A CATEGORIFICATION OF THE QUANTUM LEFSCHETZ PRINCIPLE

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ABSTRACT. The quantum Lefschetz formula explains how virtual fundamental classes (or structure sheaves) of stable maps moduli stacks behave when passing to an ambient space to the zero locus of a section. It is only valid under special assumptions (genus 0, regularity of the section and convexity of the bundle). In this note, we give a general statement at the geometric level removing these assumptions, using derived geometry. Through a study of the structure sheaves of derived zero loci we deduce a categorification of the formula in the ∞-categories of quasi-coherent sheaves. We also prove that Manolache's virtual pullbacks can be constructed as derived pullbacks, and use them to get back the classical Quantum Lefschetz formula when the hypotheses are satisfied.

CONTENTS

1. INTRODUCTION

1.1. **The quantum Lefschetz hyperplane principle.** Any quasi-smooth derived scheme is Zariski-locally presented as the (derived) zero locus of a section of a vector bundle on some smooth scheme. The Lefschetz hyperplane theorem then gives a way of understanding the cohomology of such a zero locus from the data of that of the ambient scheme and of the vector bundle. The quantum Lefschetz principle, similarly, gives the quantum cohomology, that is the Gromov–Witten theory, of the zero locus from that of the ambient scheme and the Euler class of the vector bundle.

Let X be a smooth projective variety and let E be a vector bundle on X, and consider the abelian cone stack $\mathbb{R}^0 p_* \operatorname{ev}^* E$ on $\overline{\mathscr{M}}_{g,n}(X,\beta)$, where $\operatorname{ev}\colon \overline{\mathscr{C}}_{g,n}(X,\beta) \to \overline{\mathscr{M}}_{g,n}(X,\beta)$ is the canonical evaluation map (corresponding by the isomorphism $\overline{\mathscr{C}}_{g,n}(X,\beta) \simeq \overline{\mathscr{M}}_{g,n+1}(X,\beta)$ to evaluation at the $(n+1)$th marking) and $p\colon \overline{\mathscr{C}}_{g,n} \to \overline{\mathscr{M}}_{g,n}$ is the projection. Let s be a regular section of E and $i\colon Z \hookrightarrow X$ be its zero locus. An inspection of the moduli problems (see the proof of corollary 3.2.5) reveals that the disjoint union, over all classes $\gamma \in A_1 Z$ mapped by i_* to β, of the moduli stacks of stable maps to Z of degree γ coincides with the zero locus of the induced section $\mathbb{R}^0 p_* \operatorname{ev}^* s$ of $\mathbb{R}^0 p_* \operatorname{ev}^* E$. The natural question, leading to the quantum Lefschetz theorem, is whether this identification remains true at the "virtual" level, which was conjectured by Cox, Katz and Lee in [CKL01, Conjecture 1.1]. It was indeed proved in [KKP03] for Chow homology, and the statement was lifted in [Jos10] to G_0-theory, that under assumptions on E the Gromov–Witten theory of Z is equivalent to that of X twisted by the Euler class of E, in that the following holds.

Theorem A ([KKP03, Jos10]). *For any $\gamma \in A_1 Z$ such that $i_*\gamma = \beta$, let $u_\gamma\colon \overline{\mathscr{M}}_{0,n}(Z,\gamma) \hookrightarrow \overline{\mathscr{M}}_{0,n}(X,\beta)$ denote the closed immersion. Suppose E is convex, that is $\mathbb{R}^1 p_*(C, \mu^* E) = 0$ for any stable map $\mu\colon C \to X$ from a rational (*i.e.* genus-0) stable curve $C \xrightarrow{p} S$ (so that the cone $\mathbb{R}^0 p_* \operatorname{ev}^* E$ is a vector bundle). Then*

(1)
$$\sum_{i_*\gamma=\beta} u_{\gamma,*} \left[\overline{\mathscr{M}}_{0,n}(Z,\gamma)\right]^{\mathrm{vir}} = \left[\overline{\mathscr{M}}_{0,n}(X,\beta)\right]^{\mathrm{vir}} \smile c_{\mathrm{top}}(\mathbb{R}^0 p_* \operatorname{ev}^* E) \in A_\bullet\left(\overline{\mathscr{M}}_{0,n}(X,\beta)\right),$$

and

(2)
$$\sum_{i_*\gamma=\beta} u_{\gamma,*} \left[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathscr{M}}_{0,n}(Z,\gamma)}\right] = \left[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathscr{M}}_{0,n}(X,\beta)}\right] \otimes \lambda_{-1}(\mathbb{R}^0 p_* \operatorname{ev}^* E) \in G_0\left(\overline{\mathscr{M}}_{0,n}(X,\beta)\right).$$

It was shown in [CGI+12] that the quantum Lefschetz principle as stated in (1) can be false when the vector bundle E is not convex (or as soon as g is greater than 0).The reason for this is that $\mathbb{R}^0 p_* \operatorname{ev}^* E$ no longer equals $\mathbb{R} p_* \operatorname{ev}^* E$ and the twisting Euler class should be corrected by taking into account the term $\mathbb{R}^1 p_* \operatorname{ev}^* E$: in other words, one should use the full derived pushforward and view the induced cone as a *derived* vector bundle $\mathbb{R} p_* \operatorname{ev}^* E$; this will require viewing our moduli stacks through the lens of derived geometry.

In this note we use this pilosophy to undertake the task of relaxing the hypotheses on theorem A and lifting it to a categorified (and a geometric) statement, by which we mean that:

- we will give a formula at the level of a derived ∞-category of quasicoherent sheaves,
- we will not need to fix the genus to 0,
- we will not need to assume that E is convex, or in fact a classical vector bundle (*i.e.* it can come from any object of the ∞-category $\mathfrak{Perf}(\mathcal{O}_X)$),
- we will not need to assume that the section is regular, as we can allow the target to be any derived scheme rather than a smooth scheme.

We note however that only the categorified form of the formula will hold in full generality, as the usual convexity (and genus) hypotheses are still needed to ensure coherence conditions so as to decategorify to G_0-theory.

1.2. **Derived moduli stacks and virtual classes.** In [MR18], the categorification of Gromov–Witten classes, as a lift from operators between G_0-theory groups to dg-functors between dg-categories of quasicoherent (or coherent, or perfect) $\mathcal{O}$-modules, was achieved through the use of derived algebraic geometry. Indeed, this language allows one to interpret the homological corrections appearing in classical algebraic geometry as actual geometric objects; in particular the virtual structure sheaf $\left[\mathcal{O}^{\mathrm{vir}}_{\overline{\mathscr{M}}_{0,n}(X,\beta)}\right]$ was realised as the actual structure sheaf of a derived thickening $\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)$ of the moduli stack, so that applying of the $(\infty,2)$-functor $\mathfrak{QCoh}$ produces the desired lift of Gromov–Witten theory.

The idea of viewing the virtual fundamental class as a shadow of a higher structure sheaf was introduced in [Kon95], and made more precise first in [CFK09] using the language of dg-schemes and in [Toë14, §3.1] *via* derived geometry. The derived moduli stack of stable maps $\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)$ was constructed in [CFK02] and [STV15]. Finally, [MR18] showed that the virtual structure sheaf really is given by the structure sheaf of the derived thickening, or rather its image by the isomorphism expressing that G-theory does not detect thickenings. Hence, in order to understand theorem A from a completely geometric point of view, the role of the virtual classes should indeed be played by derived moduli stacks.

We may now state the main result of this note, which addresses the question of similarly understanding the virtual statement of the quantum Lefschetz principle as a derived geometric phenomenon, and of deducing an expression for the "virtual structure sheaf" of $\coprod_\gamma \overline{\mathscr{M}}_{g,n}(Z,\gamma)$, understanding along the way the appearance of the Euler class of the bundle. In the remainder of this introduction, we shall write $\mathbb{R}u\colon \coprod_\gamma \mathbb{R}\overline{\mathscr{M}}_{g,n}(Z,\gamma) \hookrightarrow \mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)$ the canonical closed immersion (beware that $\mathbb{R}u$ is not a right derived functor, but simply a morphism of derived stacks which is a thickening of u).

Theorem B (Categorified quantum Lefschetz principle, see corollary 3.2.5 and proposition 2.2.1)**.** *Let X be a derived scheme, $\mathcal{E} \in \mathfrak{Perf}(\mathcal{O}_X)$, and s a section of $\mathbb{V}_X(\mathcal{E})$ with zero locus $Z = X \times^{\mathbb{R}}_{\mathbb{V}_X(\mathcal{E})} X$. Write $\jmath\colon \mathbb{E}^\vee \coloneqq (\mathbb{R}p_* \operatorname{ev}^* \mathcal{E})^\vee \to \mathcal{O}_{\overline{\mathscr{M}}_{g,n}(X,\beta)}$ the cosection (of modules) corresponding to $\mathbb{R}p_* \operatorname{ev}^* s$. There is an equivalence*

$$(3)\quad (\mathbb{R}u)_* \mathcal{O}_{\coprod_\gamma \mathbb{R}\overline{\mathscr{M}}_{g,n}(Z,\gamma)} \simeq \mathcal{O}_{\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)} \otimes \mathcal{S}ym\left(\operatorname{cofib}(\jmath)\right)^{\mathbb{G}_a} = \mathcal{S}ym\left(\operatorname{cofib}(\jmath)\right)^{\mathbb{G}_a}$$

in $\mathfrak{QCoh}\left(\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)\right)$, where $\operatorname{cofib}(\jmath)$ denotes the cofibre (or homotopy cokernel) of the linear morphism $\jmath$ and where the additive group $\mathbb{G}_a$ acts faithfully (allowing the taking of homotopy invariants without introducing additional stackiness).

We first notice that, in this categorified statement and unlike in the G-theoretic one, the Euler class of $\mathbb{E}^\vee$ is refined to one taking into account the section s. Nonetheless this is indeed a categorification of theorem A, as we will explain in corollary 2.2.5 and subsection 4.3. When s is the zero section, meaning that $\jmath$ is the zero morphism, then $\mathcal{S}ym(\operatorname{cofib}(\jmath)) = \mathcal{S}ym(\mathbb{E}^\vee[1]) \otimes \mathcal{O}_{\mathbb{G}_a}$, with $\mathcal{S}ym(\mathbb{E}^\vee[1]) = \bigwedge^\bullet(\mathbb{E}^\vee)$ so that in that case we do recover a categorified Euler class. In particular, passing to the G_0 groups will indeed provide an identification of the cofibres of any and all sections, and hence give back eq. (2); this is corollary 4.3.3.

The theorem will in fact come as a corollary of a geometric statement, as a translation of the fact that Euler classes (also known, in the categorified setting, as Koszul complexes) represent zero loci of sections. Indeed, we will show that the moduli stack $\coprod_\gamma \mathbb{R}\overline{\mathscr{M}}_{g,n}(Z,\gamma) = \mathcal{S}pec_{\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)}\left((\mathbb{R}u)_*\mathcal{O}_{\coprod_\gamma \mathbb{R}\overline{\mathscr{M}}_{g,n}(Z,\gamma)}\right)$ satisfies the universal property of the zero locus of $\mathbb{R}p_* \operatorname{ev}^* s$, meaning that (per corollary 3.2.5, the *geometric quantum Lefschetz principle*) it features in the cartesian square

$$\begin{tikzcd}
\coprod_{i_*\gamma=\beta} \mathbb{R}\overline{\mathscr{M}}_{g,n}(Z,\gamma) \arrow[r, "\mathbb{R}u_1"] \arrow[d, "\mathbb{R}u_2"'] \arrow[dr, phantom, "\lrcorner"] & \mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta) \arrow[d, "\mathbb{R}p_* \operatorname{ev}^* s"] \\
\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta) \arrow[r, "0_{\mathbb{E}}"'] & \mathbb{E}|_{\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)}
\end{tikzcd} \tag{4}$$

The formula eq. (3) for its relative function ring will then be a consequence of the general result proposition 2.2.1 describing zero loci of sections of vector bundles.

