

2. Regressions

Suppose we are interested in the connection between

- an outcome variable y (e.g. job satisfaction, engagement, ...)
- and a variable x which may affect y (e.g. wage, the size of bonus payments, whether the firm uses performance pay or not, ...)

Let e be a variable which describes all other determinants of y that we do not observe

Then we can denote the relationship between y and x as

$$y = f(x, e) \tag{1}$$

Key aim: Understand this function and learn about it by analyzing data

Distinction: Prediction and Causality

(i) Prediction

- Question: to what extent does knowing x allow us to *predict* y ?
- Example:
 - When we as observers see that a company uses performance pay
 - What can we predict about the job satisfaction of its employees?
 - In other words: Is employee satisfaction higher in firms that use performance pay?

(ii) Causality

- Question: to what extent does a change of x *lead to* a change of y ?
- Example:
 - A firm introduced performance pay
 - We want to know how this affected employee satisfaction
 - In other words: Did the change in performance pay *cause* a change in employee satisfaction?

These are different questions!

Further examples:

- *Education and wages*

The fact that more educated people earn more does not tell us that education causes higher earnings

- *Gender diversity and performance*

The fact that successful firms employ more women on boards does not tell us that a higher share of women causes a higher performance

Note:

- Answering the first (prediction) is typically substantially simpler than answering the second (causality)
- In the public debate (and also still in some fields in academia) these questions are often confounded
- We will start by thinking about the first question and then move to the second

The key idea of the following:

- Question: Why are regressions so important in empirical research?
- Answer:
 - Because they provide useful approximations to *conditional expectation functions*
 - And *conditional expectation functions* are a powerful tool to predict outcomes
- But:

Without further ingredients they do not automatically detect causal relationships

2.1 The Conditional Expectation Function

- Think of X_i and Y_i as random variables (where X_i may be a vector)
- We are interested in the *conditional expectation function* (CEF) of Y_i given X_i in the population

$$E[Y_i|X_i]$$

- Useful interpretation:

Think of $E[Y_i|X_i]$ as a function stating the mean of Y_i among all people who share the same value(s) of X_i

- If Y_i is discrete and takes values out of a set T

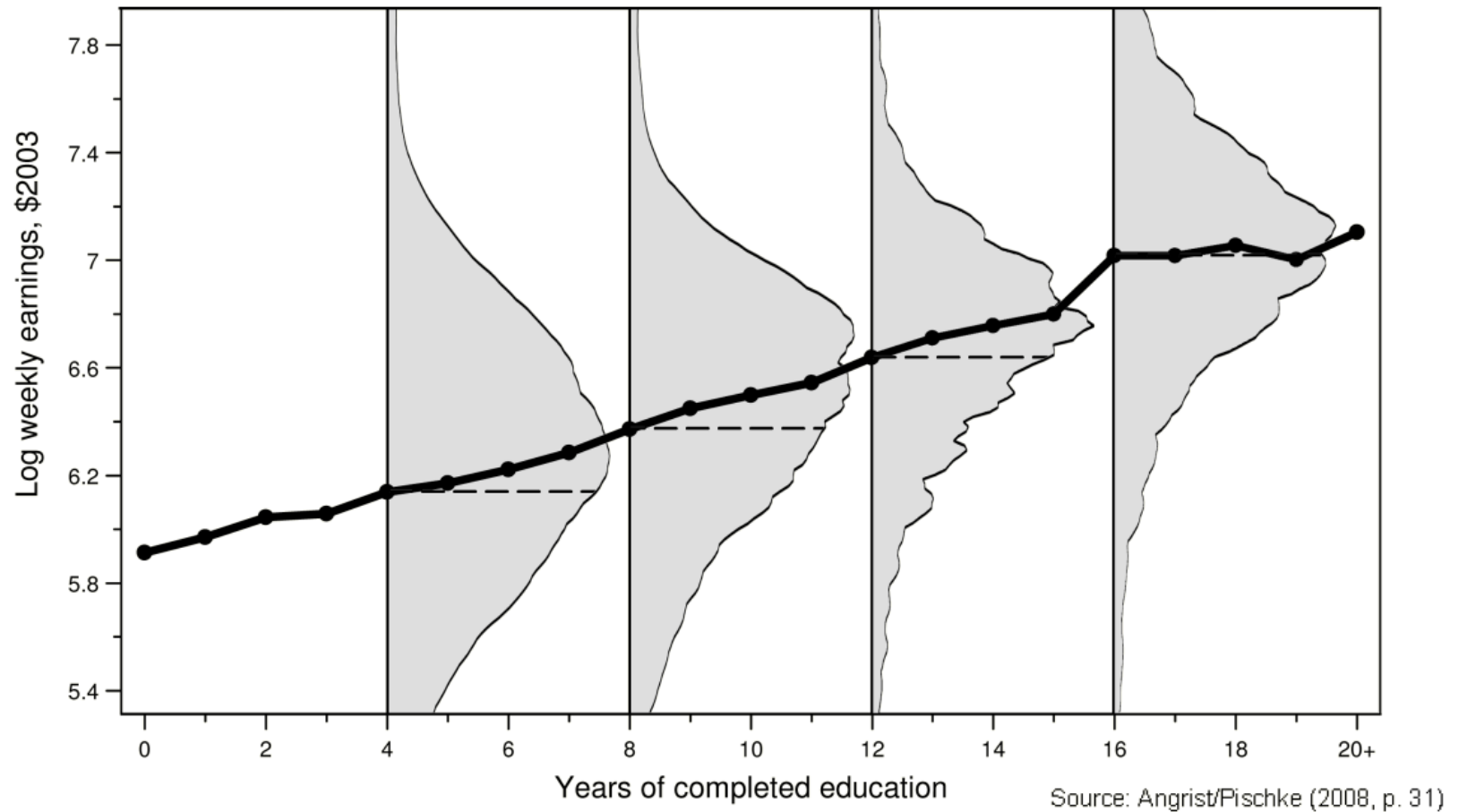
$$E[Y_i|X_i = x] = \sum_{t \in T} \Pr(Y_i = t|X_i = x) \cdot t$$

