Monoidal envelopes and Grothendieck construction for dendroidal Segal objects

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We propose a construction of the monoidal envelope of ∞-operads in the model of Segal dendroidal spaces, and use it to define cocartesian fibrations of such. We achieve this by viewing the dendroidal category as a "plus construction" of the category of pointed finite sets, and work in the more general language of algebraic patterns for Segal conditions. Finally, we rephrase Lurie's definition of cartesian structures as exhibiting the categorical fibrations coming from envelopes, and deduce a straightening/unstraightening equivalence for dendroidal spaces.

Contents

1 Introduction

Any monoidal category $\mathcal{V}^{\otimes}$ defines an operad whose colours are the objects of $\mathcal{V}$ and whose multimorphisms $C_1, \cdots, C_n \to D$ are given by the morphisms $C_1 \otimes \cdots \otimes C_n \to D$. The operads which arise in this way are said to be representable; to avoid confusion — since we shall model operads as certain presheaves — we will call them *representably monoidal*. Indeed, the tautological multimorphism $C_1, \cdots, C_n \to C_1 \otimes \cdots \otimes C_n$ (corresponding to $\mathrm{id}_{C_1 \otimes \cdots \otimes C_n}$) carries the universal property that every multimorphism with source $(C_1, \cdots, C_n)$ must factor through it, by a unique unary morphism of source $C_1 \otimes \cdots \otimes C_n$. This universal property can be seen as a cocartesianity condition.

Recall indeed that, if $p\colon \mathcal{E} \to \mathcal{B}$ is a cocartesian fibration (or opfibration), there is a factorisation system on $\mathcal{E}$ whose left class consists of p-cocartesian morphisms and whose right class consists of purely p-vertical morphisms. One can also define a notion of opfibration of multicategories, as done for example in [Her04] for generalised multicategories, and more generally of cocartesian (multi)morphisms therein, so that a multicategory is representably monoidal if and only if its morphism to the terminal operad is an opfibration. In general, an opfibration of multicagories can be thought of as a morphism whose fibres are monoidal categories: since the selection of a colour comes from the operad generated by the nodeless edge η, which only has unary morphisms, the vertical arrows are always unary ones, while the cocartesian arrows are those exhibiting tensor products.

In the homotopical setting, there are various ways of modelling ∞-operads, each coming with its advantages and drawbacks. The model preferred by [Lur17], which represents an ∞-operad by its ∞-category of operators, is biased to make cocartesian fibrations and monoidal envelopes immediately accessible, but requires entangling combinatorics to recover an operadic intuition on any construction. On the other hand, the dendroidal models, based on categories of trees as the shapes for ∞-operads, is closer to the diagramatic operadic intuition, but generally requires more work to make any construction. For exemple, cocartesian fibrations and their straightening were studied in [Heu11] in the model of dendroidal sets, a point-set model which is not manifestly model-invariant.

A more homotopically robust model is that of Segal dendroidal spaces, or more generally Segal dendroidal objects in any complete $(\infty, 1)$-category. This model makes it clear that all that is needed to model algebraic objects such as operads is a general category of "shapes" and their Segal decompositions. This philosophy was realised in [CH21] which defines a notion of algebraic pattern, a base shape category over which can be defined Segal presheaves.

Thus one would like to define a notion of cartesian operations and fibrations for Segal objects over algebraic patterns, and in addition construct the free fibration generated by an arbitrary morphism. But to do this requires a notion of direction for the operations of Segal objects: in categories, there are two directions for cartesianity, from the source or from the target (giving rise to cartesian and cocartesian morphisms), while for operads the notion of (co)cartesian morphisms which has so far been explored looks from the target (as in [CS10, Definition 5.1]), but we expect that any choice of input to separate would give a notion of cartesianity. Not all choices of direction are as good as others: for operads, the choice of the output to keep apart leads to functoriality for composition, while the other choices do not (due to the absence of a duality operation as for categories). As a consequence, one may need to restrict the study to "good" orientations; this seems likely to be impossible to find for modular operads.

In this work, we have elected to eschew the problem by replacing a perhaps more canonical definition of directability by a more practical one. The presence of a notion of direction for operations means that we can think of any composite of them as having an order of progression. This is what is captured by the objects of the simplex category $\mathbb{\Delta}$, chains of morphisms going from a beginning to an end. Hence we will base our notion of direction on this category, declaring an algebraic pattern to be *well-directed* if it can be written as the output of a certain construction involving $\mathbb{\Delta}$. The appropriate construction to consider turns out to be a variant of the plus construction suggested by Baez–Dolan and studied by, among others, [Bar18] and [Ber22].

In section 3, we define this plus construction for appropriately complete algebraic patterns. Then, in section 4, we will use it to construct the "representably monoidal" envelope of a Segal object. While the construction makes sense for general well-directed patterns, we are only able to exhibit its good monoidal properties by restricting patterns such as the one $\Omega^{\mathrm{op}^\flat}$ for operads. Finally, in section 5 we further use this envelope functor to define a straightening/unstraightening Grothendieck construction for ∞-operads; the results of this section are not new and are simply variants of results appearing in [Lur17], but this language provides a new, more operadic, point of view on them.

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2 Review of the language of algebraic patterns

2.1 Algebraic patterns and Segal objects

Definition 2.1.1 (Algebraic pattern)**.** An **algebraic pattern** is an $(\infty, 1)$-category endowed with a unique factorisation system and a selected class of objects called **elementary**. The morphisms in the left class of the factorisation system are called **inert**, and those in the right class **active**.

Notation 2.1.2. If $\mathcal{O}$ is the underlying $(\infty, 1)$-category of the algebraic pattern considered, one writes $\mathcal{O}^{\mathrm{inrt}}$ and $\mathcal{O}^{\mathrm{act}}$ for the wide and locally full sub-$(\infty, 1)$-categories whose morphisms are respectively the inert and the active morphisms.

We also denote $\mathcal{O}^{\mathrm{el}}$ the full sub-$(\infty, 1)$-category of $\mathcal{O}^{\mathrm{inrt}}$ on the selected elementary objects. For any object $O \in \mathcal{O}$, we further write $\mathcal{O}^{\mathrm{el}}_{O/} \coloneqq \mathcal{O}^{\mathrm{el}} \times_{\mathcal{O}^{\mathrm{inrt}}} \mathcal{O}^{\mathrm{inrt}}_{O/}$, the $(\infty, 1)$-category of inert morphisms from O to an elementary object.

Example 2.1.3. Segal's category $\mathbb{\Gamma}$, (a skeleton of) the opposite of the category $\mathfrak{FinSet}_*$ of pointed finite sets, admits an active-inert factorisation system where a morphism of pointed finite sets $f\colon (S, s_0) \to (T, t_0)$ is **inert** if for every $t \in T \setminus \{t_0\}$, the preimage $f^{-1}(t)$ consists of exactly one element, and **active** if the only element of S mapped to t_0 is s_0 — so that, in particular, the subcategory of active morphisms $\mathbb{\Gamma}^{\mathrm{op}^{\mathrm{act}}}$ identifies with the category of finite sets.

The induced inert-active factorisation system on $\mathbb{\Gamma}^{\mathrm{op}}$ gives rise to two algebraic pattern structures, from two choices of elementary objects. The algebraic pattern $\mathbb{\Gamma}^{\mathrm{op}^\flat}$ has as elementaries the two-element sets (isomorphic to $\langle 1 \rangle$), while the pattern $\mathbb{\Gamma}^{\mathrm{op}^\natural}$ has as elementaries the singletons (isomorphic to $\langle 0 \rangle$) and the two-element sets.

Example 2.1.4. The (non-augmented) simplicial indexing category $\mathbb{\Delta}$, identified with the category of ordered non-empty finite sets and order-preserving maps between them, admits a factorisation system in which a map is **active** if it preserves the top and bottom elements, and **inert** if it corresponds to the inclusion of a linear subset.

The induced inert-active factorisation system on $\mathbb{\Delta}^{\mathrm{op}}$ gives rise to two algebraic pattern structures: the algebraic pattern $\mathbb{\Delta}^{\mathrm{op}^\flat}$ has as elementaries the two-element sets (isomorphic to $[1]$), while the pattern $\mathbb{\Gamma}^{\mathrm{op}^\natural}$ has as elementaries the singletons (isomorphic to $[0]$) and the two-element sets.

Example 2.1.5. Of particular interest to us, the dendroidal category $\mathbb{\Omega}$ of Moerdijk–Weiss can be described as a category of (non-planar) rooted trees, with morphisms the morphisms of free (coloured, symmetric) operads generated by these trees. We shall give an alternate construction of (a sufficient subcategory of) it in section 3.

Certain particularly interesting trees can be distinguished:

the free-living edge is the tree, denoted η, consisting of one edge but no vertex (so the operad it freely generates has one colour, and only its identity unary morphism);

the corollas are the trees, denoted $\star_n$ with $n \in \mathbb{N}$, with a single vertex and $n + 1$ edges (one of which is the root) attached to it. Note that the corolla $\star_n$ with n leaves has as automorphism group the symmetric group $\mathfrak{S}_n$.

The category $\mathbb{\Omega}$ admits a factorisation system in which a morphism is **active** if it is boundary-preserving, and **inert** if it corresponds to a subtree inclusion (which, in particular, is valence-preserving on the vertices).

The induced inert-active factorisation system on $\mathbb{\Omega}^{\mathrm{op}}$ gives again rise to two algebraic pattern structures: the algebraic pattern $\Omega^{\mathrm{op}^\flat}$ has as elementaries the corollas $\star_n$, while the pattern $\mathbb{\Omega}^{\mathrm{op}^\natural}$ has as elementaries the corollas and the free-living edge η.

Definition 2.1.6 (Morphisms of algebraic patterns)**.** Let $\mathfrak{O}$ and $\mathfrak{P}$ be algebraic patterns. A **morphism of algebraic patterns** from $\mathfrak{O}$ to $\mathfrak{P}$ is an $(\infty, 1)$-functor $\mathfrak{O} \to \mathfrak{P}$ which preserves active and inert morphisms and elementary objects.

Example 2.1.7. There is a functor $\Delta \to \Omega$ which, taking the standard skeleton of Δ, sends $[n]$ to the linear tree with n nodes and $n+1$ edges. One may remark that it is fully faitful, and can also be identified with the canonical projection functor $\Omega_{/\eta} \to \Omega$. By translating the definition of inert and active morphisms for the given factorisation system on Δ^{op}, one immediately sees that this functor induces morphisms of algebraic patterns $\Delta^{\mathrm{op}\natural} \to \Omega^{\mathrm{op}\natural}$ and $\Delta^{\mathrm{op}\flat} \to \Omega^{\mathrm{op}\flat}$.

Definition 2.1.8 (Segal objects)**.** Let $\mathfrak{O}$ be an algebraic pattern. An $(\infty, 1)$-category $\mathfrak{C}$ is said to be **$\mathfrak{O}$-complete** if it admits limits of diagrams with shape $\mathfrak{O}^{\mathrm{el}}_{O/}$ for any $O \in \mathfrak{O}$.

