Report

Derivation of an equation for the enstrophy of a three dimensional flow

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1 Derivation of the linearized enstrophy equation

The vorticity $\vec{\omega}(t, \vec{x})$ of a flow velocity field $\vec{u}(t, \vec{x})$ of components u (x-direction), v (y-direction), and w (z-direction), is defined as its curl $\vec{\omega} = \nabla \times \vec{u}$, that is

$$\omega_x = \partial_x w - \partial_z v \tag{1}$$
$$\omega_y = \partial_z u - \partial_x w \tag{2}$$
$$\omega_z = \partial_x v - \partial_y u. \tag{3}$$

The incompressible, viscous, nonlinear vorticity equation in non-dimensional form is [7, Appendix C.]

$$\partial_t \vec{\omega} + \underbrace{(\vec{u} \cdot \nabla)\vec{\omega}}_{\text{Convection}} = \underbrace{(\vec{\omega} \cdot \nabla)\vec{u}}_{\text{Vortex stretching}} + \underbrace{Re^{-1}\nabla^2\vec{\omega}}_{\text{Diffusion}}, \tag{4}$$

where all quantities have been normalized with reference quantities and, consequently, the flow control parameter is the Reynolds number $Re = U_{\text{ref}} L_{\text{ref}}/\nu$ (ν is the kinematic viscosity).

Because we will be studying perturbations on the fluid flow, we will decompose the vorticity field $\vec{\omega}$ and the velocity field $\vec{v}$ into the sum of both a stationary basic flow and a perturbation component, where the tilde indicates that a quantity is the result of a perturbation:

$$\vec{\omega}_i = \vec{\Omega}_i + \tilde{\omega}_i \qquad \vec{v}_i = \vec{U}_i + \tilde{v}_i. \tag{5}$$

We will consider parallel flows that are streamwise-directed and homogeneous in x and z so that $\vec{U}(\vec{x}) = U(y)\vec{e}_x$ and $\vec{\Omega}(\vec{x}) = \nabla \times \vec{U} = \Omega(y)\vec{e}_z = -U'\vec{e}_z$.

Substituting into equation 4 gives us the following, (in Einstein's notation) :

$$\partial_t(\vec{\Omega}_i + \tilde{\omega}_i) = (\vec{\Omega}_j + \tilde{\omega}_j)\frac{\partial(\vec{U}_i + \tilde{v}_i)}{\partial x_j} - (\vec{U}_j + \tilde{v}_j)\frac{\partial(\vec{\Omega}_i + \tilde{\omega}_i)}{\partial x_j} + \frac{1}{Re}\frac{\partial^2}{\partial x_j^2}(\vec{\Omega}_j + \tilde{\omega}_j). \tag{6}$$

Because the basic flow U is an exact solution of the Navier-Stokes equations, the following equation for the basic vorticity holds:

$$\partial_t \vec{\Omega}_i = \vec{\Omega}_j \frac{\partial \vec{U}_i}{\partial x_j} - U_j \frac{\partial \vec{\Omega}_i}{\partial x_j} + \frac{1}{Re}\frac{\partial^2 \vec{\Omega}_j}{\partial x_j^2}. \tag{7}$$

Subtracting this from equation 6 gives us the following nonlinear equation for a perturbation in a 3-dimensional flow.

$$\partial_t \tilde{\omega}_i = \vec{\Omega}_j \frac{\partial \tilde{v}_i}{\partial x_j} + \tilde{\omega}_j \frac{\vec{U}_i}{\partial x_j} - \vec{U}_j \frac{\partial \tilde{\omega}_i}{\partial x_j} - \tilde{v}_j \frac{\partial \vec{\Omega}_i}{\partial x_j} + \tilde{\omega}_j \frac{\partial \tilde{v}_i}{\partial x_j} - \tilde{v}_j \frac{\partial \tilde{\omega}_j}{\partial x_j} + \frac{1}{Re}\frac{\partial^2 \tilde{\omega}_j}{\partial^2 x_j^2}. \tag{8}$$