where $\Pr(Y_i = t|X_i = x)$ is the conditional probability that $Y_i = t$ when $X_i = x$

Distinguish:

- *Population*: Complete group of potential observations for our question (for example: all working age people living in Germany, all US firms, ...)
- *A sample*: the observations that we can use for our research
 - employees who take part in a survey study like the GSOEP or LPP
 - set of firms for which we have information on management practices
 - subjects taking part in an experiment
- We can estimate the population CEF from a representative sample
 - If we for instance observe pairs (Y_i, X_i) for $i = 1, \dots, n$
 - We can estimate the conditional expectation of Y_i for a specific value of $X_i = x$ by taking the average of Y_i across observations with $X_i = x$

Example: The CEF of earnings as a function of years of education



- It is often useful to visualize data with graphs
- Particularly useful: package *Seaborn* (`import seaborn as sns`)

Examples:

- `sns.barplot(x='country', y='income', data=df)`
 - Plots one bar for each realization of x with height equal to mean of y
 - Note: illustrates the estimated CEF for categorical variables
 - Adds confidence bands: from all samples that can be drawn, the confidence interval will contain the true population mean in 95% of the cases
- `sns.relplot(x='income', y='happiness', data=df)`
 - Scatter plot where each dot is a data point
- `sns.histplot(df['wage'])`
 - Plots histogram of the variable
 - Note: `df['x']` returns a series of all observations of variable x

Your Task

Feedback Talks and Employee Engagement

- Let us use the LPP to study the association between the use of feedback/appraisal interviews and employee engagement
- Open again the notebook `LPPanalysis.ipynb`
- Import further modules
 - `import statsmodels.api as sm`
 - `import statsmodels.formula.api as smf`
 - `import seaborn as sns`
- To estimate the CEF, simply compare the mean of job engagement between employees who had an appraisal/feedback interview and those who didn't
 - Use the `enga_std` variable you generated before
 - `mmagespr` is a dummy variable which is equal to 1 if the employee had an appraisal interview and 0 otherwise
 - Note: To do this, it is convenient to use the `groupby` method
Syntax (adapt!): `df.groupby(df.country).wage.describe()`
- Visualize the CEF with a barplot
Adapt: `sns.barplot(x='country', y='income', data=df)`
- Save the notebook

Two key results (for the proofs see Angrist/Pischke (2009, pp. 32-33))

Result: CEF Decomposition Property

We can decompose Y_i such that $Y_i = E[Y_i|X_i] + \varepsilon_i$

(i) where ε_i is mean independent of X_i , that is $E[\varepsilon_i|X_i] = 0$

(ii) and therefore, ε_i is uncorrelated with any function of X_i

- Therefore: A random variable Y_i can be decomposed into a piece that is “explained by X_i ” (the Conditional Expectation Function) and a piece that remains unexplained by any function of X_i
- In the example: We can decompose the wage of a person
 - in a piece that is “explained” by education (i.e. the CEF)
 - and piece that is left over
 - and this latter piece is uncorrelated with (“orthogonal to”) any function of education

Result: CEF Prediction Property

Let $m(X_i)$ be any function of X_i . The CEF solves

$$E[Y_i|X_i] = \arg \min_{m(X_i)} E[(Y_i - m(X_i))^2]$$

so it is the best predictor of Y_i given X_i in the sense that it solves the minimum mean square error (MMSE) prediction problem.

