Grid-Based Fluids

COMS 6998 – Problem Solving for Physical Simulation

Eulerian vs. Lagrangian

Lagrangian

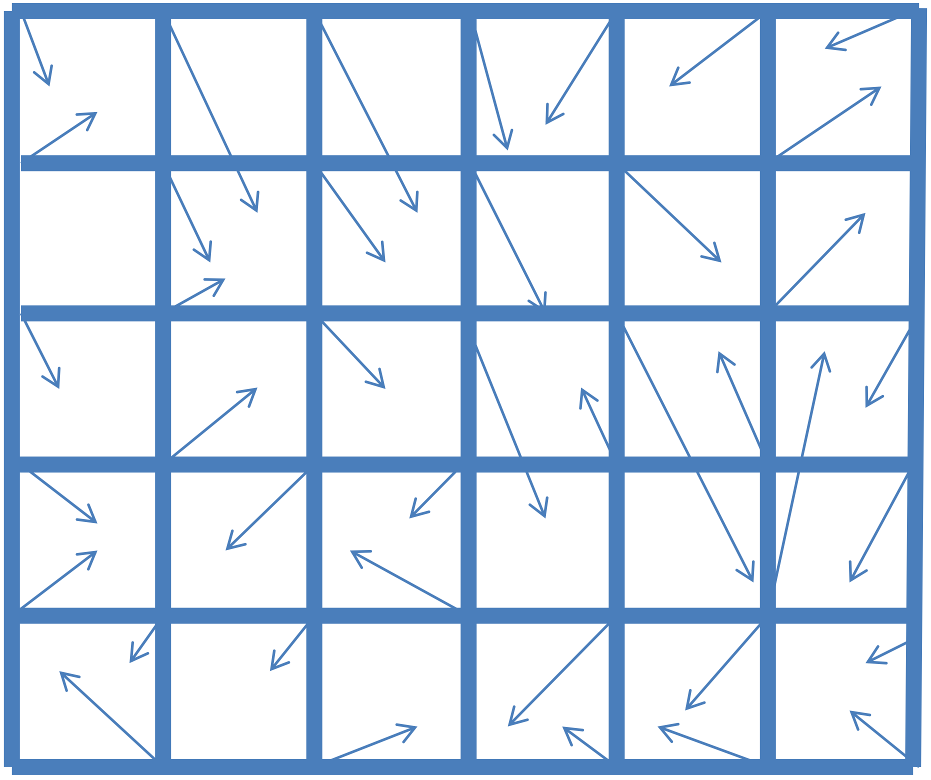
- Particles carry data samples and travel with the flow.

Eulerian

- Samples are fixed in a grid, and information flows past.

Analogy: weather station vs. weather balloon

Eulerian representation



How to evolve grid data?

Recall the material derivative.

$$\frac{D\varphi}{Dt} = \frac{\partial\varphi}{\partial t} + \boldsymbol{u} \cdot \nabla\varphi$$

For quantity $\varphi(x, t)$ under velocity field $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{t})$.

Material Derivative: Derivation

$$\frac{d}{dt}\big(\varphi(x,t)\big) = \frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial t} \qquad \text{(Chain rule)}$$

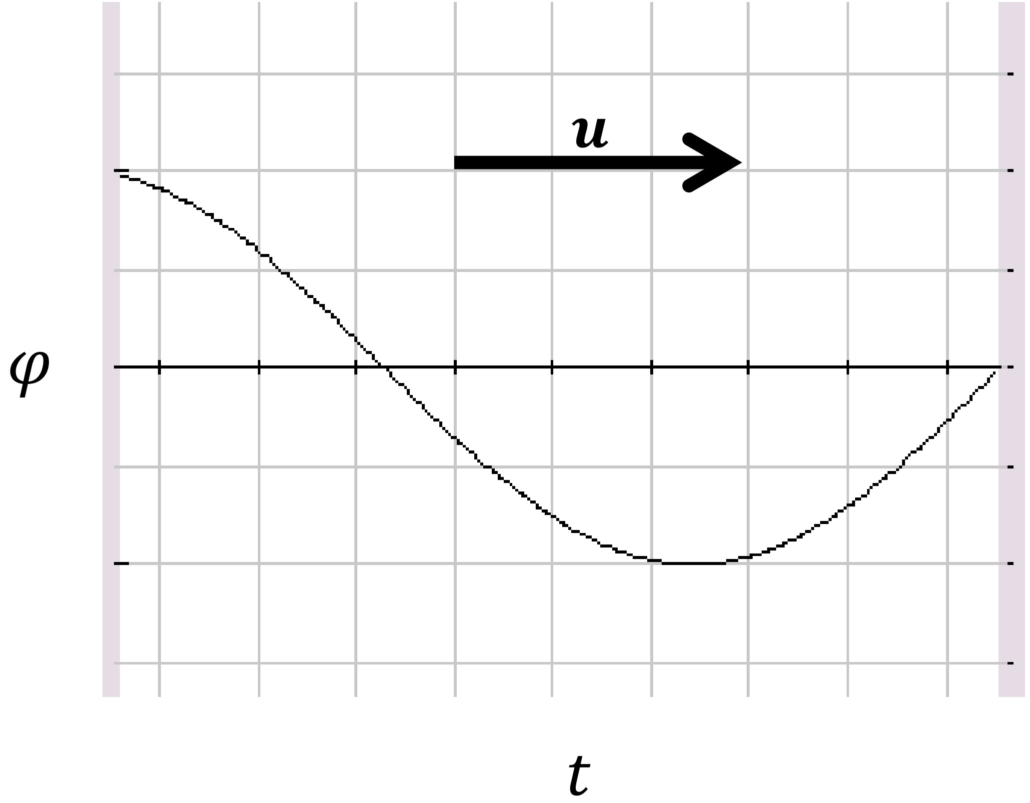
$$= \frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial x}u \qquad \text{(Definition of u)}$$

$$= \frac{\partial\varphi}{\partial t} + \nabla\varphi \cdot u \qquad \text{(Definition of gradient)}$$

$$= \frac{\partial\varphi}{\partial t} + u \cdot \nabla\varphi \qquad \text{(Rearrange)}$$

Advection term

$$\frac{D\varphi}{Dt} = \frac{\partial \varphi}{\partial t} + \boldsymbol{u} \cdot \nabla \varphi$$



Computing Advection

Different methods:

1) Particles (Lagrangian)
2) Particle-in-cell
3) Semi-Lagrangian
4) Eulerian

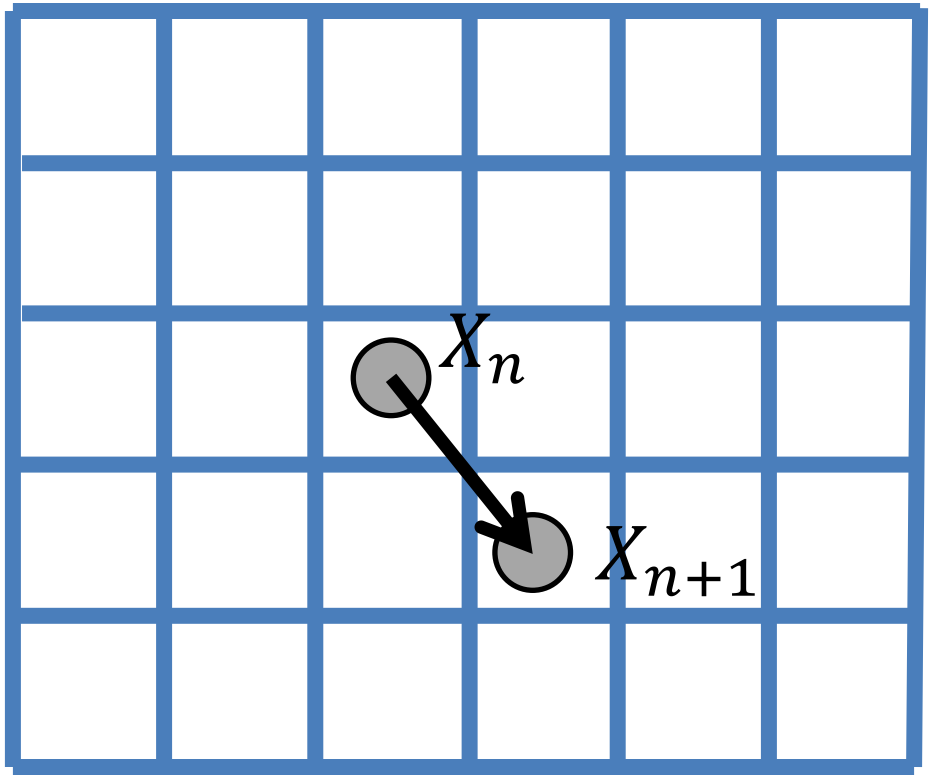
Consider $\frac{\partial \varphi}{\partial t} + u \cdot \nabla \varphi = 0.$

Particles (Lagrangian)

Interpolate velocity at particle location.