The original proof of the quantum Lefschetz principle in [KKP03] also consisted of applying an excess intersection formula to a geometric (or homological) statement, here the fact that the embedding u satisfies the compatibility condition implying that Gysin pullback along it preserves the virtual class. The situation was shed light upon in [Man12], where it was shown that, using relative perfect obstruction theories (POTs), one can construct *virtual pullbacks*, which always preserve virtual classes. The embedding u being regular, its own cotangent complex can be used as a POT to construct a virtual pullback, which evidently coincides with the Gysin pullback.

Here we will show (in section 4) that, in much the same way as for the virtual classes, the virtual pullbacks may be understood as coming from derived geometric pullbacks of coherent sheaves, so that our statement for the embedding of derived moduli stacks does imply the quantum Lefschetz formula for the virtual classes (and in fact its standard proof).

1.3. **Acknowledgements.** These results were obtained as part of my PhD thesis, and I wish to thank first my director Étienne Mann for suggesting this problem and for numerous discussions. Many thanks are also due to my co-director Cristina Manolache for helping me understand the construction of virtual pullbacks, to Benjamin Hennion for explaining to me over which base the proof of proposition 3.2.4 was to take place, and to Massimo Pippi for explaining the subtleties of coherence and boundedness for the passage to K-theory. I also thank Marc Levine who suggested that remark 4.2.6 was possible.

1.4. **Notations and conventions.** We will use freely the language of $(\infty,1)$-categories (referred to as ∞-categories), developed in a model-independent manner in [RV19], and of derived algebraic geometry, as developed for example in [TV08] and [Lur19]. The ∞-category of ∞-groupoids, also known as that of spaces in [Lur09], will be denoted $\infty-\mathfrak{Grpd}$, and similarly the ∞-category of ∞-categories is $\infty-\mathfrak{Cat}$.

We work over a fixed field k of characteristic 0; hence the ∞-category of k-module spectra can be modelled as the localisation of the category of k-dg-modules along quasi-isomorphisms, in a way compatible with the monoidal structures so that connective k-$\mathscr{E}_\infty$-algebras are modelled by k-cdgas concentrated in non-positive cohomological degrees. The ∞-category of derived stacks on the big étale ∞-site of k will simply be denoted $\mathfrak{dSt}_k$.

The fully faithful left adjoint $\iota\colon \mathfrak{St}_k \hookrightarrow \mathfrak{dSt}_k$ to the truncation ∞-functor t_0 will be omitted from notation, and the counit of the adjunction will be denoted $\jmath$, with components the closed immersions $\jmath_X\colon t_0X \hookrightarrow X$. Furthermore, we implicitly embed stacks into derived stacks; as such all construction are derived by default. In particular the symbol $\times$ will refer to the (homotopical) fibre product of derived stacks; the 1-categorical (strict, or underived) fibre product of classical stacks will be denoted $\times^1$, that is $X \times^1_Y Z = t_0(X \times_Y Z)$ for X, Y and Z classical.

By a dg-category (over k) we will mean a k-linear stable ∞-category. We shall always use cohomological indexing. For any derived stack X, one defines its G_0-theory group $G_0(X)$ as the zeroth homotopy group of the K-theory spectrum of the dg-category $\mathfrak{Coh}^b(X)$.

2. Zero loci of sections of derived vector bundles

2.1. Vector bundles in derived geometry.

Definition 2.1.1 (Total space of a quasicoherent module)**.** Let X be a derived Artin k-stack. For any quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$, the linear derived stack $\mathbb{V}_X(\mathcal{M})$ is described by the ∞-functor of points mapping an X-derived stack $\phi\colon T \to X$ to the ∞-groupoid

$$\mathrm{Map}_{\mathfrak{QCoh}(T)}(\mathcal{O}_T, \phi^*\mathcal{M}). \tag{5}$$

We call **abelian cone** over X any X-stack equivalent to the total space $\mathbb{V}_X(\mathcal{M})$ of a quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$. We shall say that $\mathbb{V}_X(\mathcal{M})$ is a **perfect** cone if $\mathcal{M}$ is perfect (equivalently, dualisable), and a **vector bundle** if $\mathcal{M}$ is locally free of finite rank (as defined in [Lur19, Notation 2.9.3.1]).

Remark 2.1.2. If $\mathcal{M}$ is a locally free $\mathcal{O}_X$-module, by [Lur19, Proposition 2.9.2.3] we may take a Zariski open cover $\coprod_i U_i \to X$ with $\mathcal{M}|_{U_i}$ free of rank r_i. We deduce from this (or from [Lur17, Remark 7.2.4.22] and [Lur19, Remark 2.9.1.2]) that any locally free module has Tor-amplitude concentrated in degree 0, and it will follow from proposition 2.1.10 that any vector bundle is smooth over its base.

Remark 2.1.3. If $\mathcal{M}$ is dualisable, with dual $\mathcal{M}^\vee$, then as pullbacks commute with taking duals we have for any $\phi\colon T \to X$

$$\begin{aligned}\mathbb{V}_X(\mathcal{M})(\phi) &= \mathrm{Map}_{\mathfrak{QCoh}(T)}(\phi^*\mathcal{M}^\vee, \mathcal{O}_T)\\ &= \mathrm{Map}_{\mathfrak{Alg}(\mathcal{O}_X)}(\mathcal{S}ym_{\mathcal{O}_X}(\mathcal{M}^\vee), \phi_*\mathcal{O}_T) = \mathcal{S}pec_X(\mathcal{S}ym_{\mathcal{O}_X}(\mathcal{M}^\vee))(\phi)\end{aligned} \tag{6}$$

where $\mathcal{S}pec_X$ denotes the non-connective relative spectrum ∞-functor. Hence the restriction of $\mathbb{V}_X$ to $\mathfrak{Perf}(\mathcal{O}_X)$ is naturally equivalent to the composite $\mathcal{S}pec_X \circ \mathcal{S}ym_{\mathcal{O}_X} \circ (-)^\vee$. In particular, if $\mathcal{M}$ is a connective module then $\mathbb{V}_X(\mathcal{M})$ is a relatively coaffine stack (while if $\mathcal{M}$ is co-connective $\mathbb{V}_X(\mathcal{M})$ is an affine derived X-scheme).

Note however that the ∞-functor $\mathcal{S}pec_X$ only becomes fully faithful when restricted to either connective $\mathcal{O}_X$-algebras (as this restriction is equivalent to the Yoneda embedding thereof) or co-connective $\mathcal{O}_X$-algebras, but not when acting on general $\mathcal{O}_X$-algebras in degrees of arbitrary positivity.

Warning 2.1.4 (Terminology). Note that our convention for derived perfect cones is dual to that used in (among others) [Toë14] (and dating back to EGA2), which defines the total space of a quasicoherent $\mathcal{O}_X$-module $\mathcal{M}$ as the X-stack whose sheaf of sections is $\mathcal{M}^\vee$, *i.e.* what we denote $\mathbb{V}_X(\mathcal{M}^\vee)$.

Example 2.1.5.
i. If X is a classical Deligne–Mumford stack and $\mathcal{M}$ is of perfect amplitude in $[-1, 0]$, the truncation $t_0(\mathbb{V}_X(\mathcal{M}[1]^\vee))$ is the abelian cone Picard stack $\mathcal{H}^1/\mathcal{H}^0(\mathcal{M}^\vee)$ of [BF97, Proposition 2.4].
ii. By [TV08, Proposition 1.4.1.6], $\mathbb{V}_X(\mathbb{T}_X) = TX = \mathbb{R}\mathcal{M}ap(k[\varepsilon], X)$ is the tangent bundle stack of X. More generally, using $k[\varepsilon_n]$ where ε_n is of cohomological degree $-n$ (so of homotopical degree n) we have the shifted tangent bundle $T[-n]X \simeq \mathbb{V}_X(\mathbb{T}_X[-n])$. Dually, one also defines the shifted cotangent stack $T^\vee[n]X = \mathbb{V}_X(\mathbb{L}_X[n])$.

Lemma 2.1.6 ([TV07, Sub-lemma 3.9],[AG14, Theorem 5.2][1])**.** *Suppose $\mathcal{M}$ is of perfect Tor-amplitude contained in $[a, b]$ (where $a, b \in \mathbb{Z}$). Then the derived stack $\mathbb{V}_X(\mathcal{M})$ is $(-a)$-geometric and strongly of finite presentation.*

Construction 2.1.7. For any derived stack X, the ∞-functor $\mathbb{V}_X$ gives a link between two functorial (in X) constructions. On the one hand we have the ∞-functor $(-)_{\text{ét}} \colon \mathfrak{dSt}_k \to \infty-\mathfrak{Cat}$ mapping a derived k-stack X to its étale ∞-topos $X_{\text{ét}}$ and a map of derived stacks $f \colon X \to Y$ to the direct image f_* of the induced geometric morphism, mapping a sheaf $\mathcal{F}$ on $\mathfrak{dSt}_{k,/X}$ to the sheaf $f_*\mathcal{F} \colon (U \to Y) \mapsto \mathcal{F}(U \times_Y X \to X)$.

On the other hand, we have the ∞-functor $\mathfrak{QCoh}(-)$ mapping a derived k-stack X to the underlying ∞-category of the dg-category $\mathfrak{QCoh}(X)$, and a map $f \colon X \to Y$ to $\mathcal{M} \mapsto f_*\mathcal{M}$ (where the direct image sheaf is considered an $\mathcal{O}_Y$-module through $f^\sharp \colon \mathcal{O}_Y \to f_*\mathcal{O}_X$). Then for any $\mathcal{M} \in \mathfrak{QCoh}(X)$, we obtain the functor of points of its total space, $\mathbb{V}_X(\mathcal{M})$, which is an étale sheaf on $\mathfrak{dSt}_{k,/X}$.

Lemma 2.1.8. *Let $\mathfrak{dSt}_k^{(\mathrm{QCA})}$ denote the wide and 2-full sub-∞-category whose 1-arrows are the QCA maps (whose fibres are quasi-compact, with affine automorphism groups of geometric points, and with classical inertia stacks of finite presentation over their truncations, see* [DG13, Definition 1.1.8]*). The ∞-functors $\mathbb{V}_X \colon \mathfrak{QCoh}(X) \to X_{\text{ét}}$ assemble into a natural transformation $\mathbb{V} \colon \mathfrak{QCoh}(-) \Rightarrow (-)_{\text{ét}}$ of ∞-functors $\mathfrak{dSt}_k^{(\mathrm{QCA})} \to \infty-\mathfrak{Cat}$.*

Proof. We must show that, for any $f \colon X \to Y$ and any $\mathcal{M} \in \mathfrak{QCoh}(X)$, we have $f_*(\mathbb{V}_X(\mathcal{M})) = \mathbb{V}_Y(f_*\mathcal{M})$. For any $\phi \colon U \to Y$, the base change along f will take place in the cartesian square

$$\begin{array}{ccc} X \times_Y U & \xrightarrow{X \times_Y \phi} & X \\ {\scriptstyle f \times_Y U}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow[\phi]{} & Y \end{array} \quad . \tag{7}$$

Then we have $\mathbb{V}_Y(f_*\mathcal{M})(U) = \mathrm{Map}_{\mathfrak{QCoh}(U)}(\mathcal{O}_U, \phi^* f_*\mathcal{M})$ while

$$\begin{aligned} f_*(\mathbb{V}_X(\mathcal{M}))(U) &= \mathrm{Map}_{\mathfrak{QCoh}(X \times_Y U)}(\mathcal{O}_{X \times_Y U}, (X \times_Y \phi)^*\mathcal{M}) \\ &\simeq \mathrm{Map}_{\mathfrak{QCoh}(X \times_Y U)}((f \times_Y U)^*\mathcal{O}_U, (X \times_Y \phi)^*\mathcal{M}) \\ &\simeq \mathrm{Map}_{\mathfrak{QCoh}(U)}(\mathcal{O}_U, (f \times_Y U)_*(X \times_Y \phi)^*\mathcal{M}). \end{aligned} \tag{8}$$

[1] The grading convention used in [AG14] is homotopical, in opposition to our cohomological convention.

