Finite volume on non-uniform grid

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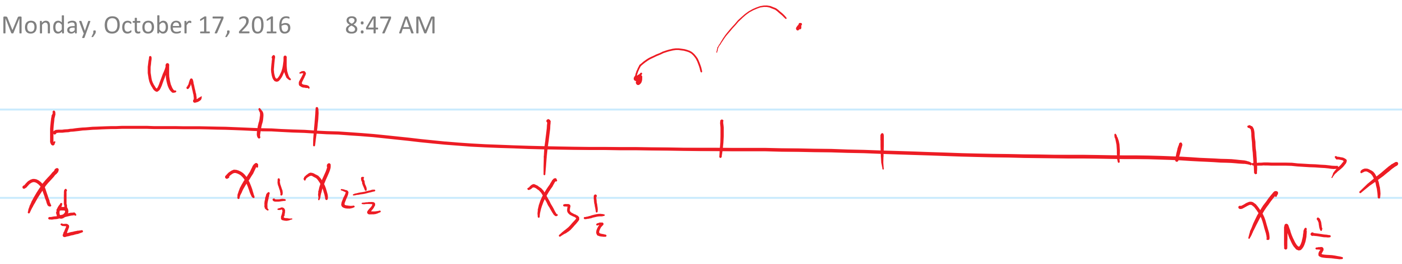

$$u_i = \frac{1}{\underbrace{x_{i+\frac12} - x_{i-\frac12}}_{\Delta x_i}} \int_{x_{i-\frac12}}^{x_{i+\frac12}} u \, dx$$

$$\frac{du_i}{dt} = \frac{1}{\Delta x_i}\left(f_{i-\frac12} - f_{i+\frac12} \right)$$

Finite volume in multiple dimensions

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$$\frac{\partial u}{\partial t} + \nabla\cdot\vec{f}(u) = g(u)$$ conservative form

$$\vec{f}(u) = \left(f_x(u),\ f_y(u),\ f_z(u)\right)$$

$$\nabla\cdot\vec{f}(u) := \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$

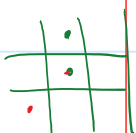
$$= \frac{df_x}{du}\frac{\partial u}{\partial x} + \frac{df_y}{du}\cdot\frac{\partial u}{\partial y} + \frac{df_z}{du}\frac{\partial u}{\partial z}$$

$$= \frac{d\vec{f}}{du}\cdot\nabla u$$ primitive form

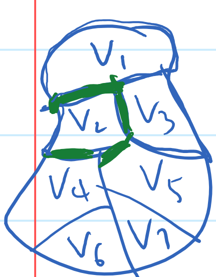
$$\nabla u := \left(\frac{\partial u}{\partial x},\ \frac{\partial u}{\partial y},\ \frac{\partial u}{\partial z}\right)$$

integrate

$$\int_V \frac{\partial u}{\partial t} + \nabla\cdot\vec{f}(u)\,dV = \int_V g(u)\,dV$$



$$\frac{\frac{d}{dt}\int_V u\,dV}{V} + \frac{\int_{\partial V}\vec{n}\cdot\vec{f}(u)\,ds}{V} = \frac{\int_V g(u)\,dV}{V}$$



$$\frac{d}{dt}u_i + \frac{1}{V_i}\int_{\partial V_i}\vec{n}\cdot\vec{f}(u)\,ds = g_i$$ exact

$$u_i := \frac{1}{V_i}\int_{V_i} u\,dV \qquad \int_{B_{ij}}\vec{n}\,ds =: \vec{n}_{ij}\cdot S_{ij}$$

Approximation

$$\frac{du_i}{dt} + \frac{1}{V_i}\sum_{j\in nbr(i)}\vec{n}_{ij}\cdot\vec{f}_{ij}\,S_{ij} = g(u_i)$$

