# Formal deformation theory

Reading seminar on DT theory

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## Section 1: Functors of Artin rings

- Functors of Artin rings
  - Artin rings as infinitesimally thickened points
  - From formal schemes to functors of Artin rings
- Tangent spaces to deformation problems
  - Properties
  - Computations of tangent spaces
- Extending deformations
  - Representability and atlases
  - Obstructions

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#### Formal neighbourhood of a closed affine subscheme

Let  $X = \operatorname{Spec} A$  be noetherian and I be an ideal of A defining  $Z = \operatorname{Spec}(A/I)$ .

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### Neighbourhood of a point

If I is prime so  $Z = \{x\}$  is (the closure of) a point, then  $\widehat{A}_I \simeq \widehat{(\mathcal{O}_{X,x})}_{\mathfrak{m}_x}$  where  $\mathfrak{m}_x = I \cdot \mathcal{O}_{X,x}$ .

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The formal neighbourhood  $\widehat{X}_Z$  is the topologically locally ringed space

$$\mathsf{Spf}(\widehat{A}_I) \coloneqq \Big( |\mathsf{Spec}(\widehat{A}_I/I)|_{\mathsf{Zar}} = |\mathsf{Spec}(A/I)|_{\mathsf{Zar}}, \ \varprojlim_n \mathscr{O}_{\mathsf{Spec}(A/I^n)} \Big).$$

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- The formal neighbourhood of (0,0) in  $\mathbb{k}[x,y]/(y^2-x^2-x^3)$  is  $\mathbb{k}[x,y]/((y-\xi)(y+\xi))$ : in  $\mathbb{k}[x,y]$ , the polynomial  $x^2(1+x)$  acquires a square-root  $\xi=x\sqrt{1+x}$  (by formal Taylor expansion).

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A ring is **Artinian** if it satisfies the descending chain condition for ideals. A **local Artin**  $\mathbb{k}$ -algebra is a local ring  $(A, \mathfrak{m}_A)$  with A Artinian and  $A/\mathfrak{m}_A \simeq \mathbb{k}$ .

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Write  $\mathfrak{Art}_{\Bbbk}$  the category of local Artin  $\Bbbk\text{-algebras}$  and local homomorphisms.

Interpretation:  $\mathfrak{Art}_{\mathbb{k}}^{op}$  is the category of (affine) "fat points".

## Complete local rings

A pair (R, I) is **complete** if  $R \to \widehat{R}_I$  is an isomorphism.

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### Every complete local k-algebra R is a pro-object in $\mathfrak{Art}_k$ .

Indeed, R is a projective limit  $\varprojlim_n R/\mathfrak{m}_R^n$  where each  $R/\mathfrak{m}_R^n$  is local noetherian with nilpotent maximal ideal, so Artinian.

## Surjections of Artin rings

A surjection of local Artin  $\varphi \colon B \twoheadrightarrow A$  is a **small extension** if  $\mathfrak{m}_B \cdot \ker(\varphi) = 0$ . It is **principal** if  $\ker(\varphi)$  is a principal ideal of B.

Remark:  $ker(\phi)$  has a canonical structure of A-module (as  $\mathfrak{m}_B \supset I$ ).

$$0 \to \mathbb{k} = (\varepsilon^n) \rightarrowtail \mathbb{k}[\varepsilon]/(\varepsilon^{n+1}) \twoheadrightarrow \mathbb{k}[\varepsilon]/(\varepsilon^n) \to 0$$
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#### **Proposition**

Every surjection of local Artin rings factors as a composite of small extensions.

For any k-algebra R and an R-module I, we let  $\mathsf{Ex}_k(R,I)$  denote the set of isomorphism classes of square-zero extensions of R by I.

►  $\mathsf{Ex}_{\Bbbk}(R,I)$  has a structure of R-module, with  $r \cdot [\widetilde{R}] := [\alpha_{r,*}\widetilde{R}]$  where  $\alpha_r \colon I \to I, i \mapsto r \cdot i$  and, for any  $a \colon I \to J$ ,  $a_*\widetilde{R} = \widetilde{R} \coprod_I J$ .

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#### Pre-deformation functors

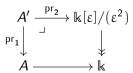
Fact: The functor of points of a formal scheme is determined by its values on k-algebras with reduced part k, *i.e.* it can be recovered from its restriction to  $\mathfrak{Art}_k$ .

▶ A **pre-deformation functor** is a precosheaf  $\mathscr{F}$  on  $\mathfrak{Art}_{\Bbbk}$ , that is a (covariant) functor  $\mathfrak{Art}_{\Bbbk} \to \mathfrak{Set}$ , such that  $\mathscr{F}(\Bbbk) \simeq *$ .