By the base-change property of [DG13, Corollary 1.4.5 (i)] (since f is QCA) the two coincide. $\square$

Remark 2.1.9. By [Toë12, Theorem 2.1] , if $f\colon X \to Y$ is quasi-smooth and proper then f_* sends perfect $\mathcal{O}_X$-modules to perfect $\mathcal{O}_Y$-modules.

Finally, we shall use the following well-known description of the cotangent complex of a perfect cone.

Proposition 2.1.10 ([AG14, Theorem 5.2]). *Let $\mathcal{M}$ be a perfect $\mathcal{O}_X$-module, and write $\pi\colon \mathbb{V}_X(\mathcal{M}) \to X$ the structure morphism. Then $\mathbb{L}_{\pi\colon \mathbb{V}_X(\mathcal{M})/X} \simeq \pi^*\mathcal{M}^\vee$.*

Proof. The equivalence is established fibrewise in [Lur17, Proposition 7.4.3.14]. $\square$

2.2. **Excess intersection formula.** In this subsection, we work with the closed embedding $u\colon T \hookrightarrow M$ of derived stacks defined as the zero locus of a section s of a perfect cone $\mathcal{S}pec_M \mathcal{S}ym_{\mathcal{O}_M}(\mathcal{F}^\vee)$ on M: we fix a perfect $\mathcal{O}_X$-module $\mathcal{F}$ and a morphism of algebras $s^\sharp\colon \mathcal{S}ym_{\mathcal{O}_M}(\mathcal{F}^\vee) \to \mathcal{O}_M$, corresponding to the cosection $\widetilde{s}\colon \mathcal{F}^\vee \to \mathcal{O}_M$ of the module $\mathcal{F}^\vee$.

Proposition 2.2.1. *The derived M-stack T may be recovered as the quotient $T \simeq \mathbb{V}_M(\mathrm{cofib}(\widetilde{s})^\vee)/\mathbb{G}_{a,M}$ for a certain 1-faithful action of additive group M-scheme $\mathbb{G}_{a,M} = \mathbb{A}^1_M = \mathcal{S}pec_M(\mathcal{O}_M[t]) = \mathcal{S}pec_M \mathcal{S}ym_{\mathcal{O}_M} \mathcal{O}_M^{\oplus 1}$ on $\mathbb{V}_M(\mathrm{cofib}(\widetilde{s})^\vee)$, that is T is the relative spectrum of the $\mathcal{O}_M$-algebra*

$$u_*\mathcal{O}_T = \mathcal{S}ym_{\mathcal{O}_M}(\mathrm{cofib}(\widetilde{s}))^{\mathbb{G}_{a,M}}. \tag{9}$$

More generally, the monad u_*u^* identifies with tensoring by $\mathcal{S}ym_{\mathcal{O}_M}(\mathrm{cofib}(\widetilde{s}))^{\mathbb{G}_{a,M}}$.

Proof. Let $\bar{s}\colon \mathbb{A}^1_M \to \mathbb{V}_M(\mathcal{F})$ be the linearisation of s, obtained as the image of $\widetilde{s}$ by $\mathbb{V}_M$. As $\mathcal{S}ym_{\mathcal{O}_M}$ is a left-adjoint it preserves colimits (by [RV19, Theorem 2.4.2]) and the function ring of $\mathbb{A}^1_M \times_M T = \mathbb{A}^1_M \times_{\mathbb{V}_M(\mathcal{F})} M$, which is by definition the tensor product (and by [Lur17, Proposition 3.2.4.7] the pushout of algebras), is equivalent to the image by $\mathcal{S}ym_{\mathcal{O}_M}$ of the $\mathcal{O}_M$-module $\mathcal{O}_M^{\oplus 1} \oplus_{\mathcal{F}^\vee} 0 = \mathrm{cofib}(\widetilde{s})$.

It ensues that $\mathbb{V}_M(\mathrm{cofib}(\widetilde{s})^\vee) \simeq T \times_M \mathbb{A}^1_M$ inherits the action of $\mathbb{A}^1_M$ on itself by translation, which is faithful. We now only need the following general result to quotient out the residual $\mathbb{A}^1_M$.

Lemma 2.2.1.1. *Let G be a group object in an ∞-category. Then the quotient of G by the translation self-action is a terminal object.*

Proof of the lemma 2.2.1.1. The simplicial diagram encoding the action is the simplicial décalage of the diagram encoding the group structure. The décalage ∞-functor is left-adjoint to the forgetful ∞-functor from split (augmented) simplicial objects to simplicial objects: the 0-th object of the unshifted diagram becomes (with its face map) the augmentation, while the forgotten degenerations provide the splitting. By [RV19, Proposition 2.3.11], the colimit of a split simplicial object is given by its augmentation. As it is necessary for a simplicial object to define a group that its 0-th stage be terminal, this proves the lemma. $\square$

In our case, the simplicial diagram encoding the action of $\mathbb{A}^1_M$ on $T \times_M \mathbb{A}^1_M$ is obviously the base-change to T of the diagram encoding the M-group structure of $\mathbb{A}^1_M$ (and in fact encodes the group T-scheme $\mathbb{A}^1_T$), so the colimit does indeed recover T.

Finally, both u_* and u^* are left-adjoints, so by the homotopical Eilenberg–Watts theorem of [Hov15] (see also [GR17a, Chapter 4, Corollary 3.3.5]) their composite $u_* u^*$ is equivalent to tensoring by $u_* u^* \mathcal{O}_M$. $\square$

Remark 2.2.1.2. One may also view T, or more accurately its function ring $u_*\mathcal{O}_T$, as being recovered by descent along the faithfully flat map $\varpi\colon \mathbb{A}^1_M \to M$. By [Lur19, Proposition D.3.3.1], quasicoherent sheaves satisfy faithfully flat descent, and the forgetful ∞-functor from $\mathscr{E}_\infty$-algebras to modules is fully faithful and (as a right-adjoint) limit-preserving, so $u_*\mathcal{O}_T$ can be reconstructed from the canonical descent datum on $\mathcal{O}_{T\times_M \mathbb{A}^1_M}$.

This is a translation in the dual world of algebras of the fact that the base-changed $T \times_M \varpi$ is an epimorphism in the ∞-topos of derived stacks and thus an effective epimorphism, which means that T can be recovered as the codescent object (*i.e.* the "quotient", or geometric realisation) of the simplicial kernel of $T \times_M \varpi$. As the action of $\mathbb{A}^1_M$ on $T \times_M \mathbb{A}^1_M$ is faithful, the simplicial object encoding it is an effective groupoid, which is thus equivalent to the simplicial kernel of its quotient map. As this quotient map is equivalent to the projection $T \times_M \varpi\colon \mathbb{A}^1_T = T \times_M \mathbb{A}^1_M \to T$, we obtain that the simplicial kernel of $T \times_M \varpi$, which is nothing but (the dual of) the descent datum for T along ϖ, is equivalent to the action groupoid of $\mathbb{A}^1_M$ on $T \times_M \mathbb{A}^1_M$ (the group structure is encoded in the choice of isomorphism of simplicial objects).

Remark 2.2.2 (Koszul complexes). Suppose $\mathcal{F}$ is locally free. Then, passing to a Zariski open cover $\coprod U_i \to M$, we may assume as in remark 2.1.2 that $\mathcal{F}|_{U_i}$ is free of rank r_i. Write $\widetilde{s}|_{U_i} = (s_\ell)_{1\leq \ell \leq r}$ in coordinates. Then we recover the Koszul complex $\bigotimes_{\ell=1}^r \operatorname{cofib}(s_\ell)$, as studied for instance in [KR19, §2.3.1].

Recall that the exterior algebra of the quasicoherent $\mathcal{O}_M$-module $\mathcal{F}$ is $\bigwedge^\bullet \mathcal{F} \coloneqq \mathcal{S}ym_{\mathcal{O}_M}(\mathcal{F}[1]) = \bigoplus_{n\geq 0} (\bigwedge^n \mathcal{F})[n]$.

Corollary 2.2.3 (Excess intersection formula)**.** *For any quasicoherent $\mathcal{O}_T$-module $\mathcal{M}$ that is the restriction (along u^*) of an $\mathcal{O}_M$-module, there is an equivalence*

$$u^* u_* \mathcal{M} = \mathcal{M} \otimes_{\mathcal{O}_T} \bigwedge^\bullet \mathcal{F}^\vee|_T. \tag{10}$$

Proof. The ∞-functor u^* is a left-adjoint so it preserves colimits, among which in particular cofibres and codescent objects. By definition, we are given an equivalence $u^*\widetilde{s} \simeq u^* 0 = 0$, so the image by u^* of eq. (9) takes the form $\mathcal{S}ym(\operatorname{cofib} 0)^{\mathbb{G}_a}$. By definition of the zero morphism, we may decompose this pushout as the composite of two amalgamated sums:

$$\begin{tikzcd} \mathcal{F}^\vee[1] \oplus \mathcal{O}_M \arrow[dr, phantom, "\lrcorner"] & \mathcal{F}^\vee[1] \arrow[l] \arrow[dr, phantom, "\lrcorner"] & 0 \arrow[l, "!"'] \\ \mathcal{O}_M \arrow[u] & 0 \arrow[l, "!"] \arrow[u, "!"] & \mathcal{F}^\vee \arrow[l, "!"] \arrow[u, "!"'] \arrow[ll, bend left, "0"] \end{tikzcd}, \tag{11}$$

so that $u^*\mathcal{O}_T \otimes_{\mathcal{O}_M} \mathcal{O}_{\mathbb{A}^1_M} = \mathcal{S}ym_{\mathcal{O}_M}(\mathcal{F}^\vee[1] \oplus \mathcal{O}_M) = \mathcal{S}ym(\mathcal{F}[-1]^\vee) \otimes_{\mathcal{O}_M} \mathcal{O}_{\mathbb{A}^1_M}$. As u^* has a structure of monoidal ∞-functor, this extends to any $\mathcal{O}_T$-module $\mathcal{M}$ in the image of u^*.

