Finite element -- Continuity of linear functional, coercivity of bilinear form, Uniqueness and Stability

$$a(u, v) = l(v) \quad \text{for all } v \in X_v$$

$$u \in X_u$$

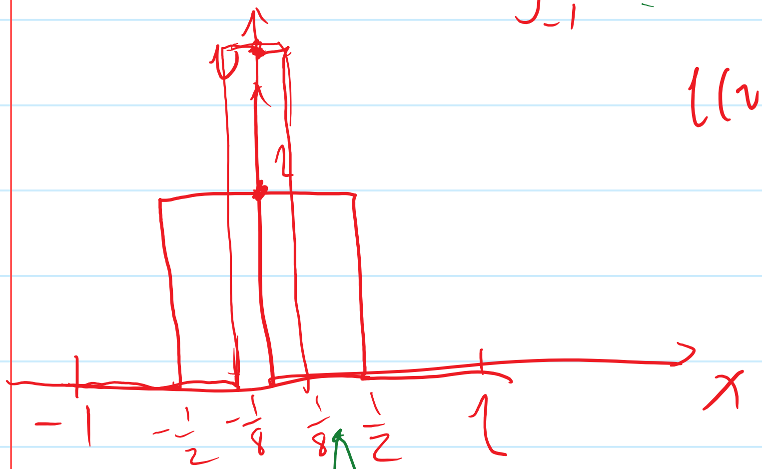
$l(v)$ is continuous iff

$$\|l\|_{X_v} = \sup_{v \in X_v} \frac{|l(v)|}{\|v\|_{X_v}} < \infty$$

or equivalently $\exists\, C$, s.t $\dfrac{\|l(v)\|}{\|v\|_{X_v}} < C \quad \forall v$

example $\delta(x)$ is a linear functional

$$l(v) = \int_{-1}^{1} \delta(x) \cdot v(x)\, dx$$

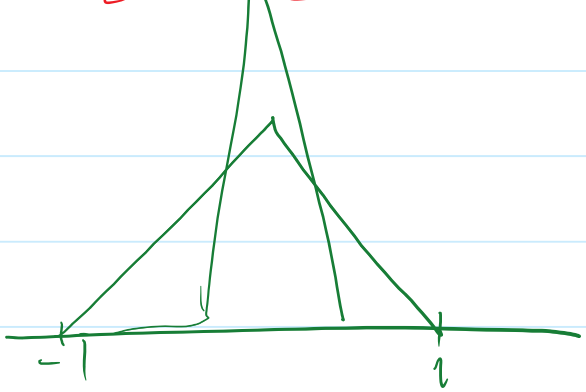


$l(v)$ is not continuous in L^2.

is continuous in H^1.

$$\|v\|_{H^1} = \int_{-1}^{1} \left(v^2 + \left(\frac{dv}{dx}\right)^2 \right) dx$$

if $\|v\|_{H^1} = 1$



Finite element -- Continuity of linear functional, coercivity of bilinear form, Uniqueness and Stability

Continuity of $a(u,v)$, $\exists\ C < \infty$

$$|a(u,v)| \le C \, \|u\| \cdot \|v\| \quad \text{for all } u, v$$

Coercivity: $\exists\ 0 < B < \infty$

s.t. $a(u,u) \ge B \cdot \|u\|^2$

---

well-posedness of weak form:

if u satisfying $\underline{a(u,v) = l(v) \quad \forall v}$

then $B\|u\|^2 \le a(u,u)$

$$\|u\| \le \frac{1}{B} \cdot \frac{a(u,u)}{\|u\|} = \frac{1}{B} \frac{l(u)}{\|u\|}$$

$$\le \frac{1}{B} \|l\|$$

if u_1, u_2 satisfying the same $a(u_1, v) = l(v)\ \forall v$

$a(u_2, v) = l(v)\ \forall v$

$$a(u_1 - u_2, v) = \underline{0}$$

$$\|u_1 - u_2\| \le \frac{1}{B} \cdot \|0\| = 0$$

Finite element -- Continuity of linear functional, coercivity of bilinear form, Uniqueness and Stability

① $\underline{a(u,u)} = \int_0^1 \left(\frac{du}{dx}\right)^2 dx$ with $\underline{u(0)=u(1)=0}$

$$\underline{\|u\|_{H^1}^2} = \int_0^1 \left(u^2 + \left(\frac{du}{dx}\right)^2\right) dx \le P' \cdot \int_0^1 \left(\frac{du}{dx}\right)^2 dx + \int_0^1 \left(\frac{du}{dx}\right)^2 dx$$