Let $\mathfrak{C}$ be an $\mathfrak{O}$-complete $(\infty, 1)$-category. A **Segal $\mathfrak{O}$-object** in $\mathfrak{C}$ is an $(\infty, 1)$-functor $\mathscr{X} \colon \mathfrak{O} \to \mathfrak{C}$ such that $\mathscr{X}|_{\mathfrak{O}^{\mathrm{inrt}}}$ is a lax extension of $\mathscr{X}|_{\mathfrak{O}^{\mathrm{el}}}$ (along the inclusion). Explicitly, this means that for any $O \in \mathfrak{O}$ the canonical map

$$\mathscr{X}(O) \to \lim_{E \in \mathfrak{O}^{\mathrm{el}}_{O/}} \mathscr{X}(E) \tag{1}$$

is invertible.

The $(\infty, 1)$-category $\mathfrak{Seg}_{\mathfrak{O}}(\mathfrak{C})$ of Segal $\mathfrak{O}$-objects in $\mathfrak{C}$ is the full sub-$(\infty, 1)$-category of $\mathfrak{C}^{\mathfrak{O}}$ spanned by the Segal objects.

Example 2.1.9.

- A Segal $\Gamma^{\mathrm{op}\flat}$-object is a commutative algebra object (or $\mathscr{E}_\infty$-algebra object).
- A Segal $\Delta^{\mathrm{op}\flat}$-object is an associative algebra object (or $\mathscr{A}_\infty$-algebra object, or $\mathscr{E}_1$-algebra), while a Segal $\Delta^{\mathrm{op}\natural}$-object is an internal category.
- A Segal $\Omega^{\mathrm{op}\flat}$-object is an internal monochromatic operad, while a Segal $\Omega^{\mathrm{op}\natural}$-object is an internal (coloured) operad.

Remark 2.1.10. Let $\mathfrak{O}$ be an algebraic pattern such that the inclusion $\mathfrak{O}^{\mathrm{el}} \hookrightarrow \mathfrak{O}^{\mathrm{inrt}}$ is codense. Then for any $O \in \mathfrak{O}$, the corepresentable $\text{よ}_{\mathfrak{O}^{\mathrm{op}}}(O) \colon \mathfrak{O} \to \infty\text{-}\mathfrak{Grpd}$ is a Segal $\mathfrak{O}$-∞-groupoid — this is an immediate consequence of the limit-preservation property of hom ∞-functors. It should be viewed as the Segal object generated by O.

Example 2.1.11. Over Δ^{op}, the Segal object $\text{よ}[n]$ corresponds to the linear category $\mathbb{n} + \mathbb{1}$ with n successive arrows.

Example 2.1.12. Over Ω^{op}, the Segal object $\text{よ}\star_n$ generated by the corolla with $n+1$ flags is also denoted $\star_n$ and called the corolla. It corresponds to the operad with $n+1$ colours $O_1, \ldots, O_{n+1}$ and, for each permutation $\sigma \in \mathfrak{S}_n$, a single operation of signature $(O_{\sigma(1)}, \ldots, O_{\sigma(n)}; O_{n+1})$.

2.2 Morphisms of algebraic patterns I: (weak) Segal fibrations

Definition 2.2.1 (Segal morphism of algebraic patterns)**.** A morphism of algebraic patterns $\mathscr{F}\colon \mathfrak{O} \to \mathfrak{P}$ is said to be a **Segal morphism** if "it preserves Segal conditions", that is if for any $\mathfrak{P}$-complete $(\infty,1)$-category $\mathfrak{C}$, the induced $(\infty,1)$-functor $\mathscr{F}^*|_{\mathfrak{Seg}_{\mathfrak{P}}}\colon \mathfrak{Seg}_{\mathfrak{P}}(\mathfrak{C}) \subset \mathfrak{C}^{\mathfrak{P}} \to \mathfrak{C}^{\mathfrak{O}}$ factors through $\mathfrak{Seg}_{\mathfrak{O}}(\mathfrak{C}) \hookrightarrow \mathfrak{C}^{\mathfrak{O}}$.

Remark 2.2.2. By [CH21, Lemma 4.5], it is enough to check Segality of a morphism with $\mathfrak{C} = \infty\text{-}\mathfrak{Grpd}$, that is to check preservation of Segal ∞-groupoids. Then the condition for $\mathscr{F}$ to be a Segal morphism can be written in a formula as: for every $\mathrm{O} \in \mathfrak{O}$, for every Segal $\mathfrak{P}$-∞-groupoid $\mathscr{X}$, the morphism of ∞-groupoids

$$\lim_{\mathfrak{P}^{\mathrm{el}}_{\mathscr{F}(\mathrm{O})/}} \mathscr{X} \to \lim_{\mathfrak{O}^{\mathrm{el}}_{\mathrm{O}/}} \mathscr{X} \circ \mathscr{F}^{\mathrm{el}} \tag{2}$$

induced by the $(\infty,1)$-functor $\mathfrak{O}^{\mathrm{el}}_{\mathrm{O}/} \to \mathfrak{P}^{\mathrm{el}}_{\mathscr{F}(\mathrm{O})/}$ is an equivalence.

Construction 2.2.3. Suppose $\mathscr{V}\colon \mathfrak{O} \to (\infty,1)\text{-}\mathfrak{Cat}$ is an $\mathfrak{O}$-monoidal $(\infty,1)$-category. Passing to the Grothendieck construction of the functor produces a cocartesian fibration $\int^{\mathrm{co}} \mathscr{V} \to \mathfrak{O}$. We shall refer to a cocartesian fibration over $\mathfrak{O}$ whose associated $(\infty,1)$-functor $\mathfrak{O} \to (\infty,1)\text{-}\mathfrak{Cat}$ satisfies the Segal conditions as a **Segal fibration** over $\mathfrak{O}$.

By [Lur17, Proposition 2.1.2.5], if $\int^{\mathrm{co}} \mathscr{V} \to \mathfrak{O}$ is a Segal $\mathfrak{O}$-fibration, the inert-active factorisation system on $\mathfrak{O}$ lifts to one on $\int^{\mathrm{co}} \mathscr{V}$, endowing it with a structure of algebraic pattern. If $\mathfrak{P} \to \mathfrak{O}$ is another algebraic pattern over $\mathfrak{O}$, we shall define a **$\mathfrak{P}$-algebra** in $\int^{\mathrm{co}} \mathscr{V}$ to be a morphism of algbraic pattern over $\mathfrak{O}$ from $\mathfrak{P}$ to $\int^{\mathrm{co}} \mathscr{V}$.

Definition 2.2.4 (Weak Segal fibration)**.** Let $\mathfrak{O}$ be an algebraic pattern. A **weak Segal $\mathfrak{O}$-fibration** (also called $\mathfrak{O}$-operad) is an $(\infty,1)$-functor $\mathscr{p}\colon \mathfrak{X} \to \mathfrak{O}$ such that:

1. for every object $\mathrm{X} \in \mathfrak{X}$, every inert arrow $\mathrm{i}\colon \mathscr{p}\mathrm{X} \to \mathrm{O}$ in $\mathfrak{O}$ admits a $\mathscr{p}$-cocartesian lift $\mathrm{i}_!\colon \mathrm{X} \to \mathrm{i}_!\mathrm{X}$;

2. for every object $\mathrm{O} \in \mathfrak{O}$, the $(\infty,1)$-functor $\mathfrak{X}_{\mathrm{O}} \to \lim_{\mathrm{E} \in \mathfrak{O}^{\mathrm{el}}_{\mathrm{O}/}} \mathfrak{X}_{\mathrm{E}}$ induced by the cocartesian morphisms over inert arrows is invertible;

3. for every $\mathrm{X} \in \mathfrak{X}$ and every choice of $\mathscr{p}$-cocartesian lift of the tautological diagram $\imath\colon \mathfrak{O}^{\mathrm{el}}_{\mathscr{p}\mathrm{X}/} \to \mathfrak{O}$ (of inert morphisms from $\mathscr{p}\mathrm{X}$) to an $\imath_!\colon (\mathfrak{O}^{\mathrm{el}}_{\mathscr{p}\mathrm{X}/})^{\triangleleft} \to \mathfrak{X}$ taking the cone point to X, for every $\mathrm{Y} \in \mathfrak{X}$, the commutative square

$$\begin{tikzcd} \mathfrak{X}(\mathrm{Y},\mathrm{X}) \arrow[r] \arrow[d] & \lim_{\mathrm{E}\in\mathfrak{O}^{\mathrm{el}}_{\mathscr{p}\mathrm{X}/}} \mathfrak{X}(\mathrm{Y},\imath_!(\mathrm{E})) \arrow[d] \\ \mathfrak{O}(\mathscr{p}\mathrm{Y},\mathscr{p}\mathrm{X}) \arrow[r] & \lim_{\mathrm{E}\in\mathfrak{O}^{\mathrm{el}}_{\mathscr{p}\mathrm{X}/}} \mathfrak{O}(\mathscr{p}\mathrm{Y},\imath(\mathrm{E})=\mathrm{E}) \end{tikzcd} \tag{3}$$

   is cartesian.

Example 2.2.5.

- A weak Segal $\mathbb{\Gamma}^{\mathrm{op}^\flat}$-fibration is the $(\infty,1)$-category of operators of an ∞-operad in the sense of [Lur17, Definition 2.1.1.10], while a weak Segal $\mathbb{\Gamma}^{\mathrm{op}^\natural}$-fibration is a generalised ∞-operad in the sense of [Lur17, Definition 2.3.2.1].
- A weak Segal $\mathbb{\Delta}^{\mathrm{op}^\flat}$-fibration is the $(\infty,1)$-category of operators of a non-symmetric ∞-operad in the sense of [GH15, Definition 2.2.6, Definition 3.1.3] while a weak Segal $\mathbb{\Delta}^{\mathrm{op}^\natural}$-fibration is virtual double ∞-category, or generalised ∞-operad in [GH15, Definition 2.4.1, Definition 3.1.13].

Definition 2.2.6 (Morphisms of weak Segal fibrations)**.** By [CH21], the source $\mathfrak{X}$ of a weak Segal $\mathfrak{O}$-fibration $p\colon \mathfrak{X} \to \mathfrak{O}$ inherits an algebraic pattern structure in which active morphisms are those lying over an active morphism in $\mathfrak{O}$, inert morphisms are the p-cocartesian morphisms lying over inert arrows of $\mathcal{O}$, and elementaries are the objects lying over elementary objects.

The $(\infty,2)$-category of weak Segal $\mathfrak{O}$-fibrations $\mathfrak{WkSegFib}_{/\mathfrak{O}}$ is the locally full sub-$(\infty,2)$-category of $(\infty,1)$-$\mathfrak{Cat}_{/\mathfrak{O}}$ spanned by the weak Segal $\mathfrak{O}$-fibrations and Segal morphisms thereof.

It can be checked directly that any Segal fibration as in construction 2.2.3 is in particular a weak Segal fibration. In fact Segal $\mathfrak{O}$-fibrations are exactly those $(\infty,1)$-functors to $\mathfrak{O}$ which are both weak Segal fibrations and cocartesian fibrations. This provides a (non-full) inclusion $(\infty,2)$-functor from the $(\infty,2)$-category of Segal fibrations into that of weak Segal fibrations.