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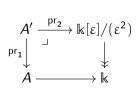
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- ▶ A pre-deformation functor **admits a differential calculus** if it preserves pullbacks along  $\mathbb{k}[\varepsilon]/(\varepsilon^2) \twoheadrightarrow \mathbb{k}$

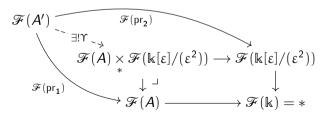


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the dashed map  $\Upsilon$  on the right is an isomorphism.

### Examples of deformation functors I

Pro-representable functors

#### (Co)Representable functors

Any local Artin  $\mathbb{k}$ -algebra A corepresents the functor  $\mathbb{A}^A \colon B \mapsto \mathsf{hom}_{\mathfrak{Art}_{\mathbb{k}}}(A,B)$ . There is a unique  $\mathbb{k} \to A$ , and representables preserve all limits.

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Suppose  $R \in \widehat{\mathfrak{Art}}_{\Bbbk}$ . The representable  $\widehat{\mathbb{A}^R}$  is left-exact so  $\widehat{\mathbb{A}^R}(S) = \varprojlim_n \widehat{\mathbb{A}^R}(S/\mathfrak{m}_S^n)$ .

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 $\mathscr{F}$  is **pro-representable** if  $\widehat{\mathscr{F}}$  is representable by  $R \in \widehat{\mathfrak{Art}}_{\Bbbk}$ .

- ► For such  $R \in \widehat{\mathfrak{Art}}_{\mathbb{K}}$ , we denote  $\mathbb{A}^R$ :  $A \mapsto \mathsf{hom}(R,A)$  the restriction of  $\widehat{\mathbb{A}^R}$ .
- **b** By Yoneda, morphisms  $\hbar^R \to \mathscr{F}$  are in bijection with formal elements of  $\mathscr{F}$  over R.

## Examples of deformation functors II

Formal neighbourhood of a point

The functor of points of a  $\mathbb{k}$ -scheme X is  $\mathscr{R}_X \colon \mathfrak{Aff}_{\mathbb{k}}^{\text{op}} = \mathfrak{Alg}_{\mathbb{k}} \to \mathfrak{Ens}, A \mapsto \text{hom}(\text{Spec}\,A, X)$ . However  $\mathfrak{Art}_{\mathbb{k}}$  is not a full subcategory of  $\mathfrak{Alg}_{\mathbb{k}}$ .

#### Lemma

Let R be a local ring. There is a bijection between morphisms  $f \colon \operatorname{Spec} R \to X$  mapping the unique closed point to  $x \in X$  and *local* homomorphisms  $f^{\sharp} \mathscr{O}_{X,x} \to R$ .

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Let X be a k-scheme and x: Spec  $k \to X$  a point. Its formal neighbourhood is

$$A \mapsto \left\{ f \in \mathsf{hom}(\mathsf{Spec}\,A,X) \mid p_A^*f \coloneqq f|_{\mathsf{Spec}\,\Bbbk} = x \right\}$$

where  $p_A^{\sharp} \colon A \to A/\mathfrak{m}_A = \mathbb{k}$  so  $f|_{\operatorname{Spec} \mathbb{k}} \colon \operatorname{Spec} k \xrightarrow{p_A} \operatorname{Spec} A \xrightarrow{f} X$ .

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## The tangent space

The **tangent set** to a pre-deformation functor  $\mathscr{F}$  is  $T_{\mathscr{F}} := \mathscr{F}(\mathbb{D})$ , where  $\mathbb{D} := \mathbb{k}[\varepsilon]/(\varepsilon^2)$ .

### Proposition

If  $\mathscr{F}$  admits a differential calculus,  $T_{\mathscr{F}}$  is a  $\Bbbk$ -vector space.

#### Construction.

 $\mathbb{D}$  is an  $\mathbb{k}$ -vector space object in  $\mathfrak{Art}_{\mathbb{k}_*/\mathbb{k}}$  with

abelian group structure from  $\mu: \mathbb{D} \times_{\mathbb{k}} \mathbb{D} \to \mathbb{D}, (a+b\varepsilon, a+b'\varepsilon) \mapsto a+(b+b')\varepsilon$ ,

scalar multiplication from  $\rho_{\lambda} \colon \mathbb{D} \to \mathbb{D}, a + b\varepsilon \mapsto a + \lambda b\varepsilon$  for  $\lambda \in \mathbb{k}$ .