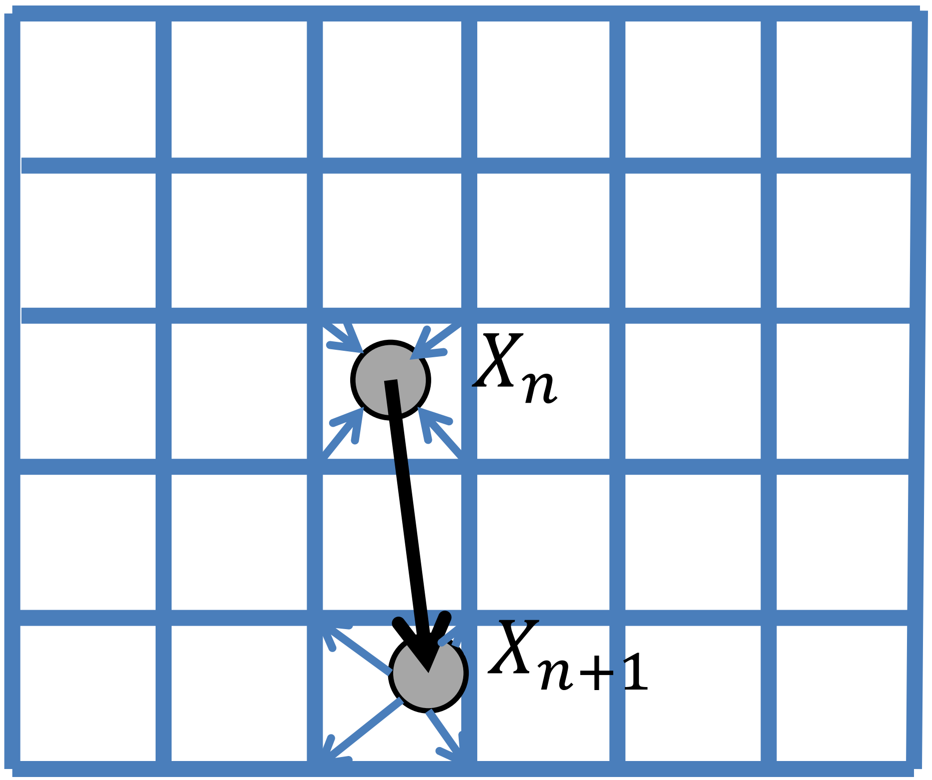
Integrate to get new position (eg. Runge-Kutta).

Does not feedback into grid.



Particle-In-Cell

1) Interpolate grid data (φ) onto particle.
2) Update particle position (Lagrangian).
3) Spread φ back onto the grid.



Eulerian

Approximate derivatives with grid-based finite differences.

$$\frac{\partial \varphi}{\partial t} + \boldsymbol{u} \cdot \nabla \varphi = 0$$

FTCS = Forward Time, Centered Space:

$$\frac{{\varphi_i}^{n+1} - {\varphi_i}^{n}}{\Delta t} + u \frac{{\varphi_{i+1}}^{n} - {\varphi_{i-1}}^{n}}{2\Delta x} = 0$$

Unconditionally Unstable!

Lax:

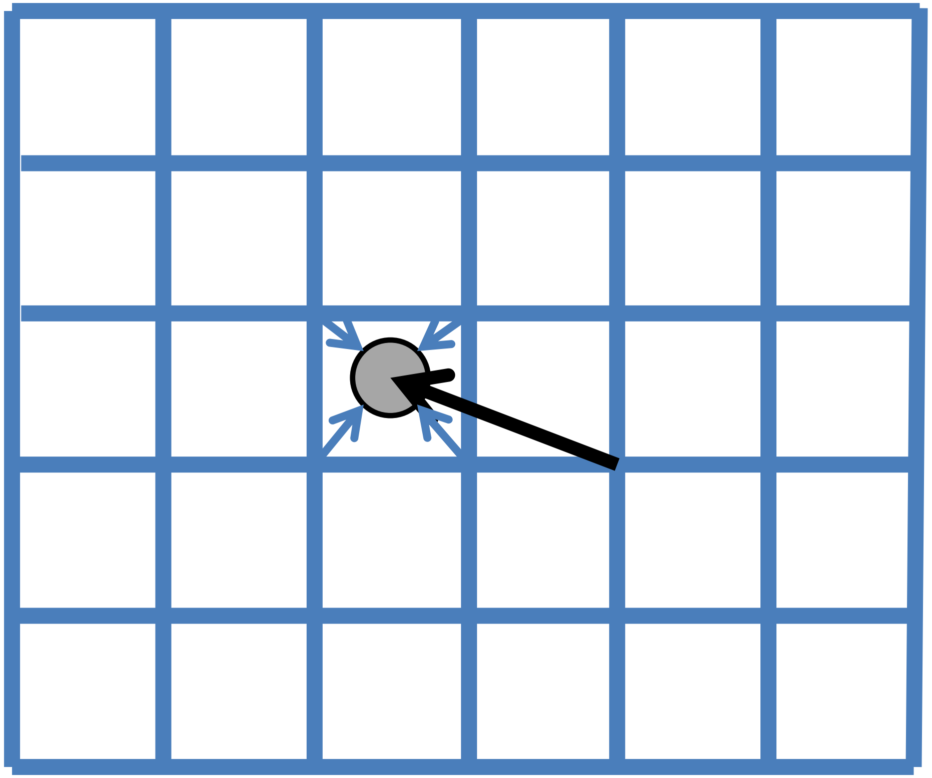
$$\frac{{\varphi_i}^{n+1} - ({\varphi_{i+1}}^{n} + {\varphi_{i-1}}^{n})/2}{\Delta t} + u \frac{{\varphi_{i+1}}^{n} - {\varphi_{i-1}}^{n}}{2\Delta x} = 0$$

Conditionally Stable!

Many methods, stability can be a challenge.

Semi-Lagrangian

Look *backwards* in time from grid points, to see where data is coming *from*.



Navier-Stokes revisited

$$\frac{\partial u}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla \cdot \nabla \boldsymbol{u} + g$$

$\frac{\partial u}{\partial t} \approx$ advection + pressure + viscosity + gravity

Use “operator splitting”: Treat each step independently.

Velocity Advection

Arbitrary quantity φ: $$\frac{\partial \varphi}{\partial t} + u \cdot \nabla \varphi = 0$$

Velocity $\boldsymbol{u}$: $$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = 0$$

Advect velocity as if it were any other quantity.

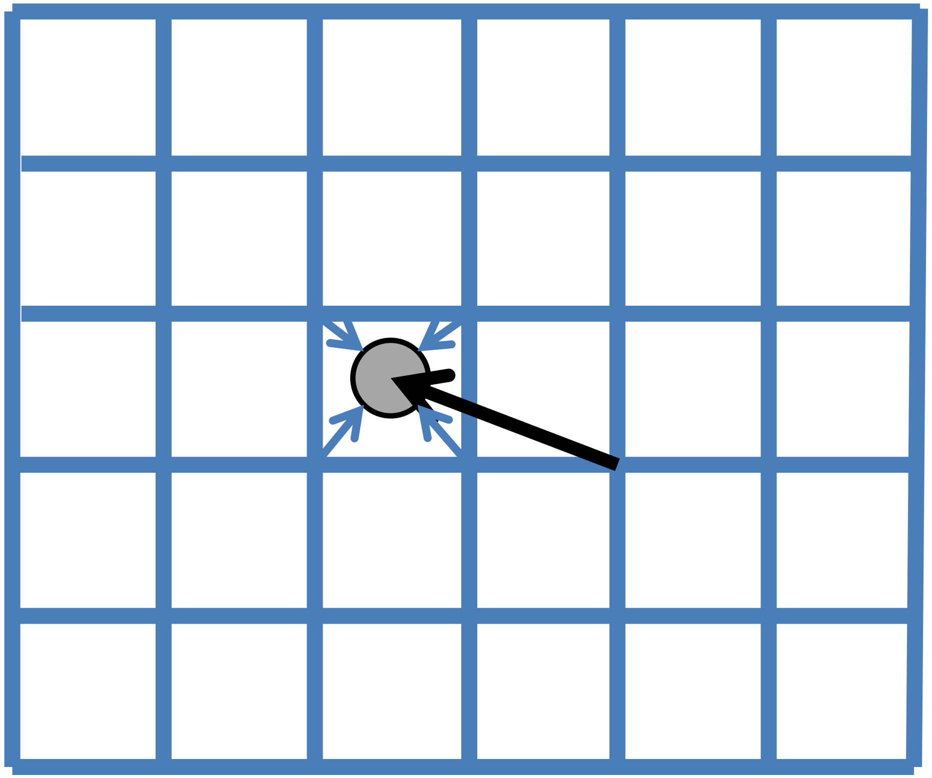
Velocity Advection

Semi-Lagrangian most common.

- Stable
- Easy to implement
- Intuitive

Drawbacks:

- Numerical dissipation



Incompressibility

Advection or other steps might introduce compression/expansion.