If we assume the perturbation to be small, we can assume that all tilde quantities are likewise small. Therefore the terms $\tilde{\omega}_j \dfrac{\partial \tilde{v}_i}{\partial x_j}$ and $\tilde{v}_j \dfrac{\partial \tilde{\omega}_i}{\partial x_j}$ are negligible, and we get the following linearized equation for the vorticity of a perturbation:

$$\partial_t \tilde{\omega}_i = \Omega_j \frac{\partial \tilde{v}_i}{\partial x_j} + \tilde{\omega}_j \frac{\partial \vec{U}_i}{\partial x_j} - \vec{U}_j \frac{\partial \tilde{\omega}_i}{\partial x_j} - \tilde{v}_j \frac{\partial \Omega_i}{\partial x_j} + \frac{1}{Re} \frac{\partial^2 \tilde{\omega}_j}{\partial^2 x_j^2}. \tag{9}$$

Now we can derive equations for each component $\partial_t \tilde{\omega}_i$ of the perturbation vorticity.

Because $\vec{\Omega}$ only has components in the z-direction, $\vec{U}$ only has components in the x-direction, and $\vec{U}$ is constant in the x and y directions, many of the above terms become zero, and therefore we get the following three equations for each component ω_i:

$$\partial_t \tilde{\omega}_x = \Omega \partial_z \tilde{u} + \tilde{\omega}_y U' - U \partial_x \tilde{\omega}_x + \frac{1}{Re} \left(\partial_x^2 \tilde{\omega}_x + \partial_y^2 \tilde{\omega}_x + \partial_z^2 \tilde{\omega}_x \right), \tag{10}$$

$$\partial_t \tilde{\omega}_y = \Omega \partial_z \tilde{v} - U \partial_x \tilde{\omega}_y + \frac{1}{Re} \left(\partial_x^2 \tilde{\omega}_y + \partial_y^2 \tilde{\omega}_y + \partial_z^2 \tilde{\omega}_y \right), \tag{11}$$

$$\partial_t \tilde{\omega}_z = \Omega \partial_z \tilde{w} - \tilde{v} \Omega' - U \partial_x \tilde{\omega}_z + \frac{1}{Re} \left(\partial_x^2 \tilde{\omega}_z + \partial_y^2 \tilde{\omega}_z + \partial_z^2 \tilde{\omega}_z \right). \tag{12}$$

The perturbation's volume enstrophy is defined as the squared size of its vorticity,

$$\tilde{\xi}(t) = \frac{1}{2} \iiint \left(\tilde{\omega}_x^2 + \tilde{\omega}_y^2 + \tilde{\omega}_z^2 \right) \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z. \tag{13}$$

The enstrophy temporal evolution is thus

$$\frac{\mathrm{d}\tilde{\xi}(t)}{\mathrm{d}t} = \frac{1}{2} \iiint \left(\partial_t \tilde{\omega}_x^2 + \partial_t \tilde{\omega}_y^2 + \partial_t \tilde{\omega}_z^2 \right) \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z. \tag{14}$$

The first object of the rest of this report will be to explicitly derive an equation for $\partial_t \xi(t)$ so the qualitative behavior of enstrophy for three-dimensional perturbations can be determined.

The linearized evolution equation per the perturbation's enstrophy in the physical space is obtained by means of a scalar product of equations 10-12 by the perturbation vorticity:

$$\begin{aligned} \partial_t \tilde{\xi} + U \partial_x \tilde{\xi} = & - U' \left[\tilde{\omega}_x \partial_z \tilde{u} + \tilde{\omega}_y \partial_z \tilde{v} + \tilde{\omega}_z \partial_z \tilde{w} \right] \\ & + \frac{1}{Re} \left[\tilde{\omega}_x \nabla^2 \tilde{\omega}_x + \tilde{\omega}_y \nabla^2 \tilde{\omega}_y + \tilde{\omega}_z \nabla^2 \tilde{\omega}_z \right] \\ & + \tilde{\omega}_x \tilde{\omega}_y U'' + \tilde{v} \tilde{\omega}_z U''. \end{aligned} \tag{15}$$

2 Transformation to the Fourier space

In the following, we will consider wave solutions within the framework of the non-modal theory. In the linear context, we can consider only individual Fourier components at a time. That is, any physical quantity $\tilde{q}$ is represented as $\tilde{q}(t,\vec{x}) = \mathcal{R}\{\hat{q}(t,y)e^{i\alpha x + i\beta z}\}$.