2.3 Morphisms of algebraic patterns II: (co)enrichable structures

To finish this section, we state (a variant of) some definitions[1] of the forthcoming paper [CH22].

First recall, as motivation, the following definition from [CH20b].

Definition 2.3.1 (Cartesian pattern)**.** A **cartesian pattern** is an algebraic pattern $\mathfrak{O}$ endowed with a morphism of patterns $|-|\colon \mathfrak{O} \to \mathbb{\Gamma}^{\mathrm{op}^\flat}$ such that for any $O \in \mathfrak{O}$, the morphism $\mathfrak{O}^{\mathrm{el}}_{O/} \to \mathbb{\Gamma}^{\mathrm{op}^\flat,\mathrm{el}}_{|O|/}$ is an equivalence.

Remark 2.3.2. Viewing $\mathbb{\Gamma}^{\mathrm{op}}$ as (the standard skeleton of) the category of pointed finite sets, one verifies that any $\langle n\rangle \in \mathbb{\Gamma}^{\mathrm{op}}$ admits exactly n inert morphisms to the unique elementary $\langle 1\rangle$, the pointed morphisms $\rho_i\colon \langle n\rangle \to \langle 1\rangle$ mapping i to 1 and every other element of $\langle n\rangle$ to the base-point. The condition of being a cartesian pattern then means that, for any $O \in \mathfrak{O}$, the $(\infty,1)$-category $\mathfrak{O}^{\mathrm{el}}_{O/}$ must be equivalent to the discrete set of the (essentially unique) lifts $\rho_{i,!}$ of the ρ_i. In particular, the Segal condition for a precosheaf $\mathcal{X}$ on $\mathfrak{O}$ is constrained to being the finite product condition

$$\mathcal{X}(O) \simeq \prod_{i=1}^{|O|} \mathcal{X}(\rho_{i,!}O). \tag{4}$$

[1]which were presented in the seminar talk available at `https://www.msri.org/seminars/25057`

We recall from [Lur17] the definition of semi-inert arrows in Γ^{op}. A map of pointed finite sets $f\colon (S, s_0) \to (T, t_0)$ is **semi-inert** if for any $t \in T \setminus \{t_0\}$, there is at most one element in $f^{-1}(t)$.

Definition 2.3.3. Let $(\mathcal{O}, |\cdot|)$ be a coenrichable algebraic pattern. An arrow f of $\mathcal{O}$ is **semi-inert** if $|f|$ is a semi-inert arrow of Γ^{op}.

Example 2.3.4. In Δ^{op}, the semi-inert morphisms are those corresponding to the cellular morphisms of Δ as defined in [Hau17] and [Hau16], that is maps of totally ordered sets $f\colon S \to S'$ such that for all $s \in S$, $f(\operatorname{succ} s) \leq \operatorname{succ}(f(s))$.

For $(\Delta^{\mathrm{op}})^n$, we also recover the cellular morphisms of [Hau17], those maps all of whose components are cellular in Δ^{op}.

Remark 2.3.5. As observed in [Hau17], an arrow $f\colon O \to O'$ of $\mathcal{O}$ is semi-inert if and only if, for any elementary E and any inert morphism $k\colon O' \rightarrowtail E$, the composite $O \xrightarrow{k \circ f} E$ is semi-inert.

Example 2.3.6. The active maps $\lambda_i\colon \langle 1\rangle \rightsquigarrow \langle n\rangle$ of remark 2.3.8 (in Γ^{op}) are semi-inert. It directly follows that, in any coenrichable pattern $\mathcal{O}$, their lifts $E_i^O \rightsquigarrow O$ are semi-inert.

Definition 2.3.7 (Cocartesian pattern)**.** A **cocartesian pattern** is an algebraic pattern $\mathcal{O}$ endowed with a morphism of patterns $|{-}|\colon \mathcal{O} \to \Gamma^{\mathrm{op}\flat}$ such that for any $O \in \mathcal{O}$, the morphism $\mathcal{O}^{\mathrm{act}}_{\mathrm{el}/O} \to \Gamma^{\mathrm{op}\flat,\text{½-inrt},\mathrm{act}}_{\mathrm{el}/|O|}$ is an equivalence, where $\mathcal{O}^{\text{½-inrt},\mathrm{act}}_{\mathrm{el}/O}$ means the $(\infty,1)$-category of active and semi-inert morphisms from an elementary object to O.

Remark 2.3.8. In $\Gamma^{\mathrm{op}\flat}$, any object $\langle n\rangle$ admits again precisely n active morphisms from the unique elementary object $\langle 1\rangle$, the functions $\lambda_i\colon \langle 1\rangle \rightsquigarrow \langle n\rangle$ sending $1 \in \langle 1\rangle$ to $i \in \langle n\rangle$ for any $1 \leq i \leq n$. Hence any object O in a cocartesian pattern admits exactly $|O|$ active morphisms from elementaries, denoted $E_i^O \rightsquigarrow O$.

Example 2.3.9. The functor $* \to \Gamma^{\mathrm{op}}$ picking the object $\langle 1\rangle$ defines a structure of cocartesian pattern on the terminal algebraic pattern.

Example 2.3.10. There is a functor $\Delta^{\mathrm{op}} \to \Gamma^{\mathrm{op}}$ mapping $[n]$ to $\langle n\rangle$ and sending an arrow of Δ^{op} corresponding to $\phi\colon [n] \to [m]$ in Δ to $|\phi|\colon \langle m\rangle \to \langle n\rangle$ given by

$$|\phi|(i) = \begin{cases} j & \text{if } \phi(j-1) < i \leq \phi(j) \\ * & \text{otherwise.} \end{cases} \tag{5}$$

It can be checked directly that this is a structure of cocartesian pattern on $\Delta^{\mathrm{op}\flat}$.

Example 2.3.11. A functor $\Omega^{\mathrm{op}} \to \Gamma^{\mathrm{op}} \simeq \mathfrak{FinSet}_*$ is defined in [CH20a, Definition 4.1.16] in the following way. A tree T with set of vertices $V(T)$ is mapped to the freely pointed set $V(T)_+$. A morphism $T' \leftarrow T$ in Ω^{op} is mapped to the pointed morphism $V(T')_+ \to V(T)_+$ which sends a vertex $v \in V(T')$ to the unique vertex of T whose image subtree contains v, or to the basepoint if there is no such vertex. By [loc. cit., Lemma 4.1.18], this functor preserves the inert-active factorisation system, and thus defines a morphism of algebraic patterns.

In proposition 3.2.8 we will provide another construction of this coenrichable structure from the point of view of Ω as obtained from the plus construction.

Definition 2.3.12 (Coenrichable pattern)**.** An **coenrichable pattern** is an algebraic pattern $\mathfrak{O}$ equipped with a morphism of patterns $|{-}|\colon \mathfrak{O} \to \mathbb{F}^{\mathrm{op}\natural}$ such that $(\mathfrak{O}\times_{\mathbb{F}^{\mathrm{op}\natural}}\mathbb{F}^{\mathrm{op}\flat}, |{-}|\times_{\mathbb{F}^{\mathrm{op}\natural}} \mathbb{F}^{\mathrm{op}\flat})$ is a cocartesian pattern.

When $(\mathfrak{O}, |{-}|)$ is an coenrichable pattern, we let $\mathfrak{O}^\flat \coloneqq \mathfrak{O} \times_{\mathbb{F}^{\mathrm{op}\natural}} \mathbb{F}^{\mathrm{op}\flat}$ denote the associated cocartesian pattern.

Thanks to [CH21, Corollary 5.5], we see that the fibre product $\mathfrak{O}^\flat = \mathfrak{O} \times_{\mathbb{F}^{\mathrm{op}\natural}} \mathbb{F}^{\mathrm{op}\flat}$ appearing in the definition of coenrichable patterns has a very simple description: it consists of the category $\mathfrak{O}$ equipped with its same factorisation system, and the choice of only those elementary objects living over $\langle 1\rangle \in \mathbb{F}^{\mathrm{op}}$ (*i.e.* excluding those over $\langle 0\rangle$).

Example 2.3.13. All the $\natural$-decorated patterns.

Remark 2.3.14 (Momentous pattern). Let us say that an algebraic pattern $(\mathfrak{O}, |{-}|)$ over $\mathbb{F}^{\mathrm{op}\natural}$ is **momentous** if any active morphism from a $\flat$-elementary admits an essentially unique inert retraction.

It then follows that a momentous pattern that is enrichable is coenrichable, and vice-versa. Momentous patterns which are (co)enrichable are closely related to (an ∞-categorical version of) the hypermoment categories of [Ber22].

3 The plus construction and its Segal objects

3.1 Categories of patterned trees

The following construction is a variant of one due to [Bar18] in the setting of operator categories, inspired by the plus construction or "slice operads" of [BD98], and which was also studied in [Ber22] in the setting of hypermoment categories and [BM21] for operadic categories, and used in [CHH18].

Construction 3.1.1. Let $\mathfrak{O}$ be a coenrichable algebraic pattern. Consider the restriction $\underline{\text{よ}}_{(\infty,1)\text{-}\mathfrak{Cat}}(\mathfrak{O}^{\mathrm{act}})|_{\Delta}\colon [n] \mapsto \mathfrak{O}^{\mathrm{act}[n]}$ to $\Delta \subset (\infty,1)\text{-}\mathfrak{Cat}$ of the $(\infty,2)$-functor represented by $\mathfrak{O}^{\mathrm{act}}$ and let $\Delta^{\mathrm{pre}}_{\mathfrak{O}} \to \Delta$ be its Grothendieck construction. Note that $\Delta^{\mathrm{pre}}_{\mathfrak{O}}$ is equivalent to $\Delta_{/\mathfrak{O}^{\mathrm{act}}}$ — where Δ is seen as a subcategory of $(\infty,1)\text{-}\mathfrak{Cat}$ — or also to $\Delta_{/\mathscr{N}_\bullet\mathfrak{O}^{\mathrm{act}}}$, where $\mathscr{N}_\bullet\colon (\infty,1)\text{-}\mathfrak{Cat} \hookrightarrow \infty\text{-}\mathfrak{Grpd}^{\Delta^{\mathrm{op}}}$ is a nerve functor (for example incarnating $(\infty,1)$-categories as complete Segal ∞-groupoids, or as quasicategories, to the reader's preference). Thus an

object of $\Delta^{\mathrm{pre}}_{\mathfrak{O}}$ consists of a pair $([n], O_\bullet)$ where $[n] \in \Delta$ and $O_\bullet\colon [n] \to \mathfrak{O}^{\mathrm{act}}$ is a linear diagram in $\mathfrak{O}^{\mathrm{act}}$, that is a sequence $O_0 \rightsquigarrow O_1 \rightsquigarrow \cdots \rightsquigarrow O_n$ of active morphisms in $\mathfrak{O}$, while a

morphism $([n], O_\bullet) \to ([n'], O'_\bullet)$ consists of a pair $(\phi, f_\bullet)$ where $\phi\colon [n] \to [n']$ is a map in Δ and $f_\bullet\colon O_\bullet \Rightarrow O'_{\phi(\bullet)} = O'_\bullet \circ \phi$ is a natural transformation of $[n]$-shaped diagrams in $\mathfrak{O}^{\mathrm{act}}$.