From [Tong et al 2003]
[Discrete Multiscale Vector Field Decomposition]()

Helmholtz-Hodge Decomposition

Input Velocity field

Curl-Free
(irrotational)

Divergence-Free
(incompressible)

=

+

$$u = \nabla p + u^{div\_free}$$

Incompressibility

$$\boldsymbol{u}^{div\_free} = \boldsymbol{u} - \nabla p \qquad (1)$$

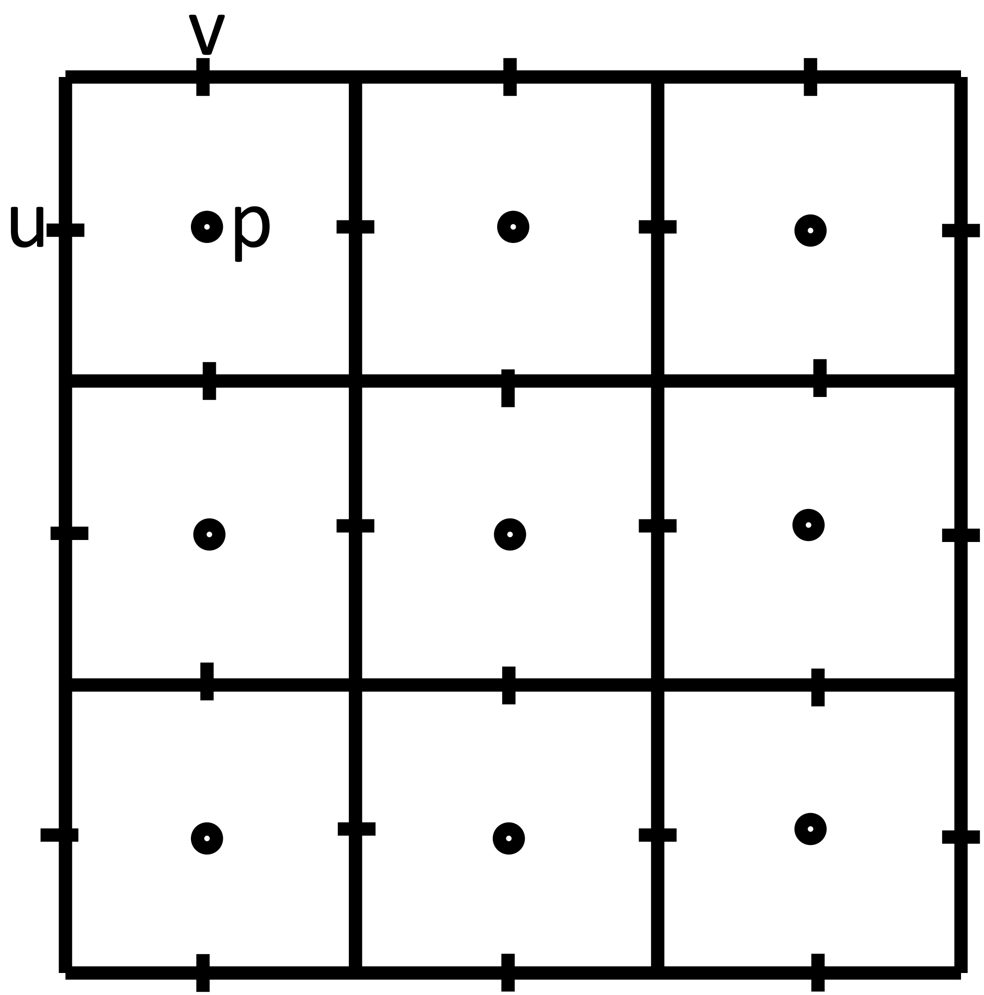
$$\nabla \cdot \boldsymbol{u}^{div\_free} = 0 \qquad (2)$$

$$\Downarrow$$

$$\nabla \cdot \nabla p = \nabla \boldsymbol{u} \qquad (3)$$

Solve (3), then plug into (1) to find new incompressible velocity field.

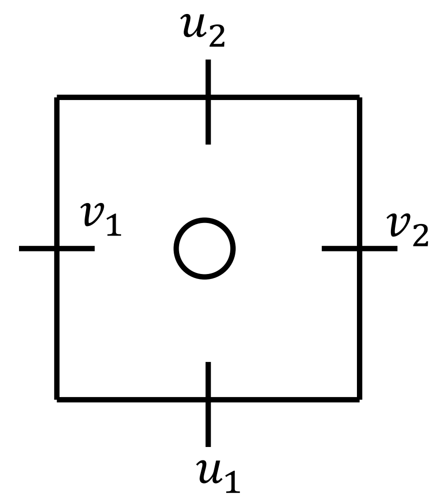
Incompressibility – Staggered Grids



Incompressibility – Staggered Grids

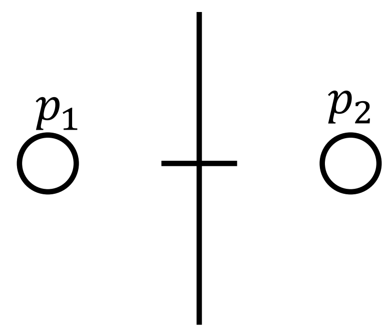
Divergence:

$$\nabla \cdot u = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \approx \frac{u_2 - u_1 + v_2 - v_1}{\Delta x}$$



Gradient:

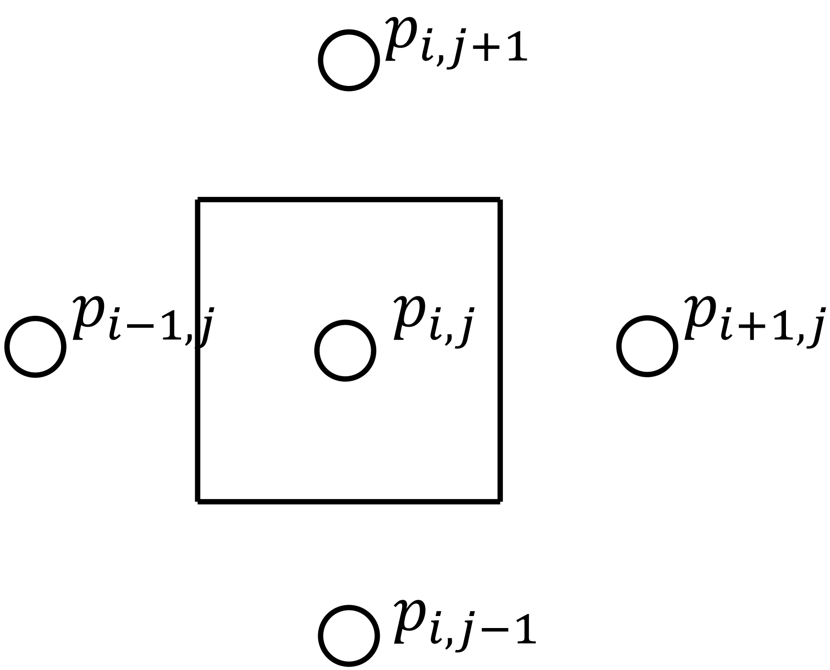
$$\nabla_x p = \frac{\partial p}{\partial x} \approx \frac{p_2 - p_1}{\Delta x}$$



Incompressibility – Staggered Grids

Laplacian (divergence of gradient):

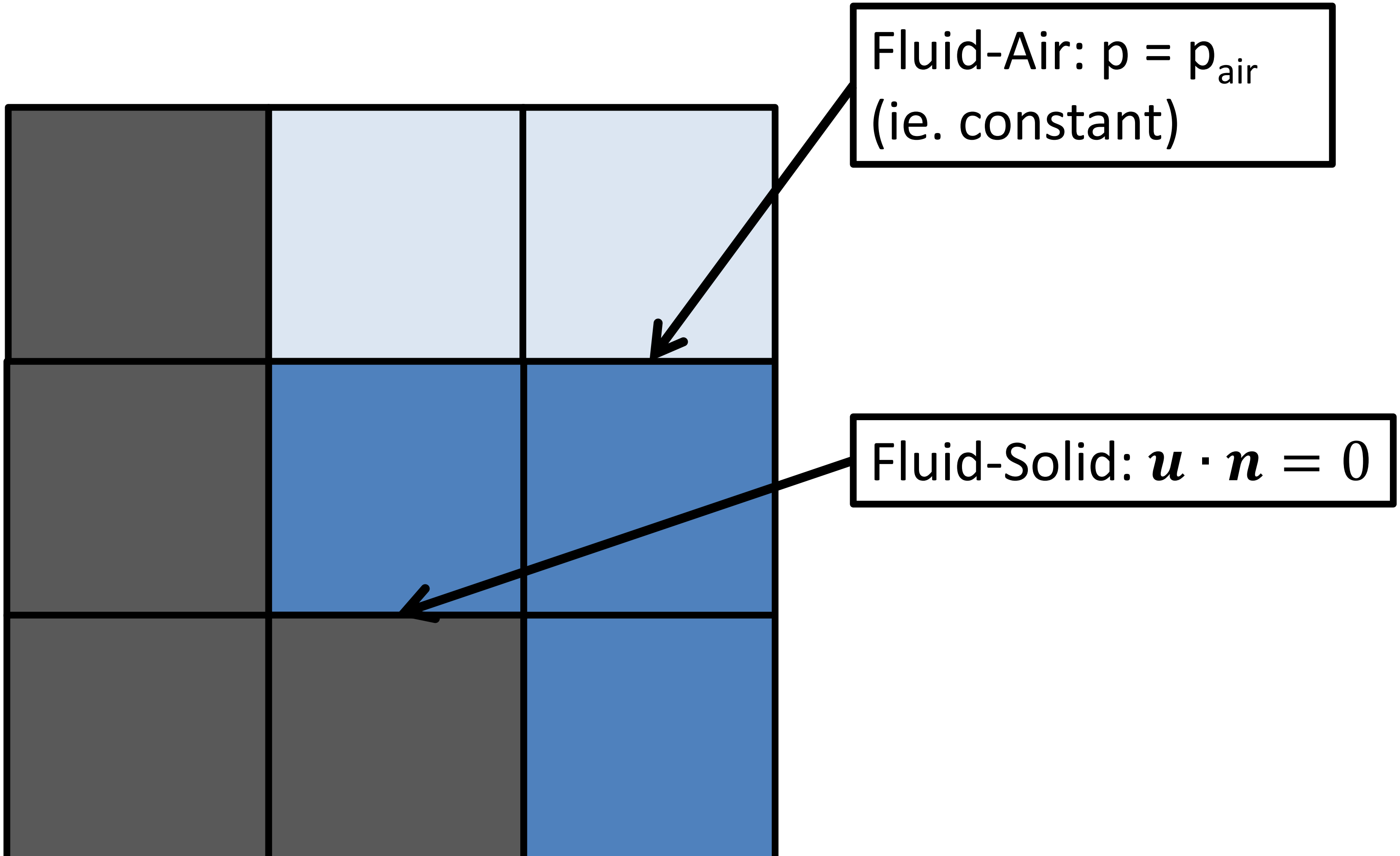
$$\nabla \cdot \nabla p = \nabla \cdot \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}\right) = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \approx \frac{\frac{p_{i+1,j}-p_{i,j}}{\Delta x} - \frac{p_{i,j}-p_{i-1,j}}{\Delta x}}{\Delta x} + \frac{\frac{p_{i,j+1}-p_{i,j}}{\Delta x} - \frac{p_{i,j}-p_{i,j-1}}{\Delta x}}{\Delta x}$$