$$\vec{f}_{ij} \approx f_{numerical}(u_i, u_j)$$

Different types of meshes

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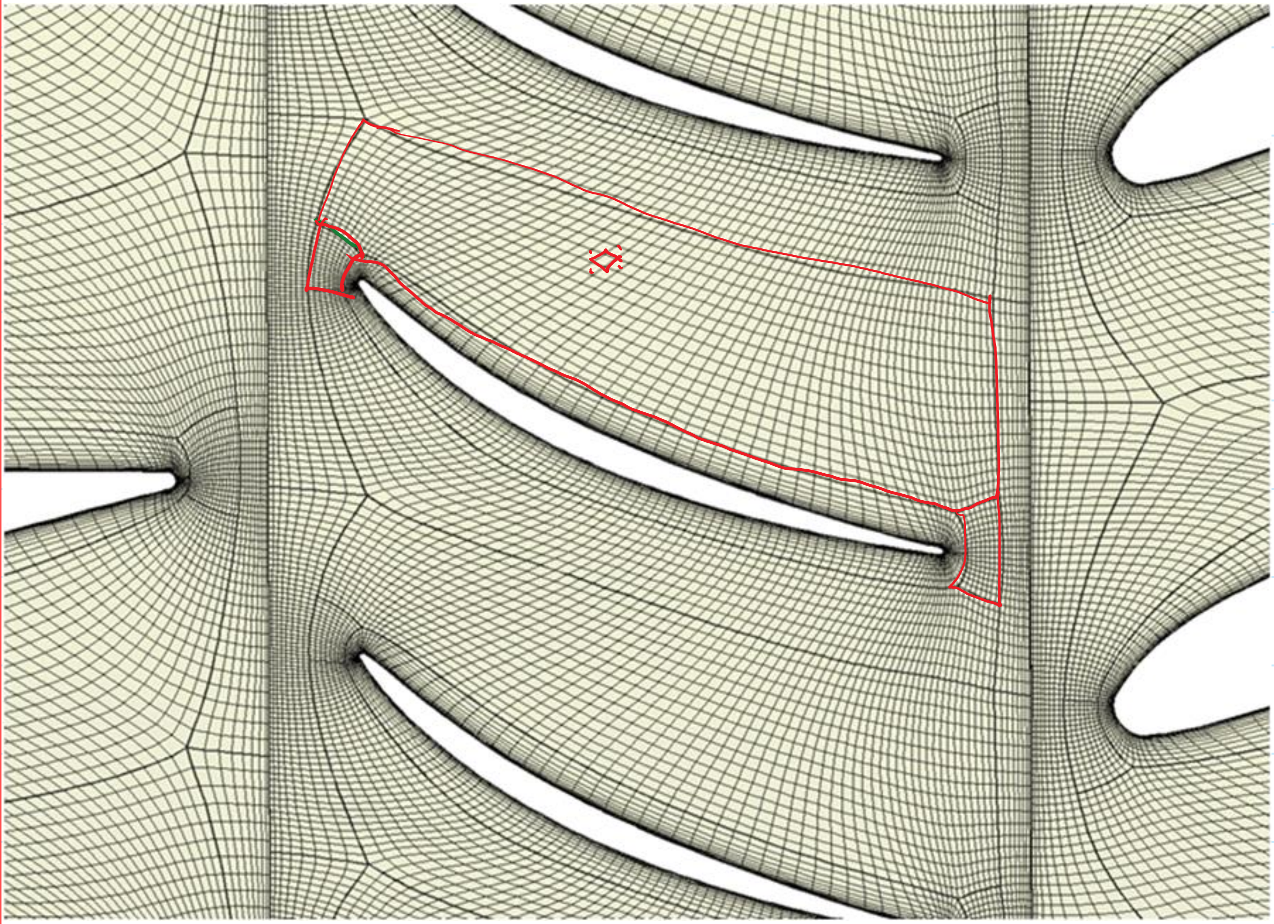

Different types of meshes

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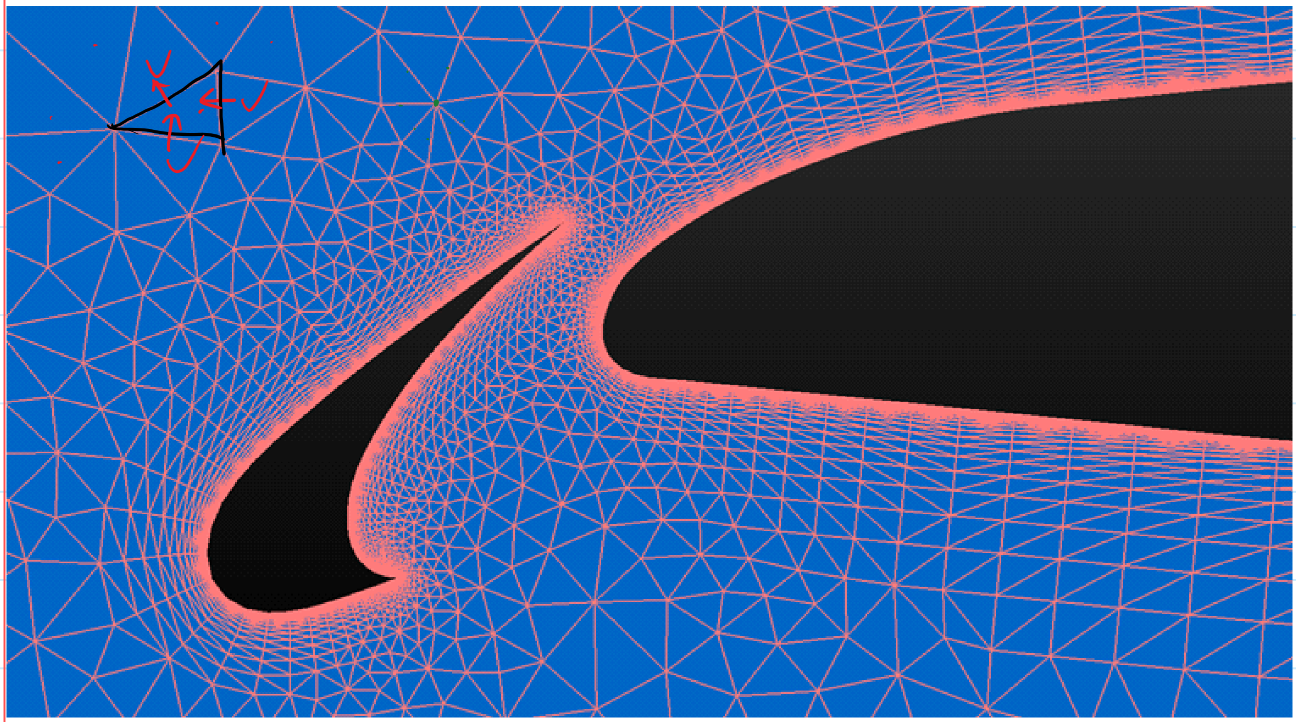