We define $\Delta_{\mathcal{O}}$ to be the wide and locally full (*i.e.* containing all objects and all higher morphisms between a selection of 1-morphisms) sub-$(\infty,1)$-category of $\Delta_{\mathcal{O}}^{\mathrm{pre}}$ on those morphisms $(\phi, f_\bullet)$ such that

- $f_\bullet$ is component-wise semi-inert (in addition to active), that is each $f_i \colon O_i \rightsquigarrow O'_{\phi(i)}$, for $i \in [n]$, is semi-inert, and
- $f_\bullet$ is a cartesian (or equifibred) natural transformation, that is for any morphism $i < j$ in $[n]$ the naturality square

$$\begin{tikzcd} O_i \arrow[r, squiggly, "f_i"] \arrow[d, squiggly, "O_{i<j}"'] & O'_{\phi(i)} \arrow[d, squiggly, "O'_{\phi(i<j)}"] \\ O_j \arrow[r, squiggly, "f_j"'] & O'_{\phi(j)} \end{tikzcd} \tag{6}$$

  is cartesian in $\mathcal{O}^{\mathrm{act}}$.

Remark 3.1.2. Given fixed $\phi \colon [m] \to [n]$ in Δ and $([n], P_\bullet)$ in $\Delta_{\mathcal{O}}$ over $[n]$, morphisms $(\phi, f_\bullet) \colon ([m], O_\bullet) \to ([n], P_\bullet)$ in $\Delta_{\mathcal{O}}$ lifting ϕ are essentially determined, if they exist, by their underlying arrow $f_m \colon O_m \to P_{\phi(m)}$ at the terminal object $m \in [m]$. Indeed, for each $i \in [m]$ the object O_i and arrow f_i are required by the pullback condition in eq. (6) to be the base-change of O_m and f_m along $i \leq m$.

Definition 3.1.3. Let $\mathcal{O}$ be a coenrichable algebraic pattern. The $(\infty,1)$-category $\Delta_{\mathcal{O}}$ is called the $(\infty,1)$-category of **$\mathcal{O}$-forests**.

The full sub-$(\infty,1)$-category $\Delta_{\mathcal{O}}^{(1)}$ of $\Delta_{\mathcal{O}}$ on the $([n], O_\bullet)$ such that $O_n \in \mathcal{O}^{\flat,\mathrm{el}}$ is called the $(\infty,1)$-category of **$\mathcal{O}$-trees**.

Example 3.1.4. For the terminal algebraic pattern $*$, the $(\infty,1)$-categories of $*$-forests and of $*$-trees both recover the simplex category Δ.

Example 3.1.5. For $\mathbb{\Gamma}^{\mathrm{op}\flat}$, we obtain a category of forests, *i.e.* disjoint unions of trees, with level structures, whose tree-like subcategory is identified in [Ber22] with a full subcategory of Ω. Indeed, recalling that the active subcategory of $\mathbb{\Gamma}^{\mathrm{op}}$ is equivalent to the category of finite sets, the object O_n is to be thought of as the set of roots of the forest, and each O_i is the set of leaves at level $n - i$. The morphisms in $(\mathbb{\Gamma}^{\mathrm{op}})^{\mathrm{act}}$ give partitions of the leaves at levels ℓ corresponding to the node (recognised by its unique output leaf at level $\ell + 1$) to which they lead.

Example 3.1.6. For $\Delta^{\mathrm{op}\flat}$, we similarly obtain a category of planar trees (or rather, forests thereof) with level structures.

3.2 The pattern structure on the plus construction

Lemma 3.2.1. *The ∞-functor $\mathcal{d}_{\mathcal{O}} \colon \Delta_{\mathcal{O}} \subset \Delta_{\mathcal{O}}^{\mathrm{pre}} \to \Delta$ is a cartesian fibration.*

Proof. We first observe as in [CHH18] that, due to the pullback condition in eq. (6), a morphism $(\phi, f_\bullet)\colon ([m], O_\bullet) \to ([n], P_\bullet)$ in $\Delta_{\mathcal{O}}$ is $d_{\mathcal{O}}$-cartesian if and only if $f_\bullet$ is a natural equivalence. Indeed, saying that $(\phi, f_\bullet)\colon ([m], O_\bullet) \to ([n], P_\bullet)$ is $d_{\mathcal{O}}$-cartesian is saying that

$$\begin{tikzcd} \Delta_{\mathcal{O},/([m],O_\bullet)} \arrow[r, "(\phi,f_\bullet)\circ-"] \arrow[d] & \Delta_{\mathcal{O},/([n],P_\bullet)} \arrow[d] \\ \Delta_{/[m]} \arrow[r, "\phi\circ-"'] & \Delta_{/[n]} \end{tikzcd} \tag{7}$$

is a cartesian square.

Let then $([k], L_\bullet)$ be any object of $\Delta_{\mathcal{O}}$, and take the fibre of eq. (7) at $([k], L_\bullet)$, which we decompose as follows:

$$\begin{tikzcd} \Delta_{\mathcal{O}}(([k],L_\bullet),([m],O_\bullet)) \arrow[r] \arrow[d] & \Delta_{\mathcal{O}}(([k],L_\bullet),([n],P_\bullet)) \arrow[d] \\ \Delta^{\mathrm{pre}}_{\mathcal{O}}(([k],L_\bullet),([m],O_\bullet)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner \text{ iff } f_\bullet \text{ invertible}"] & \Delta^{\mathrm{pre}}_{\mathcal{O}}(([k],L_\bullet),([n],P_\bullet)) \arrow[d] \\ \Delta([k],[m]) \arrow[r] & \Delta([k],[n]). \end{tikzcd} \tag{8}$$

Since $\Delta^{\mathrm{pre}}_{\mathcal{O}} \to \Delta$ is constructed as a Grothendieck construction, the lower square is cartesian if and only if $f_\bullet$ invertible. It is then enough to observe that the condition for a morphism in $\Delta^{\mathrm{pre}}_{\mathcal{O}}$ to be in $\Delta_{\mathcal{O}}$ is detected by postcomposition with a componentwise invertible morphism of $\Delta_{\mathcal{O}}$, which is clear.

This characterisation now makes it easy to determine cartesian lifts. Let $\phi\colon [m] \to [n]$ be a morphism in Δ, and let $([n], O_\bullet)$ be a lift of $[n]$ in $\Delta_{\mathcal{O}}$. We define a cartesian lift of $([n], O_\bullet)$ along ϕ to be $([m], O_{\phi(\bullet)})$ with $(\phi, \mathrm{id}_{O_\bullet \circ \phi})$. ☐

Corollary 3.2.2. *Say that an arrow $(\phi, f_\bullet)$ in $\Delta_{\mathcal{O}}$ is inert if ϕ is inert in Δ and active if ϕ is active in Δ and $f_\bullet$ is an equivalence. Then $({\Delta_{\mathcal{O}}}^{\mathrm{op\,inrt}}, {\Delta_{\mathcal{O}}}^{\mathrm{op\,act}})$ defines a factorisation system on ${\Delta_{\mathcal{O}}}^{\mathrm{op}}$.*

Proof. By [Lur17, Proposition 2.1.2.5], since $d_{\mathcal{O}}\colon \Delta_{\mathcal{O}} \subset \Delta^{\mathrm{pre}}_{\mathcal{O}} \to \Delta$ is a cartesian fibration, the factorisation system on its base can be lifted as required. ☐

Definition 3.2.3. Let $\mathcal{O}$ be a coenrichable algebraic pattern. Its **plus construction** ${\Delta_{\mathcal{O}}}^{\mathrm{op}\natural}$ is the $(\infty, 1)$-category ${\Delta_{\mathcal{O}}}^{\mathrm{op}}$ equipped with the inert-active factorisation of corollary 3.2.2 and as elementary objects those $([n], O_\bullet)$ with $[n] \in \Delta^{\mathrm{op}\natural,\mathrm{el}}$ (*i.e.* either $[0]$ or $[1]$) and $O_n \in \mathcal{O}^{\mathrm{el}}$.

Its **tree-like plus construction** is the algebraic pattern induced on ${\Delta^{(1)}_{\mathcal{O}}}^{\mathrm{op}}$.

We shall say that an algebraic pattern is **well-directed** if it can be written as the plus construction of some coenrichable algebraic pattern.

Example 3.2.4. The main result of [CHH18] shows that ${\Delta_{\Gamma^{\mathrm{op}\natural}}}^{\mathrm{op}\natural}$ is Morita-equivalent to $\Omega^{\mathrm{op}\natural}$ in the sense that their $(\infty,1)$-categories of Segal ∞-groupoids are equivalent. We will henceforth conflate operad objects with Segal ${\Delta_{\Gamma^{\mathrm{op}\natural}}}^{\mathrm{op}\natural}$-objects.

Scholium 3.2.5. *The Segal condition for a Segal ${\Delta_{\mathfrak{O}}}^{\mathrm{op}\natural}$-object $\mathscr{X}$ can be explicitly written as the following set of conditions:*

level decomposition *For any* $([n], O_\bullet)$, *the canonical arrow*

$$\mathscr{X}\big([n], O_\bullet\big) \to \mathscr{X}\big([1], (O_0 \rightsquigarrow O_1)\big) \underset{\mathscr{X}([0],(O_1))}{\times} \cdots \underset{\mathscr{X}([0],(O_{n-1}))}{\times} \mathscr{X}\big([1], (O_{n-1} \rightsquigarrow O_n)\big) \tag{9}$$

is an equivalence.

root $\mathfrak{O}$-Segal decomposition *For any height-0 forest of the form* $([0], O_0)$, *the canonical map*

$$\mathscr{X}\big([0], (O_0)\big) \to \prod_{i=1}^{|O_0|} \mathscr{X}\big([0], (E_i^{O_0})\big) \tag{10}$$

is an equivalence (where $E_i^O \rightsquigarrow O$ is the lift of O along λ_i from example 2.3.6).

shrub $\mathfrak{O}$-Segal decomposition *For any* $\big([1], (O_0 \rightsquigarrow O_1)\big)$, *the canonical arrow*

$$\mathscr{X}\big([1], (O_0 \rightsquigarrow O_1)\big) \to \prod_{i=1}^{|O_1|} \mathscr{X}\big([1], (O_{0,i} \rightsquigarrow E_i^{O_1})\big) \tag{11}$$

is an equivalence, where $O_{0,i}$ is the fibre product $O_0 \times^{\mathrm{act}}_{O_1} E_i^{O_1}$ — with $\times^{\mathrm{act}}$ indicating that it is a fibre product in $\mathfrak{O}^{\mathrm{act}}$.