The enstrophy density in the Fourier space is

$$\partial_t \|\hat{\omega}_x\|^2 = 2\Re\left[i\beta\Omega\bar{\omega}_x\hat{u} + \hat{\omega}_y\omega_x U' - \cancel{i\alpha U\|\hat{\omega}_x\|^2} + \frac{1}{Re}\left(\bar{\omega}_x\partial_y^2\hat{\omega}_x - k^2\|\hat{\omega}_x\|^2\right)\right], \tag{16}$$

$$\partial_t \|\hat{\omega}_y\|^2 = 2\Re\left[i\beta\Omega\bar{\omega}_y\hat{v} - \cancel{i\alpha U\|\hat{\omega}_y\|^2} + \frac{1}{Re}\left(\bar{\omega}_y\partial_y^2\hat{\omega}_y - k^2\|\hat{\omega}_y\|^2\right)\right], \tag{17}$$

$$\partial_t \|\hat{\omega}_z\|^2 = 2\Re\left[i\beta\Omega\bar{\omega}_z\hat{w} - \cancel{i\alpha U\|\hat{\omega}_z\|^2} - \bar{\omega}_z\hat{v}\Omega' + \frac{1}{Re}\left(\bar{\omega}_z\partial_y^2\hat{\omega}_z - k^2\|\hat{\omega}_z\|^2\right)\right], \tag{18}$$

where the bar indicates the complex conjugate, the color green indicates that a term comes from the vortex-stretching component, and the color red indicates that a term comes from the convection component.

In the Fourier space, the three components of the linearized volume enstrophy equation can be written as follows, after application of the no-slip boundary conditions for the perturbation velocity field:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1}\|\hat{\omega}_x\|^2\,\mathrm{d}y = \Re\left[\int_{-1}^{1}\left(-i\beta U'\bar{\omega}_x\hat{u} + U'\bar{\omega}_x\hat{\omega}_y\right)\mathrm{d}y\right] + \frac{1}{Re}\Re\left[\bar{\omega}_x\partial_y\hat{\omega}_x\right]_{-1}^{1} + \frac{1}{Re}\int_{-1}^{1}\left(-k^2\|\hat{\omega}_x\|^2 - \|\partial_y\hat{\omega}_x\|^2\right)\mathrm{d}y, \tag{19}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1}\|\hat{\omega}_y\|^2\,\mathrm{d}y = \Re\left[\int_{-1}^{1} -i\beta U'\bar{\omega}_y\hat{v}\,\mathrm{d}y\right] + \frac{1}{Re}\int_{-1}^{1} -k^2\|\hat{\omega}_y\|^2 - \|\partial_y\hat{\omega}_y\|^2\,\mathrm{d}y \tag{20}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1}\|\hat{\omega}_z\|^2\,\mathrm{d}y = \Re\left[\int_{-1}^{1}\left(-i\beta U'\bar{\omega}_z\hat{w} + U''\bar{\omega}_z\hat{v}\right)\mathrm{d}y\right] + \frac{1}{Re}\Re\left[\bar{\omega}_z\partial_y\hat{\omega}_z\right]_{-1}^{1} + \frac{1}{Re}\int_{-1}^{1}\left(-k^2\|\hat{\omega}_z\|^2 - \|\partial_y\hat{\omega}_z\|^2\right)\mathrm{d}y. \tag{21}$$

The last term in each of the above three equations is dissipative and will always be negative, thus contributing towards a decrease of enstrophy over time. The wall terms $[\bar{\omega}_y\partial_y\hat{\omega}_i]_{-1}^{1}$ are indefinite sign and lead to either a generation or a dissipation of enstrophy. Although the second to last terms in equations 19 and 21 came from the Laplacian term of the original equation for ω_i, its sign cannot be determined. Likewise, the other terms can also be positive or negative, and so further investigation is needed in order to determine in what way enstrophy of the flow changes over time. Of note, the second to last terms in equations 19 and 21, which do not have an analogous term in equation 20 become zero due to the no-slip boundary conditions.