Of course, this can also be obtained more directly from the fact that the leftmost diagram below is the image by $\mathcal{S}pec_M \mathcal{S}ym_{\mathcal{O}_M}$ of the rightmost one:

(12)
$$\begin{tikzcd} T \arrow[r, "u"] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "0_{\mathbb{V}_M(\mathcal{F})}"] \\ M \arrow[r, "0_{\mathbb{V}_M(\mathcal{F})}"'] & \mathbb{V}_M(\mathcal{F}) \end{tikzcd} \qquad \begin{tikzcd} u_*\mathcal{O}_T \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[l] \\ 0 \arrow[u] & \mathcal{F}^\vee \arrow[l, "!"] \arrow[u, "!"'] \end{tikzcd} .$$

□

Remark 2.2.4 (Lie-theoretic interpretation). The excess intersection formula can also be seen as coming from the study of the $\mathscr{L}_\infty$-algebroid associated with the closed embedding u. Indeed, we are studying the geometry of a closed sub-derived stack $T \subset M$, which can be understood through that of its formal neighbourhood $\widehat{M}_T = M \times_{M_{dR}} T_{dR}$. This is a formally algebraic derived stack (see [CG18, section 4.1] or [GR17b, Chapter 1, Definition 7.1.2] for details) which is a formal thickening of T. By [GR17b, Chapter 5, Theorem 2.3.2], the ∞-category of formal thickenings of T is equivalent to that of groupoid objects in formally algebraic derived stacks over T (*via* the ∞-functor sending a thickening $T \to \mathcal{F}$ to its simplicial kernel, or Čech nerve), and following the philosophy of formal moduli problems it can be considered as a model for the ∞-category of $\mathscr{L}_\infty$-algebroids.

We have the sequence of adjunctions $u^* \dashv u_* \dashv u^!$, implying that the comonad u^*u_* is left-adjoint to the monad $u^!u_*$. Let us write $T \xrightarrow{\widehat{u}} \widehat{M}_T \xrightarrow{p} M$ the factorisation of u, so that $u^!u_* = \widehat{u}^!p^!p_*\widehat{u}_*$. Note that $p\colon M \times_{M_{dR}} T_{dR} \to M$ is the canonical projection, and as both T_{dR} and M_{dR} are étale over Spec k it is also an étale morphism, and we recover $\widehat{u}^!\widehat{u}_*$. Following [GR17b, Chapter 8, 4.1.2], the monad $u^!u_*$ becomes the universal enveloping algebra of the $\mathscr{L}_\infty$-algebroid associated with u, endowed with the Poincaré–Birkhoff–Witt filtration. As the ∞-functor of assciated graded is conservative when restricted to (co)connective filtrations, we only need an expression for the associated graded of the PBW filtration. The result is then nothing but the PBW isomorphism of [GR17b, Chapter 9, Theorem 6.1.2] stating that for any regular embedding of derived stacks $u\colon T \hookrightarrow M$, the monad $\widehat{u}^!\widehat{u}_*$ on $\mathfrak{Coh}^b(T)$ is equivalent to tensoring by $\mathcal{S}ym_{\mathcal{O}_T}(\mathbb{T}_{\widehat{u}})$, and $\mathbb{T}_{\widehat{u}} = \mathbb{T}_u$ since p is étale. Passing back to the adjoint, we do obtain that u^*u_* is equivalent to tensoring with $\mathcal{S}ym_{\mathcal{O}_T}(\mathbb{T}_u^\vee)$.

A similar equivalence between the Hopf comonad u^*u_* and tensoring by the jet algebra (the dual of the universal enveloping algebra) of $\mathbb{T}_u$ was established in [CCT14, Theorem 1.3] using the model of dg-Lie algebroids for $\mathscr{L}_\infty$-algebroids (see [CG18, Proposition 4.3, Theorem 4.11] for a precise statement of the equivalence between dg-Lie algebroids and formally algebraic derived stacks as models for $\mathscr{L}_\infty$-algebroids). However this approach does not provide the PBW theorem needed to identify the jet algebra of $\mathbb{T}_u$ with $\mathcal{S}ym(\mathbb{L}_u)$.

Finally, it is easy to see from proposition 2.1.10 that the base-change property of cotangent complexes and the fibre sequence associated with the composition $\varpi \circ s = \mathbb{1}$ imply $\mathbb{L}_u = u^*\mathbb{L}_s = u^*s^*\mathbb{L}_\varpi[1] = u^*\mathcal{F}^\vee[1]$.

Then, when u is quasi-smooth so that $u_*\mathcal{O}_T$ is coherent, conservativity of the restriction of u^* to $\mathfrak{Coh}^b(M)_T$ gives another reason for the equivalence $u_*\mathcal{O}_Z \simeq (\mathrm{cofib}(\widehat{s}))^{\mathbb{G}_a}$.

Although it is not possible to directly relate s and the zero section at the geometric level and to obtain an expression of $u_*\mathcal{O}_T$ in terms of the Euler class of $\mathcal{F}^\vee$, passing to G-theory a homotopy between the maps they induce always does exist, and hence we recover the classical formulation of the quantum Lefschetz hyperplane formula.

We recall the notation of the G-theoretic Euler class of a locally free $\mathcal{O}_M$-module $\mathcal{G}$ of finite rank: $\lambda_{-1}(\mathcal{G}) \coloneqq [\bigwedge^\bullet \mathcal{G}] = \sum_{i\geq 0}[\bigwedge^i \mathcal{G}[i]] = \sum_i (-1)^i[\bigwedge^i \mathcal{G}] \in G_0(M)$.

Corollary 2.2.5 ([Kha19b, Lemma 2.1])**.** *Suppose $\mathcal{F}$ is a vector bundle. There is an equality of G-theory operators*

$$u_*u^* = (-)\otimes\lambda_{-1}(\mathcal{F}^\vee)\colon G_0(M)\to G_0(M). \tag{13}$$

Proof. We first note that, by definition, $\mathcal{F}$ being locally free of finite rank means that it is (flat-locally) almost perfect, which makes it bounded, and flat, which makes it of Tor-amplitude concentrated in $[0]$ and implies that $\mathcal{F}^\vee[1]$ has Tor-amplitude in $[-1,0]$ so that its symmetric algebra is still bounded and thus in $\mathfrak{Coh}^b(M)$, defining an element of $G_0(M)$.

By [Kha19b, Lemma 1.3], the fibre sequence $\mathcal{O}_M \to \mathrm{cofib}(\widetilde{s}) \to \mathcal{F}^\vee[1]$ implies that $[\mathcal{S}ym^n_{\mathcal{O}_M}(\mathrm{cofib}\,\widetilde{s})] = \oplus_{i=0}^n[\mathcal{S}ym^{n-i}(\mathcal{O}_M)\otimes\mathcal{S}ym^i(\mathcal{F}^\vee[1])]$ for all $n\geq 0$. By the $\mathbb{A}^1$-invariance of G-theory (or equivalently by pass to the $\mathbb{G}_{a,M}$-invariants) we may remove the symmetric algebra of $\mathcal{O}_M$, which gives the result. □

3. The geometric Lefschetz principle

3.1. **Review of the derived moduli stack of stable maps.** Let X be a target derived scheme. We denote $\pi_{g,n}\colon \mathfrak{C}_{g,n}\to\mathfrak{M}_{g,n}$ the universal curve over the moduli stack of prestable curves of genus g with n markings.

Remark 3.1.1. Note that we can allow X to be a derived scheme without any change to the usual theories of stable maps to X, as the moduli problem for prestable curves parametrises *flat* families, whose fibres over a derived stack must still be classical. More precisely, [Lur04, Theorem 8.1.3] shows (see also [PY20, Proposition 4.5] for a precise proof of the non-archimedean analogue) that the obvious extension of the moduli problem for prestable curves to a derived moduli problem is representable by a classical DM stack.

Remark 3.1.2. The authors of [MR18] used instead of $\mathfrak{M}_{g,n}$ the collection, indexed by the semigroup $\mathrm{Eff}(X)$ of effective curve classes in X, of Costello's moduli stacks $\mathfrak{M}_{g,n,\beta}$ parameterising prestable curves with irreducible components decorated by a decomposition of β, and satisfying a stability condition. The main advantage of these moduli stacks over $\mathfrak{M}_{g,n}$ is that the forgetful morphism $\mathfrak{M}_{g,n+1,\beta}\to\mathfrak{M}_{g,n,\beta}$ is the universal curve. This was necessary in [MR18] to obtain the moduli stacks of stable maps from the brane action of the operad $(\mathfrak{M}_{g,\bullet+1,\beta})$. As this property is not needed in this note, we use the more common stack of prestable curves.

Note however that the morphism $\mathfrak{M}_{g,n,\beta}\to\mathfrak{M}_{g,n}$ is étale, so either choice of moduli stack of curves could be used to define the (derived) moduli stack of stable maps to X.

Fix $\beta \in A_1X$, and let $\overline{\mathscr{M}}_{g,n}(X,\beta)$ be the moduli stack of stable maps to X of class β. It can be constructed tautologically, as an $\mathfrak{M}_{g,n}$-stack, as an open in the mapping stack $\mathcal{M}ap_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, X \times \mathfrak{M}_{g,n})$.

The ∞-category of derived stacks (or any of its slices), as an ∞-topos, is also cartesian closed, with internal hom denoted $\mathbb{R}\mathcal{M}ap(-,-)$. As the inclusion of (classical) stacks into derived stacks is fully faithful, the derived mapping stack $\mathbb{R}\mathcal{M}ap_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, X \times \mathfrak{M}_{g,n})$ is a derived thickening of $\mathcal{M}ap_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, X \times \mathfrak{M}_{g,n})$.

Proposition 3.1.3 ([MR18, (4.3.4)])**.** *Let* M *be a base derived stack and* $C \to M$ *and* $D \to M$ *be two* M*-derived stacks. Then*

$$\mathbb{T}_{\mathbb{R}\mathcal{M}ap_{/M}(C,D)/M} = \varpi_* \operatorname{ev}^* \mathbb{T}_{D/M} \tag{14}$$

where $\varpi\colon C \times_M \mathbb{R}\mathcal{M}ap_{/M}(C,D) \to \mathbb{R}\mathcal{M}ap_{/M}(C,D)$ *is the projection and* $\operatorname{ev}\colon \mathbb{R}\mathcal{M}ap_{/M}(C,D) \to D$ *is the evaluation map.*

Corollary 3.1.4. *If* X *is smooth,* $\mathbb{R}\mathcal{M}ap_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, X \times \mathfrak{M}_{g,n})$ *is a quasi-smooth derived stack.* □

Finally, by [TV08, Corollary 2.2.2.10] the (small) Zariski ∞-sites of a derived stack M and of its truncation t_0M are equivalent (and in particular 1-sites). It ensues as in [STV15, Proposition 2.1] that any open substack U of t_0M lifts uniquely to an open sub-derived stack of $\mathbb{R}U \subset M$ such that $\mathbb{R}U \times_M t_0M = U$ (so in particular $\mathbb{R}U$ is a derived thickening of U).

Definition 3.1.5. The **derived moduli stack** $\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)$ of genus-g, n-pointed **stable maps to** X of class β is the open sub-derived stack of $\mathbb{R}\mathcal{M}ap_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, X \times \mathfrak{M}_{g,n})$ corresponding to the open substack $\overline{\mathscr{M}}_{g,n}(X,\beta) \subset \mathcal{M}ap_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, X \times \mathfrak{M}_{g,n})$.

3.2. **Identification of the derived moduli stacks.** Let X be a derived scheme and $\mathcal{E} \in \mathfrak{Perf}(\mathcal{O}_X)$ a perfect $\mathcal{O}_X$-module, giving the perfect cone $E = \mathbb{V}_X(\mathcal{E})$. Let s be a section of E, and denote $Z = X \times_{s,E,0} X \subset X$ its (derived) zero locus.