Then, as  ${\mathcal F}$  preserves the relevant fibre products, define

$$+: \mathscr{F}(\mathbb{D}) \times \mathscr{F}(\mathbb{D}) \xrightarrow{\Upsilon^{-1}} \mathscr{F}(\mathbb{D} \times_{\mathbb{k}} \mathbb{D}) \xrightarrow{\mathscr{F}(\mu)} \mathscr{F}(\mathbb{D})$$
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If  $\varphi \colon \mathscr{F} \to \mathscr{G}$  is a transformation,  $d\varphi \coloneqq \varphi_{\mathbb{D}} \colon T_{\mathscr{F}} \to T_{\mathscr{G}}$  is called its **differential**.

## Example 0: Pro-representable functors

If  $\mathscr{F} = \mathscr{R}^R$  is pro-representable,  $T_{\mathscr{R}^R} = T_{R/\Bbbk,\mathfrak{m}_R} \simeq T_{R/\Bbbk}$  where  $\mathfrak{m}_R$  is the unique closed point of Spec R (so the tangent space is the tangent module).

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### Proof.

We use the characterisation of  $T_{R/\Bbbk,\mathfrak{m}_R}$  as the  $\Bbbk$ -dual of  $\Omega^1_{R/\Bbbk,\mathfrak{m}_R}=\Omega^1_{R/\Bbbk}\otimes_R \Bbbk$ , so

$$T_{R/\Bbbk,\mathfrak{m}_R} \simeq \mathsf{hom}_{\Bbbk}(\Omega^1_{R/\Bbbk} \otimes_R \Bbbk, \Bbbk) \simeq \mathsf{hom}_R(\Omega^1_{R/\Bbbk}, \Bbbk)$$

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Recall that k-linear derivations from R to an R-module M are in bijection with maps of R-augmented k-algebras into the square-zero extension  $R \oplus M$ 

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Recall that k-linear derivations from R to an R-module M are in bijection with maps of R-augmented k-algebras into the square-zero extension  $R \oplus M$ :

$$T_{R/\Bbbk} \simeq \operatorname{\mathsf{hom}}_{\Bbbk,/R}(R, R \oplus \Bbbk) \simeq \operatorname{\mathsf{hom}}_{\Bbbk,/\Bbbk}(R, \Bbbk \oplus \Bbbk) = \operatorname{\mathsf{hom}}_{\Bbbk,/\Bbbk}(R, \mathbb{D}).$$

If  $\mathscr{F} = \mathscr{h}^R$  is pro-representable,  $T_{\mathscr{H}^R} = T_{R/\Bbbk,\mathfrak{m}_R} \simeq T_{R/\Bbbk}$  where  $\mathfrak{m}_R$  is the unique closed point of Spec R (so the tangent space is the tangent module).

### Proof.

We use the characterisation of  $T_{R/\Bbbk,\mathfrak{m}_R}$  as the  $\Bbbk$ -dual of  $\Omega^1_{R/\Bbbk,\mathfrak{m}_R}=\Omega^1_{R/\Bbbk}\otimes_R \Bbbk$ , so

$$T_{R/\Bbbk,\mathfrak{m}_R} \simeq \mathsf{hom}_{\Bbbk}(\Omega^1_{R/\Bbbk} \otimes_R \Bbbk, \Bbbk) \simeq \mathsf{hom}_R(\Omega^1_{R/\Bbbk}, \Bbbk) \simeq \mathsf{Der}_{\Bbbk}(R, \Bbbk).$$

Recall that k-linear derivations from R to an R-module M are in bijection with maps of R-augmented k-algebras into the square-zero extension  $R \oplus M$ :

$$T_{R/\Bbbk} \simeq \mathsf{hom}_{\Bbbk,/R}(R, R \oplus \Bbbk) \simeq \mathsf{hom}_{\Bbbk,/\Bbbk}(R, \Bbbk \oplus \Bbbk) = \mathsf{hom}_{\Bbbk,/\Bbbk}(R, \mathbb{D}).$$

Finally, a local morphism  $R \to \mathbb{D}$  is exactly a morphism of k-augmented k-algebras.

# Extending tangent modules

Recall that the tangent module to a  $\mathbb{k}$ -algebra R is  $T_{R/\mathbb{k}} = \hom_R(\Omega_{R/\mathbb{k}}, R) = \mathrm{Der}_{\mathbb{k}}(R, R)$ .