Now using substitutions from equations 27 and 28 , which are derived from the continuity equations and the definition of enstrophy, we can eqpress the enstrophy solely in terms of the velocity components u, v, and w, the wavenumber k, the stream flow U, the Reynolds number Re, and the y-component of the vorticity, ω_y.

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1}\|\hat{\omega}_x\|^2\,\mathrm{d}y = \Re\left[\int_{-1}^{1} U'\left(\frac{i\beta}{k^2}\partial_y^2\bar{v} + \frac{i\alpha}{k^2}\partial_y\bar{\omega}_y - i\beta\bar{v}\right)(\hat{\omega}_y - i\beta\hat{u})\,\mathrm{d}y\right] \\
+ \frac{1}{Re}\Re\left[\left(\frac{i\beta}{k^2}\partial_y^2\bar{v} + \frac{i\alpha}{k^2}\partial_y\hat{\omega}_y - \cancel{i\beta\bar{v}}\right)\left(\frac{i\beta}{k^2}\partial_y^3\hat{v} + \frac{i\alpha}{k^2}\partial_y^2\hat{\omega}_y - \cancel{i\beta\partial_y\hat{v}}\right)\right]_{-1}^{1} \\
+ \frac{1}{Re}\int_{-1}^{1}\Bigg(-k^2\left(\frac{i\beta}{k^2}\partial_y^2\hat{v} + \frac{i\alpha}{k^2}\partial_y\hat{\omega}_y - i\beta\hat{v}\right)\left(\frac{i\beta}{k^2}\partial_y^2\bar{v} + \frac{i\alpha}{k^2}\partial_y\bar{\omega}_y - i\beta\bar{v}\right) \\
- \left(\frac{i\beta}{k^2}\partial_y^3\hat{v} + \frac{i\alpha}{k^2}\partial_y^2\hat{\omega}_y - i\beta\partial_y\hat{v}\right)\left(\frac{i\beta}{k^2}\partial_y^3\bar{v} + \frac{i\alpha}{k^2}\partial_y^2\bar{\omega}_y - i\beta\partial_y\bar{v}\right)\Bigg)\,\mathrm{d}y,
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1}\|\hat{\omega}_z\|^2\,\mathrm{d}y = \Re\left[\int_{-1}^{1}(\hat{v}U'' - i\beta U'\hat{w})\left(i\alpha\bar{v} - \frac{i\alpha}{k^2}\partial_y^2\bar{v} + \frac{i\beta}{k^2}\partial_y\bar{\omega}_y\right)\mathrm{d}y\right] \\
+ \frac{1}{Re}\Re\left[\left(\frac{-i\alpha}{k^2}\partial_y^2\bar{v} + \frac{i\beta}{k^2}\partial_y\bar{\omega}_y + \cancel{i\alpha\bar{v}}\right)\left(\frac{-i\alpha}{k^2}\partial_y^3\hat{v} + \frac{i\beta}{k^2}\partial_y^2\hat{\omega}_y + \cancel{i\alpha\partial_y\hat{v}}\right)\right]_{-1}^{1} \\
+ \frac{1}{Re}\int_{-1}^{1}\Bigg(-k^2\left(i\alpha\hat{v} - \frac{i\alpha}{k^2}\partial_y^2\hat{v} + \frac{i\beta}{k^2}\partial_y\hat{\omega}_y\right)\left(i\alpha\bar{v} - \frac{i\alpha}{k^2}\partial_y^2\bar{v} + \frac{i\beta}{k^2}\partial_y\bar{\omega}_y\right) \\
- \left(i\alpha\partial_y\hat{v} - \frac{i\alpha}{k^2}\partial_y^3\hat{v} + \frac{i\beta}{k^2}\partial_y^2\hat{\omega}_y\right)\left(i\alpha\partial_y\bar{v} - \frac{i\alpha}{k^2}\partial_y^3\bar{v} + \frac{i\beta}{k^2}\partial_y^2\bar{\omega}_y\right)\Bigg)\,\mathrm{d}y.
\end{aligned} \tag{23}
$$