5-point stencil:

	1	
1	-4	1
	1	

Boundaries


Fluid-Air: $p = p_{air}$
(ie. constant)
Fluid-Solid: $\boldsymbol{u} \cdot \boldsymbol{n} = 0$

Solving $\nabla \cdot \nabla p = \nabla \boldsymbol{u}$

A Poisson equation:

- Sparse, positive definite linear system of equations.
- One equation per cell, cells globally coupled.
- Typically, solve with conjugate gradient or multigrid.

Viscosity

[Carlson et al. 2003]

Viscosity

PDE: $$\frac{\partial \boldsymbol{u}}{\partial t} = \frac{\mu}{\rho}\nabla \cdot \nabla \boldsymbol{u}$$

Discretized in time:

$$\boldsymbol{u}_{\boldsymbol{new}} = \boldsymbol{u}_{old} + \frac{\Delta t \mu}{\rho}\nabla \cdot \nabla \boldsymbol{u}_*$$

If $\boldsymbol{u}_*$ is $\boldsymbol{u}_{old}$, explicit integration.

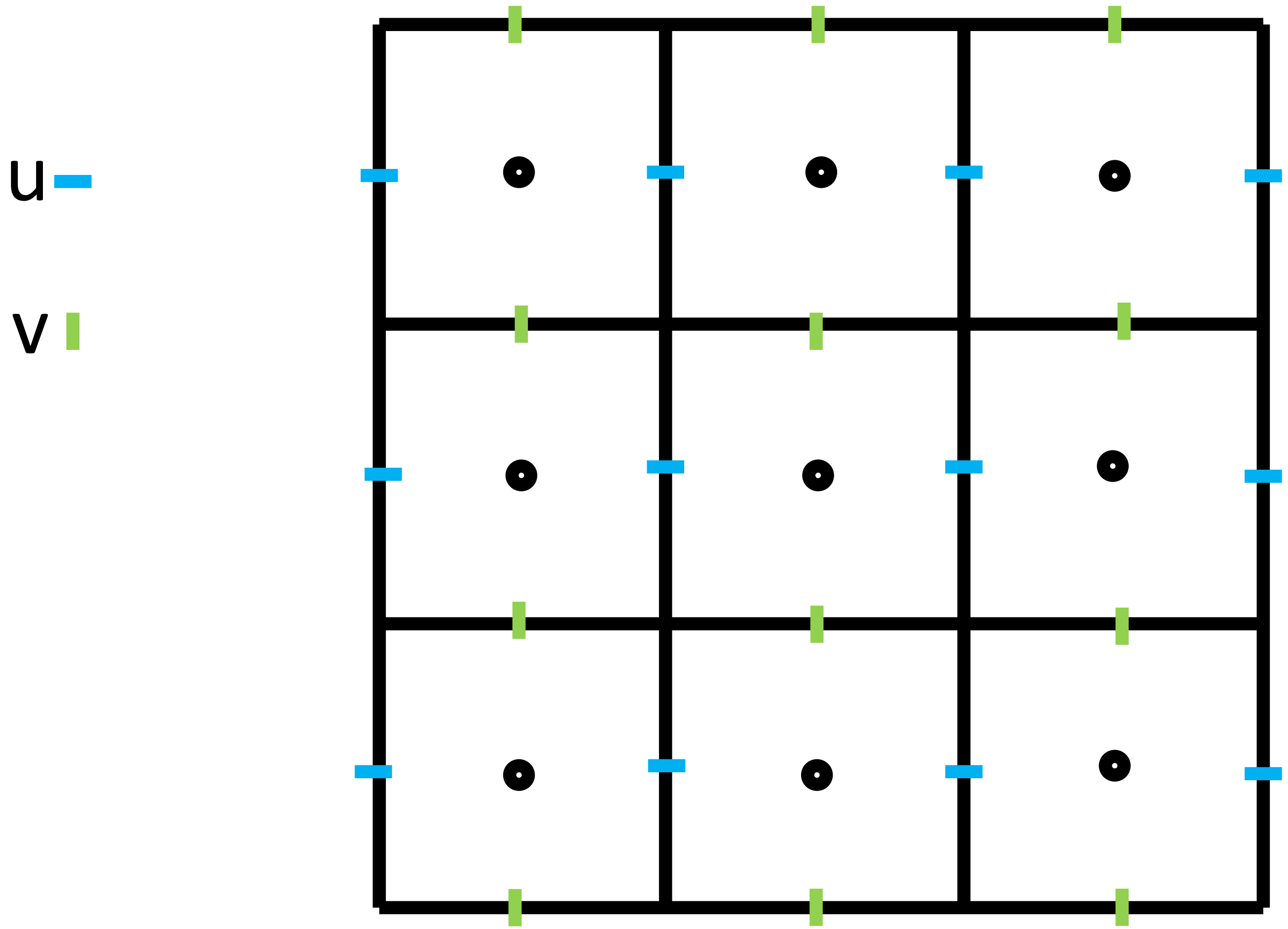
– No need to solve linear system.

If $\boldsymbol{u}_*$ is $\boldsymbol{u}_{new}$, implicit integration.

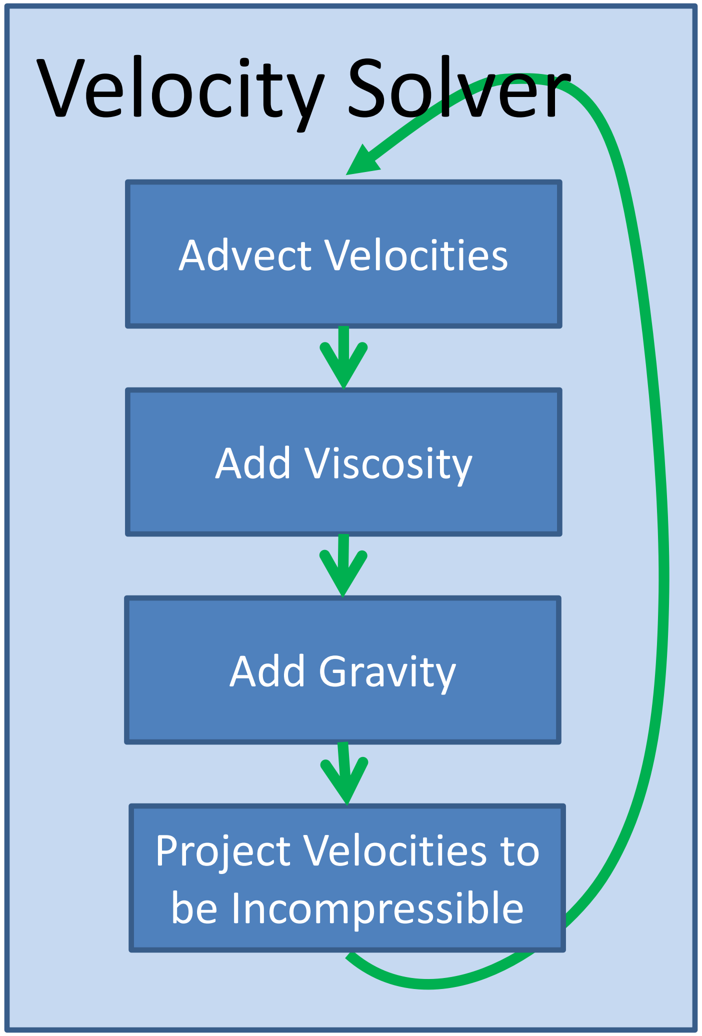
– Stable for high viscosities.