- The CEF is a very useful predictor: If I observe other related variables and “plug them into the CEF”, the value of the CEF comes close to the true value of the outcome variable
- We want a function (call it $m(X_i)$) that gives us a good prediction for Y_i
$$\hat{Y}_i = m(X_i)$$
- Important criterion: The distance between \hat{Y}_i and Y_i should be small
- The result now states: When we use the quadratic distance $(Y_i - m(X_i))^2$, then the CEF is the best function we can find

Therefore:

- The CEF provides a natural summary of empirical relationships
 - It gives the population average of Y_i for the group of people having the same X_i
 - It describes the best (MMSE) predictor of Y_i given X_i
 - It allows to decompose variance in the data (see appendix)
- If I know the CEF, I can make predictions which value Y_i would take for different values of X_i
(Note: in the population; not in the sense of a causal change in Y_i because of a change of X_i !)

But: What is connection between the CEF and regression analysis and machine learning?

- In the following: regressions and other machine learning algorithms are tools to approximate the CEF

2.2 Regression and Conditional Expectations

- Typically, we will not know the functional form of the CEF when Y is a continuous variable
- But we can try to approximate it
- Start with simple case of two variables and consider the linear function

$$Y_i = \beta_0 + \beta_1 X_i$$

- Now determine β_0 and β_1 such that

$$(\beta_0, \beta_1) = \arg \min_{b_0, b_1} E[(Y_i - b_0 - b_1 X_i)^2]$$

- Let us call this the *Population Regression Function (PRF)*
- Of all possible linear functions of X_i – which one gives us the least (quadratic) deviation from Y_i in expected terms?

$$(\beta_0, \beta_1) = \underset{b_0, b_1}{\operatorname{argmin}} E[(Y_i - b_0 - b_1 X_i)^2]$$

First order conditions

$$\begin{aligned} E[2(Y_i - b_0 - b_1 X_i)] &= 0 \\ E[2(Y_i - b_0 - b_1 X_i)X_i] &= 0 \end{aligned}$$

Hence,

$$\begin{aligned} b_0 &= E[Y_i] - b_1 E[X_i] \\ b_1 E[X_i^2] &= E[X_i Y_i] - b_0 E[X_i] \end{aligned}$$

such that

$$\begin{aligned} b_1 &= \frac{E[Y_i X_i]}{E[X_i^2]} - (E[Y_i] - b_1 E[X_i]) \frac{E[X_i]}{E[X_i^2]} \\ \Leftrightarrow b_1 &= \frac{E[Y_i X_i] - E[Y_i]E[X_i]}{E[X_i^2] - (E[X_i])^2} \end{aligned}$$

Hence, in the **bivariate case**

$$\beta_1 = \frac{E[Y_i X_i] - E[Y_i]E[X_i]}{E[X_i^2] - (E[X_i])^2} = \frac{Cov[Y_i, X_i]}{V[X_i]}$$

- This is the population version of OLS regression for the bivariate case

We can do the same in the **multivariate case**

- We can approximate the CEF with a multivariate linear function

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \dots + \beta_k X_{ik}$$

- Proceeding analogously to the bivariate case, we obtain
(then β and X_i are vectors)

$$\beta = E[X_i X_i']^{-1} E[X_i Y_i]$$

From a Sample to the Population

- So far, we spoke about whole populations but in reality, we (typically) do not know the population parameters
- We work with samples (subsets) of a population, but we want to say something about the population
- That is we want to estimate the population parameters β using a sample
- And we want to have an idea how good these estimates are

We want to

- obtain the estimated coefficients $\hat{\beta}$
- and learn about the precision of these estimates

The Bivariate Case: We want to estimate the parameter $\beta_1 = \frac{Cov[Y_i, X_i]}{V[X_i]}$

