Second order scheme using Godunov Numerical Flux

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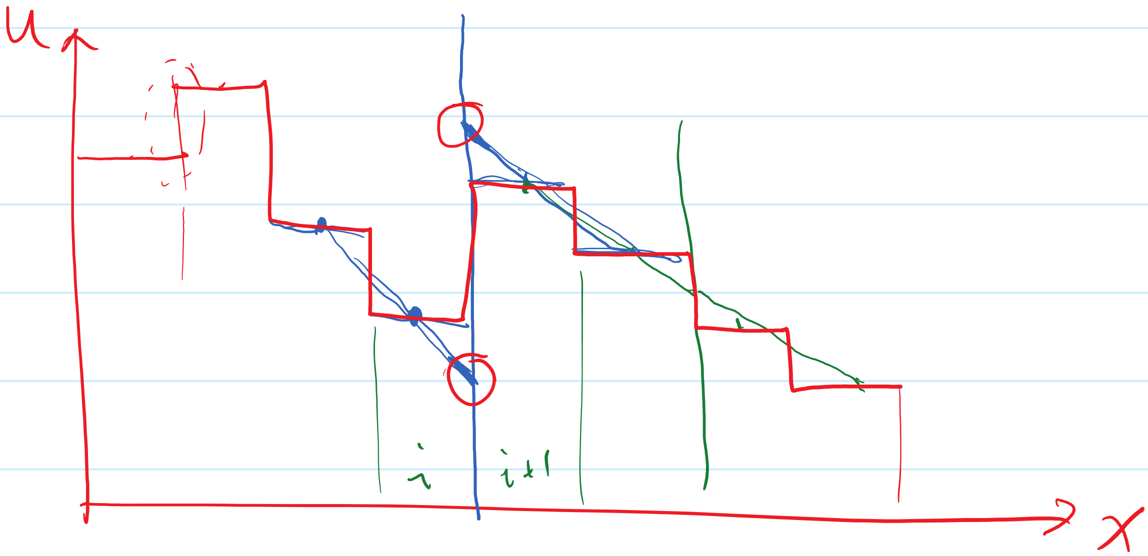
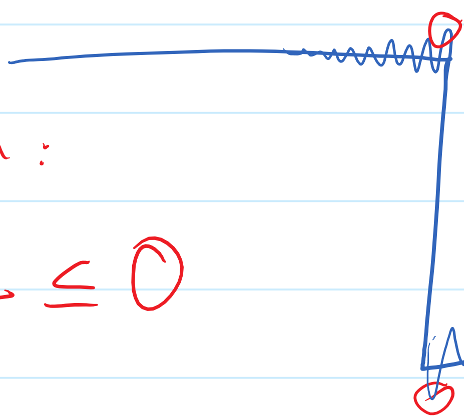


how? $\begin{cases} U^L_{i+\frac{1}{2}} \text{ using } u_i,\ u_{i-1} \\ U^R_{i+\frac{1}{2}} \text{ using } u_{i+1},\ u_{i+2} \end{cases}$

$$f_{i+\frac{1}{2}} = \text{Godunov}\left(U^L_{i+\frac{1}{2}},\ U^R_{i+\frac{1}{2}}\right)$$



Gibbs phenomenon

$$TV := \int_a^b \left|\frac{\partial u}{\partial x}\right| dx$$

criteron:

$$\frac{dTV}{dt} \leq 0$$

$$TV := \sum_{i=i_1}^{i_2} \left|\frac{u_i - u_{i-1}}{\Delta x}\right| \Delta x$$

Godunov's order barrier theorem

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Linear one-step second-order accurate numerical schemes for the convection equation

$$\frac{\partial \varphi}{\partial t} + c\frac{\partial \varphi}{\partial x} = 0, \quad t > 0, \quad x \in \mathbb{R} \qquad (10)$$

$\frac{dTV}{dt} \le 0$

cannot be monotonicity preserving unless

$$\sigma = |c| \frac{\Delta t}{\Delta x} \in \mathbb{N}, \qquad (11)$$

only feasible for constant c.