Terms with $\hat{v}, \partial_y\hat{v}$, and $\hat{\omega}_y$ and their complex conjugates are zero when evaluated at the boundary, because by the no-slip condition, all velocity components are zero at the boundary. By this property and the equations contained in 25, we also know that $\partial_y\hat{v}$ and $\partial_y\bar{v}$ are zero at the boundary, and so the terms above with a slash are zero.

3 Extending to the case of the plane Couette and Poiseuille flows

This formulation for linearized enstrophy holds for the general case of the flow of an incompressible, viscous fluid. In the case of Couette flow $U(y) = y$ and so $U' = 1$ and $U'' = 0$. In the case of the plane Poiseuille flow, instead $U(y) = 1 - y^2$, $U' = -2y$, and $U'' = -2$.

4 The relationship between enstrophy and kinetic energy

Now we can write the enstrophy in terms of the kinetic energy E, where

$$E = \frac{1}{2}\int_{-1}^{1} \left(\|\hat{u}\|^2 + \|\hat{v}\|^2 + \|\hat{w}\|^2\right) \mathrm{d}y = \frac{1}{2k^2}\int_{-1}^{1} \left(\|\partial_y \hat{v}\|^2 + k^2\|\hat{v}\|^2 + \|\hat{\omega}_y\|^2\right) dy. \tag{24}$$

By using the following equations derived from the continuity equations (see [1] and the definition 2),

$$\hat{u} = \frac{i}{k^2}(\alpha\partial_y\hat{v} - \beta\hat{\omega}_y), \tag{25}$$

$$\hat{w} = \frac{i}{k^2}(\beta\partial_y\hat{v} + \alpha\hat{\omega}_y), \tag{26}$$

and by substituting these values for $\hat{u}$ and $\hat{w}$ into equations 1 and 3, we get the following equations for enstrophy within the Fourier space:

$$\hat{\omega}_x = \frac{i\beta}{k^2}\partial_y^2\hat{v} + \frac{i\alpha}{k^2}\partial_y\hat{\omega}_y - i\beta\hat{v}, \tag{27}$$

$$\hat{\omega}_z = i\alpha\hat{v} - \frac{i\alpha}{k^2}\partial_y^2\hat{v} + \frac{i\beta}{k^2}\partial_y\hat{\omega}_y. \tag{28}$$

Using the following expressions for the square of the magnitude of each enstrophy component, we can then substitute these values into the equation for enstrophy in the Fourier space.

$$\begin{aligned}\|\hat{\omega}_x\|^2 = \hat{\omega}_x\bar{\omega}_x &= \left(\frac{i\beta}{k^2}\partial_y^2\hat{v} + \frac{i\alpha}{k^2}\partial_y\hat{\omega}_y - i\beta\hat{v}\right)\left(-\frac{i\beta}{k^2}\partial_y^2\bar{v} - \frac{i\alpha}{k^2}\partial_y\bar{\omega}_y + i\beta\bar{v}\right) \\ &= \frac{\beta^2}{k^4}\|\partial_y^2\hat{v}\|^2 + 2\Re\left[\frac{\alpha\beta}{k^4}\partial_y^2\bar{v}\partial_y\hat{\omega}_y\right] - \frac{\beta^2\bar{v}\partial_y^2\hat{v}}{k^2} + \frac{\alpha^2\|\partial_y\hat{\omega}_y\|^2}{k^4} - 2\Re\left[\alpha\beta\partial_y\hat{\omega}_y\bar{v}\right] - \frac{\beta^2\hat{v}\partial_y^2\bar{v}}{k^2} + \beta^2\|\hat{v}\|^2,\end{aligned} \tag{29}$$