▶ For any morphism  $S \rightarrow R$ , there is an exact sequence

$$0 \longrightarrow \operatorname{Der}_{S}(R,I) \longrightarrow \operatorname{Der}_{\Bbbk}(R,I) \longrightarrow \operatorname{Der}_{\Bbbk}(S,I) \otimes_{S} R$$

$$\vdash \operatorname{Ex}_{S}(R,I) \longrightarrow \operatorname{Ex}_{\Bbbk}(R,I) \longrightarrow \operatorname{Ex}_{\Bbbk}(S,I) \otimes_{S} R,$$

hence one also writes  $\operatorname{Ex}_{\Bbbk}(R,R) \eqqcolon \operatorname{H}^1(T_{R/\Bbbk}^{\bullet}) \ (= \operatorname{Ext}^1_R(\mathbb{L}\Omega^{1,ullet}_{R/\Bbbk},R)).$ 

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- As a quasicoherent  $\mathcal{O}_{\operatorname{Spec} R}$ -module,  $\operatorname{Ex}_{\Bbbk}(R,R)$  is supported on the singular locus of  $\operatorname{Spec} R \to \operatorname{Spec} \Bbbk$ .
- ▶ If R is reduced, then  $\operatorname{Ex}_{\Bbbk}(R,I) \simeq \operatorname{Ext}^1_R(\Omega^1_{R/\Bbbk},I)$ .

# Action of the tangent space

### Lemma

For any principal extension  $(t) \rightarrowtail \widetilde{A} \xrightarrow{p} A$ ,  $T_{\mathscr{F}}$  acts on the fibres of  $\mathscr{F}(\widetilde{A}) \xrightarrow{\mathscr{F}p} \mathscr{F}(A)$ .

### Proof.

▶  $\mathbb{D}$  acts on  $\widetilde{A}$  by  $\mathbb{D} \times_{\mathbb{k}} \widetilde{A} \xrightarrow{\operatorname{act}} \widetilde{A}$ ,  $(\alpha + \beta \varepsilon, a) \mapsto a + \beta t$  where t generates (t).

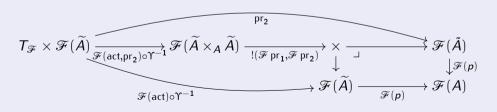
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- ► The action preserves the fibres:



## Proposition

If  $\mathscr{F}=\mathscr{R}^R$  is pro-representable, the action of  $T_{\mathscr{R}^R}$  is transitive and free on non-empty fibres.

## Proof.

Let  $\varpi \colon \widetilde{A} \twoheadrightarrow A$  be a principal extension with kernel  $I \simeq \mathbb{k}$  and  $\varphi \in \mathcal{R}^R(A) = \text{hom}(R,A)$ . Fix  $\widetilde{\varphi} \colon R \to \widetilde{A}$  in the fibre.

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In any category with products, an internal group action such that  $G \times X \xrightarrow{(act,pr_2)} X \times X$  is an isomorphism is called a **torsor**.

 $\Rightarrow$  Every choice of "base-point"  $x: * \to X$  trivialises the torsor by  $G \times * \simeq X \times *$ .

Let  $\mathscr G$  be a sheaf of groups and  $\mathscr X$  a  $\mathscr G$ -sheaf. If  $\mathscr X(U)=\emptyset$ , then  $(\mathscr G\times\mathscr X)(U)=\emptyset$  and the torsor condition is trivially satisfied over U.

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Let  $\coprod_{\alpha} U_{\alpha} \twoheadrightarrow X$  be an open cover, and  $(\gamma_{\alpha\beta})$  a Čech cocycle. For any open  $V \subset X$ , define

$$\mathcal{S}(V) := \{ (g_{\alpha} \in \mathcal{G}(V \times_{X} U_{\alpha})) \mid \gamma_{\alpha\beta}g_{\beta} = g_{\alpha} \}$$

with the obvious restriction maps and  $\mathscr{G}$ -action. Any coboundary gives a trivial torsor.

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# Contents - Section 2: Tangent spaces to deformation problems

- Functors of Artin rings
- Tangent spaces to deformation problems
  - Properties
  - Computations of tangent spaces
- Extending deformations

## Flatness over the dual numbers

Given a k-scheme X and a k-algebra A, we write  $X_A = X \otimes_k A = X \times_{\operatorname{Spec} k} \operatorname{Spec} A$ .