Proof. The only part that bears commenting upon is the shrub decomposition. Recall that the inert maps in $\Delta_{\mathfrak{O}}$ are simply those whose projection to Δ is inert, so the relevant inert morphisms in $\Delta_{\mathfrak{O}}$ are those from the elementaries over $[1] \in \Delta$. If we consider such a morphism, represented by

$$\begin{tikzcd} P \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & O_0 \arrow[d, rightsquigarrow] \\ E \arrow[r, rightsquigarrow] & O_1. \end{tikzcd} \tag{12}$$

Since $\mathfrak{O}$ is coenrichable, $E \rightsquigarrow O_1$ must be one of the lifts $E_i^{O_1}$, and then the equifibred condition implies that P is the pullback $O_0 \times_{O_1} E_i^{O_1}$. □

Proposition 3.2.6 ([CHH18, Lemma 2.11])**.** *Forests and trees have the same Segal objects: the $(\infty,1)$-functor $\mathfrak{Seg}_{{\Delta_{\mathfrak{O}}}^{\mathrm{op}\natural}}(\infty\text{-}\mathfrak{Grpd}) \to \mathfrak{Seg}_{{\Delta_{\mathfrak{O}}^{(1)}}^{\mathrm{op}\natural}}(\infty\text{-}\mathfrak{Grpd})$ of restriction along the inclusion $\Delta_{\mathfrak{O}}^{(1)} \hookrightarrow \Delta_{\mathfrak{O}}$ is an equivalence of $(\infty,1)$-categories.*

Proof. For $([n], O_\bullet)$ be an $\mathbb{O}$-forest, and denote the decomposition of $|O_n|$ into its fibres. We then write any forest as a union of trees, and use the forest decomposition condition of scholium 3.2.5. $\square$

Construction 3.2.7. We define a **corolla** of an $\mathbb{O}$-forest $F \in \Delta_{\mathbb{O}}$ to be an equivalence class of inert morphisms $F \to E$ where E is elementary.

We can now define a functor $|-|^{\mathrm{op}} \colon \Delta_{\mathbb{O}} \to \mathbb{F}$ by counting the numbers of corollas in an $\mathbb{O}$-forest, as in [CH20a, Definition 2.2.10].

Proposition 3.2.8. *The functor* $|-| \colon {\Delta_{\mathbb{O}}}^{\mathrm{op}} \to \mathbb{F}^{\mathrm{op}}$ *gives a structure of coenrichable pattern on* ${\Delta_{\mathbb{O}}}^{\mathrm{op}\natural}$.

Proof. By direct verification; this follows essentially by definition of corollas. $\square$

Proposition 3.2.9 ([Bar18])**.** *There is an equivalence of* $(\infty, 1)$*-categories between* ${\Delta_{\mathbb{O}}}^{\mathrm{op}\flat}$*-Segal objects and weak Segal* $\mathbb{O}$*-fibrations.*

Proof (Sketch of construction). The proof is a direct generalisation of that of [HM22, Theorem 3.1.1], which is quite technically involved, so we do not reproduce it here. However, since it will be useful in section 5, we recall the construction of the adjunction.

We first explain what is, for our purposes, the most important part: how to get from Segal objects to weak fibrations.

We start by constructing an ∞-functor $\omega^* \colon \infty\text{-}\mathfrak{Grpd}^{{\Delta_{\mathbb{O}}}^{\mathrm{op}}} \to \infty\text{-}\mathfrak{Grpd}^{{\Delta_{/\mathbb{O}}}^{\mathrm{op}}}$. Recall that presheaves on $\Delta_{\mathbb{O}}$ can be seen equivalently as presheaves on $\infty\text{-}\mathfrak{Grpd}^{{\Delta_{\mathbb{O}}}^{\mathrm{op}}}$ satisfying the representability condition of being colimit-preserving. We now define an ∞-functor $\omega \colon \Delta_{/\mathbb{O}} \to \infty\text{-}\mathfrak{Grpd}^{{\Delta_{\mathbb{O}}}^{\mathrm{op}}}$ as the composition $\imath^* \circ \text{よ}_{\Delta_{/\mathbb{O}}}$ where $\imath^*$ is restriction (or inverse image) of presheaves along the canonical ∞-functor $\imath \colon \Delta_{\mathbb{O}} \hookrightarrow \Delta_{/\mathbb{O}^{\mathrm{act}}} \xrightarrow{(\mathbb{O}^{\mathrm{act}} \to \mathbb{O})_!} \Delta_{/\mathbb{O}}$.

We thus get an inverse image ∞-functor $\omega^* \colon \infty\text{-}\mathfrak{Grpd}^{{\infty\text{-}\mathfrak{Grpd}^{{\Delta_{\mathbb{O}}}^{\mathrm{op}}}}^{\mathrm{op}}} \to \infty\text{-}\mathfrak{Grpd}^{{\Delta_{/\mathbb{O}}}^{\mathrm{op}}}$, which we can restrict to the (full) sub-$(\infty, 1)$-category of those presheaves which are representable and satisfy the Segal condition. To finish, recall also (from [Lur09, Corollary 5.1.6.12]) that $\infty\text{-}\mathfrak{Grpd}^{{\Delta_{/\mathbb{O}}}^{\mathrm{op}}} \simeq {\infty\text{-}\mathfrak{Grpd}^{\Delta^{\mathrm{op}}}}_{/\mathscr{N}_\bullet \mathbb{O}}$, so we eventually have

$$\omega^* \colon \mathfrak{Seg}_{{\Delta_{\mathbb{O}}}^{\mathrm{op}\natural}}(\infty\text{-}\mathfrak{Grpd}) \to {\infty\text{-}\mathfrak{Grpd}^{\Delta^{\mathrm{op}}}}_{/\mathscr{N}_\bullet \mathbb{O}}. \tag{13}$$

In the case where $\mathbb{O} = \mathbb{F}^{\mathrm{op}\natural}$, our functor ω corresponds to the one described in [HHM16, §5.1]. By [HM22, §3.3], this functor does land first in the subcategory $\mathfrak{Seg}_{\Delta^{\mathrm{op}\natural}}(\infty\text{-}\mathfrak{Grpd})_{/\mathscr{N}_\bullet \mathbb{O}}$ which we interpret as $(\infty, 1)\text{-}\mathfrak{Cat}_{/\mathbb{O}}$, and in fact in the further subcategory of those ∞-functors to $\mathbb{O}$ which are weak Segal fibrations.

To go the other way, we consider $\varpi \colon \Delta_{\mathbb{O}} \xrightarrow{\text{よ}} \infty\text{-}\mathfrak{Grpd}^{{\Delta_{\mathbb{O}}}^{\mathrm{op}}} \to \mathfrak{WkSegFib}_{/\mathbb{O}}$ and the profunctor it corepresents: then $(\omega^*)^{-1} \colon (\mathfrak{E} \to \mathbb{O}) \mapsto \hom_{\mathfrak{WkSegFib}_{/\mathbb{O}}}(\varpi(-), \mathfrak{E}) \colon \Delta_{\mathbb{O}} \to \infty\text{-}\mathfrak{Grpd}$. $\square$

4 Monoidal envelopes and Grothendieck opfibrations

4.1 Construction of the monoidal envelope functor

Let $\mathcal{O}$ be an operad. Its monoidal envelope is constructed as a monoidal category $\mathscr{E}nv(\mathcal{O})$ which has as set of objects the free monoid generated by the colours of $\mathcal{O}$, whose elements are denoted as $C_1 \otimes \cdots \otimes C_n$ or simply $C_1 \cdots C_n$. If $C_1 \otimes \cdots \otimes C_n$ is such a string of colours of $\mathcal{O}$ and D is one colour, a morphism $C_1 \otimes \cdots \otimes C_n \to D$ is given by a multimorphism $C_1, \ldots, C_n \to D$ in $\mathcal{O}$. If $C_1 \otimes \cdots \otimes C_n$ and $D_1 \otimes \cdots \otimes D_m$ are two such strings of colours of $\mathcal{O}$, to defines a morphism $C_1 \otimes \cdots \otimes C_n \to D_1 \otimes \cdots \otimes D_m$ one needs to further select of partition of the inputs $(C_1, \ldots, C_n)$ into m (possibly empty) parts.

The above is so far just a description of the underlying category $\mathfrak{Env}(\mathcal{O})$ of the envelope of $\mathcal{O}$; to define it as a monoidal category, or representably monoidal operad, one must also define multimorphisms of higher arity in the operadic structure. Let $(C^i_1 \otimes \cdots \otimes C^i_{p_i})_{i\in[\![1,n]\!]}$ be n colours of $\mathscr{E}nv(\mathcal{O})$ and let $D_1 \otimes \cdots \otimes D_m$ be a further colour. By the representability condition, a multimorphism $(C^1_1 \cdots C^1_{p_1}, \ldots, C^n_1 \cdots C^n_{p_n}) \to D_1 \otimes \cdots \otimes D_m$ is given by a morphism $C^1_1 \otimes \cdots \otimes C^n_{p_n} \to D_1 \otimes \cdots \otimes D_m$, that is a partition of the entries and a collection of multimorphisms to each D_i.

In the dendroidal model, one simply defines the object $\mathscr{E}nv(\mathcal{O})(\star_n)$ of all n-ary morphisms without specifying their sources and targets. To describe this, it becomes useful to reverse the thinking: taking a family of multimorphisms of $\mathcal{O}$, we can ask how to interpret it as a multimorphism in $\mathscr{E}nv(\mathcal{O})$. If $C_1, \cdots, C_r$ is the union of the domains of the multimorphisms in the family considered, the decomposition in family provides a partition of p indexed by the targets; however, from the point of view of $\mathscr{E}nv(\mathcal{O})$, this partition is completely artificial as it is only used to construct a morphism whose target may consist of several colours. Thus it must be forgotten. Meanwhile, if the family is to be interpreted as a multimorphism of specified arity n in $\mathscr{E}nv(\mathcal{O})$, the set of colours $(C_i)_{i\in[\![1,r]\!]}$ must be endowed with a partition into n parts.

In a formula, we have that

$$\mathscr{E}nv(\mathcal{O})(\star_n) = \coprod_{\substack{r\in\mathbb{N} \\ \lambda \text{ partition of } r \text{ in } n}} \prod_{i=1}^{m} \mathcal{O}(\star_{\lambda(i)}). \tag{14}$$

Recall that partitions are the same thing as active morphisms in $\mathbb{\Gamma}^{\mathrm{op}}$. We can then interpret the coproduct on partitions as a colimit over morphisms in $(\mathbb{\Gamma}^{\mathrm{op}})^{\mathrm{act}}$.

Viewed in this way, this formula is very reminiscent to the one computing oplax extensions, with one difference: the colimit is taken only of the trees whose height is that of a corolla, *i.e.* compatibly with the projection to $\mathbb{\Delta}$. To that end, the notion of extension needs to be refined to a "fibrewise" one.

To define the necessary fibrewise extensions, we need to briefly work in the generality of formal ∞-category theory introduced in [RV22], that is in the framework of ∞-cosmoi, modelling the $(\infty, 2)$-category of ∞-categories. Recall from [RV22, Proposition 9.1.8, Theorem 9.3.3] that (pointwise) oplax extensions along an ∞-functor can be expressed as colimits weighted by the conjoint of this ∞-functor. We will define fibrewise extensions similarly, replacing this conjoint by a relative version.