For a fixed QCA morphism $\pi\colon \mathfrak{C} \to \mathfrak{M}$ of derived k-stacks, we consider the universal map from a base-change of $\mathfrak{C}$ over the derived mapping $\mathfrak{M}$-stack $\mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})$:

$$\begin{tikzcd} \mathfrak{C} \underset{\mathfrak{M}}{\times} \mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) \arrow[r, "\operatorname{ev}"] \arrow[d, "p"'] & X \\ \mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) & \end{tikzcd} \tag{15}$$

Let $\mathbb{E} := p_* \operatorname{ev}^* E = \mathbb{V}_{\mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, X\times\mathfrak{M})}(p_* \operatorname{ev}^* \mathcal{E})$ be the induced abelian (and perfect by remark 2.1.9 if π is quasi-smooth) cone over $\mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})$, and $\sigma := p_* \operatorname{ev}^* s$ its induced section. Write also $0_{\mathbb{E}}\colon \mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) \to \mathbb{E}$ for the zero section.

Theorem 3.2.1. *There is an equivalence of* $\mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})$*-derived stacks*

$$\mathbb{R}\mathcal{M}ap_{\mathfrak{M}}(\mathfrak{C}, Z \times \mathfrak{M}) \simeq \mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) \underset{\sigma,\mathbb{E},0_{\mathbb{E}}}{\times} \mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}), \tag{16}$$

that is the diagram

$$\begin{tikzcd} \mathbb{R}\mathcal{M}ap_{\mathfrak{M}}(\mathfrak{C}, Z \times \mathfrak{M}) \arrow[r, "u_1"] \arrow[d, "u_2"'] \arrow[dr, phantom, "\lrcorner" very near start] & \mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) \arrow[d, "\sigma = p_* \mathrm{ev}^* s"] \\ \mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) \arrow[r, "0_{\mathbb{E}}"'] & \mathbb{E} = p_* \mathrm{ev}^* E \end{tikzcd} \tag{17}$$

is cartesian.

The theorem will follow directly from some formal results.

Lemma 3.2.2. *Let $\mathfrak{V}$ be a presentable symmetric monoidal ∞-category and $\mathfrak{C}$ a $\mathfrak{V}$-enriched ∞-category. Then the hom ∞-profunctor of $\mathfrak{C}$ commutes with (weighted) limits in both variables (co- and contravariantly).*

Proof. We will deduce this result as a direct consequence of the abstract characterisation of limits; hence the only non-formal part of the proof is to ensure that we have an ∞-cosmos of $\mathfrak{V}$-enriched ∞-categories in order to access the virtual equipment of ∞-profunctors (or bimodules).

Since $\mathfrak{V}$ is presentable, by [Lur09, Proposition A.3.7.6] and [Lur17, Remark 4.1.8.9] it can be presented as the ∞-categorical localisation of a combinatorial simplicial symmetric monoidal model category $\mathfrak{V}^{\mathrm{mod}}$. Then by the rectification results of [Hau15, Theorem 5.8] the ∞-category of $\mathfrak{V}$-enriched ∞-categories is itself the localisation of the simplicial model category of $\mathfrak{V}^{\mathrm{mod}}$-enriched categories[2]. This allows us to use [RV19, Proposition E.1.1] to conclude that there is indeed an ∞-cosmos of $\mathfrak{V}$-∞-categories, and we shall speak of weighted (co)-limits as defined in terms of the associated virtual proarrow equipment.

We need to check that for any object C, seen equivalently as $C \colon * \to \mathfrak{C}$, the $\mathfrak{V}$-functor $\mathrm{Map}(C, \mathbb{1}_{\mathfrak{C}})$ commutes with limits. A functor $\mathscr{L} \colon \mathfrak{A} \to \mathfrak{C}$ is a limit of $\mathscr{D} \colon \mathfrak{I} \to \mathfrak{C}$ weighted by the profunctor $\mathscr{W} \colon \mathfrak{I} \nrightarrow \mathfrak{A}$ if and only if its companion $\mathscr{L}_* \colon \mathfrak{A} \nrightarrow \mathfrak{C}$ defines a lax extension of $\mathscr{D}_*$ through $\mathscr{W}$. Given such a limit, we must thus check that there is a cell α filling the diagram to a lax extension diagram. But the functor $\mathrm{Map}(C, \mathscr{D})$ is nothing but the functor $\mathcal{Y}^{-1}(\mathscr{D}_* \otimes C^*) \colon \mathfrak{I} \simeq \mathfrak{I} \times *^{\mathrm{op}} \to \mathfrak{V}$ corresponding to the profunctor $\mathscr{D}_* \otimes C^* \colon \mathfrak{I} \nrightarrow *$ (either, in our context of $\mathfrak{V}$-categories, by the definition of $\mathfrak{V}$-profunctors or, more abstractly, by the Yoneda structure $\mathcal{Y}$ induced as in [Roa19, Proposition 5.6] by the augmented virtual proarrow equipment), where $\otimes$ denotes the composition of profunctors, also interpreted as the tensor product of bimodules (which is written in the opposite order as usual

[2]To use the results of [Hau15], we must choose $\mathfrak{V}^{\mathrm{mod}}$ to further be a left proper tractable biclosed monoidal model category satisfying the monoid axiom

composition, *i.e.* following the pictorial order of arrows). As C^* and $\mathscr{D}_*$ are composable ([RV19, Proposition 12.4.7]), we can obtain the desired cell from that expressing $\mathscr{L}$ as a limit and from the composition cell:

(18)
$$\begin{tikzcd} \mathfrak{I} \arrow[r, "\mathscr{W}"] \arrow[d, equal] & \mathfrak{A} \arrow[r, "\mathscr{y}^{-1}(\mathscr{L}_* C^*)_*"] \arrow[d, phantom, "\Downarrow\alpha"] & \mathfrak{V} \arrow[d, equal] \\ \mathfrak{I} \arrow[rr, "\mathscr{y}^{-1}(\mathscr{D}_* C^*)_*"'] & & \mathfrak{V} \end{tikzcd} = \begin{tikzcd} \mathfrak{I} \arrow[r, "\mathscr{W}"] \arrow[d, equal] & \mathfrak{A} \arrow[r, "\mathscr{L}_*"] \arrow[d, phantom, "\Downarrow"] & \mathfrak{C} \arrow[r, "\mathscr{y}^{-1}(C^*)_*"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\mathbb{1}"] & \mathfrak{V} \arrow[d, equal] \\ \mathfrak{I} \arrow[rr, "\mathscr{D}_*"'] \arrow[d, equal] & & \mathfrak{C} \arrow[r, "\mathscr{y}^{-1}(C^*)_*"'] \arrow[d, phantom, "\Downarrow\text{compos}"] & \mathfrak{V} \arrow[d, equal] \\ \mathfrak{I} \arrow[rrr, "\mathscr{y}^{-1}(\mathscr{D}_* C^*)_*"'] & & & \mathfrak{V} \end{tikzcd}.$$

Its universal property follows from those of the aforementioned universal cells.

The proof straightforwardly dualises to show that $\mathrm{Map}_{\mathfrak{C}}(\mathbb{1}_{\mathfrak{C}}, C)$ sends colimits (*i.e.* limits in the opposite $\mathfrak{V}$-category $\mathfrak{C}^{\mathrm{op}}$) to limits in $\mathfrak{V}$. □

In our case, we apply this lemma to the ∞-category $\mathfrak{dSt}_k$, which as an ∞-topos is cartesian closed and thus self-enriched, and we find that $\mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, Z \times \mathfrak{M})$ is equivalent to the fibre product

$$\mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) \underset{\mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, E \times \mathfrak{M})}{\times} \mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}), \tag{19}$$

with structure morphisms induced by s and the zero section of E. Hence, in order to prove theorem 3.2.1 we only need to identify the two derived stacks over which the fibre products are taken (as well as the two pairs of structure maps), the derived stack of maps to the abelian cone E and the induced cone $\mathbb{E} = \not{p}_* \operatorname{ev}^* E$.

Remark 3.2.3. In our context of a cartesian closed ∞-category, the internal hom ∞-functor is further characterised as a right adjoint to taking cartesian product, so the fact that it preserves limits follows more directly from [RV19, Theorem 2.4.2].

Proposition 3.2.4. *There is an equivalence of* $\mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})$*-derived stacks*

$$\mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, E \times \mathfrak{M}) \simeq \mathbb{E}. \tag{20}$$

Proof. Let $a\colon S \to \mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})$ be an $\mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})$-stack, with corresponding family $C_a = a^*\mathfrak{C} = S \times_{\mathfrak{M}} \mathfrak{C} \to S$ (where we implicitly push the structure maps forward along $\mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) \to \mathfrak{M}$). Note that, as $\not{p}\colon \mathfrak{C} \times_{\mathfrak{M}} \mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) \to \mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})$ is just projection onto the first factor, we have

$$\begin{aligned} \not{p}^{-1}(a) &= S \times_{\mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M})} \left(\mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) \times_{\mathfrak{M}} \mathfrak{C}\right) \\ &= S \times_{\mathfrak{M}} \mathfrak{C} =: C_a, \end{aligned} \tag{21}$$

as seen in the cartesian diagram

$$\begin{tikzcd} C_a = S \underset{\mathfrak{M}}{\times} \mathfrak{C} \arrow[r, "\tilde{a}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) \underset{\mathfrak{M}}{\times} \mathfrak{C} \arrow[r] \arrow[d, "\not{p}"] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{C} \arrow[d] \\ S \arrow[r, "a"'] & \mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, X \times \mathfrak{M}) \arrow[r] & \mathfrak{M} \end{tikzcd}. \tag{22}$$

By lemma 2.1.8, as $\pi\colon \mathfrak{C} \to \mathfrak{M}$ was supposed QCA and QCA morphisms are stable by base-change, we have

$$\begin{aligned}
\mathbb{E}(a) &= \mathbb{V}_{\mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C},X\times\mathfrak{M})}(p_* \operatorname{ev}^* \mathcal{E})(a) \\
&= p_* \mathbb{V}_{\mathfrak{C}\times_{\mathfrak{M}}\mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C},X\times\mathfrak{M})}(\operatorname{ev}^* \mathcal{E})(a) \\
&= \mathbb{V}_{\mathfrak{C}\times_{\mathfrak{M}}\mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C},X\times\mathfrak{M})}(\operatorname{ev}^* \mathcal{E})(C_a) \\
&= \operatorname{Map}_{\mathfrak{Perf}(\mathcal{O}_{C_a})}(\mathcal{O}_{C_a}, \widetilde{a}^* \operatorname{ev}^* \mathcal{E}) = \operatorname{Map}_{/X}(C_a, E),
\end{aligned} \tag{23}$$

where $\operatorname{ev}\circ\widetilde{a}\colon C_a \to X$ is the map from a family of curves to X classified by a.

