Nonlinear Conservation Laws, Primitive, Conservative and Integral Forms

Monday, September 26, 2016 9:03 AM

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

$f(u) \equiv u$

$f(u) = u^2/2$

Burgers Equation

Primitive

$$\frac{\partial u}{\partial t} + \left(\frac{df}{du}\right)\frac{\partial u}{\partial x} = 0 \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

Conservative

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \qquad \frac{\partial u}{\partial t} + \frac{\partial u^2/2}{\partial x} = 0$$

Integral form.

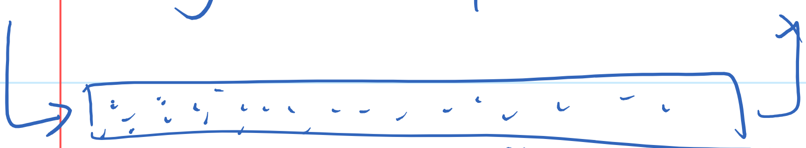
$$\int_a^b \left(\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x}\right) dx = 0$$

$$\frac{d}{dt}\int_a^b u\,dx + f(u)\Big|_a^b = 0$$

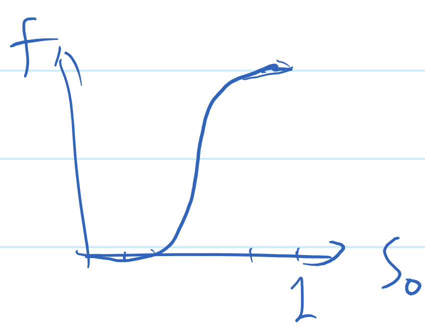
$$\frac{d}{dt}\int_a^b u\,dx = f(u(a)) - f(u(b))$$

$$\frac{d}{dt}\int_a^b u\,dx + \frac{u(b)^2}{2} - \frac{u(a)^2}{2} = 0$$

Buckley-Leverett Equation



$$\frac{\partial S_o}{\partial t} + \frac{\partial f(S_o)}{\partial x} = 0$$



Nonlinear Conservation Laws, Primitive, Conservative and Integral Forms

Monday, September 26, 2016 9:03 AM

Primitive
$$\frac{\partial u}{\partial t} + \left(\frac{d\vec{f}}{du}\right) \cdot \nabla u = 0$$

Conservative
$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{f}(u) = 0$$

Integral
$$\int_{\Omega} \left(\frac{\partial u}{\partial t} + \nabla \cdot \vec{f}(u) \right) dV = 0$$

$$\frac{d}{dt}\int_{\Omega} u \, dV + \int_{\partial\Omega} \vec{f}(u) \cdot \vec{n} \, dS = 0$$

$$\frac{d}{dt}\int_{\Omega} u \, dV = \int_{\partial\Omega} \vec{f}(u) \cdot (-\vec{n}) \, dS$$

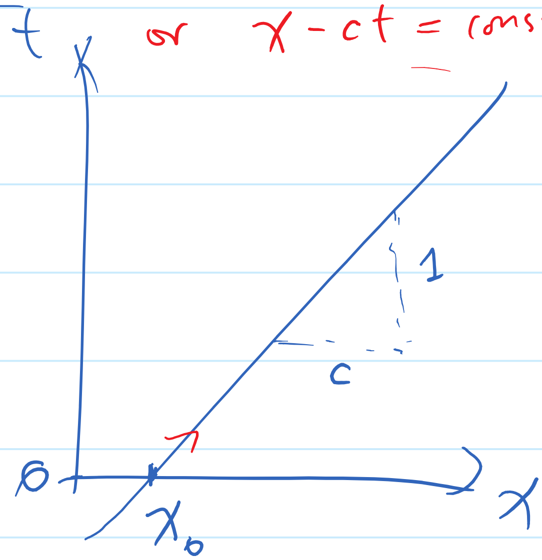
Scalar conservation law: Behavior of smooth solution

Monday, September 26, 2016 9:03 AM

$$\frac{\partial u}{\partial t} + \left(\frac{df}{du}\right)\frac{\partial u}{\partial x} = 0$$

$x = x_0 + ct$

or $x - ct = \text{constant}$



$$\frac{df}{du}(u)$$

$$u(x_0 + ct, t)$$

$$\frac{du(x_0+ct, t)}{dt}$$

$$= \frac{\partial u}{\partial x} \cdot \frac{d(x_0+ct)}{dt} + \frac{\partial u}{\partial t} \cdot 1$$

$$c = \frac{df}{du}\Big|_{x = x_0,\, t=0}$$

$$= c \cdot \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0$$

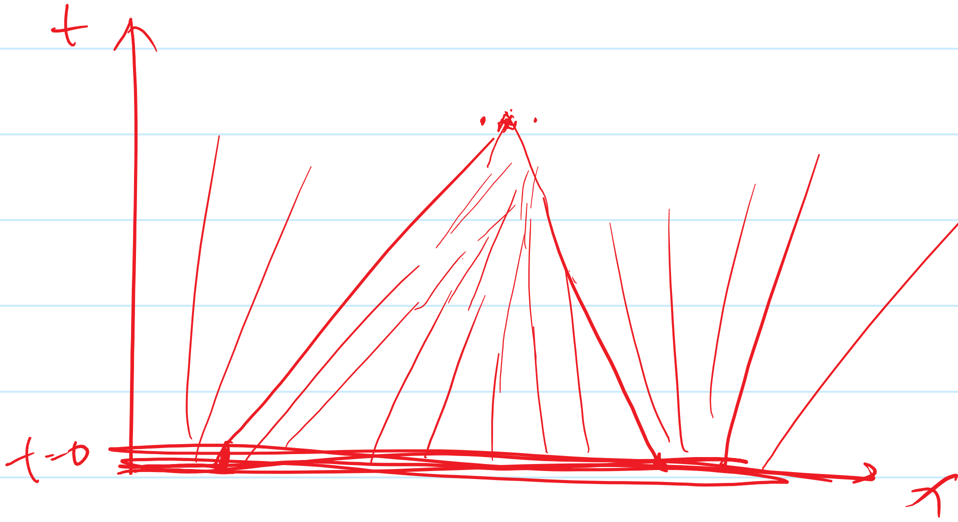
$$f(u) = \frac{u^2}{2} \qquad \frac{df}{du} = u$$