Different types of meshes

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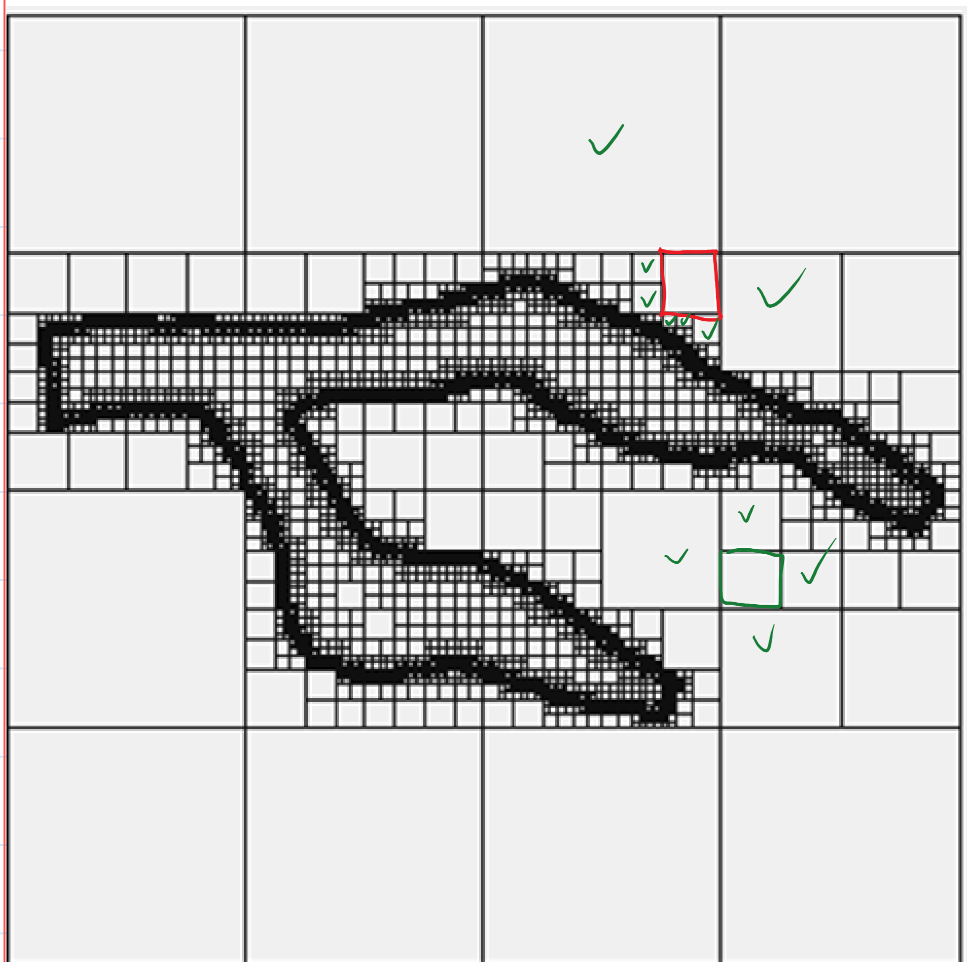

Finite volume in multiple dimensions -- data structure

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Every interface must know its neighboring cells.