### Differential modules

An  $\mathcal{O}_{X_{\mathbb{D}}}$ -module consists of:

- ightharpoonup a sheaf on  $|X|_{Z_{ar}}$
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Modules over  $\mathscr{O}_X[\varepsilon]/(\varepsilon^2)$  correspond to **differential**  $\mathscr{O}_X$ -modules: pairs  $\mathscr{F}=(\mathscr{F}_0,\psi)$  where  $\mathscr{F}_0$  is an  $\mathscr{O}_X$ -module and  $\psi\colon \mathscr{F}_0\to \mathscr{F}_0$  such that  $\psi\circ\psi=0$ .

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### **Flatness**

Suppose  $A woheadrightarrow \mathbb{k}$  is a principal extension. A quasicoherent  $\mathscr{O}_{X_A}$ -module  $\mathscr{F}$  is flat over A iff  $\mathscr{F} \otimes \mathscr{O}_X = \mathscr{F}|_X$  is flat over  $\mathbb{k}$  and  $\mathscr{F} \otimes \mathfrak{m}_A \mathscr{O}_{X_A} \xrightarrow{\cong} \mathfrak{m}_A \mathscr{F}$ .

Remark: When  $A = \mathbb{D} = \mathbb{k}[\varepsilon]/(\varepsilon^2)$ , then  $\varepsilon \mathcal{O}_{X_{\mathbb{D}}} \simeq \mathcal{O}_X$ , so  $\mathscr{F}_0$  splits (over  $\mathbb{k}$ , not  $\mathcal{O}_X$ ).

# Example 1: Deformations of a scheme

Let  $X_0$  be a finite type  $\Bbbk$ -scheme. A deformation of  $X_0$  over  $A \in \mathfrak{Att}_{\Bbbk}$  is  $X \to \operatorname{Spec}(A)$  flat and surjective with an isomorphism  $\vartheta \colon X \otimes_A \Bbbk \xrightarrow{\cong} X_0$ . A morphism  $(X, \vartheta) \to (X', \vartheta')$  is  $f \colon X \to X'$  such that  $\vartheta' \circ (f \otimes_A \Bbbk) = \vartheta$ .

▶ The functor of deformations of  $X_0$  is  $\mathscr{Def}_{X_0} \colon A \mapsto \{A\text{-deformations of } X_0\}/\simeq$ . Functoriality is by taking pullbacks of families: if  $f \colon \operatorname{Spec} B \to \operatorname{Spec} A$ , then  $(\mathscr{Def}_{X_0}(f^{\sharp}))(X) = f^*X \coloneqq X \otimes_A B$ .

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- ▶ An A-deformation  $X \to \operatorname{Spec} A$  is **locally trivial** if there is a cover  $\coprod_{\alpha} U_{\alpha} \twoheadrightarrow X_0$  such that  $X|_{U_{\alpha}} \simeq U_{\alpha} \otimes_{\Bbbk} A$ .
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- ▶ Subfunctor  $\mathscr{D}ef_{X_0}^{triv}$  of locally trivial deformations.

#### Lemma

Every deformation of a smooth affine k-scheme  $X_0 = \operatorname{Spec} R_0$  is trivial.

### Proof.

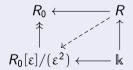
Let  $X\in\mathscr{D}\!\mathit{ef}_{X_0}(\mathbb{D}).$  Note first that X is affine,  $X=\operatorname{Spec} R,$  and smooth over  $\mathbb{D}$  (by flatness).

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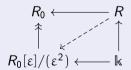
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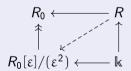
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As  $\mathbb{k} \to R$  is smooth, the dashed lift exists. It has an inverse from the bilinear  $R_0 \times \mathbb{D} \to R$ . Triviality over  $A \in \mathfrak{Art}_{\mathbb{k}}$  is by induction on  $\dim_{\mathbb{k}}(A)$ : decompose in successive extensions.

### Theorem

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Indeed, in an affine chart  $U_{\alpha\beta} = \operatorname{Spec} B$ , an automorphism  $B \otimes_{\Bbbk} \mathbb{D} \xrightarrow{\simeq} B \otimes_{\Bbbk} \mathbb{D}$  reducing to  $\operatorname{id}_B$  modulo  $\varepsilon$  is of the form  $b \mapsto b + \delta(b)\varepsilon$ . By before  $\delta$  must be a  $\mathbb{D}$ -derivation  $B \otimes_{\Bbbk} \mathbb{D} \to B$ , and  $\operatorname{Der}_{\mathbb{D}}(B \otimes_{\Bbbk} \mathbb{D}, B) \simeq \operatorname{Der}_{\Bbbk}(B, B)$ .