$$\begin{aligned}\|\hat{\omega}_z\|^2 = \hat{\omega}_z\bar{\omega}_z &= \left(i\alpha\hat{v} - \frac{i\alpha\partial_y^2\bar{v}}{k^2} + \frac{i\beta\partial_y\hat{\omega}_y}{k^2}\right)\left(-i\alpha\bar{v} + \frac{i\alpha\partial_y^2\hat{v}}{k^2} - \frac{i\beta\partial_y\hat{\omega}_y}{k^2}\right) \\ &= \alpha^2\|\hat{v}\|^2 - \frac{\alpha^2\hat{v}\partial_y^2\bar{v}}{k^2} + 2\Re\left[\frac{\alpha\beta\hat{v}\partial_y\bar{\omega}_y}{k^2}\right] - \frac{\alpha^2\bar{v}\partial_y^2\hat{v}}{k^2} + \frac{\alpha^2\|\partial_y^2\hat{v}\|^2}{k^4} - 2\Re\left[\alpha\beta\partial_y\hat{\omega}_y\partial_y^2\hat{v}\right] + \frac{\beta^2\|\partial_y\hat{\omega}_y\|^2}{k^4}.\end{aligned} \tag{30}$$

Eventually, the wave enstrophy becomes

$$\hat{\xi} = \frac{1}{2k^2}\int_{-1}^{1} \left(\|\partial_y^2\hat{v}\|^2 - 2\Re[\bar{v}\partial_y^2\hat{v}] + \|\partial_y\hat{\omega}_y\|^2 + k^4\|\hat{v}\|^2 + k^2\|\hat{\omega}_y\|^2\right) dy. \tag{31}$$

Using the equality $\Re\left[\bar{v}\partial_y^2\hat{v}\right] = \Re\left[\partial_y(\bar{v}\partial_y\hat{v}) - \|\partial_y\hat{v}\|^2\right]$, we arrive at the following equation:

$$\frac{\hat{\xi}}{k^2} = \frac{1}{2k^2}\int_{-1}^{1}\left(\frac{\|\partial_y^2\hat{v}\|^2}{k^2} + 2\|\partial_y\hat{v}\|^2 + \frac{\|\partial_y\hat{\omega}_y\|^2}{k^2} + k^2\|\hat{v}\|^2 + \|\hat{\omega}_y\|^2\right)dy - 2\Re\left[\cancel{\int_{-1}^{1}\partial_y(\bar{v}\partial_y\hat{v})dy}\right]. \tag{32}$$

The last term of 32 reduces to $-2\Re\left[\bar{v}\partial_y\hat{v}\right]_{-1}^{1}$, and because at least one of $\hat{v}$ or $\bar{v}$ is zero at the boundary, the entire term becomes zero. Using equation 24, we can see that this implies

$$\frac{\hat{\xi}}{k^2} = E + \frac{1}{2k^2}\int_{-1}^{1}\left(\frac{\|\partial_y^2\hat{v}\|^2}{k^2} + \frac{\|\partial_y\hat{\omega}_y\|^2}{k^2} + \|\partial_y\hat{v}\|^2\right)dy. \tag{33}$$

5 Final remarks

This work could be further continued by using the method of Synge [5], [6], [4], [2, Chap.3], [3], and [1], as well as substitutions from the Squire and Orr-Sommerfeld equations, in order to determine conditions for the monotonic decay of the perturbation's enstrophy, which would provide useful information on the potential onset of nonlinear coupling and on conditions which can lead to the transition to turbulence.

By inspiration from the 2-dimensional case, Synge's method can be applied to the set of equations 19-21. Of note, equations 19 and 21 differ in structure from 20 because they contain nonlinear boundary terms which depend only on the vorticity fluctuation. Substituting 20 as well as 22 and 23 (the simplified versions of 19 and 21) into equation 14 gives us an equation for enstrophy in only two variables, v and ω_y, which can be used to determine the monotonic decay region for the integral enstrophy in the three-dimensional situation.

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