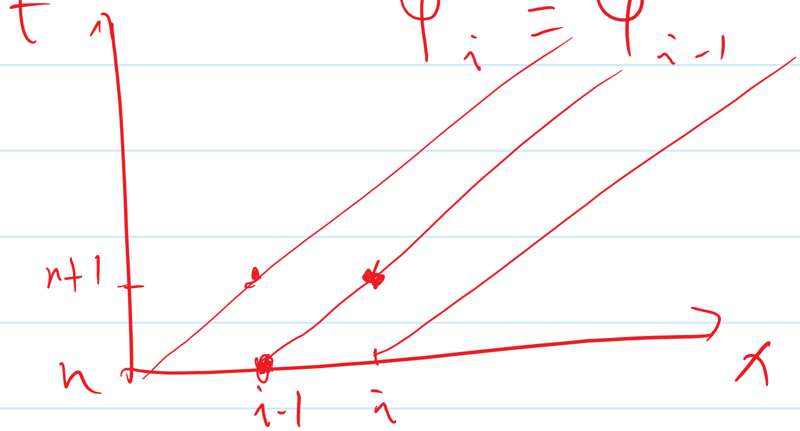
where σ is the signed Courant–Friedrichs–Lewy condition (CFL) number.

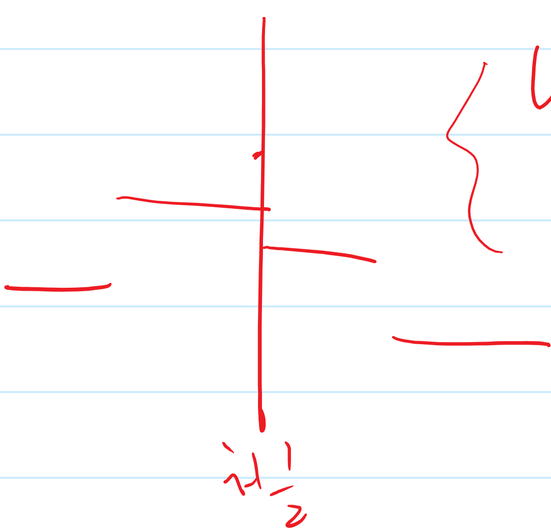
$\sigma = 1$

$$\frac{\varphi_i^{n+1} - \varphi_i^n}{\Delta t} + c\,\frac{\varphi_i^n - \varphi_{i-1}^n}{\Delta x} = 0$$

$c\,\Delta t = \Delta x$

$\varphi_i^{n+1} = \varphi_{i-1}^n$





$$\begin{cases} u_{i+\frac{1}{2}}^{L} = \text{nonlinear}(u_i, u_{i-1}) \\ u_{i+\frac{1}{2}}^{R} = \text{nonlinear}(u_{i+1}, u_{i+2}) \end{cases}$$

Conditions for total variation diminishing scheme

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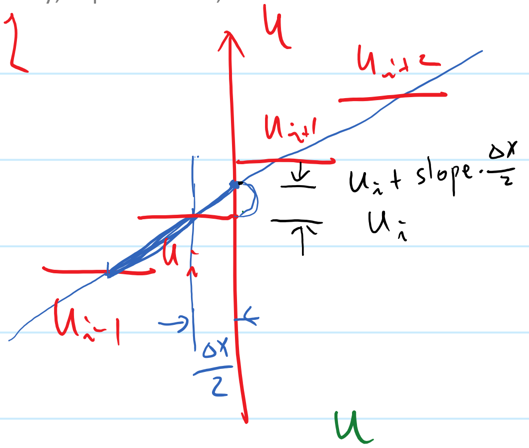
Case 1