- We have a sample of size N and thus observe (Y_i, X_i) for $i = 1, \dots, N$
- We can estimate
 - $Cov[Y_i, X_i]$ by the sample covariance $\frac{1}{N} \sum_{i=1}^N (X_i - \bar{X})(Y_i - \bar{Y})$
 - $V[X_i]$ by the sample variance $\frac{1}{N} \sum_{i=1}^N (X_i - \bar{X})^2$
- And this leads to the OLS estimator $\hat{\beta} = \frac{\frac{1}{N} \sum_{i=1}^N (X_i - \bar{X})(Y_i - \bar{Y})}{\frac{1}{N} \sum_{i=1}^N (X_i - \bar{X})^2}$

Multivariate Case: We want to estimate $\beta = E[X_i X_i']^{-1} E[X_i Y_i]$

- We observe (Y_i, X_i') for $i = 1, \dots, N$, that is
 - $(Y_1, X_{10}, X_{11}, X_{12}, \dots, X_{1K-1})$,
 - $(Y_2, X_{20}, X_{21}, X_{22}, \dots, X_{2K-1}), \dots$
- We can estimate $E[X_i X_i']$ by $\frac{1}{N} \sum_{i=1}^N X_i X_i'$ and $E[X_i Y_i]$ by $\frac{1}{N} \sum_{i=1}^N X_i Y_i$
- And this leads to the OLS estimator $\hat{\beta} = \left[\sum_{i=1}^N X_i X_i' \right]^{-1} \sum_{i=1}^N X_i Y_i$

- We can use the module statsmodels & it is convenient to use “formulas”
- Import: `import statsmodels.formula.api as smf`
- If you have a DataFrame df containing variables y, x1 and x2 and you want to regress y (dependent variable) on x1 and x2 (indep. variables):
`reg = smf.ols('y ~ x1 + x2', data=df).fit()`
- And show the results with
`print(reg.summary())`
- Note: one can also directly get nice regression tables (as reported in research papers) with different specifications with `summary_col` from `statsmodels.iolib.summary2` import `summary_col`
- Example:
 - `reg1 = smf.ols('y ~ x1', data=df).fit()`
 - `reg2 = smf.ols('y ~ x1 + x2', data=df).fit()`
 - `print(summary_col([reg1, reg2], stars=True))`

- New variables can be created by `df['newvarname'] = ...`
- You can also generate new variables and compute their value as a function of existing variables:

```
df['salesPerEmp'] = df['sales']/df['emp']
```

- A Boolean variable takes values *True* or *False*
 - A condition such as `(x>5)` gives back the value `True` when it's true and `False` otherwise
- A Boolean variable can be used like a dummy variable, i.e. a variable which takes only values 0 and 1
- A dummy variable can thus be created using a condition
 - Hence, `df['dummy'] = (df['X']==5)` creates a dummy variable (column) that takes value `True` if the variable X is equal to 5

Bloom and Van Reenen (2007), Bloom and Van Reenen (2012) study survey data

- Evaluate whether differences in the use management practices can explain productivity differences between firms
- Use an interview-based evaluation tool to assess 18 basic management practices
- Run the survey in many industries and countries
- Interviewers give a score from 1-5 on the 18 practices
- Compute a management score computed from the surveys
- Study the association between
 - the management score and
 - the financial success of the companies (e.g. sales, ROCE)

Management Practice Dimensions

(examples, see Bloom und Van Reenen (2010, p. 206))

- Introduction of modern manufacturing techniques
- Rationale for introduction of modern manufacturing techniques
- Performance tracking
- Performance dialogue
- Consequence management
- Target time horizon
- Targets are stretching
- Managing human capital
- Promoting high performers
- Attracting human capital

Your Task

Association between Management Practices & Performance

- Use data from Bloom, Genakos, Sadun and Van Reenen. “Management Practices Across Firms and Countries.” The Academy of Management Perspectives, 26, no. 1 (2012): 12-33.
- Start a new notebook (you can copy the first part with the imports and adapt from the previous exercise, but save it under a different name)
- Read the data into a DataFrame
 - `path_to_data =`
`'https://raw.githubusercontent.com/dsliwka/EEMP2022/main/datasets/AMP_Data.csv'`
 - `df = pd.read_csv(path_to_data)`
- The data set for instance contains variables `management` (the management score across practices) and financial KPI `roce` (=EBIT/Capital employed)
- Type `df` to show the DataFrame
- Inspect the data set