Viscosity

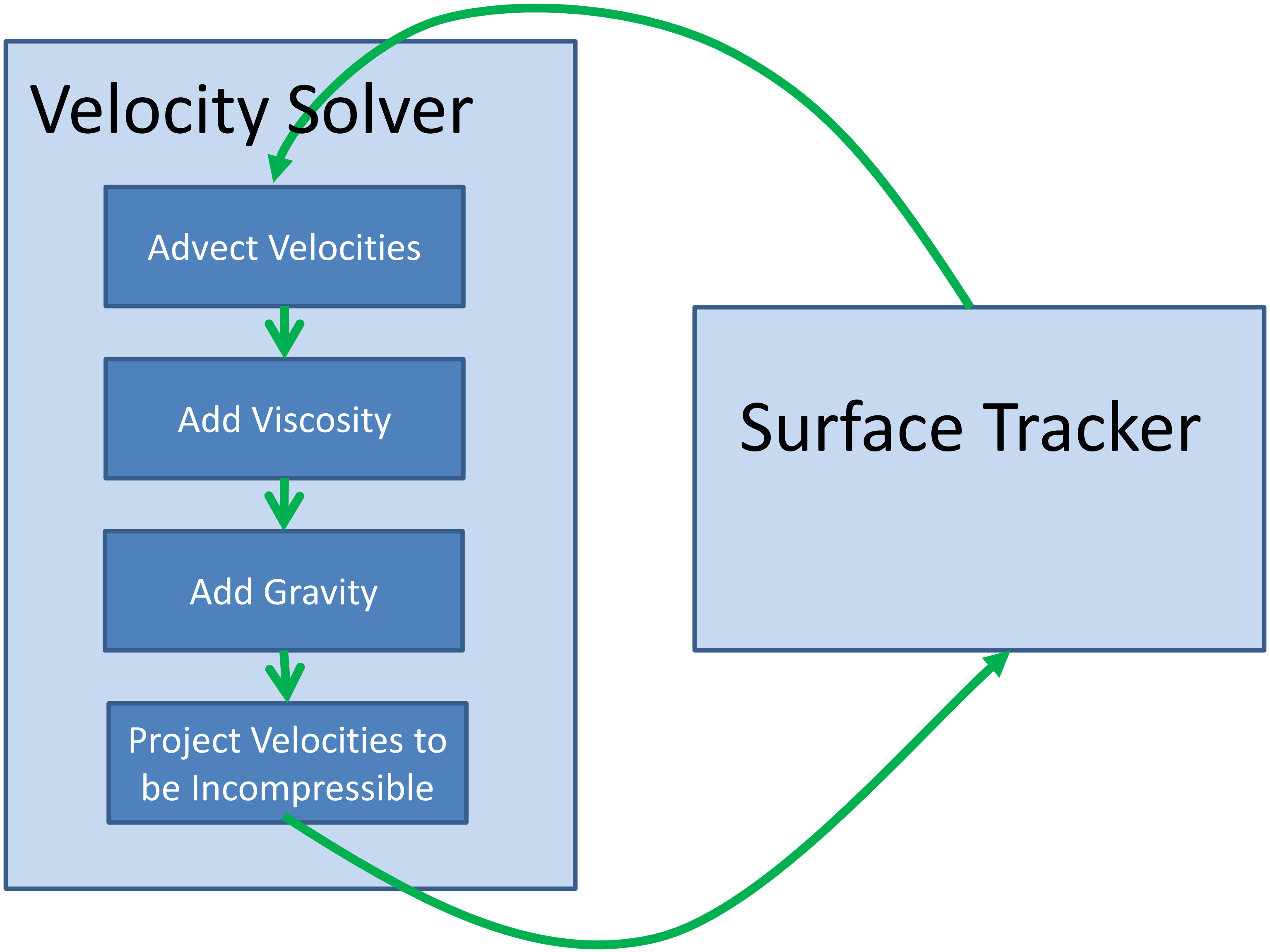
Applies smoothing to each velocity component (ie. u, v, w) independently.



The Big Picture



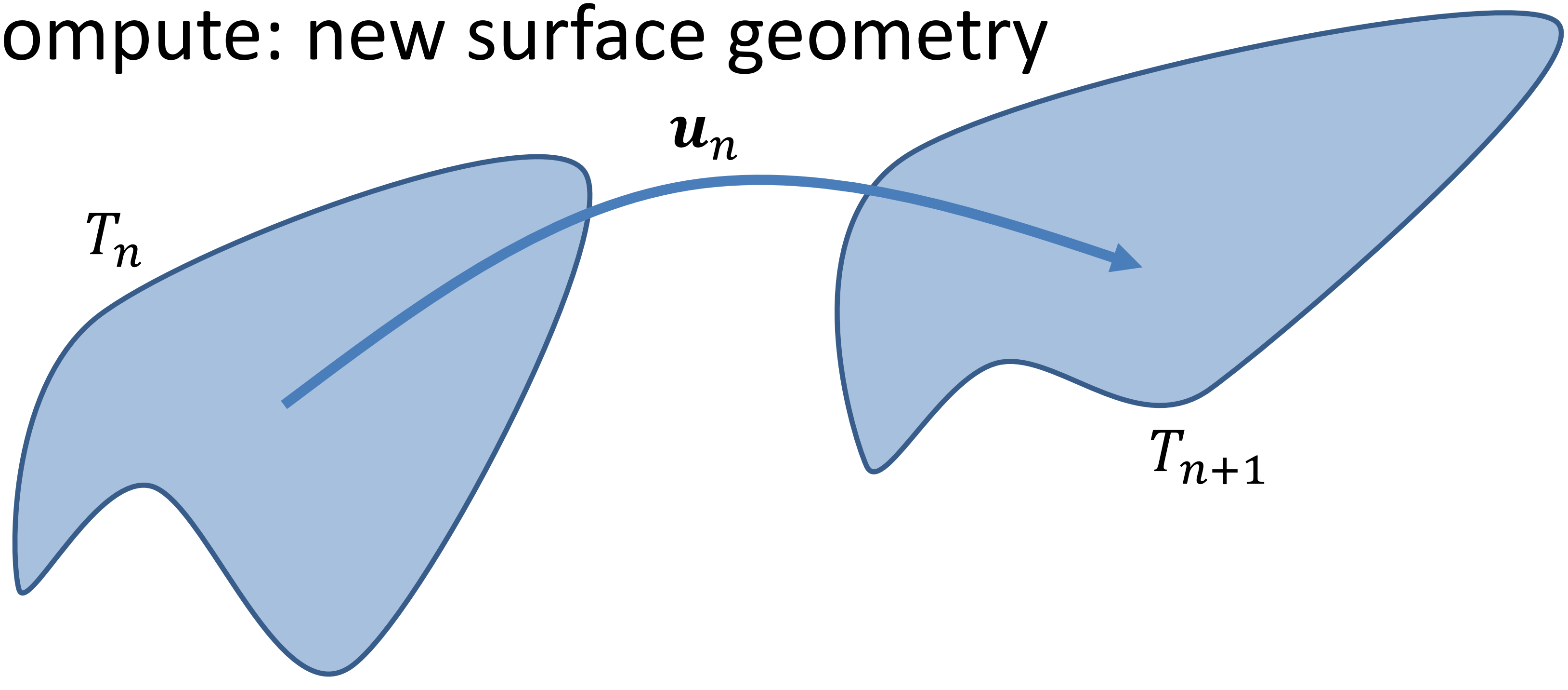
What about liquids?