Meanwhile, we have by definition

$$\begin{aligned}
&\mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, E \times \mathfrak{M})(a) \\
=&\operatorname{Map}_{\mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C},X\times\mathfrak{M})}(S, \mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, E \times \mathfrak{M})) \\
\simeq&\operatorname{Map}_{/\mathfrak{M}}(S, \mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, E \times \mathfrak{M})) \underset{\operatorname{Map}_{/\mathfrak{M}}(S,\mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C},X\times\mathfrak{M}))}{\times} \{a\}.
\end{aligned} \tag{24}$$

Indeed, from[Lur09, Lemma 6.1.3.13] the standard categorical arguments[3] show that for any morphism $p\colon M' \to M$ in an ∞-category and any cospan $S \to M' \leftarrow T$ over M' we have $\operatorname{Map}_{/M'}(S,T) \simeq \operatorname{Map}_{/M}(S,T) \times_{\operatorname{Map}_{/M}(S,M')} \{p\}$; and we can compute

$$\begin{aligned}
&\mathbb{R}\mathcal{M}ap_{/\mathfrak{M}}(\mathfrak{C}, E \times \mathfrak{M})(a) \\
\simeq&\operatorname{Map}_{/\mathfrak{M}}(S \times_{\mathfrak{M}} \mathfrak{C}, E \times \mathfrak{M}) \underset{\operatorname{Map}_{/\mathfrak{M}}(S\times_{\mathfrak{M}}\mathfrak{C},X\times\mathfrak{M})}{\times} \{a\} \\
=&\operatorname{Map}_{/X\times\mathfrak{M}}(C_a, E \times \mathfrak{M}) = \operatorname{Map}_{/X}(C_a, E).
\end{aligned} \tag{25}$$

□

This completes the proof of theorem 3.2.1. □

In our setting, we will have $\mathfrak{M} = \mathfrak{M}_{g,n}$, $\mathfrak{C} = \mathfrak{C}_{g,n}$, and the QCA morphism $\pi\colon \mathfrak{C} \to \mathfrak{M}$ is $\pi_{g,n}$ the universal curve over the moduli stack of prestable curves of genus g with n marked points. One may also wish to replace $\mathfrak{M}_{g,n}$ by an appropriate moduli stack of twisted curves to accommodate for orbifold targets; however we have not done so as we believe that the current notion of morphism of twisted curves is not suitable for the derived stacks we need as targets.

We will also write $p_{g,n} = p$, $\operatorname{ev}_{g,n} = \operatorname{ev}$ and $\mathbb{E}_{g,n} = \mathbb{E} = (p_{g,n})_* \operatorname{ev}^*_{g,n} E$.

Corollary 3.2.5 (Geometric quantum Lefschetz principle)**.** *Fix a class $\beta \in A_1X$. There is an equivalence of $\mathbb{R}\mathcal{M}ap_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, Z \times \mathfrak{M}_{g,n})$-derived stacks*

$$\coprod_{i_*\gamma=\beta} \mathbb{R}\overline{\mathscr{M}}_{g,n}(Z,\gamma) \simeq \mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta) \underset{\mathbb{E}|_{\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)}}{\times} \mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta). \tag{26}$$

Proof. Note first that, as Zariski-open immersions are stable by pullbacks, both $\coprod_{i_*\gamma=\beta} \mathbb{R}\overline{\mathscr{M}}_{g,n}(Z,\gamma)$ and $\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta) \times_{\mathbb{E}} \mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)$ are open sub-derived stacks of $\mathbb{R}\mathcal{M}ap_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, Z \times \mathfrak{M}_{g,n})$, so by [STV15, Proposition 2.1] to show that they are equal it is enough to show that their truncations define identical substacks of $t_0\,\mathbb{R}\mathcal{M}ap_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, Z \times \mathfrak{M}_{g,n})$.

[3] suggested to the author by Benjamin Hennion

As a ring is discrete if and only if the ∞-groupoids of morphisms toward it are, the truncation of such a derived mapping stack with (discrete) source flat over the discrete base is $\mathcal{M}ap_{/\mathfrak{M}_{g,n}}(\mathfrak{C}_{g,n}, t_0(Z \times \mathfrak{M}_{g,n}))$ (see [TV08, Theorem 2.2.6.11, hypothesis (1)]), and similarly $t_0\left(\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta) \times_{\mathbb{E}} \mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)\right) = \overline{\mathscr{M}}_{g,n}(X,\beta) \times^{s}_{t_0\mathbb{E}} \overline{\mathscr{M}}_{g,n}(X,\beta)$. In addition, the truncation ∞-functor commutes with colimits so $t_0\left(\coprod_{i_*\gamma=\beta} \mathbb{R}\overline{\mathscr{M}}_{g,n}(Z,\gamma)\right) = \coprod_{i_*\gamma=\beta} \overline{\mathscr{M}}_{g,n}(t_0 Z, \gamma)$.

We now compare the two stacks pointwise. For any $S \to \mathfrak{M}_{g,n}$ (with corresponding prestable genus-g curve $C_S \to S$), we have that $\left(\coprod_{i_*\gamma=\beta} \overline{\mathscr{M}}_{g,n}(t_0 Z,\gamma)\right)(S) = \coprod_{i_*\gamma=\beta} \overline{\mathscr{M}}_{g,n}(t_0 Z,\gamma)(S)$ is tautologically the disjoint union (over $\gamma \in i_*^{-1}(\beta)$) of the groupoids of S-indexed families of stable maps from C_S to $t_0 Z$ of class γ, and (27)

$$\left(\overline{\mathscr{M}}_{g,n}(X,\beta) \underset{t_0\mathbb{E}}{\overset{1}{\times}} \overline{\mathscr{M}}_{g,n}(X,\beta)\right)(S) \simeq \overline{\mathscr{M}}_{g,n}(X,\beta)(S) \underset{t_0\mathbb{E}|_{\overline{\mathscr{M}}_{g,n}(X,\beta)}(S)}{\overset{1}{\times}} \overline{\mathscr{M}}_{g,n}(X,\beta)(S)$$

with $t_0\mathbb{E}|_{\overline{\mathscr{M}}_{g,n}(X,\beta)}(S) = \hom(C_S, E)$. An object of the latter 2-fibre product consists of a pair of stable maps f_1, f_2 from C_S to X and an automorphism φ of C_S such that $s \circ f_1 = 0 \circ f_2 \circ \varphi$, while an automorphism $(f_1, f_2, \varphi) \simeq (f_1, f_2, \varphi)$ is given by a pair of automorphisms of C_S compatible with all the data; or automorphisms of the two stable maps compatible with φ (so that when φ is not $\mathbb{1}_{C_S}$ the notion of automorphism is more rigid than the usual automorphisms of a single stable map). In particular, the obvious functor $\coprod_{i_*\gamma=\beta} \overline{\mathscr{M}}_{g,n}(t_0 Z,\gamma)(S) \to \left(\overline{\mathscr{M}}_{g,n}(X,\beta) \times^1_{t_0\mathbb{E}} \overline{\mathscr{M}}_{g,n}(X,\beta)\right)(S)$ sending a stable map $f\colon C_S \to Z$ to $(i_{Z\hookrightarrow X} \circ f, i_{Z\hookrightarrow X} \circ f, \mathbb{1}_{C_S})$ is clearly fully faithful, and in fact an equivalence. $\square$

We may now apply proposition 2.2.1 to deduce a proof of theorem B. In the next section 4, we will see how we can recover from this and corollary 2.2.5 the classical (virtual) quantum Lefschetz formula.

Remark 3.2.6. Further evidence for this geometric form of the quantum Lefschetz principle can also be found by comparing the tangent complexes. Let us write temporarily $M(X)$ and $M(Z)$ for the moduli stacks of stable maps $\mathbb{R}\overline{\mathscr{M}}_{g,b}(X,\beta)$ and $\coprod_{i_*\gamma=\beta} \mathbb{R}\overline{\mathscr{M}}_{g,n}(Z,\gamma)$, and $M'(Z)$ for the zero locus $M(X) \times_{\mathbb{E}} M(X)$. The universal property of the latter stack induces a canonical morphism denoted $\Upsilon\colon M(Z) \to M'(Z)$ such that $u'_i \circ \Upsilon = u_i$ for $i = 1, 2$ where $u_i\colon M(Z) \hookrightarrow M(X)$ and $u'_i\colon M'(Z) \hookrightarrow M(X)$ are the canonical arrows (as in eq. (17)).

We know from proposition 3.1.3 that $\mathbb{T}_{M(X)/\mathfrak{M}_{g,n}} \simeq p_{g,n,*} \operatorname{ev}^*_{g,n} \mathbb{T}_X|_{M(X)}$. There is a fibre sequence $i_1^*\mathbb{L}_X \to \mathbb{L}_Z \to \mathbb{L}_{i_1\colon Z/X}$, and as Z sit by definition in a cartesian square we have that $\mathbb{L}_{i_1} = i_2^*\mathbb{L}_{X/E} = \mathcal{E}^\vee[1]|_Z$ (where once again we have written $i_{1,2}\colon Z \hookrightarrow X$ the two canonical inclusions). As both pushforward and pullback preserve fibre sequences, we obtain finally that $\mathbb{T}_{M(Z)/\mathfrak{M}_{g,n}}$ is the fibre of the morphism $p_{g,n,*} \operatorname{ev}^*_{g,n} \mathbb{T}_X|_{M(Z)} \to p_{g,n,*} \operatorname{ev}^*_{g,n} \mathcal{E}|_{M(Z)}$.

Following the same logic, writing $M'(Z)$ for the zero locus, we see that $\mathbb{T}_{M'(Z)}$ is the fibre of $\mathbb{T}_{M(X)} = p_{g,n,*} \operatorname{ev}^*_{g,n} \mathbb{T}_X|_{M(X)} \to p_{g,n,*} \operatorname{ev}^*_{g,n} \mathcal{E}|_{M'(Z)}$. But we have seen that $\Upsilon^* \circ u_i'^{,*} = u_i^*$ so it is clear that $\Upsilon^*\mathbb{T}_{M'(Z)} \simeq \mathbb{T}_{M(Z)}$.

As it is sufficient and necessary for a morphism of derived stacks to be an equivalence that it induce an isomorphism on the truncation and that its (co)tangent complex vanish, this is another way of proving theorem 3.2.1.

Example 3.2.7. Let $(X, f\colon X \to \mathbb{A}^1)$ be a Landau–Ginzburg model, from which we deduce the perfect cone $T^\vee X = \mathbb{V}_X(\mathbb{L}_X)$ and section $d_{dR} f$, whose zero locus is by definition the critical locus $\mathbb{R}\operatorname{Crit}(f)$ (which is the intersection of two Lagrangians in a 0-shifted symplectic derived stack and thus carries a canonical (-1)-shifted symplectic form). Then the derived moduli stack of stable maps to $\mathbb{R}\operatorname{Crit}(f)$ is the zero locus of the induced section of

$$p_* \operatorname{ev}^* T^\vee X = \mathbb{V}_{\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)}(p_*(\operatorname{ev}^* \mathbb{L}_X)) \tag{28}$$

But notice that

$$T^\vee \mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta) \simeq \mathbb{V}_{\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)}((p_* \operatorname{ev}^* \mathbb{T}_X)^\vee) \simeq \mathbb{V}_{\mathbb{R}\overline{\mathscr{M}}_{g,n}(X,\beta)}(p_! \operatorname{ev}^* \mathbb{L}_X) \tag{29}$$

where $p_!\colon \mathcal{F} \mapsto p_*(\mathcal{F}^\vee)^\vee \simeq p_*(\mathcal{F} \otimes \omega_p)$ is the left adjoint to p^* (by [Lur19, Proposition 6.4.5.3]), so $\mathbb{R}\overline{\mathscr{M}}_{g,n}(\mathbb{R}\operatorname{Crit}(f), \beta)$ cannot be expected to carry a (-1)-shifted symplectic structure if (g, n) differs from $(0, 1)$ or $(1, 0)$.