- Inspect the data in more detail by plotting graphs, for instance use
 - `sns.histplot(df.xvar)` to plot a histogram of a variable `xvar`
 - `sns.relplot(x='xvar', y='yvar', data=df)` for a scatter plot
 - `sns.regplot(x='xvar', y='yvar', data=df)` for a scatter plot that includes a regression line
- Now run a regression of `roce` as dependent variable on management
 - Recall the syntax (adapt!):
 - ```
reg = smf.ols('yvar ~ xvar1 + xvar2',
data=df).fit()
print(reg.summary())
```
- Interpret your result
- Save your notebook as `ManagementPractices` to reuse it later

## 2.3 Dummy Variables

When  $X_i$  is a single dummy variable that only takes value 0 or 1

- Then  $E[Y_i|X_i = 0]$  is a constant and  $E[Y_i|X_i = 1]$  is another constant and the CEF is fully characterized by these constants:

$$E[Y_i|X_i] = \underbrace{E[Y_i|X_i = 0]}_{\beta_0} + X_i \cdot \underbrace{(E[Y_i|X_i = 1] - E[Y_i|X_i = 0])}_{\beta_1}$$

is a linear function of  $X_i$

- When I have precise estimates of the PRF, I have a precise estimate of  $E[Y_i|X_i]$

### Note:

- The PRF exactly describes the CEF
- Linearity is not an assumption but a fact
- This is a very common data structure, for instance in an experiment:  
 $X_i$  indicates whether somebody is in the treatment instead of the control group



- Open again the notebook `LPPanelanalysis.ipynb`
- Estimate a regression of (std.) engagement on the `mmagespr` dummy
- Compare the constant term (intercept) and coefficient of `mmagespr` with the conditional means computed in the last exercise. What do you see?
- Inspect the robustness of the connection between engagement and the use of appraisal interviews
- To do so, estimate a multivariate regression adding the following further explanatory variables (variable names in parentheses):
  - Age (`alter`)
  - Manager (dummy `mleitung`)
  - Temporary contract (dummy `mbef`)
  - Part time work (dummy `maz_voll_teil`)
  - Works from home (dummy `mheim`)
  - Training (dummy `mwb`)

## 2.4 Interaction Terms

- Sometimes we expect that the conditional expectation function  $E[Y_i | X_{i1}, X_{i2}]$  is not additively separable such that it can sensibly be approximated by a population regression  $Y_i = \alpha + \beta_1 X_{i1} + \beta_2 X_{i2}$
- Then we may want to allow for the possibility that the effect of  $X_{i1}$  depends on the size of  $X_{i2}$ , for instance
  - The effect of performance pay on job satisfaction may depend on gender
  - The effect of a training may depend on experience
- In experiments we might consider a setting in which  $X_{i1}$  is a treatment dummy and  $X_{i2}$  is a specific characteristic of a treated object and we may want to study *heterogenous treatment effects*
- For instance, the object is a
  - person and the characteristic is the age, gender, or experience
  - firm and the characteristic is the size, industry, region, ...

- When expecting that the effect of  $X_{i1}$  depends on the size of  $X_{i2}$  researchers typically estimate a regression

$$Y_i = \alpha + \beta_1 \cdot X_{i1} + \beta_2 \cdot X_{i2} + \beta_3 \cdot X_{i1} \cdot X_{i2} + \varepsilon_i$$

- We thus include an *interaction term* and approximate the CEF by a linear function from  $\mathbb{R}^2 \rightarrow \mathbb{R}$
- Note: Never forget to include both variables as well as their interaction
- If we estimate a regression of this form, the effect of  $X_{i1}$  on  $Y_i$  is approximately

$$\frac{\partial E[Y_i | X_{i1}, X_{i2}]}{\partial X_{i1}} \approx \beta_1 + \beta_3 \cdot X_{i2}$$

- $\beta_3$  thus estimates the extent to which the effect of  $X_{i1}$  depends on  $X_{i2}$

- Sometimes we want to use only a subset of the DataFrame, for instance if we want to run a regression only on a subset of the data
- Pandas has different methods for subset selection
- For instance, one could use the *indexing operator* `[]` to select columns
  - `df['age']` gives back a series that contains only column age
  - `df[['age', 'wage']]` gives a DataFrame including only columns age & wage from the initial DataFrame df
- If we put a condition in the brackets, then rows are selected that satisfy this condition
  - `df[df['age'] > 50]` gives back a DataFrame containing only rows (observations) where age is larger than 50
  - We can use `&` (for and) and `|` (for or):
  - `df[(df['age'] > 50) | (df['age'] < 30)]` gives back a DataFrame that contains only observations where age < 30 or > 50