Surface Tracker

Given: liquid surface geometry, velocity field, timestep

Compute: new surface geometry



Surface Tracker

Ideally:

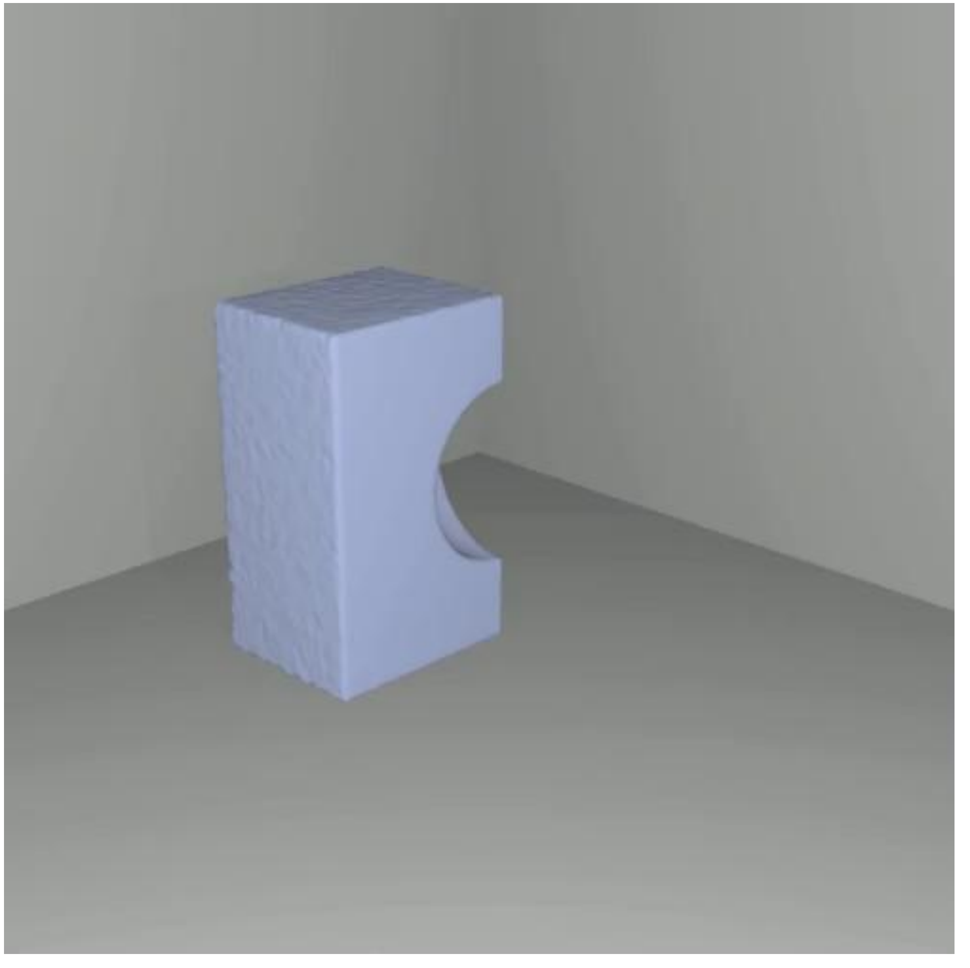
- Efficient
- Accurate
- Handles merging/splitting (topology changes)
- Conserves volume
- Retains small features
- Smooth surface for rendering
- Provides convenient geometric operations
- Easy to implement…

Very hard (impossible?) to do all of these at once.

Surface Tracking Options

1. Particles
2. Level sets
3. Volume-of-fluid (VOF)
4. Triangle meshes
5. Hybrids (many of these)

Particles

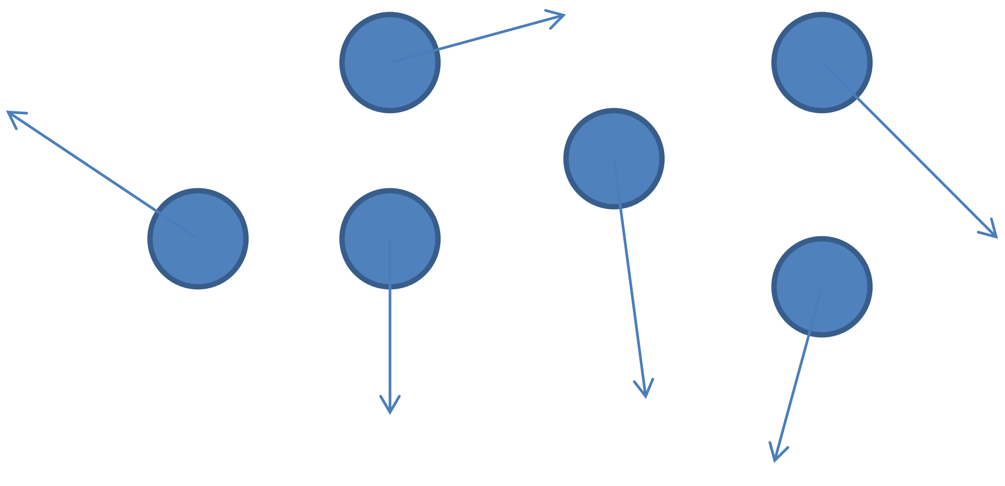


[Zhu & Bridson 2005]

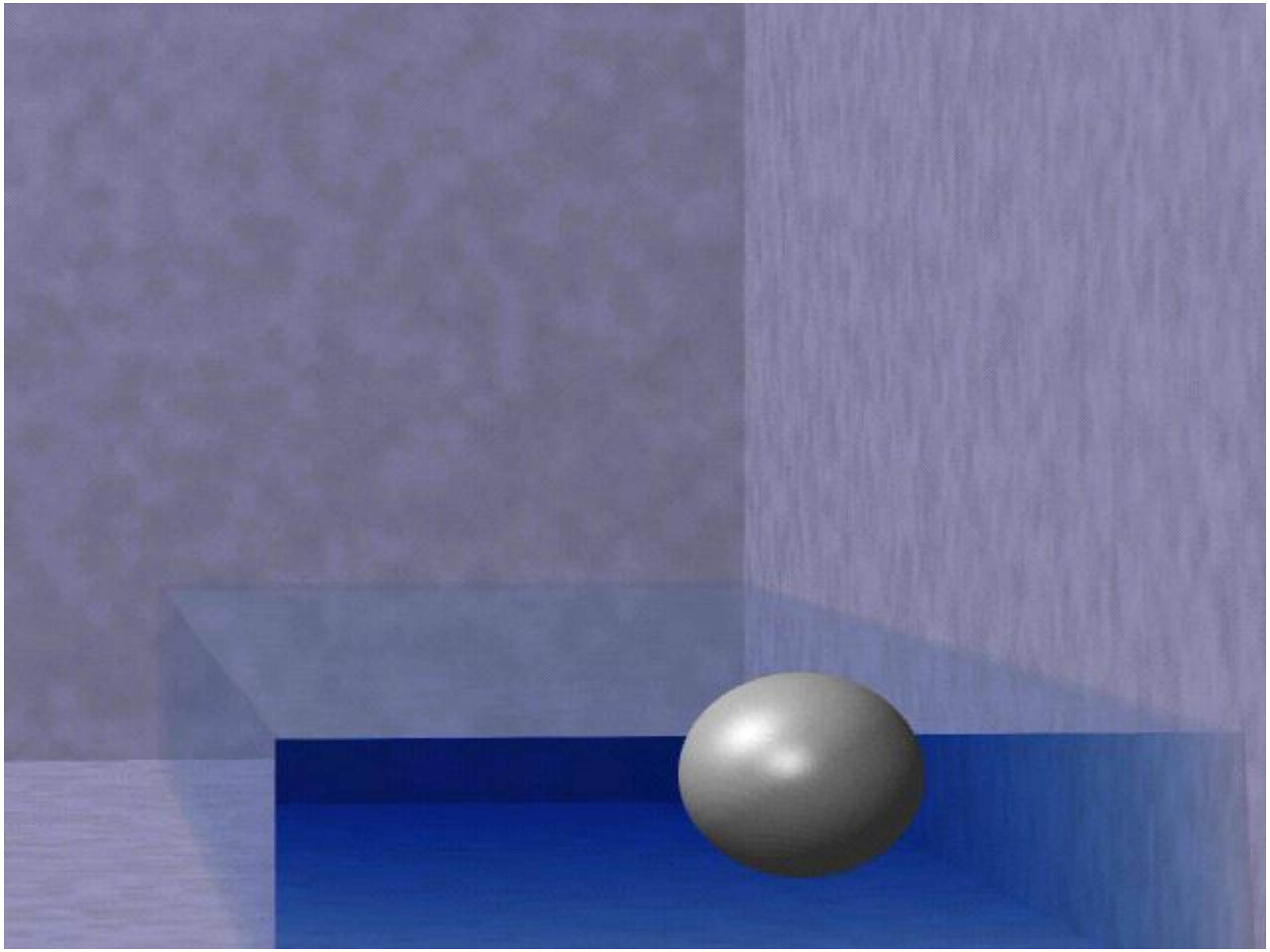
Particles

Perform passive Lagrangian advection on each particle.

Need to reconstruct a surface, as for SPH.