It is also possible to go the other way, that is to obtain a Landau–Ginzburg model from our general setting. If $\varpi\colon E^\vee \to X$ is the dual of the perfect cone with section s, then the section $\varpi^* s$ of $\varpi^* E$ can be paired with the tautological section t of $\varpi^* E^\vee$, defining a function $w_s = \langle s, t\rangle$ on the total space $E^\vee$. By [Isi12, Corollary 3.8], if X is smooth, there is an equivalence $\mathfrak{Coh}^b(Z) \simeq \mathfrak{Sing}(\mathbb{R}\operatorname{Zero}(w_s)/\mathbb{G}_m)$ with the $\mathbb{G}_m$-equivariant dg-category of singularities of $\mathbb{R}\operatorname{Zero}(w_s)$ (where $\mathbb{G}_m$ acts by rescaling on the fibres of $E^\vee$). However we only have $Z = \mathbb{R}\operatorname{Crit}(w_s)$ if Z is smooth (see [CJW19, Lemma 2.2.2] in the regular and underived case).

4. Functoriality in intersection theory by the categorification of virtual pullbacks

We have obtained a categorified form of the quantum Lefschetz principle, which in the cases where $\mathbb{E} = p_* \operatorname{ev}^* E$ is a vector bundle we can by corollary 2.2.5 decategorify by passing to the G_0-theory groups of the derived moduli stacks. To show that our statement is indeed a categorification of the quantum Lefschetz principle, it remains to compare it with the virtual statement, in the G_0-theory of the truncated moduli stacks. As explained in the introduction, this will be obtained through an appropriate construction of virtual pullbacks. These were defined in [Man12] (and in [Qu18] for G_0-theory) from perfect obstruction theories. Following the understanding of virtual classes and the constructions of [MR18], we will give an alternate construction from derived thickenings. To ensure consitency, we show in subsection 4.2 that our construction coincides with that of [Qu18] when both are defined, and we use it in subsection 4.3 to get back the virtual form of the quantum Lefschetz formula.

Remark 4.0.1. The derived origin of virtual pullbacks was already considered in [Sch11, Section 7], where it is shown that any morphism of DM stacks which is the classical truncation of a morphism of derived DM stacks, with the induced obstruction theory, carries the compatibility necessary for the construction of a virtual pullback. However, the origin of the virtual classes and their precise relation to derived thickenings was still considered mysterious, and no direct construction of the virtual pullbacks from derived algebraic geometry was given.

4.1. **Definition from derived geometry.** Let $f\colon X \to Y$ be a quasi-smooth morphism of derived stacks, that is its cotangent complex $\mathbb{L}_{f\colon X/Y}$ is of perfect Tor-amplitude in $[-1, 0]$.

Remark 4.1.1. By [GR17a, Chapter 4, Lemma 3.1.3], as the quasi-smooth morphism f is of finite Tor-amplitude, the pullback of quasicoherent sheaves f^* maps $\mathfrak{Coh}^b(Y)$ to $\mathfrak{Coh}^b(X)$. As we work in G-theory, which is the K-theory of the stable ∞-category of bounded coherent sheaves, the notation f^* will be understood in this section to mean the restriction of the pullback operation to coherent sheaves.

Recall that, due to the theorem of the heart and [Lur19, Corollary 2.5.9.2 with $n = 0$], the closed embedding $\jmath_X\colon t_0X \hookrightarrow X$ induces an isomorphism $\jmath_{X,*}\colon G_0(t_0X) \xrightarrow{\simeq} G_0(X)$ in G-theory, with inverse $(\jmath_{X,*})^{-1}(\mathcal{G}) = \sum_{i\geq 0}(-1)^i[\pi_i(\mathcal{G})]$.

It is therefore natural to define the virtual pullback along t_0f to be given by the actual pullback along f, intertwined with these isomorphisms.

However we wish to consider the virtual pullback as a bivariant class, that is defined as a collection of maps $G_0(Y') \to G_0(X \times_Y S) = G_0(X \times^1_Y Y')$ indexed by all t_0Y-schemes $Y' \to t_0Y$, or more generally by all derived Y-schemes $Y' \to Y$. Then the virtual pullback we defined should be the map corresponding to the t_0Y-scheme $\mathbb{1}_{t_0Y}\colon t_0Y = t_0Y$.

We recall that we use the notation $\times^1$ (a fibre product decorated by 1) to differentiate the strict (1- or 2-categorical) fibre products of classical stacks from the implictly ∞-categorical fibre products of derived stacks.

Definition 4.1.2. The **bivariant virtual pullback** along f is the collection, indexed by all Y-schemes $a\colon Y' \to Y$, of maps $(t_0f)^{!,a}_{\mathrm{DAG}}\colon G_0(t_0Y') \to G_0(t_0(Y' \times_Y X))$ defined as follows.

For a morphism of schemes $a\colon Y' \to Y$, we have the diagram

$$
\begin{tikzcd}
t_0\left(Y' \underset{Y}{\times} X\right) \simeq t_0Y' \overset{1}{\underset{t_0Y}{\times}} t_0X \arrow[r, hook, "\jmath_{Y'\times_Y X}"] \arrow[dr, bend right] & Y' \underset{Y}{\times} X \arrow[r, "\widetilde{f}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y' \arrow[d, "a"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}
\tag{30}
$$

Then we set $(t_0f)^{!,a}_{\mathrm{DAG}} := (\jmath_{Y'\times_Y X,*})^{-1} \circ \widetilde{f}^* \circ \jmath_{Y',*}$.

Lemma 4.1.3. *The virtual pullback only depends on $t_0a\colon t_0Y' \to t_0Y$. That is, for any $a_1, a_2\colon Y'_1, Y'_2 \to Y$ with $t_0a_1 = t_0a_2$, the virtual pullbacks $f^{!,a_1}_{\mathrm{DAG}}$ and $f^{!,a_2}_{\mathrm{DAG}}$ induced by a_1 and a_2 are equal.*

Proof. For any $a\colon Y' \to Y$, we compare the virtual pullbacks induced by a and $t_0Y' \xrightarrow{t_0(a)} t_0Y \xrightarrow{\jmath_Y} Y$.

(31)
$$
\begin{tikzcd}
t_0Y' \overset{1}{\underset{t_0Y}{\times}} t_0X \arrow[rr, hook] \arrow[dr, hook, bend right] & & Y' \times_Y X \arrow[rr, "\widetilde{f}"] \arrow[dd] & & Y' \arrow[dd, "a"] \\
& t_0(Y') \times_Y X \arrow[ur, "i"] \arrow[rr, "\widehat{f}"'] \arrow[dr] & & t_0Y' \arrow[ur, hook] \arrow[dr, "\jmath_Y \circ t_0 a"'] & \\
& & X \arrow[rr, "f"'] & & Y
\end{tikzcd}
$$

The back square is cartesian and its side $\jmath_{Y'}$ is a closed immersion and thus proper, so the base-change formula gives $\widetilde{f}^* \circ \jmath_{Y',*} = i_*\widehat{f}^*$. Commutativity of the leftmost triangle implies that $i_*\jmath_{t_0Y'\times_Y X,*} = \jmath_{Y'\times_Y X,*}$, and as both closed immersions involved induce isomorphisms in G-theory, we have $(\jmath_{Y'\times_Y X,*})^{-1}i_* = (\jmath_{t_0Y'\times_Y X,*})^{-1}$. Putting the ingredients together, we finally obtain that

(32)
$$f^{!,a}_{\mathrm{DAG}} \coloneqq (\jmath_{Y'\times_Y X,*})^{-1}\widetilde{f}^*\jmath_{Y',*} = (\jmath_{Y'\times_Y X,*})^{-1}i_*\widehat{f}^* = (\jmath_{t_0Y'\times_Y X,*})^{-1}\widehat{f}^* \eqqcolon f^{!,\jmath_Y\circ t_0 a}_{\mathrm{DAG}}.$$

□

Remark 4.1.4 (Functoriality). The virtual pullbacks satisfy obvious functoriality properties. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two composable arrows, and let $a\colon Z' \to Z$ be a Z-scheme. We have the commutative diagram

(33)
$$
\begin{tikzcd}
t_0(Z' \times_Z X) \arrow[dr, hook, "\jmath_{Z'\times_Z X}"'] & t_0(Z' \times_Z X) \arrow[dr, hook, "\jmath_{Z'\times_Z Y}"] & & \\
& Z' \times_Z X \arrow[r, "\widetilde{f}"] \arrow[d] & Z' \times_Z Y \arrow[r, "\widetilde{g}"] \arrow[d, "b"] & Z' \arrow[d, "a"] \\
& X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}
$$

It follows by associativity of fibre products that

(34)
$$
\begin{aligned}
(t_0f)^{!,b}_{\mathrm{DAG}} \circ (t_0g)^{!,a}_{\mathrm{DAG}} &= (\jmath_{(Z'\times_Z Y)\times_Y X,*})^{-1} \circ \widetilde{f}^* \circ (\jmath_{Z'\times_Z Y,*}) \circ (\jmath_{Z'\times_Z Y,*})^{-1} \circ \widetilde{g}^* \circ \jmath_{Z',*} \\
&= (\jmath_{Z'\times_Z X,*})^{-1} \circ \widetilde{gf}^* \circ \jmath_{Z',*} \eqqcolon (t_0(gf))^{!,a}_{\mathrm{DAG}}.
\end{aligned}
$$

4.2. **Comparison with the construction from obstruction theories.**

Construction 4.2.1 (Virtual pullbacks from perfect obstruction theories)**.** Let $g\colon V \to W$ be a morphism of Artin stacks of Deligne–Mumford type (*i.e.* relatively DM) endowed with a perfect obstruction theory $\varphi\colon E \to \mathbb{L}_{g\colon V/W}$, inducing the closed immersion $\varphi^\vee\colon \mathfrak{C}_{g\colon V/W} \hookrightarrow \mathfrak{E}$, where $\mathfrak{E} = t_0(\mathbb{V}_V(E[1]^\vee))$ is the vector bundle (Picard) stack associated with E and $\mathfrak{C}_g$ is the intrinsic normal cone of g (constructed in [BF97]). As in [MR18] we define a derived thickening $\mathbb{R}^\varphi V$ of V as the derived intersection

$$
\begin{tikzcd}
\mathbb{R}^{\varphi} V \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{C}_g \arrow[d, hook, "\varphi^{\vee}"] \\
V \arrow[r, hook, "0_{\mathfrak{E}}"'] & \mathfrak{E}
\end{tikzcd}
\tag{35}
$$

Note that the arrow p is a retract of $\jmath_V$, and provides a splitting of the induced perfect obstruction theory $\jmath_V^*\mathbb{L}_{\mathbb{R}^\varphi V} \to \mathbb{L}_V$. We may use it to define a map of derived stacks $\mathbb{R}^\varphi g \colon \mathbb{R}^\varphi V \xrightarrow{p} V \xrightarrow{f} W$ which is a derived thickening of g.

We also recall the construction of the virtual pullback $g^!_\varphi$, or $g^!_{\mathrm{POT}}$, from the perfect obstruction φ, defined in [Man12] for Chow homology then [Qu18] for G_0-theory.