- For categorical variables statsmodels formulae can automatically generate dummy variables for each category with the `C()` operator:

```
smf.ols('Wage ~ age + C(Region)', data=df).fit()
```

- Interaction terms can also be directly generated with `*`

```
smf.ols('Wage ~ age * female', data=df).fit()
```

- Note: when using `*` statsmodels also includes the two interacted variables separately
- Furthermore: You can use functions (from numpy) to transform variables directly in the regression equation

```
smf.ols('np.log(Wage) ~ age * female', data=df).fit()
```

- Note: the function `np.log(x)` computes the log of `x`

## Your Task

## Association between Management Practices & Performance

- Open your ManagementPractices.py file
- Research question: Is a management practice scoring that has been developed in one countries is equally predictive for performance in a country with a different culture?
- Background: the B/vR scoring has been developed in the UK
- Your task: Find out whether the management score is equally predictive for ROCE in China as compared to the UK
- First create a dummy variable `ChinaD` that indicates whether an observation is from China (inspect variable `country`)
- Then create a data frame that only includes data from the UK and China:  

```
dfn=df[(df["country"]=="China")|(df["country"]=="Great Britain")]
```
- Now rerun your regression of ROCE on management interacting management with `ChinaD` (do not forget to run it on the `dfn` DataFrame!)
- Interpret your results

## 2.5 Estimating Non-linear functions

- In some applications we have reason to believe that the CEF is non-linear
- For instance, wages may first increase in age and then decrease
- Many applied researchers then start by estimating a quadratic function

$$Y_i = \alpha + \beta_1 \cdot X_i + \beta_2 \cdot X_i^2 + \varepsilon_i$$

- Hence, we approximate the CEF with a quadratic function
- This can also be useful when we suspect that the CEF is concave or convex
- But be careful when interpreting  $\beta_1$ : this is no longer the slope parameter but

$$\frac{\partial E[Y_i|X_i]}{\partial X_i} \approx \beta_1 + \beta_2 \cdot 2X_i$$

- Sign of  $\beta_2$  estimates the second derivative of the function, as

$$\frac{\partial^2 E[Y_i|X_i]}{\partial X_i^2} \approx 2\beta_2$$

- Open again the notebook `LPPanalysis.ipynb`
- Generate a new variable `alter2` which is  $\text{alter}^2$   
To do so you can either compute `alter*alter` or `alter**2`
- Now regress engagement on `alter` and `alter2`
- How do you interpret the results?
- Hint: You can also graphically inspect the connection (but think about the interpretation first!) using

```
sns.regplot(y='enga_std', x='age', data=df,
 x_bins=10, order=2)
```

- `x_bins` specifies that not each observation is plotted as a dot but neighboring observations are averaged in bins (here 10)
- `order=2` specifies that the regression plot fits a polynomial of order 2 which is a parabola



- Sometimes researchers replace the dependent variable with its logarithm

$$\ln Y_i = \alpha + \beta \cdot X_i + \varepsilon_i$$

- Part of reason: Logs less sensitive to outliers & may reduce heteroscedasticity (→ statistical tests)
- But more importantly: logs sometimes lead to convenient interpretations
- When  $X_i$  is a dummy variable our CEF is fully captured by a regression &

$$- \ln Y_{i1} = \alpha + \beta + \varepsilon_i$$

$$- \ln Y_{i0} = \alpha + \varepsilon_i$$

- Then 
$$\beta = \ln Y_{i1} - \ln Y_{i0} = \ln \frac{Y_{i1}}{Y_{i0}}$$

- Such that 
$$\frac{Y_{i1}}{Y_{i0}} = \exp(\beta) \approx 1 + \beta$$

→ The coefficient  $\beta$  is approximately equal to the percentage change in the outcome variable (approximation is okay for small enough  $\beta$  (like  $\beta < 0.2$ ))

→ The outcome is unaffected by the units in which  $Y_i$  is measured

## Hence:

- Regression provides the best linear predictor for the dependent variable; the CEF provides the best unrestricted predictor
- Even if the CEF is non-linear, regressions provide the best linear approximation
- A/P: This *“lines up with our view of empirical work as an effort to describe essential features of statistical relationships without necessarily trying to pin them down exactly”*
- Furthermore
  - Imposing linearity reduces complexity
  - A linear function is summarized in a few parameters that often have accessible interpretations
- But: there is danger of oversimplification
  - Other machine learning techniques allow to relax assumption of linearity or on specific functional forms
  - May allow to come closer to the true CEF in complex data