# Tangent to deformations of a scheme

## Proposition

If  $X_0$  is of finite type over  $\mathbb{K}$ , then  $T_{\mathscr{Def}_{X_0}} \simeq \mathsf{Ex}_{\mathbb{K}}(\mathscr{O}_{X_0}, \mathscr{O}_{X_0})$ . In particular, if  $X_0$  is reduced,  $T_{\mathscr{Def}_{X_0}} \simeq \mathsf{Ext}^1_{\mathscr{O}_{X_0}}(\Omega^1_{X_0}, \mathscr{O}_X)$ .

### Proof.

Let X be a first-order deformation. Since  $\mathscr{O}_X$  is flat over  $\mathbb{k}$  it splits as a  $\mathbb{k}$ -linear extension  $0 \to \varepsilon \mathscr{O}_X \rightarrowtail \mathscr{O}_X \twoheadrightarrow \mathscr{O}_{X_0} \to 0$ , with  $\varepsilon \mathscr{O}_X = \mathscr{O}_{X_0}$ . Hence we get a  $\mathbb{k}$ -linear self-extension of  $\mathscr{O}_{X_0}$ .

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- ▶ Given an extension  $0 \to \mathcal{O}_{X_0} \xrightarrow{i} \mathcal{A} \xrightarrow{p} \mathcal{O}_{X_0} \to 0$ , one can endow  $\mathcal{A}$  with a  $\mathbb{D} \otimes_{\Bbbk} \mathcal{O}_{X_0}$ -algebra structure by having  $i \circ p$  act as  $\varepsilon$ .



## Example of the projective spaces

The Euler sequence  $0 \to \mathcal{O}_{\mathbb{P}^n_{\Bbbk}} \to \mathcal{O}_{\mathbb{P}^n_{\Bbbk}}(1)^{\oplus n+1} \to \mathcal{T}_{\mathbb{P}^n_{\Bbbk}/\Bbbk} \to 0$  implies that  $\mathsf{H}^1(\mathbb{P}^n_{\Bbbk}, \mathcal{T}_{\mathbb{P}^n_{\Bbbk}/\Bbbk}) = 0$  for  $n \geq 1$ , so  $\mathbb{P}^n_{\Bbbk}$  is rigid.

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## Computation for the affine line

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# Example of the projective spaces

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- ▶ Let  $\delta_0$ :  $\mathbb{k}[x] \to \mathbb{k}[x]$  be the derivation defined by  $x \mapsto \delta(x)$  and  $\delta_1$ :  $\mathbb{k}[x^{-1}] \to \mathbb{k}[x^{-1}]$  defined by  $\delta_1(x^{-1}) = -\delta(x^{-1})$ . Then  $\delta_0 \otimes_{\mathbb{k}} \mathbb{k}[x^{-1}] \delta_1 \otimes_{\mathbb{k}} \mathbb{k}[x]$  gives back  $\delta$ .

Hence the Čech cocycle defined by  $\delta$  is a coboundary, and defines a trivial deformation.

Let  $\mathcal{F}_0$  be a quasicoherent sheaf on  $X_0$ , flat over k.

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### Restrictions and inductions of sheaves

The principal extension  $0 \to \Bbbk \hookrightarrow \mathbb{D} \twoheadrightarrow \Bbbk \to 0$  induce maps  $p: X_{\mathbb{D}} \to X_0$  and  $i: X_0 \to X_{\mathbb{D}}$ .

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#### Theorem

If  $\mathscr{F}_0$  is locally free,  $T_{\mathscr{D}ef_{\mathscr{F}_0}}\simeq \check{H}^1(X_0,\mathscr{H}om(\mathscr{F}_0,\mathscr{F}_0)).$ 

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# Tangent to deformations of a quasicoherent sheaf

## Proposition

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- ▶ Given an  $\mathcal{O}_{X_0}$ -extension of  $\mathcal{F}_0$  by  $\mathcal{F}_0$ , there is again a unique  $\mathcal{O}_{X_{\mathbb{D}}}$ -module structure restricting to  $\mathcal{F}_0$ : for  $0 \to \mathcal{F}_0 \xrightarrow{i} \mathcal{E} \xrightarrow{p} \mathcal{F}_0 \to 0$  the nilpotent endomorphism is  $i \circ p \colon \mathcal{F} \to \mathcal{F}$ .