Construction 4.1.1 (Relative comma ∞-category)**.** In an ∞-cosmos $\mathfrak{K}$, consider a cocorrespondence in the sliced ∞-cosmos $\mathfrak{K}_{/B}$:

$$\begin{tikzcd} & \mathfrak{E} \arrow[d, two heads] & \\ \mathfrak{F} \arrow[ur, "f"] \arrow[r, bend right, two heads, "p"'] & \mathfrak{B} & \mathfrak{G} \arrow[ul, "g"'] \arrow[l, bend left, two heads, "q"] \end{tikzcd} \tag{15}$$

We let $f \downarrow_{/\mathfrak{B}} g$ denote the comma object in $\mathfrak{K}_{/B}$, and call it the **relative comma ∞-category** over B.

Remark 4.1.2. By [RV22, Proposition 1.2.22, (iv)–(vi)], the relative comma ∞-category can be constructed as

$$f \downarrow_{/\mathfrak{B}} g \simeq (\mathfrak{F} \times_{\mathfrak{B}} \mathfrak{G}) \underset{(\mathfrak{E}\times_{\mathfrak{B}}\mathfrak{E})}{\times} (\mathfrak{B} \times_{\mathfrak{B}^2} \mathfrak{E}^2) \tag{16}$$

(where the map $\mathfrak{B} \to \mathfrak{B}^2$ is the diagonal). That is, informally, an object of $f \downarrow_{/\mathfrak{B}} g$ consists of a triple (F, G, α) where F and G are objects of $\mathfrak{F}$ and $\mathfrak{G}$ respectively and $\alpha\colon f(F) \to g(G)$ is an arrow in $\mathfrak{E}$, such that all data live above the same object of $\mathfrak{B}$ (*i.e.* there is an object $B \in \mathfrak{B}$ and isomorphisms $pF \xrightarrow{\simeq} B$, $qG \xrightarrow{\simeq} B$, and $\alpha \overset{\simeq}{\Rightarrow} \mathrm{id}_B$).

Lemma 4.1.3. *The canonical projection $f \downarrow_{/\mathfrak{B}} g \to \mathfrak{F} \times \mathfrak{G}$ is a discrete two-sided fibration in $\mathfrak{K}$.*

Proof. As in [RV22, Proposition 7.4.6]. □

Definition 4.1.4 (Fibrewise (op)lax extensions)**.** Let $\mathfrak{E} \xrightarrow{p} \mathfrak{B}$ and $\mathfrak{F} \xrightarrow{q} \mathfrak{B}$ be two ∞-categories defined over a base $\mathfrak{B}$, and let $\mathscr{K}\colon \mathfrak{E} \to \mathfrak{F}$ be an ∞-functor defined over $\mathfrak{B}$. Let $\mathscr{D}\colon \mathfrak{E} \to \mathfrak{G}$ be an $(\infty, 1)$-functor, so that we have the solid diagram

$$\begin{tikzcd} \mathfrak{E} \arrow[rr, "\mathscr{D}"] \arrow[dr, "\mathscr{K}"'] \arrow[ddr, "p"'] & & \mathfrak{G} \\ & \mathfrak{F} \arrow[ur, dashed] \arrow[d, bend left, "q"] & \\ & \mathfrak{B} & \end{tikzcd} \tag{17}$$

A **fibrewise lax extension** of $\mathscr{D}$ along $\mathscr{K}$ (relative to $\mathfrak{B}$) is a limit

$$\mathrm{Lex}_{\mathscr{K}/\mathfrak{B}}\, \mathscr{D} \coloneqq \left\{ \mathscr{K}_*^{/\mathfrak{B}}, \mathscr{D} \right\} : \mathfrak{F} \to \mathfrak{G} \tag{18}$$

of $\mathscr{D}$ weighted by the ∞-profunctor $\mathscr{K}_*^{/\mathfrak{B}}\colon \mathfrak{E} \nrightarrow \mathfrak{F}$ associated with the relative comma ∞-category $\mathrm{id}_{\mathfrak{F}} \downarrow_{/\mathfrak{B}} \mathscr{K}$.

A **fibrewise oplax extension** of $\mathscr{D}$ along $\mathscr{K}$ (relative to $\mathfrak{B}$) is a colimit

$$\operatorname{Opex}_{\mathscr{K}/\mathfrak{B}} \mathscr{D} \coloneqq \mathscr{K}^*_{/\mathfrak{B}} \star \mathscr{D}\colon \mathfrak{F} \to \mathfrak{G} \tag{19}$$

of $\mathscr{D}$ weighted by the ∞-profunctor $\mathscr{K}^*_{/\mathfrak{B}}\colon \mathfrak{F} \nrightarrow \mathfrak{E}$ corresponding to the relative comma ∞-category $\mathscr{K} \downarrow_{/\mathfrak{B}} \mathrm{id}_{\mathfrak{F}}$.

Remark 4.1.5. We have the explicit formulae, deduced from [RV22, Lemma 9.5.5], computing fibrewise extensions: the fibrewise oplax extension of $\mathscr{D}$ along $\mathscr{K}$, evaluated at F, is

$$\operatorname{Opex}_{\mathscr{K}/\mathfrak{B}} \mathscr{D}(\mathsf{F}) = \operatorname*{colim}_{\substack{\mathscr{K}(\mathsf{E})\to\mathsf{F} \\ \mathsf{E}\in\mathfrak{E}_{\varphi(\mathsf{F})}}} \mathscr{D}(\mathsf{E}), \tag{20}$$

and the fibrewise lax extension of $\mathscr{D}$ along $\mathscr{K}$, evaluated at F, is

$$\operatorname{Lex}_{\mathscr{K}/\mathfrak{B}} \mathscr{D}(\mathsf{F}) = \lim_{\substack{\mathsf{F}\to\mathscr{K}(\mathsf{E}) \\ \mathsf{E}\in\mathfrak{E}_{\varphi(\mathsf{F})}}} \mathscr{D}(\mathsf{E}). \tag{21}$$

Definition 4.1.6 (Envelope)**.** Let $\mathcal{X}\colon {\Delta_{\mathfrak{O}}}^{\mathrm{op}} \to \infty\text{-}\mathfrak{Grpd}$ be a precosheaf on ${\Delta_{\mathfrak{O}}}^{\mathrm{op}}$. Its **envelope** is the precosheaf

$$\mathscr{E}nv(\mathcal{X}) = \imath^* \operatorname{Opex}_{\imath/\Delta^{\mathrm{op}}} \mathcal{X} \tag{22}$$

where $\imath^*$ denotes the $(\infty,1)$-functor of (fibrewise) restriction along $\imath\colon {\Delta_{\mathfrak{O}}}^{\mathrm{op}} \hookrightarrow (\Delta_{\mathfrak{O}}^{\mathrm{pre}})^{\mathrm{op}}$, right-adjoint to fibrewise oplax extension.

Remark 4.1.7. From the formula for fibrewise extensions (eq. (20)), we see that the value taken by the envelope of $\mathcal{X} \in \mathfrak{Seg}_{{\Delta_{\mathfrak{O}}}^{\mathrm{op}\natural}}(\mathfrak{C})$ at an $\mathfrak{O}$-tree $\mathsf{T}_\bullet$ of length $[n]$ is computed by

$$\mathscr{E}nv(\mathcal{X})(\mathsf{T}_\bullet) = \operatorname*{colim}_{\substack{\mathsf{B}_\bullet \rightsquigarrow \mathsf{T}_\bullet \\ \in((\Delta_{\mathfrak{O}}^{\mathrm{pre}})_{[n]})_{/\mathsf{T}_\bullet}}} \prod_{i=1}^{|\mathsf{B}_n|} \mathcal{X}(\mathsf{B}_{\bullet,i}) \tag{23}$$

where $(\mathsf{B}_{\bullet,i})_i$ denotes the forest decomposition in fibres (as in eq. (11)), and where we denote the indexing arrows as $\rightsquigarrow$ to emphasise that we view them as families of n active arrows of $\mathfrak{O}$.

Remark 4.1.8. In the case $\mathfrak{O} = \Gamma^{\mathrm{op}\flat}$, we may understand the envelope in the following way. Recall that an active morphism in Γ^{op} can be seen as giving a partition of its source indexed by its target (possibly with empty parts). Then the colimit creates several copies of $\mathcal{X}(\star_n)$, each equipped with a new partition specifying how to distribute its inputs.

To work with envelopes in an intuitive way, it will then be convenient to introduce the following terminology.

Definition 4.1.9 ($\mathfrak{O}$-Partitions)**.** Let $\mathfrak{O}$ be an algebraic pattern, and let O and P be objects of $\mathfrak{O}$. An $\mathfrak{O}$-partition of O by P, or simply a P**-partition of** O when there is no ambiguity, is an active morphism $O \rightsquigarrow P$.

Example 4.1.10. If $\mathbb{1}$ denotes the terminal ${\Delta_{\mathfrak{O}}}^{\mathrm{op}\natural}$-Segal ∞-groupoid, then its envelope $\mathscr{E}\mathit{nv}(\mathbb{1})$ evaluates, on any forest $T_\bullet$ of length n, the the groupoidal localisation of $(\Delta_{\mathfrak{O}}^{\mathrm{pre}})_{[n]/T_\bullet}$.

Proof. By definition, $\mathscr{E}\mathit{nv}(\mathbb{1})(T_\bullet)$ is given by the colimit

$$\mathscr{E}\mathit{nv}(\mathbb{1})(T_\bullet) = \operatorname*{colim}_{B_\bullet \rightsquigarrow T_\bullet} \mathbb{1} \tag{24}$$

(where the last $\mathbb{1}$ is the terminal ∞-groupoid).

Now $\mathscr{E}\mathit{nv}(\mathbb{1})(T_\bullet)$ is an ∞-groupoid so it is equivalent to its localisation, and furthermore the localisation functor $\mathscr{L}\colon (\infty,1)\text{-}\mathfrak{Cat} \to \infty\text{-}\mathfrak{Grpd}$, being left-adjoint to the inclusion $\infty\text{-}\mathfrak{Grpd} \hookrightarrow (\infty,1)\text{-}\mathfrak{Cat}$, preserves colimits, so we may compute the colimit on the right-hand side in $(\infty,1)\text{-}\mathfrak{Cat}$ and then localise.

As a colimit of a constant diagram of $(\infty,1)$-categories, the colimit in $(\infty,1)\text{-}\mathfrak{Cat}$ is equivalent to the tensor of the constant value $\mathbb{1}$ by the indexing $(\infty,1)$-category $(\Delta_{\mathfrak{O}}^{\mathrm{pre}})_{[n]/T_\bullet}$.

Lemma 4.1.10.1. *For any object W and any change-of-enrichment ∞-functor $\mathscr{L}$ with a fully faithful right-adjoint $\mathscr{R}$, $\mathscr{L}(W \otimes X) \simeq (\mathscr{L}W) \otimes X$.*

Proof.

$$\begin{aligned} \hom((\mathscr{L}W) \otimes X, Y) &\simeq \hom(\mathscr{L}W, Y^X) \simeq \hom(W, \mathscr{R}(Y^X)) \simeq \hom(W, (\mathscr{R}Y)^X) \\ &\simeq \hom(W \otimes X, \mathscr{R}Y) \simeq \hom(\mathscr{L}(W \otimes X), Y) \end{aligned} \tag{25}$$

□

$(\Delta_{\mathfrak{O}}^{\mathrm{pre}})_{[n]/T_\bullet}[^{-1}] \otimes \mathbb{1}$.