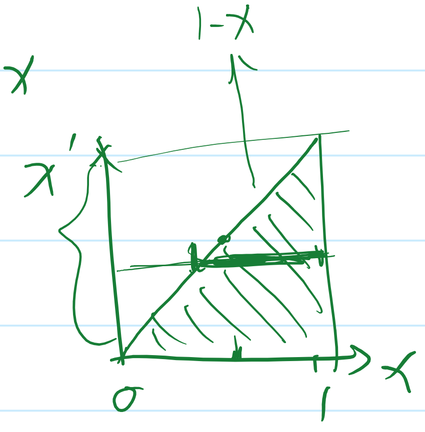
Poincare's inequality.

$$\int_0^1 (u-\bar{u})^2 dx \le P \cdot \int_0^1 \left(\frac{du}{dx}\right)^2 dx$$

$$\bar{u} = \int_0^1 u \, dx$$

$$\bar{u} = \int_0^1 \int_0^x \left(\frac{du}{dx'}\right) dx' \cdot dx$$

$$= \int_0^1 \left(\frac{du}{dx}\right)(1-x)\, dx$$



Neumann, $\frac{du}{dx} = 0$ at $x = 0, 1$

$$\underline{a(u,u)} = \int_0^1 \left(\frac{du}{dx}\right)^2 dx$$

$$\underline{\|u\|^2} = \int_0^1 u^2 + \left(\frac{du}{dx}\right)^2 dx$$

$a(u,v) = l(v)$ not well-posed

$$\int_0^1 \frac{\partial^2 u}{\partial x^2} dx = \frac{\partial u}{\partial x}\Big|_0^1 = 0 = -\int_0^1 f \, dx$$

Finite element -- Continuity of linear functional, coercivity of bilinear form, Uniqueness and Stability

$a(u,v)$ $\qquad u \in R^n$

$= u^T A v$ $\qquad v \in R^n$

$n \times n$ matrix

coercivity $\qquad u^T \underline{A} u \geq \beta \cdot \|u\| \cdot \|u\|$

$u^T U \Sigma V^T u$

Finite element -- A priori and a posteriori error estimate

$u, \quad u^h$

u satisfies $\quad a(u, v) = l(v)$ for all $v \in X$

u^h satisfies $\quad a(u^h, v^h) = l(v^h)$ for all $v^h \in X_h$

$$X_h \subset X$$

e.g $X = H^1$ continuous

$X_h = \{$ piecewise linear functions $\}$

a priori : $\|u-u^h\|$ estimated in terms of a, l, and u,

a posteriori $\|u-u^h\|$ estimated . . . a, l, $\underline{u^h}$

$$\sim \frac{\partial^4 u}{\partial x^4} \cdot \Delta x^3 \frac{1}{12} + O(\Delta x^4) \quad \text{a priori}$$

Finite element -- Minimization error in energy norm

Energy norm is $a(u,u)$ when a is symmetric

$$a(u,v) = a(v,u)$$

$$\underbrace{B\,\|u\|^2 \le}_{\text{coercitivity}} a(u,u) \underbrace{\le C\,\|u\|^2}_{\text{continuity}} \quad \text{equivalent}$$

$a(u-u^h, u-u^h)$ compared with $a(u-w^h, u-w^h)$ for another $w^h \in X^h$

$$a(u-w^h, u-w^h) = a(u-w^h, u-u^h) + a(u-w^h, u^h-w^h)$$

$$\boxed{\begin{aligned} a(u, u^h-w^h) &= l(u^h-w^h) \\ a(u^h, u^h-w^h) &= \end{aligned}}$$