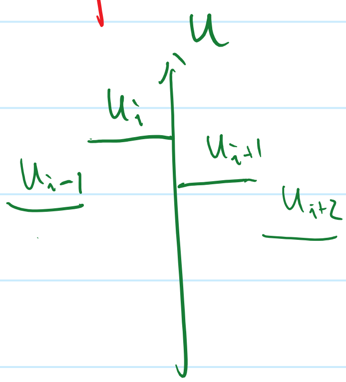


$$u_i - u_{i-1} = u_{i+1} - u_i = u_{i+2} - u_{i+1}$$

$$U^{L,R}_{i+\frac{1}{2}} = \frac{u_i + u_{i+1}}{2}$$

$$U^{L}_{i+\frac{1}{2}} = u_i + \underbrace{\frac{u_i - u_{i-1}}{\Delta x}}_{\text{slope}}\frac{\Delta x}{2}\cdot\phi$$

Case II



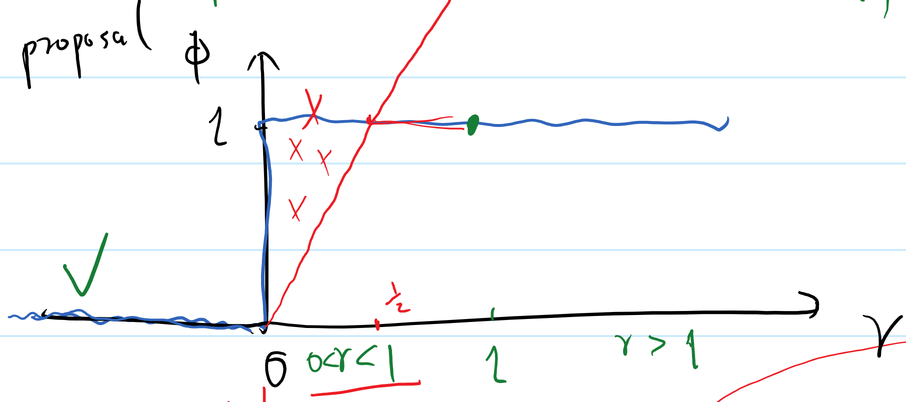
$$U^{L}_{i+\frac{1}{2}} = u_i + \frac{u_i - u_{i-1}}{\Delta x}\frac{\Delta x}{2}\cdot\phi$$

$\phi(r)$ is 1 for smooth solution $\quad r = 1$

0 for local extrema $\quad r < 0$

$\phi(r)$ depends on $\quad r = \frac{u_{i+1} - u_i}{u_i - u_{i-1}}$

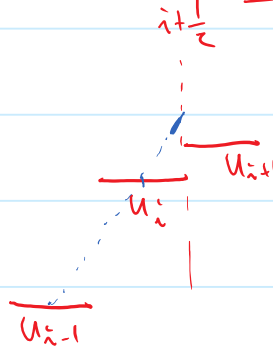
proposal



$$U^{L}_{i+\frac{1}{2}} = u_i + \frac{u_i - u_{i-1}}{\cancel{\Delta x}}\frac{\cancel{\Delta x}}{2}\cdot\phi \le u_{i+1}$$