□

Definition 4.1.11 (Reduced pattern)**.** A coenrichable pattern $\mathfrak{O}$ is **reduced** if $\mathfrak{O}_0^{\mathrm{el}}$ is terminal for active morphisms.

In particular, when $\mathfrak{O}$ is reduced, evaluating example 4.1.10 on the elementaries, we find that:

- $\mathscr{E}\mathit{nv}(\mathbb{1})(\eta)$ is the localisation of $\mathfrak{O}^{\mathrm{act}}$;
- for any corolla $C_O = \big([1], (O \rightsquigarrow 1)\big)$, an object of $\mathscr{E}\mathit{nv}(\mathbb{1})(C_O)$ consists of an active morphism $P_0 \rightsquigarrow P_1$ of $\mathfrak{O}$ (a P_1-partition of P_0) along with an O-partition $P_0 \rightsquigarrow O$.

4.2 Opfibrations and representable monoidality

Proposition 4.2.1. *For any Segal $\Delta_{\mathfrak{O}}{}^{\mathrm{op}\natural}$-object $\mathscr{X}$, its envelope $\mathscr{E}nv(\mathscr{X})$ is a Segal $\Delta_{\mathfrak{O}}{}^{\mathrm{op}\natural}$-object.*

Proof. The Segal condition requires comparing

$$\mathscr{E}nv(\mathscr{X})(O_n) = \underset{B_n \to O_n}{\mathrm{colim}}\ \mathscr{X}(B_n) = \underset{B_n \to O_n}{\mathrm{colim}}\ \underset{B_n \to E}{\lim}\ \mathscr{X}(E) \tag{26}$$

and

$$\underset{O_n \to E}{\lim}\ \mathscr{E}nv(\mathscr{X})(E) = \underset{O_n \to E}{\lim}\ \underset{B_{0,1} \to E}{\mathrm{colim}}\ \mathscr{X}(B_{0,1}). \tag{27}$$

Their equality is the condition of distributivity of limits over colimits as made explicit in [CH21, Definition 7.12], and using [CH21, Corollary 7.17] which shows that ∞-$\mathfrak{Grpd}$ is admissible (or even any $(\infty, 1)$-topos since $\mathfrak{O}$ is coenrichable).

Alternatively, we can use the explicit form of the Segal conditions given in scholium 3.2.5, which can be checked explicitly. □

For the cocartesian properties of the envelope, we now need to specialise to the case $\mathfrak{O} = \Gamma^{\mathrm{op}\flat}$. Indeed, following [Her04, Theorem 2.4], we will characterise cocartesian fibrations of Segal $\mathfrak{P}$-objects (for $\mathfrak{P} = \Delta_{\mathfrak{O}}{}^{\mathrm{op}\natural}$ a well-directed algebraic pattern) in terms of cocartesian fibrations of the underlying $\mathfrak{P}_{\mathfrak{P}_0/}$-objects of their envelopes. This presupposes having already a good understanding of cocartesian fibrations of $\mathfrak{P}_{\mathfrak{P}_0/}$-objects, which is only the case when $\mathfrak{P}_{\mathfrak{P}_0/}$ is $\Delta^{\mathrm{op}\natural}$, in particular when $\mathfrak{P} = \Omega^{\mathrm{op}\natural}$ (and $\mathfrak{P}_0 = \{\eta\}$).

Definition 4.2.2 (Linearisable pattern)**.** A coenrichable algebraic pattern $\mathfrak{O} \xrightarrow{|-|} \Gamma^{\mathrm{op}\natural}$ is linearisable if the collection of $\flat$-elementary objects is multiterminal in $\mathfrak{O}^{\mathrm{act}}$.

Proposition 4.2.3. *For any coenrichable pattern $\mathfrak{O}$ whose elementary objects are rigid, the tree-like plus construction $\Delta_{\mathfrak{O}}^{(1)\,\mathrm{op}\natural}$ is linearisable.*

Proof. Let $([n], O_\bullet)$ be an object of $\Delta_{\mathfrak{O}}$. By the condition for tree-like objects, O_n must be in $\mathfrak{O}^{\flat,\mathrm{el}}$. Recall from remark 3.1.2 that a morphism to $([n], O_\bullet)$ is determined by the data of a morphism to $[n]$ in Δ and a (semi-inert and active) morphism to O_n.

Now the elementaries of $\Delta_{\mathfrak{O}}{}^{\mathrm{op}\flat}$ are of the form $([1], (E_0 \rightsquigarrow E_1))$ with $E_1 \in \mathfrak{O}^{\flat,\mathrm{el}}$. There is only one active morphism $[1] \rightsquigarrow [n]$, which is specified by $(0 < 1) \mapsto (0 < n)$. Finally, an active morphism in $\Delta_{\mathfrak{O}}{}^{\mathrm{op}\flat}$ must have its morphism to O_n be an isomorphism, so E_1 is also forced to be O_n (while E_0 was already seen to be equivalent to O_0). □

Remark 4.2.4. In a coenrichable pattern $\mathfrak{O}$, the $\flat$-elementaries lie over $\langle 1\rangle \in \Gamma^{\mathrm{op}}$. An active and semi-inert morphism to $\langle 1\rangle$ can have as source only $\langle 1\rangle$ or $\langle 0\rangle$. So if $\mathfrak{O}$ is linearisable, for any object O the unique active morphism to an elementary will only be semi-inert in $|O|$ is $\langle 1\rangle$ or $\langle 0\rangle$, in which case it *will* be so because there are no non-semi-inert morphisms $\langle 0\rangle \to \langle 1\rangle$ or $\langle 1\rangle \to \langle 1\rangle$.

Lemma 4.2.5. *Suppose $\mathbb{O}$ is linearisable. The slice $(\Delta_{\mathbb{O}})_{/\eta}$ is equivalent to $\Delta \times \mathbb{O}^{\flat,\mathrm{el}}$, itself equivalent to* $\operatorname{colim}_{\mathbb{O}^{\flat,\mathrm{el}}} \Delta$.

Proof. Consider a morphism $T \to \eta$ from some tree $T = ([n], O_\bullet)$, consisting of a map $\phi\colon [n] \to [0]$ and a transformation $O_\bullet \to \phi^* \mathbb{1}$. $\square$

By [Bar22, Proposition 2.37], there is a canonical pattern structure on a coslice of a pattern. The equivalence of categories underlies one of algebraic patterns.

Definition 4.2.6 (Underlying $\mathbb{O}^{\flat,\mathrm{el}}$-category)**.** Suppose $\mathbb{O}$ is linearisable, and denote $\not{p}\colon (\Delta_{\mathbb{O}})^{\mathrm{op}}_{\eta/} \to \Delta_{\mathbb{O}}{}^{\mathrm{op}}$. The **underlying category** object of a Segal $\Delta_{\mathbb{O}}{}^{\mathrm{op}\natural}$-object $\mathscr{X}$ is its restriction along $\not{p}$, where we note that that $\mathfrak{Seg}_{\operatorname{colim}_{\mathbb{O}^{\flat,\mathrm{el}}} \Delta^{\mathrm{op}\natural}}(\mathfrak{C}) \simeq \lim_{\mathbb{O}^{\flat,\mathrm{el}}} \mathfrak{Seg}_{\Delta^{\mathrm{op}\natural}}(\mathfrak{C})$.

Remark 4.2.7. The functor $\not{p}$ clearly admits unique liftings of inert morphisms, and it is extendable, so by [CH21, Proposition 7.13] the functor has a left-adjoint (given by oplax extension), which is fully-faithful and has as its essential images the presheaves with empty value on any non-linear tree.

Definition 4.2.8 (Cocartesian fibration)**.** A morphism $\mathscr{X} \to \mathscr{B}$ of Segal $\Delta_{\mathbb{F}^{\mathrm{op}\flat}}{}^{\mathrm{op}\natural}$-$\infty$-groupoids is a **cocartesian fibration of Segal objects** if $\mathfrak{Env}(\mathscr{X}) \to \mathfrak{Env}(\mathscr{B})$ is a cocartesian fibration (of $(\infty,1)$-categories), where $\mathfrak{Env}(\mathscr{X})$ denotes the underlying $(\infty,1)$-category of the ∞-operad $\mathscr{E}nv(\mathscr{X})$.

A Segal object $\mathscr{X}$ is **representably monoidal** if the unique morphism $\mathscr{X} \to \mathbb{1}$ is a cocartesian fibration of Segal objects.

Proposition 4.2.9. *For any Segal object $\mathscr{X}$, $\mathscr{E}nv(\mathscr{X})$ is a representably monoidal.*

Proof. Let $\phi\colon O_0 \rightsquigarrow O_1$ be a morphism in $\mathfrak{Env}(\mathbb{1})$, seen as an $\mathbb{O}$-partition of O_0 by O_1, and let $C \in \mathfrak{Env}(\mathscr{E}nv(\mathscr{X}))$ lying over O_0. Under the identification

$$
\begin{aligned}
\mathscr{E}nv\big(\mathscr{E}nv(\mathscr{X})\big)(\eta) &\simeq \operatorname*{colim}_{P \in \mathbb{O}^{\mathrm{act}}} \prod_{i=1}^{|P|} \mathscr{E}nv(X)(\eta) \\
&\simeq \operatorname*{colim}_{P \in \mathbb{O}^{\mathrm{act}}} \prod_{i=1}^{|P|} \operatorname*{colim}_{Q_i \in \mathbb{O}^{\mathrm{act}}} \prod_{j=1}^{|Q_i|} \mathscr{X}(\eta),
\end{aligned} \tag{28}
$$

we write C as

$$
(C_{1,1} \cdots C_{1,|Q_1|}) \cdots (C_{|O_0|,1} \cdots C_{|O_0|,|Q_{|O_0|}|}) \tag{29}
$$

for some choice of $(Q_i)_{i=1}^{|O_0|}$.

We set

$$
\phi_! C = (C_{1,1} \cdots C_{n_1,i_{n_1}}) \cdots (C_{n_{m-1}+1,1} \cdots C_{n_m,i_{n_m}}) \tag{30}
$$

which lies over $\langle m \rangle$.

A morphism $C \to \phi_! C$ in $\mathfrak{Env}(\mathscr{E}nv(\mathscr{X}))$ is given by a moprhism $C_{1,1} \cdots C_{n_m,i_{n_m}} \to C_{1,1} \cdots C_{n_m,i_{n_m}}$ in $\mathfrak{Env}(\mathscr{X})$ along with a partition of $\sum_{k=1}^{m} i_{n_k}$ into n parts. We define the lift $C \to \phi_! C$ of ϕ to be given by the identity arrow of $C_{1,1} \cdots C_{n_m,i_{n_m}}$ along with the partition exhibited in eq. (29). This lift is cocartesian. $\square$

Thus the construction $\mathscr{X} \mapsto \mathscr{Env}(\mathscr{X})$ defines an $(\infty, 1)$-functor $\mathscr{Env}\colon \infty\text{-}\mathfrak{Oprd} \to \mathfrak{Mon}(\infty,1)\!-\!\mathfrak{Cat}$.