# Section 3: Extending deformations

- Functors of Artin rings
  - Artin rings as infinitesimally thickened points
  - From formal schemes to functors of Artin rings
- Tangent spaces to deformation problems
  - Properties
  - Computations of tangent spaces
- Extending deformations
  - Representability and atlases
  - Obstructions

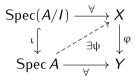
# Contents - Section 3: Extending deformations

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### Formal smoothness

A morphism of schemes  $\varphi: X \to Y$  is formally smooth when it has the right lifting property against square-zero immersions, *i.e.* if for any square-zero immersion

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## Formally smooth morphism

A morphism of pre-deformation functors  $\varphi \colon \mathscr{F} \to \mathscr{G}$  is **formally smooth** if for every small surjection  $A \twoheadrightarrow B$ , the map  $\mathscr{F}(A) \to \mathscr{F}(B) \times_{\mathscr{G}(B)} \mathscr{G}(A)$  is surjective.

 $\mathscr{F}$  is formally smooth if  $\mathscr{F} \to *$  is so, *i.e.*  $\mathscr{F}(A) \twoheadrightarrow \mathscr{F}(B)$  for every  $A \twoheadrightarrow B$ .

### Versal families

A pre-deformation functor  $\mathscr{F}$  is pro-representable iff there is an isomorphism  $\mathscr{R}_R \to \mathscr{F}$ , which by Yoneda corresponds to a formal family  $\widehat{\xi} \in \widehat{\mathscr{F}}(R)$ . Such a family is called **universal**.

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## Atlases for deformation problems

A **semi-universal family** for  $\mathscr{F}$  is a formal element  $\widehat{\xi} \in \widehat{\mathscr{F}}(R)$  such that the corresponding  $\mathscr{Z}^R \to \mathscr{F}$  is smooth and its differential is an isomorphism  $T_{R/\Bbbk} \stackrel{\cong}{\to} T_{\mathscr{F}}$ .

Remark: If  $\phi: \mathscr{E} \to \mathscr{F}$  is smooth, it is surjective (thus so is its differential).

### Versal families

A pre-deformation functor  $\mathscr{F}$  is pro-representable iff there is an isomorphism  $\mathscr{R}_R \to \mathscr{F}$ , which by Yoneda corresponds to a formal family  $\widehat{\xi} \in \widehat{\mathscr{F}}(R)$ . Such a family is called **universal**.

## Atlases for deformation problems

A semi-universal family for  $\mathscr F$  is a formal element  $\widehat{\xi} \in \widehat{\mathscr F}(R)$  such that the corresponding  $\mathscr R^R \to \mathscr F$  is smooth and its differential is an isomorphism  $T_{R/\Bbbk} \xrightarrow{\cong} T_{\mathscr F}$ .

Remark: If  $\phi \colon \mathscr{E} \to \mathscr{F}$  is smooth, it is surjective (thus so is its differential).

## Proposition

If  $(R,\widehat{\xi})$  and  $(S,\widehat{\psi})$  are two semi-universal families, there is an isomorphism  $(R,\widehat{\xi})\simeq (S,\widehat{\psi})$  inducing a uniquely determined  $T_{R/\Bbbk}\simeq T_{S/\Bbbk}$ .

If  $(R, \hat{\xi})$  and  $(S, \hat{\psi})$  are universal, the isomorphism between them is unique.

# Schlessinger's criterion

## Definition (deformation functor)

A pre-deformation functor  $\mathcal{F}$  is a **deformation functor** if

- 1. for every small surjection  $p \colon \widetilde{A} \twoheadrightarrow A$  and every  $f \colon B \to A$ , the map  $\Upsilon_{p,f} \colon \mathscr{F}(\widetilde{A} \times_A B) \to \mathscr{F}(\widetilde{A}) \times_{\mathscr{F}(A)} \mathscr{F}(B)$  is surjective
- 2.  $\mathscr{F}$  admits a differential calculus, *i.e.*  $\Upsilon_{p,f}$  is bijective when  $p: \mathbb{D} \to \mathbb{k}$ .
- $\mathscr{F}$  is **homogeneous** if the maps  $\Upsilon_{p,f}$  are always bijective.

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## Theorem (Schlessinger)

- A pre-deformation functor  $\mathscr{F}$  admits a semi-universal formal family if and only if it is a deformation functor with finite-dimensional tangent space.
- ▶ It has a universal formal family iff it is in addition homogeneous.

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Remark: For every  $\mathbb{k} \hookrightarrow \widetilde{A} \twoheadrightarrow A$ ,  $T_{\mathscr{F}}$  acts transitively on the fibres of  $\mathscr{F}(\widetilde{A}) \to \mathscr{F}(A)$ .