Go through all (i, j), compute $\vec{f}_{ij} = f_{numerical}(u_i, u_j)$

Also must store $\vec{n}_{ij} \cdot S_{ij}$

compute $\vec{f}_{ij} \cdot \vec{n}_{ij} S_{ij}$

set all $\frac{du_i}{dt} = 0$

$$\begin{cases} \frac{du_i}{dt} = \frac{du_i}{dt} - \frac{1}{V_i} \vec{f}_{ij} \cdot \vec{n}_{ij} S_{ij} \\ \frac{du_j}{dt} = \frac{du_j}{dt} + \frac{1}{V_j} \vec{f}_{ij} \vec{n}_{ij} S_{ij} . \end{cases}$$

Total: $= \sum_i V_i \frac{du_i}{dt} = 0$

$$\begin{cases} \vec{n}_{ij} = - \vec{n}_{ji} \\ S_{ij} = S_{ji} \end{cases}$$

Systems of conservation laws

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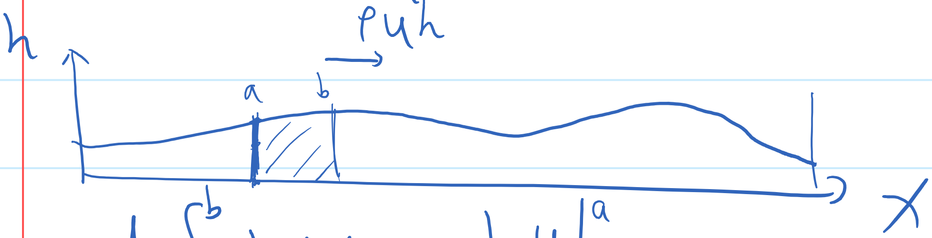
$$\begin{cases} \dfrac{\partial u_1}{\partial t} + \dfrac{\partial f_1(u_1, u_2)}{\partial x} = 0 \\[2ex] \dfrac{\partial u_2}{\partial t} + \dfrac{\partial f_2(u_1, u_2)}{\partial x} = 0 \end{cases}$$

Shallow water equation



$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_a^b h\,dx = h\cdot u \Big|_b^a \\[2ex] \dfrac{d}{dt}\displaystyle\int_a^b \rho u h\,dx = \rho h u^2 + \dfrac{h^2}{2}\cdot\rho g \Big|_b^a \end{cases}$$

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial hu}{\partial x} = 0 \\[2ex] \dfrac{\partial (\rho u h)}{\partial t} + \dfrac{\partial}{\partial x}\left(\rho h u^2 + \dfrac{\rho g h^2}{2}\right) = 0 \end{cases}$$

$$m := \rho u h$$

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial (m/\rho)}{\partial x} = 0 \\[2ex] \dfrac{\partial m}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{m^2}{\rho h} + \dfrac{\rho g}{2} h^2\right) = 0 \end{cases}$$

Systems of conservation laws -- examples

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Scalar: characteristics: $\dfrac{df}{du}$

System: $\begin{pmatrix} \dfrac{df_h}{dh} & \dfrac{df_m}{dh} \\ \dfrac{df_h}{dm} & \dfrac{df_m}{dm} \end{pmatrix}$

primitive form.

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{df_h}{\partial h} \cdot \dfrac{\partial h}{\partial x} + \dfrac{df_h}{\partial m} \cdot \dfrac{\partial m}{\partial x} = 0 \\[2ex] \dfrac{\partial m}{\partial t} + \dfrac{\partial f_m}{\partial h} \cdot \dfrac{\partial h}{\partial x} + \dfrac{df_m}{\partial m} \cdot \dfrac{\partial m}{\partial x} = 0 \end{cases}$$

$$f_h = \frac{m}{\rho} \qquad f_m = \frac{m^2}{\rho h} + \frac{\rho g}{2} h^2$$

$$\begin{pmatrix} \dfrac{\partial f_h}{\partial h} & \dfrac{\partial f_m}{\partial h} \\[2ex] \dfrac{\partial f_h}{m} & \dfrac{\partial f_m}{\partial m} \end{pmatrix} = \begin{pmatrix} 0 & -\dfrac{m^2}{\rho h^2} + \rho g h \\[2ex] \dfrac{1}{\rho} & 2\dfrac{m}{\rho h} \end{pmatrix}$$

eigenvalues: $u \pm \sqrt{gh}$

Systems of conservation laws

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$$
\begin{cases}
\dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial x}(\rho u) = 0 \\[2ex]
\dfrac{\partial (\rho u)}{\partial t} + \dfrac{\partial}{\partial x}(\rho u^2 + P) = 0 \\[2ex]
\dfrac{\partial (\rho E)}{\partial t} + \dfrac{\partial}{\partial x}(\rho u E + P u) = 0
\end{cases}
$$

$\underline{P(\rho, u, E)}$