Level sets

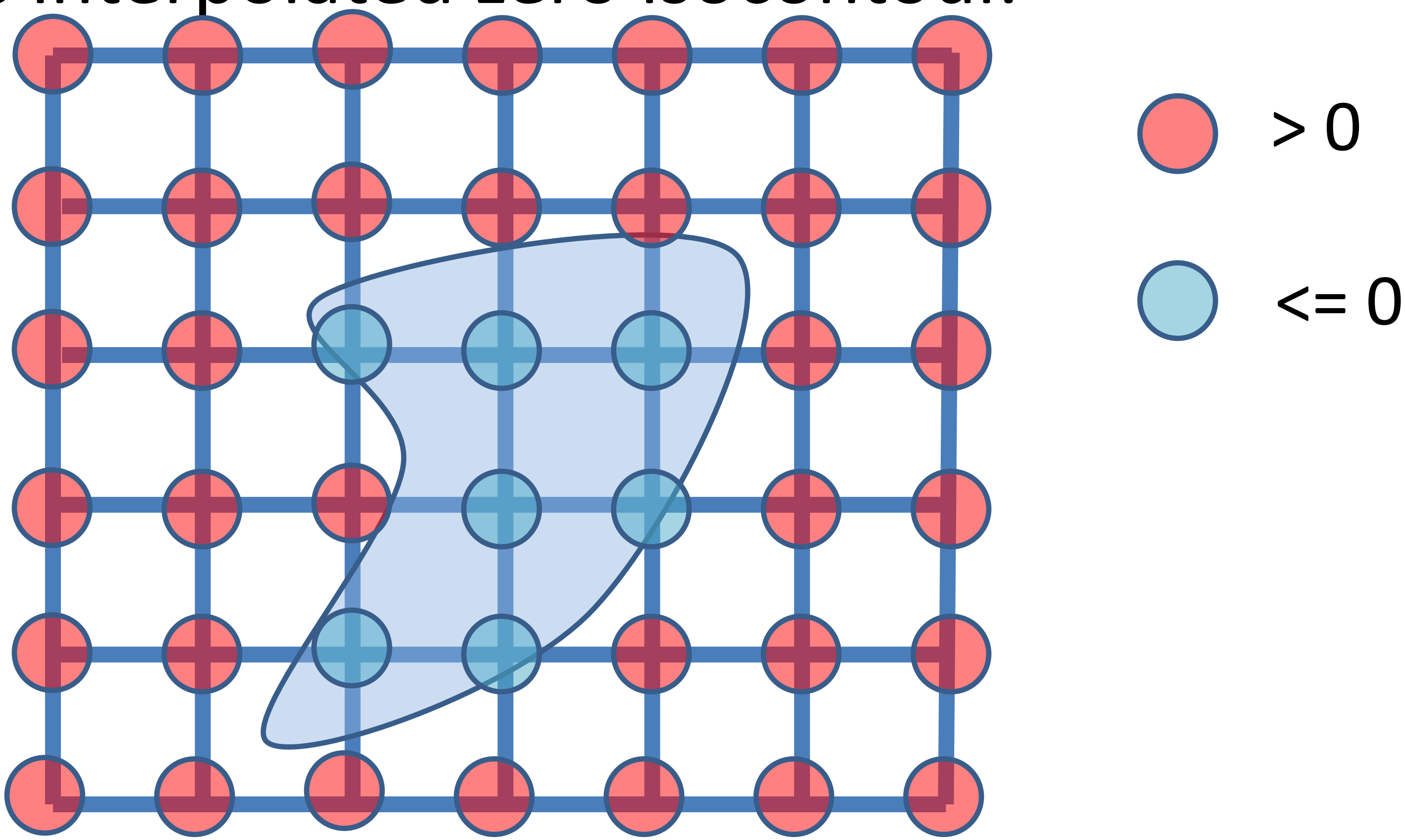


[Losasso et al. 2004]

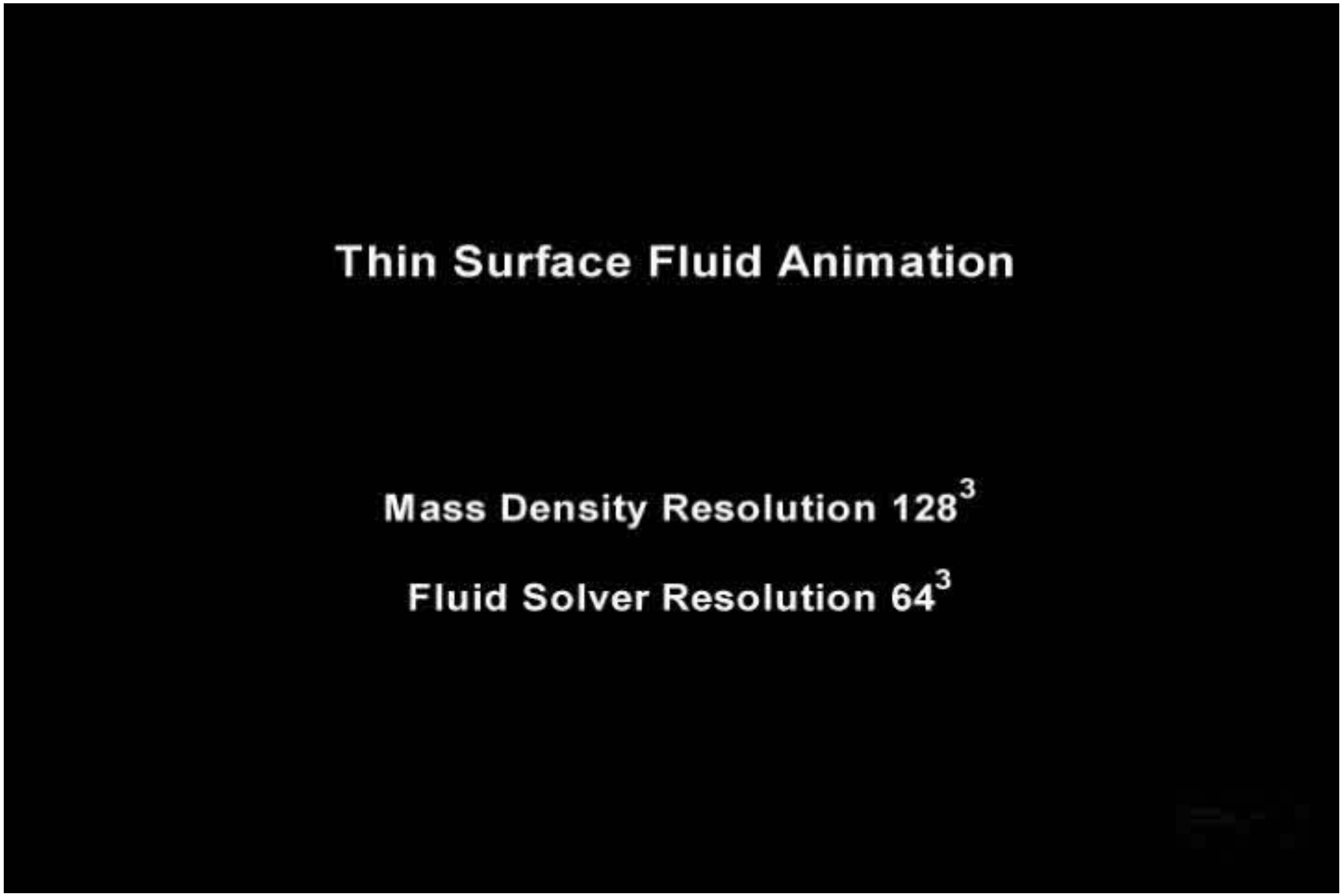
Level sets

Each grid point stores *signed* distance to the surface (inside <= 0, outside > 0).

Surface is interpolated zero isocontour.



Volume of fluid

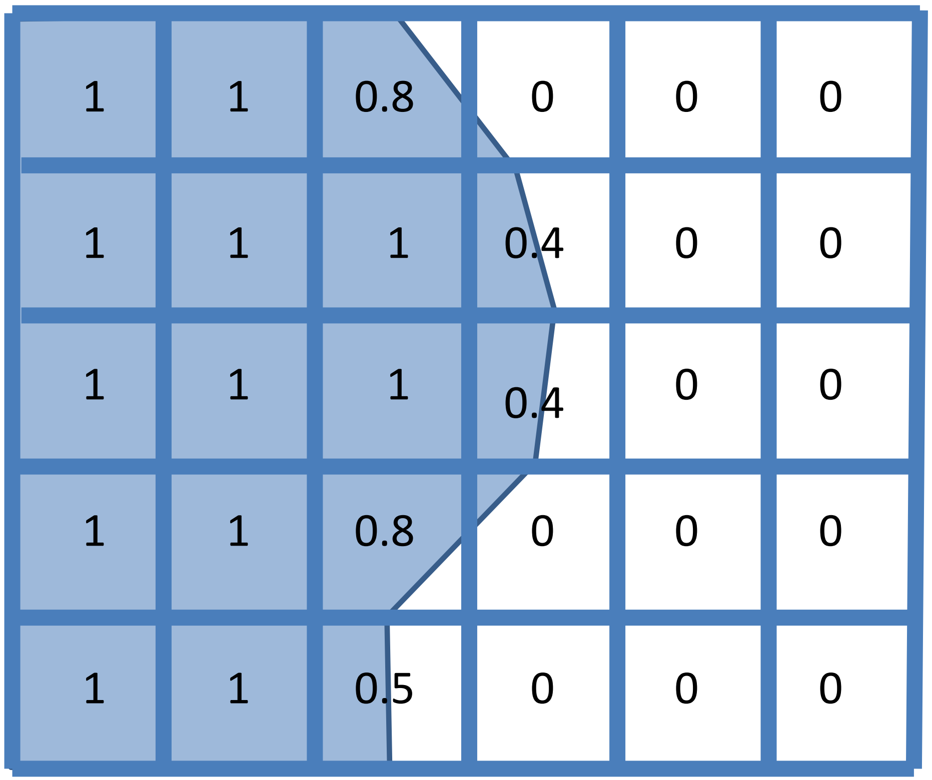


[Mullen et al 2007]