$u < 0$

$u = 0$

$u > 0$

$$x = x_0 + \frac{df}{du} t$$ are called characteristic lines



$\frac{\partial f(u)}{\partial x}$? over discontinuity ?

Use Integral form

$$\frac{d}{dt}\int_a^b u\,dx + f(u)\Big|_b - f(u)\Big|_a = 0$$

discontinuity at $c(t)$, $S_s := \frac{dc}{dt}$

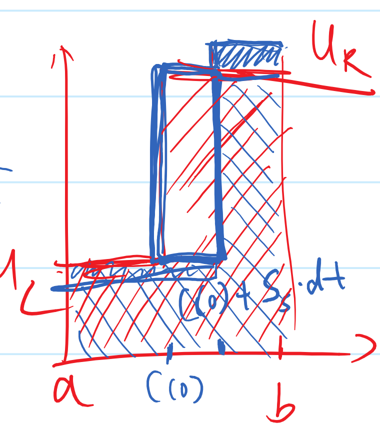
$a = c(0) - \varepsilon$ $\qquad u(a) = u_L$

$b = c(0) + \varepsilon$ $\qquad u(b) = u_R$

At $t=0$ $\quad \int_a^b u\,dx = (u_L + u_R)\,\varepsilon$

At $t = dt$ $\quad \int_a^b u\,dx = -(u_R - u_L)\,S_s\,dt$

$$\frac{d}{dt}\int_a^b u\,dx = (u_L - u_R)\,S_s + \int_a^b u\,dx\Big|_{t=0}$$



Scalar conservation law: Shocks

Monday, September 26, 2016 9:03 AM

$$\frac{d}{dt}\int_a^b u\,dx = f(u)\Big|_{x=a} - f(u)\Big|_{x=b}$$

$$\| \qquad\qquad\qquad \|$$

$$(u_L - u_R)\, S_s = f(u_L) - f(u_R)$$

$$S_s = \frac{f(u_L) - f(u_R)}{u_L - u_R} \not\approx \frac{df}{du}$$

Burgers:

$$S_s = \frac{u_L^2/2 - u_R^2/2}{u_L - u_R}$$

$$= \frac{\frac{1}{2}(u_L + u_R)(u_L - u_R)}{u_L - u_R}$$

$$= \frac{u_L + u_R}{2}$$

Scalar conservation law: Shocks -- A paradox

Monday, September 26, 2016 9:03 AM

$$C_s = \frac{f_L - f_R}{u_L - u_R} = \frac{\Delta f}{\Delta u}$$

$$\frac{\partial u}{\partial t} + \frac{\partial u^2/2}{\partial x} = 0 \qquad C_s = \frac{u_L + u_R}{2}$$

$$u\left(\frac{\partial u}{\partial t} + \frac{\partial \frac{u^2}{2}}{\partial x}\right) = 0 \qquad \frac{\partial}{\partial t}\left(\frac{u^2}{2}\right) + \frac{\partial}{\partial x}\left(\frac{u^3}{3}\right) = 0$$

$$V := \frac{u^2}{2}$$

$$\left\{\begin{array}{l} u\dfrac{\partial u}{\partial t} = \dfrac{\partial u^2/2}{\partial t} \\ u\dfrac{\partial u^2/2}{\partial x} = \dfrac{1}{3}\dfrac{\partial u^3}{\partial x} \end{array}\right.$$

$$f(V) = \frac{u^3}{3} = \frac{(2V)^{3/2}}{3}$$

$$C_s = \frac{\Delta\left(\frac{(2V)^{3/2}}{3}\right)}{\Delta V} \neq \frac{u_L + u_R}{2}$$

$$\frac{\partial u^3}{\partial x} = \frac{\partial u \cdot u^2}{\partial x} = u \cdot \frac{\partial u^2}{\partial x} + u^2 \frac{\partial u}{\partial x}$$

$$= \frac{\partial u \cdot u \cdot u}{\partial x} = 3u^2 \frac{\partial u}{\partial x}$$

$$u^2 \frac{\partial u}{\partial x} = \frac{1}{3}\frac{\partial u^3}{\partial x}$$

$$\frac{\partial u^3}{\partial x} = u\frac{\partial u^2}{\partial x} + \frac{1}{3}\frac{\partial u^3}{\partial x}$$

$$u \cdot \frac{\partial u^2}{\partial x} = \frac{2}{3}\frac{\partial u^3}{\partial x}$$

Finite Volume

Monday, September 26, 2016 9:03 AM

$$\frac{d}{dt}\int_a^b u\,dx = (b-a)\cdot\frac{d\,\bar{u}_{a,b}}{dt}$$

$$= f_a - f_b$$