk. This allows us to assess how credit events affect overall credit of loan portfolios.

Another example of Monte Carlo simulations outside of academia would be in medicine.[20] Monte Carlo can be used for modelling how radiation has been absorbed into a patient's tissue which can help scientists optimise imaging protocols as well as assessing image quality. The reason Monte Carlo is a good choice for medicinal image processing is due to its ability to simulate the different acquisition parameters from the imaging protocols.

As a final example of how Monte Carlo can be used, we move to engineering (environmental engineering to be more specific). Monte Carlo simulations can model air quality[21] to predict concentration of pollutants in the atmosphere. This can help assess the emissions from given factors and allow us to identify where pollutants seem to be most concentrated by simulating their diffusion. Ideally, this would lead to new measures being established to reduce these highly concentrated pollutant areas and keep air quality high.

7.3 Random Numbers

Random numbers are as they sound, a generation of random numbers distributed over a given range.

7.3.1 In Academia

One application of random numbers is through the simulation of noise[22] via random numbers. By simulating noise via random numbers we can learn how noise may be affecting results as well as spotting any systematic errors that can be avoided in the future.

A well known use of random numbers is radioactive decay counters (Geiger counters). We can use random number generator algorithms to help make radiation detection more efficient.[23] Sometimes not all radiation is detected via a detector, we can use random numbers to model the Geiger counters detection efficiency and from that create a probability distribution that the radiation interacting with the detector produces a measurable signal.

In neuroscience research we can use random numbers to model network connectivity in the brain.[24] Random numbers can be used to model the unpredictability observed in neural systems as well as the variability of them.

7.3.2 Outside Academia

In cryptography, we may see random numbers used to generate high-security passwords.[25] Due to the nature of random numbers having a uniform distribution with no noticeable pattern, we can generate a long password which should theoretically be hard to reproduce. Even when producing new random passwords in cryptography, it would take millions of iterations to produce a number with a similar pattern to that of the original one, even if we were to set initial conditions to help it do so.

We may also encounter random numbers in logistics and transportation.[26] As logistics and transportation involve a lot of random processes, we can incorporate random numbers which allow prediction of the performance under uncertain conditions. Say if a vehicle breaks down, we could still model potential arrival times with random numbers. Logistics and transportation is also fit for random number models as we can use random algorithms to make efficient vehicle routes. Random number algorithms are very fast in converging so this optimises the solution.

The financial market is once again, a good fit for the random number modelling. We can predict how stock prices will evolve based on random variables by creating trading algorithms.[27] Financial markets implement randomness such as through mean-reversion strategies. This can help the algorithm exploit inefficiencies and optimise for future strategies.