Let $a \colon W' \to W$ and write $g' \colon V' \to W'$ the base-change of g. Recall that one may define a deformation space $\mathfrak{D}_{V'}W'$ over $\mathbb{P}^1_k$, with general fibre W' giving the open immersion $j \colon W' \times \mathbb{A}^1_k \hookrightarrow \mathfrak{D}_{V'}W'$, and special fibre $\mathfrak{C}_{g'}$ giving the complementary closed immersion $i \colon \mathfrak{C}_{g'} \times \{\infty\} \hookrightarrow \mathfrak{D}_{V'}W'$. It follows that there is an exact sequence of abelian groups $G_0(\mathfrak{C}_{g'}) \to G_0(\mathfrak{D}_{V'}W') \to G_0(W' \times \mathbb{A}^1) \to 0$ (coming from the fibred sequence of G-theory spectra). Furthermore, as (by excess intersection) i^*i_* is equivalent to tensoring by the symmetric algebra on the conormal bundle of $\mathfrak{C}_{g'}$ in $\mathfrak{D}_{V'}W'$ and as the latter is trivial, we have $i^*i_* = 0$, inducing a map $G_0(W' \times \mathbb{A}^1) \to G_0(\mathfrak{C}_{g'})$: concretely, any section $j^{*,-1}$ of j^* gives the same map when post-composed with i^* so we do have a well-defined map $i^*j^{*,-1}$. The specialisation map $\mathrm{sp} \colon G_0(W') \to G_0(\mathfrak{C}_{g'})$ is then defined by precomposing it by $\mathrm{pr}^* \colon G_0(W') \to G_0(W' \times \mathbb{A}^1)$. Finally, the cartesian square defining V' induces by [Man12, Proposition 2.26] a closed immersion $c \colon \mathfrak{C}_{g'} \hookrightarrow a^*\mathfrak{C}_g = V' \times_V \mathfrak{C}_g$, and the virtual pullback $g^{!,a}_\varphi$ along g is constructed as the composite

$$
g^{!,a}_\varphi \colon G_0(W') \xrightarrow{\mathrm{sp}} G_0(\mathfrak{C}_{g'}) \xrightarrow{c_*} G_0(a^*\mathfrak{C}_g) \xrightarrow{(a^*\varphi^\vee)_*} G_0(a^*\mathfrak{E} = V' \times_V \mathfrak{E}) \xrightarrow{0^*_{a^*\mathfrak{E}}} G_0(V'). \tag{36}
$$

Lemma 4.2.2. *The virtual pullback $(t_0\mathbb{R}^\varphi g)^!_{\mathrm{DAG}}$ as defined above for the map $\mathbb{R}^\varphi g$ coincides with the virtual pullback $g^!_\varphi$ of* [Man12, Qu18]*: for any $a \colon W' \to W$, we have $(t_0\mathbb{R}^\varphi g)^{!,a}_{\mathrm{DAG}} = g^{!,a}_\varphi \colon G_0(W') \to G_0(V')$.*

Proof. We adapt the results of [Jos10, Proposition 3.5] to the more general case of a morphism that need not be a regular embedding.

Let again $a \colon W' \to W$ and write $g' \colon V' \to W'$ the base-change of g. We now review our construction of the virtual pullback from derived thickenings from the point of view of the perfect obstruction theory. The map $g^{!,a}_{\mathrm{DAG}}$ of definition 4.1.2 is computed in the following way: we define a derived thickening $\mathbb{R}^\varphi V'$ of $V' = V \times^1_W W'$ as $V' \times_{\mathfrak{E}} \mathfrak{C}_g$; note that we have $\mathbb{R}^\varphi V' = V' \times_V \mathbb{R}^\varphi V$ and writing $p' \colon \mathbb{R}^\varphi V' \to V'$ we obtain a derived thickening $\mathbb{R}^\varphi g' = g' \circ p' \colon \mathbb{R}^\varphi V' \to W'$ of g'. Then $g^{!,a}_{\mathrm{DAG}}$ is the pullback along $\mathbb{R}^\varphi g'$ followed by the inverse of $\jmath_{\mathbb{R}^\varphi V', *}$.

We also note that the fibred product $V' \times_{a^*\mathfrak{E}} (a^*\mathfrak{C}_g)$ is the base-change of $V \times_{\mathfrak{E}} \mathfrak{C}_g$ along $a' \colon V' \to V$, so the square

$$\begin{tikzcd} \mathbb{R}^{\varphi} V' \arrow[r, "q'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner"] & a^*\mathfrak{C}_g \arrow[d, hook, "a^*\varphi^{\vee}"] \\ V' \arrow[r, hook, "0_{a^*E}"'] & a^*\mathfrak{E} \end{tikzcd} \tag{37}$$

is cartesian. As p' is proper, we have $0^*_{a^*\mathfrak{E}}(a^*\varphi^{\vee})_* = p'_* q'^{,*}$; concomitantly, as p' is a retract of $\jmath_{\mathbb{R}^{\varphi}V'}$ we have in G-theory $(\jmath_{\mathbb{R}^{\varphi}V',*})^{-1} = p'_*$. We conclude that the virtual pullback of [Qu18] coincides with $(\jmath_{\mathbb{R}^{\varphi}V',*})^{-1} \circ q'^{,*} \circ c_* \circ \mathrm{sp}$, and thus it only remains to check that the latter part specialises to $(\mathbb{R}^{\varphi} g')^* = p'^{,*} \circ g'^{,*}$. But the deformation space $\mathfrak{D}_{V'}W'$ provides exactly an interpolation between $g' \colon V' \to W'$ and $V' \hookrightarrow \mathfrak{C}_{g'}$, so by transporting this comparison along the $\mathbb{A}^1$-invariance of G-theory the lemma is proved. □

Recall that for any quasi-smooth morphism $f \colon X \to Y$ of derived Artin stacks, by [STV15, Proposition 1.2] the canonical map $\varphi \colon j_X^* \mathbb{L}_f \to \mathbb{L}_{t_0 f}$ is a perfect obstruction theory.

Proposition 4.2.3. *Let $f \colon X \to Y$ be a quasi-smooth relatively DM map of derived Artin stacks. The virtual pullback $(t_0 f)^!_{\mathrm{DAG}}$ defined with derived geometry is equal to $(t_0 \mathbb{R}^{\varphi} t_0 f)^!_{\mathrm{DAG}}$, and thus to the virtual pullback $(t_0 f)^!_{\varphi}$ of* [Man12, Qu18]*, induced by the obstruction theory $\varphi \colon j_X^* \mathbb{L}_f \to \mathbb{L}_{t_0 f}$.*

Proof. The proof is similar to the one given in [MR18, Proposition 4.3.2] for the comparison of the virtual classes defined from perfect obstruction theories and derived geometry, which mainly followed [LS12]: one constructs a deformation to the normal bundle of the closed immersion $\jmath_X \colon t_0 X \hookrightarrow X$, and finally uses that G-theory is $\mathbb{A}^1$-invariant. □

We shall henceforth simply write $(t_0 f)^!$ for the virtual pullback along f.

Example 4.2.4 (Virtual classes). Suppose $Y = \mathrm{Spec}(k)$ so $f \colon X \to \mathrm{Spec}(k)$ is the structure morphism. The virtual structure sheaf of $t_0 X$ is $\left[\mathcal{O}^{\mathrm{vir}}_{t_0 X}\right] = f^{!,\mathbb{1}_X}([\mathcal{O}_{\mathrm{Spec}(k)}]) = (\jmath_{X,*})^{-1}([\mathcal{O}_X])$.

Example 4.2.5. Suppose that the classical map g is already a quasi-smooth immersion, so that $\mathbb{1}_{\mathbb{L}_g}$ is a perfect obstruction theory. Then the virtual pullback is given by the Gysin pullback $g^!$, studied in details for example in [Jos10].

Remark 4.2.6 (Virtual pullbacks in generalised motivic homology theories). Our construction of virtual pullbacks only relies on the fact that G-theory is insensitive to the non-reduced structure, and the identification with the classical definition requires simply the specialisation morphism and, more generally, the $\mathbb{A}^1$-invariance. These ingredients are present in motivic homotopy theory (by construction for the $\mathbb{A}^1$-invariance, and by [Kha19a, Corollary 3.2.9] for the insentivity to derived structures), so the virtual pullbacks in motivic cohomology theories also admit the derived geometric interpretation.

4.3. Recovering the quantum Lefschetz formula.

Proposition 4.3.1. *With the notations of subsection 2.2, if $\mathcal{F}$ is a vector bundle then* $(t_0 u)_* \left[\mathcal{O}_T^{\mathrm{vir}}\right] = \left[\mathcal{O}_M^{\mathrm{vir}}\right] \otimes \lambda_{-1}(\pi_0 \mathcal{F}^\vee)$.

Proof. By naturality of the transformation $\jmath$, we have $(t_0 u)_* = (\jmath_{M,*})^{-1} u_* \jmath_{T,*}$ so that $(t_0 u)_* (t_0 u)^! = (\jmath_{M,*})^{-1} u_* u^* \jmath_{M,*} = (\jmath_{M,*})^{-1}(\jmath_{M,*}(-) \otimes \lambda_{-1}(\mathcal{F}^\vee))$ by corollary 2.2.5. Hence $(t_0 u)_* \left[\mathcal{O}_T^{\mathrm{vir}}\right] = (t_0 u)_* (t_0 u)^! \left[\mathcal{O}_M^{\mathrm{vir}}\right] = (\jmath_{M,*})^{-1}(\lambda_{-1}(\mathcal{F}^\vee))$.

By [Lur19, Corollary 25.2.3.3], as $\mathcal{F}^\vee$ is flat over $\mathcal{O}_M$ so are its exterior powers $\bigwedge^n(\mathcal{F}^\vee)$. In particular, by [TV08, Proposition 2.2.2.5. (4)] they are strong $\mathcal{O}_M$-modules, meaning that $\pi_i(\bigwedge^n \mathcal{F}^\vee) \simeq \pi_i(\mathcal{O}_M) \otimes_{\pi_0(\mathcal{O}_M)} \pi_0(\bigwedge^n \mathcal{F}^\vee)$ for all natural integers i and n, and we conclude that

$$
\begin{aligned}
(t_0 u)_* \left[\mathcal{O}_T^{\mathrm{vir}}\right] &= \sum_{i \geq 0} (-1)^i \sum_{n \geq 0} (-1)^n \left[\pi_i \left(\bigwedge\nolimits^n \mathcal{F}^\vee\right)\right] \\
&= \sum_{i \geq 0} (-1)^i [\pi_i(\mathcal{O}_M)] \otimes \sum_{n \geq 0} (-1)^n \left[\bigwedge\nolimits^n \pi_0(\mathcal{F}^\vee)\right]
\end{aligned}
\tag{38}
$$

as required. □

Remark 4.3.2. In the setting of the quantum Lefschetz principle, the only cases in which $\mathbb{E}_{g,n}$ is a vector bundle are when E is convex, that is $\mathbb{R}^1 p_* f^* \mathcal{E} = 0$ for any stable map $(p\colon C \to S, f\colon C \to X)$ from a rational curve C, and thus the genus is $g = 0$, which is the setting in which the quantum Lefschetz principle is already known. We conclude that it is not possible to relax the hypotheses for the quantum Lefschetz principle in G_0-theory, and that the more general version is thus only valid in its categorified form.

One may also notice that as the cotangent complex of u is $p_* \mathrm{ev}^* \mathcal{E}^\vee[1]$, which has Tor-amplitude in $[-2, 0]$ (in fact $[-2, -1]$) unless the above conditions are satisfied, so that u is not quasi-smooth and the virtual pullback along it cannot be defined.

Corollary 4.3.3. *If $\mathbb{E}_{0,n} = p_{0,n;*} \mathrm{ev}_{0,n}^* E$ is a vector bundle (that is if E is convex), the G_0-theoretic quantum Lefschetz formula of theorem A holds:*

$$
(t_0 u)_* \sum_{i_* \gamma = \beta} \left[\mathcal{O}^{\mathrm{vir}}_{\mathscr{M}_{0,n}(Z,\gamma)}\right] = \left[\mathcal{O}^{\mathrm{vir}}_{\mathscr{M}_{0,n}(X,\beta)}\right] \otimes \lambda_{-1}(\pi_0 p_{0,n;*} \mathrm{ev}_{0,n}^* \mathcal{E}^\vee).
\tag{39}
$$

□

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