Remark 4.2.10 (Monoidal structure on monoidal ∞-categories). Using the idea from the proof of proposition 4.2.9, we can construct a product on the colours of a representably monoidal ∞-operad $\mathscr{X}$. First note that (regardless of monoidality) the colours of $\mathscr{Env}(\mathscr{X})$ in the image of the unit map $\mathscr{X} \to \mathscr{Env}(\mathscr{X})$ are exactly those whose image under the morphism $\mathscr{Env}(\mathscr{X}) \to \mathscr{Env}(\mathbb{1})$ is 1.

Consider an n-uple of colours of $\mathscr{X}$, given by n morphisms $C_1, \cdots, C_n\colon \mathfrak{L}\eta \to \mathscr{X}$. Those define a morphism $(C_1, \cdots, C_n)\colon \mathfrak{L}\eta \to \mathscr{Env}(\mathscr{X})$, whose image lies over the colour n of $\mathscr{Env}(\mathbb{1})$. Now since $\mathscr{X}$ is representably monoidal, the morphism $n \to 1$ in $\mathscr{Env}(\mathbb{1})$ has a cocartesian lift from $(C_1, \cdots, C_n)$, whose target is then a colour $C_1 \otimes \cdots \otimes C_n$ of $\mathscr{X}$. Clearly, the same construction can be applied to obtain a product of morphisms as well, with appropriate functoriality.

Theorem 4.2.11. *The $(\infty, 1)$-functor $\mathscr{Env}$ is left-adjoint to the inclusion $\mathfrak{Mon}(\infty,1)\!-\!\mathfrak{Cat} \hookrightarrow \infty\text{-}\mathfrak{Oprd}$.*

Proof. By the construction of the envelope, we have a unit morphism $\eta_{\mathscr{X}}\colon \mathscr{X} \to \mathscr{Env}(\mathscr{X})$ for any ∞-operad $\mathscr{X}$. We need to construct a counit $\varepsilon_{\mathscr{V}}\colon \mathscr{Env}(\mathscr{V}) \to \mathscr{V}$ for any representably monoidal ∞-operad $\mathscr{V}$, which is a morphism of monoidal ∞-categories.

This morphism is provided by the construction of remark 4.2.10: since $\mathscr{V}$ is representably monoidal, it admits a monoidal product, and its envelope simply corresponds to adding a second level a parenthesising to the products.

We will construct $\varepsilon_{\mathscr{V}}$ componentwise, as a natural transformation of ∞-functors ${\Delta_{\Gamma^{\mathrm{op}\flat}}}^{\mathrm{op}} \to \infty\text{-}\mathfrak{Grpd}$. Let $T_\bullet$ be a tree. By the formula eq. (23), giving a map $\mathscr{Env}(\mathscr{V})(T_\bullet) \to \mathscr{V}(T_\bullet)$ is equivalent to giving, for each morphism $B_\bullet \rightsquigarrow T_\bullet$, a map $\mathscr{X}(B_\bullet) \to \mathscr{X}(T_\bullet)$. But recall that $B_\bullet \rightsquigarrow T_\bullet$ can be interpreted as a $T_\bullet$-partition of $B_\bullet$. Following the previous remark, we can use the product to simply reorganise the parenthesising levels according to the partition, which produces the desired morphism.

Finally, it is directly checked that ε and η satisfy the triangular equalities, so that they do exhibit an adjunction. $\square$

5 Cartesian monoidal structures and application to the Grothendieck construction

5.1 Cartesian monoidal structures

We follow the idea used independently in [BGS20] and [DK19] to define cartesian monoidal ∞-categories.

Construction 5.1.1 (Anti-plus construction)**.** We let $\overline{\Delta}_{\mathfrak{O}}^{\mathrm{pre}}$ denote the Grothendieck construction of $\mathfrak{L}_{(\infty,1)\text{-}\mathfrak{Cat}}(\mathfrak{O}^{\mathrm{act}})|_{\Delta} \circ (-)^{\mathrm{op}}$, and define a locally full sub-$(\infty, 1)$-category $\overline{\Delta}_{\mathfrak{O}}$ by imposing the same conditions as in the plus construction.

Scholium 5.1.2. *There is an equivalence of $(\infty, 1)$-categories between Segal ${\overline{\Delta}_{\mathfrak{O}}}^{\mathrm{op}\natural}$-objects and $(\infty, 1)$-functors $\mathscr{p}\colon \mathfrak{P} \to \mathfrak{O}^{\mathrm{op}}$ such that $\mathscr{p}^{\mathrm{op}}$ is a weak Segal fibration.*

Definition 5.1.3 (Unitality)**.** Let $\mathfrak{O}$ be a reduced pattern. A $\overline{\Delta}_{\mathfrak{O}}{}^{\mathrm{op}\natural}$-object $\mathscr{X}$ is **anti-unital** if it is local with respect to the morphism $\star_\emptyset \leftarrow \eta$.

Remark 5.1.4. The $(\infty,1)$-category of anti-unital Segal $\overline{\Delta}_{\mathfrak{O}}{}^{\mathrm{op}\natural}$-objects in an $(\infty,1)$-category $\mathfrak{C}$ is evidently a localisation of $\mathfrak{Seg}_{\overline{\Delta}_{\mathfrak{O}}{}^{\mathrm{op}\natural}}(\mathfrak{C})$. It can also be seen a category of Segal objects for a different pattern.

The pattern for anti-unital $\overline{\Delta}_{\mathfrak{O}}{}^{\mathrm{op}\natural}$-objects is the same as $\overline{\Delta}_{\mathfrak{O}}{}^{\mathrm{op}\natural}$, with $\star_\emptyset$ removed from the elementaries.

Lemma 5.1.5. *The morphism of algebraic patterns* $\overline{\Delta}_{\mathfrak{O}}{}^{\mathrm{op}\natural}{}_{\eta/} \to \overline{\Delta}_{\mathfrak{O}}{}^{\mathrm{op}\natural,\mathrm{unit.}}$ *has unique lifting of active morphisms.*

Corollary 5.1.6. *The restriction morphism* $\mathfrak{Seg}_{\overline{\Delta}_{\mathfrak{O}}{}^{\mathrm{op}\natural,\mathrm{unit.}}} \to \mathfrak{Seg}_{\overline{\Delta}_{\mathfrak{O}}{}^{\mathrm{op}\natural}{}_{\eta/}}$ *(taking underlying ∞-categories) admits a right-adjoint (by lax extension).*

We now give an alternate construction of the cartesian structure, in the spirit of [Lur17].

Construction 5.1.7. Let $\mathfrak{U}_{\leq -1}$ be the subobject classifier ((-1)-truncated universe) in the $(\infty,1)$-topos $\infty\text{-}\mathfrak{Grpd}^{\Delta_{\mathfrak{O}}{}^{\mathrm{op}}}$. Define a presheaf $\widetilde{\mathscr{Env}}(\mathfrak{C}^\times)$ over $\mathscr{Env}(\mathbb{1})$ by the specification $\hom(\mathcal{K}, \widetilde{\mathscr{Env}}(\mathfrak{C}^\times)) \simeq \hom(\mathfrak{K} \times_{\mathfrak{Env}(\mathbb{1})} \mathfrak{Env}(\mathfrak{U}_{\leq -1}), \mathfrak{C})$. Note that the canonical (mono)morphism $\texttt{true}\colon \mathbb{1} \rightarrowtail \mathfrak{U}_{\leq -1}$ induces a transformation $\widetilde{\mathscr{Env}}(\mathfrak{C}^\times) \to \mathfrak{C}$.

When $\mathcal{K}$ is $\text{よ}\eta$, and the transformation $\eta \to \mathscr{Env}(1)$ selects an object O of $\mathfrak{O}^{\mathrm{act}}$, we obtain functors from the poset of subobjects of O to $\mathfrak{C}$. We define $\mathscr{Env}(\mathfrak{C}^\times)$ to be the subfunctor of $\widetilde{\mathscr{Env}}(\mathfrak{C}^\times)$ on the functors as above which exhibit their image as product of the elementaries in their source.

Proposition 5.1.8 ([Lur17, Proposition 2.4.1.5.])**.** *The structure morphism* $\mathscr{Env}(\mathfrak{C}^\times) \to \mathfrak{Env}(\mathbb{1})$ *is the image of a cocartesian fibration of* $\Delta_{\mathfrak{O}}{}^{\mathrm{op}\natural}$*-objects.*

5.2 Straightening cocartesian fibrations

Definition 5.2.1 (Weak cartesian structure)**.** Let $\mathscr{X}$ be a Segal $\Delta_{\mathfrak{O}}{}^{\mathrm{op}\natural}$-∞-groupoid, and let $\mathfrak{D}$ be a Segal $\Delta_{\mathfrak{O}}{}^{\mathrm{op}\natural}{}_{\eta/}$-∞-groupoid. A **lax cartesian structure** from $\mathscr{O}$ to $\mathfrak{D}$ is a morphism $\mathfrak{Env}(\mathscr{X}) \to \mathfrak{D}$ that takes decompositions to products.

It is a **strong** cartesian structure if it induces an equivalence of underlying $(\infty,1)$-categories.

Lemma 5.2.2 ([Lur17, Proposition 2.4.1.5.])**.** *The transformation* $\mathfrak{Env}(\mathfrak{C}^\times) \to \mathfrak{C}$ *is a strong cartesian structure.*

Proposition 5.2.3 ([Lur17, Proposition 2.4.1.7])**.** *There is an equivalence of* $(\infty,1)$*-categories between lax cartesian structures from* $\mathscr{X}$ *to* $\mathfrak{D}$ *and morphisms* $\mathscr{X} \to \mathfrak{D}^\times$*.*

Remark 5.2.4. Consider a lax cartesian structure $\varphi\colon \mathfrak{Env}(\mathscr{X}) \to \mathfrak{D}$ in the case where $\mathfrak{D}$ is $(\infty,1)$-$\mathfrak{Cat}$ (or ∞-$\mathfrak{Grpd}$). It is classified by a cocartesian fibration $p\colon \Phi \coloneqq \int^{\mathrm{co}} \varphi \to \mathfrak{Env}(\mathscr{X})$. We will say that a cocartesian fibration over $\mathfrak{Env}(\mathscr{X})$ is **lax cartesian** if the ∞-functor $\mathfrak{Env}(\mathscr{X}) \to (\infty,1)$-$\mathfrak{Cat}$ that it classifies is a lax cartesian structure.

Lemma 5.2.5. *A cocartesian fibration over $\mathscr{E}nv(\mathcal{X})$ is lax cartesian monoidal if and only if it is in the essential image of the functor $\mathscr{E}nv$.*

Proof. By [HK21] or [BHS22], a cocartesian fibration over $\mathscr{E}nv(\mathcal{X})$ is in the image of $\mathscr{E}nv$ if and only if it is equifibred (or cartesian) as a natural transformation. $\square$

Corollary 5.2.6. *For every Segal ${\Delta_{\mathbb{O}}}^{\mathrm{op}\natural}$-object $\mathcal{X}$, there is an equivalence of $(\infty, 1)$ categories between cocartesian fibrations over $\mathcal{X}$ and morphisms $\mathcal{X} \to (\infty, 1)\text{-}\mathfrak{Cat}^{\times}$.* $\square$

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