# Contents - Section 3: Extending deformations

- Functors of Artin rings
- 2 Tangent spaces to deformation problems
- Extending deformations
  - Representability and atlases
  - Obstructions

### Extensions and obstruction calculus

## Definition (obstruction spaces)

An **obstruction theory** for a pre-deformation functor  $\mathscr{F}$  is a  $\Bbbk$ -vector space  $\upsilon$  with, for every small extension  $(E)\colon 0\to I\hookrightarrow \widetilde{A}\twoheadrightarrow A\to 0$  a map  $o_{(E)}\colon \mathscr{F}(A)\to \upsilon\otimes_{\Bbbk} I$  which is functorial in morphisms of extensions, and such that when  $A=\Bbbk$  (so  $\mathscr{F}(A)=*$ ),  $o_{(E)}(*)=0$ .

We get for every  $A \in \mathfrak{Art}_{\mathbb{k}}$ ,  $\xi \in \mathscr{F}(A)$ , and every  $\mathbb{k}$ -vector space I a  $\mathbb{k}$ -linear map  $o_{(-)}(\xi) \colon \mathsf{Ex}_{\mathbb{k}}(A,I) \to \mathfrak{v} \otimes I$ .

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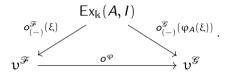
#### Lemma

If  $(\upsilon,(o_{(E)})_{(E)})$  is an obstruction theory for  $\mathscr{F}$ , then whenever an element  $\xi\in\mathscr{F}(A)$  lifts along a small extension  $(E)\colon\widetilde{A}\twoheadrightarrow A$  to  $\widetilde{\xi}\in\mathscr{F}(\widetilde{A})$  we have  $o_{(E)}(\xi)=0$ .

An obstruction theory is called **complete** if vanishing of the obstruction  $o_{(E)}(\xi)$  is equivalent to the existence of a lift along (E).

### Universal obstruction theories

If  $\mathscr{F},\mathscr{G}$  are pre-deformation functors endowed with complete obstruction theories  $(v^{\mathscr{F}},(o_E^{\mathscr{F}}))$  and  $(v^{\mathscr{F}},(o_E^{\mathscr{F}}))$ , an **obstruction map** for  $\phi\colon\mathscr{F}\to\mathscr{G}$  is a linear map  $o^{\phi}\colon v^{\mathscr{F}}\to v^{\mathscr{F}}$  such that for every  $\xi\in\mathscr{F}(A)$  and every I, the following triangle commutes:



### Theorem (Fantechi-Manetti)

Every deformation functor admits an initial obstruction theory, which is furthermore complete.

The universal obstruction theory for  $\mathscr{R}^R$  is  $H^1(T^{ullet}_{R/\Bbbk,\mathfrak{m}_R}) \coloneqq \mathsf{Ex}_\Bbbk(R,\Bbbk)$ .

### Obstructions and smoothness

### Proposition

Let  $\phi\colon \mathscr{F} \to \mathscr{G}$  be a morphism of deformation functors. If  $\phi$  has an injective obstruction map and  $d\phi$  is surjective then  $\phi$  is smooth.

 $\Rightarrow$  A deformation functor is smooth if and only if it has 0 as obstruction theory.

For a scheme  $X_0$  of finite type over  $\Bbbk$ ,  $H^2(X_0, \mathcal{T}_{X_0})$  is an obstruction space for  $\mathscr{Def}_{X_0}^{triv}$ . Hence if  $X_0$  is smooth and  $H^2(X_0, \mathcal{T}_{X_0}) = 0$ , then  $\mathscr{Def}_{X_0}$  is smooth of dimension  $\dim H^1(X_0, \mathcal{T}_{X_0})$ .

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### Dimension from obstructions

- ▶ If  $\mathscr{F} \to \mathscr{C}$  is smooth, then any obstruction theory for  $\mathscr{C}$  gives one for  $\mathscr{F}$ ; in particular any obstruction theory on a deformation functor induces one on a semi-universal family.
- ▶ If v is an obstruction theory for  $\mathbb{A}^R$ , then  $\dim_{\mathbb{k}}(T_{R/\mathbb{k}}) \ge \dim_{\mathrm{Krull}}(R) \ge \dim_{\mathbb{k}}(T_{R/\mathbb{k}}) \dim_{\mathbb{k}}(v)$ .

# Other questions in deformation theory

## Stacky aspects

▶ Automorphisms of deformations as obstructions to pro-representability

## Formal aspects

- Effectivity of deformations
- ► Algebraisability of formal families

### Cohomological aspects

- Extensions and obstructions from the cotangent complex
- (Derived) Deformation problems and differential graded Lie algebras

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