$$\frac{u_i - u_{i-1}}{2}\phi \le u_{i+1} - u_i$$

$$\phi \le 2\frac{u_{i+1} - u_i}{u_i - u_{i-1}} = 2r$$



Flux limiters

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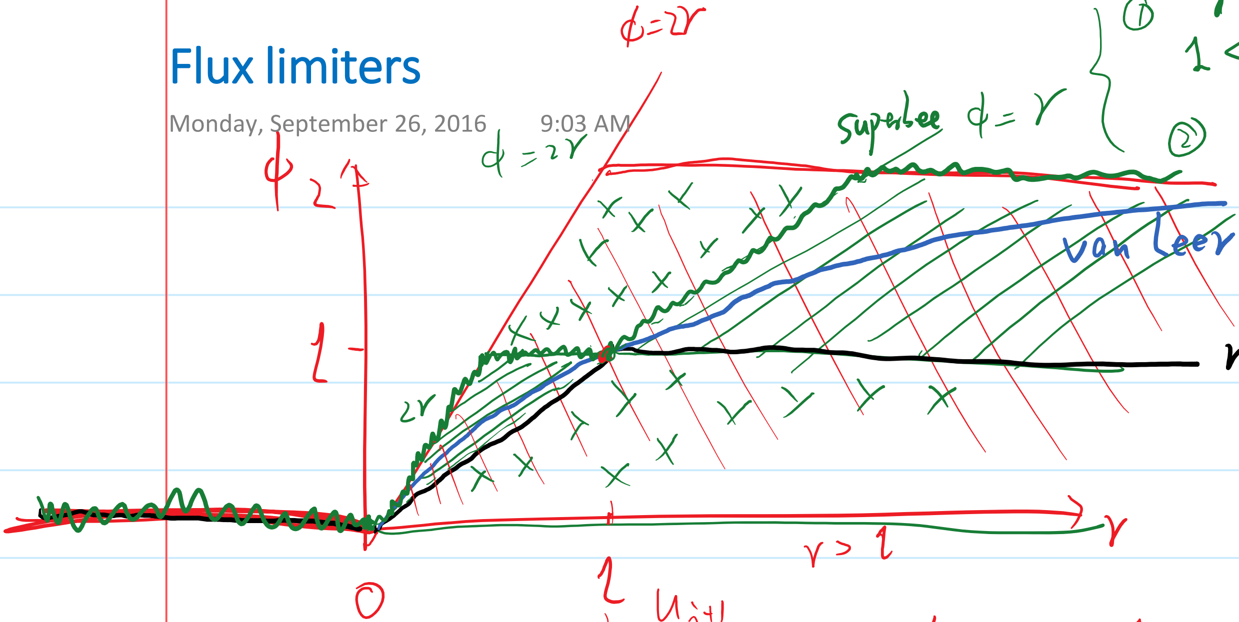
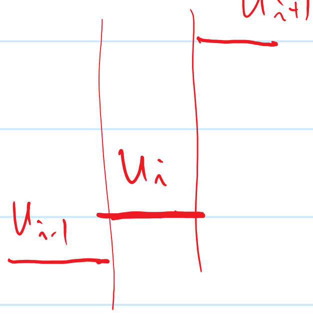


① $\phi > 1$:
$1 < \phi < \min(r, 2)$

② $0 < r < 1$
$r < \phi < \min(2r, 1)$

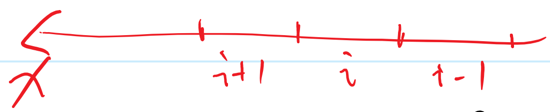
$$r := \frac{u_{i+1} - u_i}{u_i - u_{i-1}}$$



$$(u_i - u_{i-1})\,\phi(r) = (u_{i+1} - u_i)\,\phi\left(\frac{1}{r}\right)$$

symmetry

$$2r \geq \phi(r) = r\,\phi\left(\frac{1}{r}\right)$$



$$\phi_{minmod}(r) = \begin{cases} 0 & r \leq 0 \\ r & 0 \leq r \leq 1 \\ 1 & r \geq 1 \end{cases}$$

$$\phi_{superbee}(r) = \begin{cases} 0 & r \leq 0 \\ 2r & 0 \leq r \leq \frac{1}{2} \\ 1 & \frac{1}{2} \leq r \leq 1 \\ r & 1 \leq r \leq 2 \\ 2 & r \geq 2 \end{cases}$$

$$\phi_{vanLeer} = \begin{cases} 0 & r \leq 0 \\ 2r \ldots \end{cases}$$

$$\phi_{vanLeer} = \begin{cases} 0 & r \le 0 \\ \dfrac{2r}{1+r} & r \geq 0 \end{cases}$$

Flux limiters

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$$u^L_{i+\frac{1}{2}} = u_i + \frac{(u_i - u_{i-1})}{2}\,\phi\left(\frac{u_{i+1}-u_i}{u_i - u_{i-1}}\right)$$

$$u^R_{i+\frac{1}{2}} = u_{i+1} + \frac{u_{i+1}-u_{i+2}}{2}\,\phi\left(\frac{u_i - u_{i+1}}{u_{i+1}-u_{i+2}}\right)$$