$$= a(u-w^h, u-u^h) + a(u^h-w^h, u^h-w^h)$$

$$a(u-w^h, u-u^h) = a(u-u^h, u-u^h) + a(u^h-w^h, u-u^h)$$

$$= a(u-u^h, u-u^h) + \cancel{a(u^h-w^h, 0)}$$

$$a(u-w^h, u-w^h) = a(u-u^h, u-u^h) + a(u^h-w^h, u^h-w^h)$$

$$\ge a(u-u^h, u-u^h) + B\,\|u^h-w^h\|^2$$

$$a(u-u^h, u-u^h) \le a(u-w^h, u-w^h) \quad \text{for all } w^h$$

Finite element -- Interpolation and interpolation error

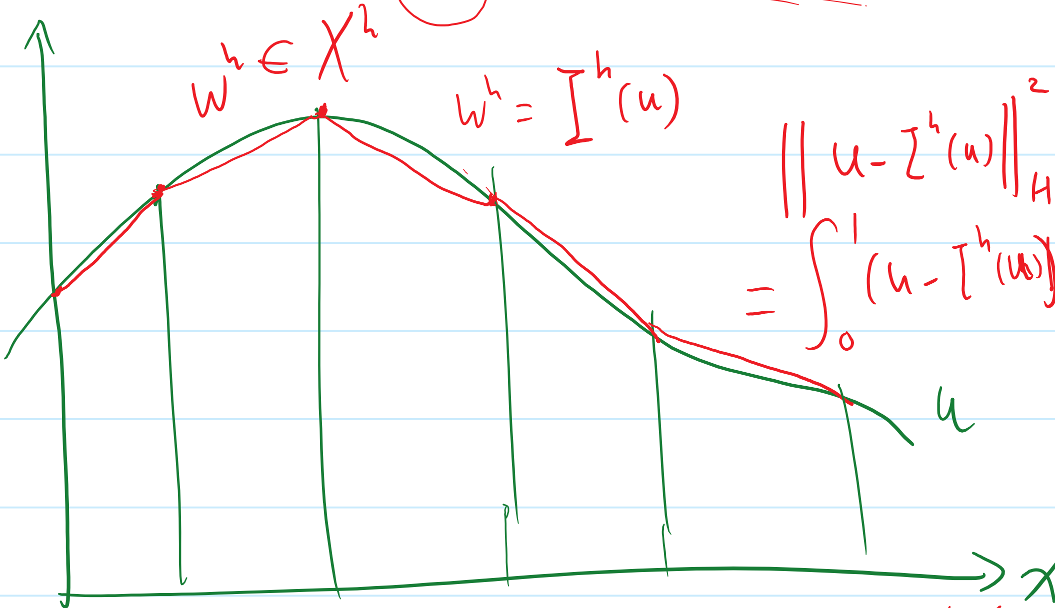
$$\|u-u^h\|_{H^1} \le A \|u-w^h\|_{H^1} \text{ for all } w^h \in X^h$$

$$\|u-u^h\|_{H^1}^2 \le \frac{1}{B} \cdot a(u-u^h, u-u^h)$$

$$\le \frac{1}{B} \cdot a(u-w^h, u-w^h)$$

$$\le \frac{C}{B} \cdot \|u-w^h\|_{H^1}^2$$

$$\|u-u^h\|_{H^1}^2 \le \frac{C}{B} \inf_{w^h \in X^h} \|u-w^h\|_{H^1}$$



$$\|u - I^h(u)\|_{H^1}^2 = \int_0^1 (u - I^h(u))^2 - \left(\frac{du}{dx} - \frac{dI^h(u)}{dx}\right)^2 dx$$

$h = \max |x_{i+1} - x_i|$

$$\frac{d I^h(u)}{dx} = \frac{u_{i+1} - u_i}{x_{i+1} - x_i}$$

$$\left.\frac{du}{dx}\right|_\xi = \frac{u_{i+1} - u_i}{x_{i+1} - x_i} \qquad x_i \le \xi \le x_{i+1}$$

$O(h)$

$$\left.\frac{du}{dx}\right|_\eta = \left.\frac{du}{dx}\right|_\xi + \left.\frac{d^2u}{dx^2}\right|_\xi (\eta - \xi) + O(\eta - \xi)^2$$

$$\left.\frac{du}{dx}\right|_{\eta} = \left.\frac{du}{dx}\right|_{\xi} + \left.\frac{d^2u}{dx^2}\right|_{\xi}(\eta-\xi) + O(\eta-\xi)^2$$

Finite element -- Error bound in H1 norm

$$\left(u - I^h(u)\right)(\xi) = \int_{x_i}^{\xi_i} \underbrace{\left(\frac{du}{dx} - \frac{dI^h(u)}{dx}\right)}_{O(h)} dx$$

$$= O(h^2)$$

$$\|u - I^h(u)\|^2 = \int \underbrace{\left(u - I^h(u)\right)^2}_{O(h^2)} + \left(\frac{d}{dx}\underbrace{\left(u - I^h(u)\right)}_{O(h)}\right)^2 dx$$

$$= O(h^2)$$

$$\|u - u^h\|^2 \le \frac{C}{\beta}\|u - I^h(u)\|^2 \le O(h^2)$$

Finite element -- Error bound in L2 norm

$$\|u-u^h\|_{L_2}^2 \le \|u-u^h\|_{H^1}^2 \sim O(h^2)$$