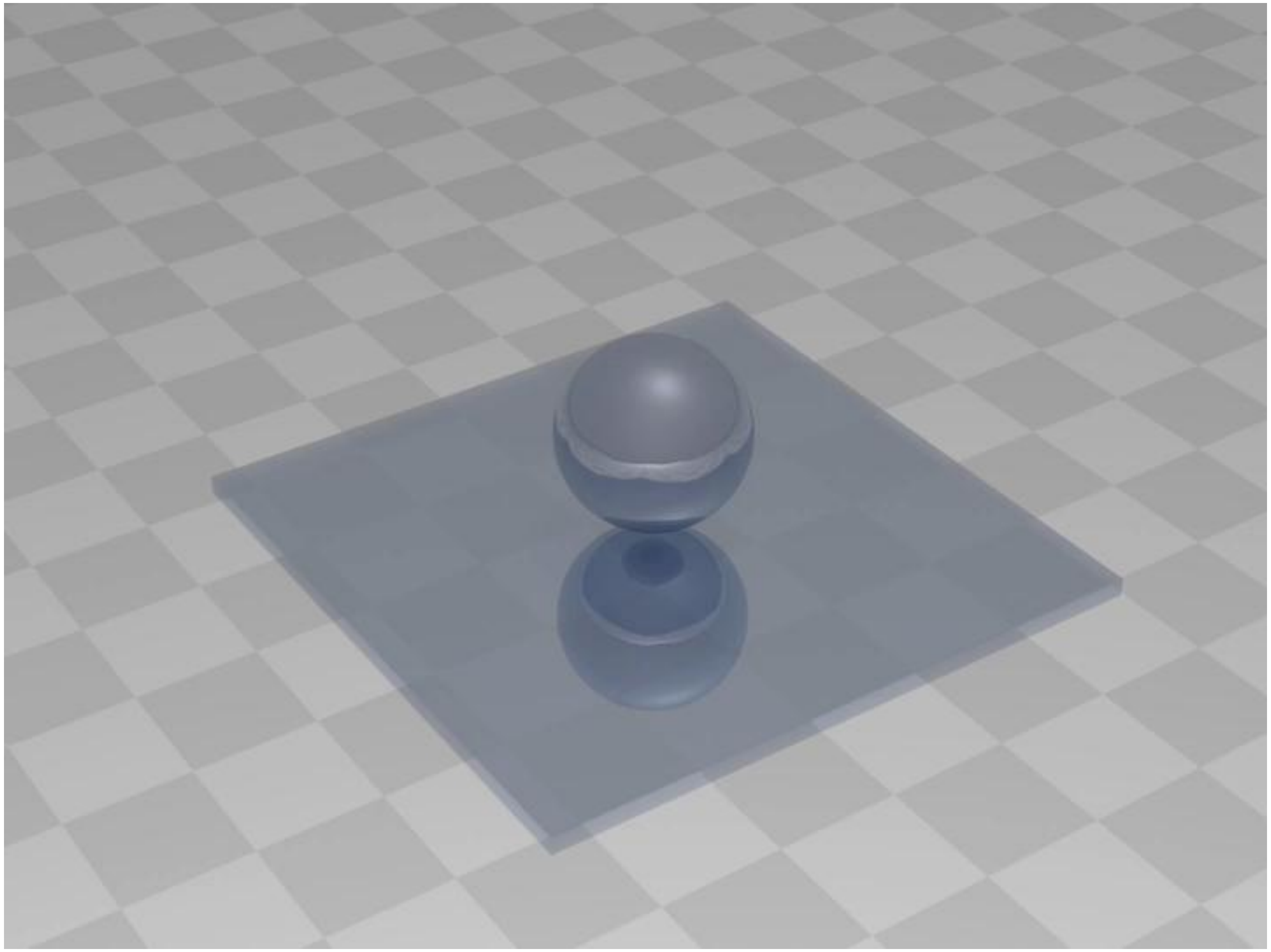
Volume-Of-Fluid

Each cell stores fraction $f \in [0,1]$ indicating how empty/full it is.

Surface is transition region, $f \approx 0.5$.



Meshes

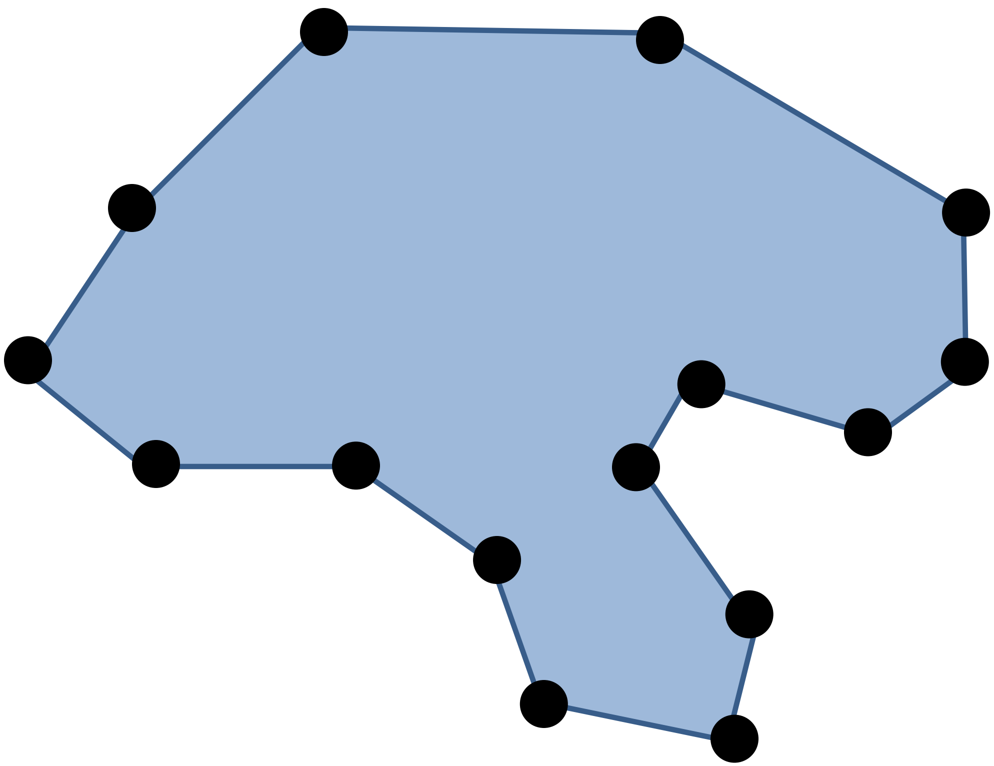


[Brochu et al 2010]

Meshes

Store a triangle mesh.

Advect its vertices, and correct for collisions.



Reference Material

SIGGRAPH course notes: Fluid Simulation for Computer Animation
- http://www.cs.ubc.ca/~rbridson/fluidsimulation/

Basics:
- Stable Fluids (Stam 1999)
  - http://www.dgp.toronto.edu/people/stam/reality/Research/pdf/ns.pdf
- Practical Animation of Liquids (Foster & Fedkiw 2001)
  - http://physbam.stanford.edu/~fedkiw/papers/stanford2001-02.pdf

High Viscosity Liquids:
- Melting and Flowing (Carlson et al. 2003)
  - http://www.cc.gatech.edu/~turk/melting/melting.html
- Accurate Viscous Free Surfaces… (Batty & Bridson 2008)
  - http://www.cs.ubc.ca/~rbridson/docs/batty-sca08-viscosity.pdf

Reference Material

Better advection:

- Visual Simulation of Smoke (Fedkiw et al. 2001)
  - http://physbam.stanford.edu/~fedkiw/papers/stanford2001-01.pdf
- An Unconditionally Stable MacCormack Method (Selle et al. 2006)
  - http://physbam.stanford.edu/~fedkiw/papers/stanford2006-09.pdf
- Animating Sand as a Fluid (Zhu & Bridson 2005)
  - http://www.cs.ubc.ca/~rbridson/docs/zhu-siggraph05-sandfluid.pdf

Better incompressibility:

- Using the particle levelset method and a second order accurate pressure boundary condition for free surface flows (Enright et al. 2003)
  - http://physbam.stanford.edu/~fedkiw/papers/stanford2003-03.pdf
- A fast variational framework for accurate solid-fluid coupling (Batty et al. 2007)
  - http://www.cs.ubc.ca/nest/imager/tr/2007/Batty_VariationalFluids/

Reference Material

Surface Tracking:

- A Fast and Accurate Semi-Lagrangian Particle level set method (Enright et al. 2005)
  - http://physbam.stanford.edu/~fedkiw/papers/stanford2003-10.pdf
- Physics-Based Topology Changes for Thin Fluid Features (Wojtan et al. 2010)
  - http://pub.ist.ac.at/group_wojtan/thin_fluid_features/thin_fluid_features.html
- Reconstructing Surfaces of Particle-Based Fluids using Anisotropic Kernels (Yu & Turk 2010)
  - http://www.cc.gatech.edu/~turk/my_papers/sph_surfaces.pdf
- Robust Topological Operations for Dynamic Explicit Surfaces (Brochu & Bridson 2009)
  - http://www.cs.ubc.ca/labs/imager/tr/2009/eltopo/eltopo.html

Disclaimer: This is definitely not an exhaustive list!