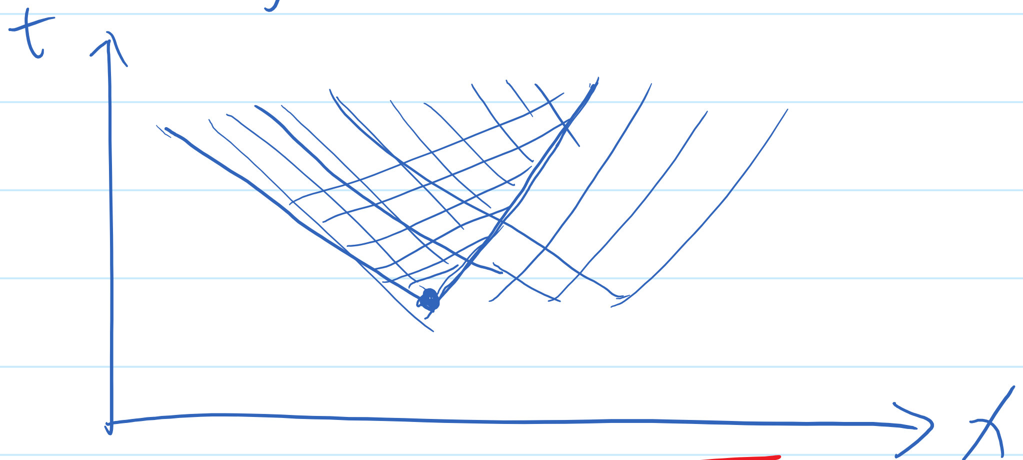
Systems of conservation laws -- flux reconstruction via numerical dissipation

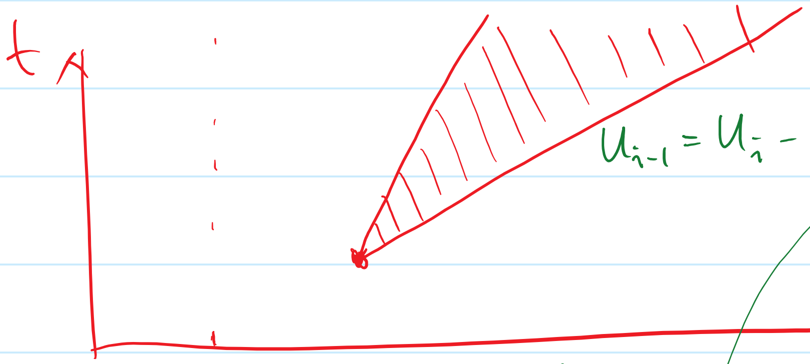
Monday, October 17, 2016 8:47 AM

$$\frac{\partial \vec{u}}{\partial t} + \left(\frac{\partial \vec{f}}{\partial \vec{u}}\right)\cdot\nabla \vec{u} = 0$$

eigenvalues $\longrightarrow$ characteristic speeds



shallow water eqn: $c = u \pm \sqrt{gh}$



$$U_{i-1} = U_i - \frac{\partial u}{\partial x}\cdot\Delta x + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

upwind — 1st order

$$\frac{dU_i}{dt} + \frac{U_i - U_{i-1}}{\Delta x} = 0$$

higher order

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \frac{\Delta x}{2}\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

upwinding if $c<0$. $\frac{dU_i}{dt} + c\frac{U_{i+1} - U_i}{\Delta x} = 0$

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \Delta x\cdot|c|\cdot\frac{\partial^2 u}{\partial x^2}$$

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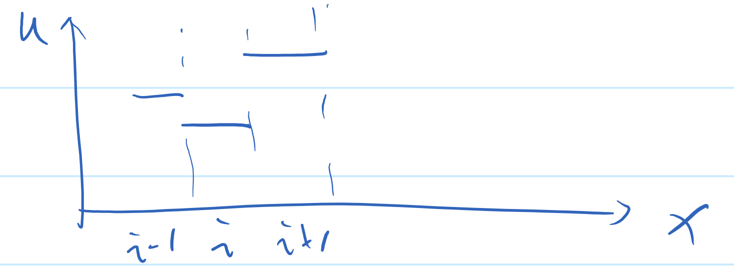
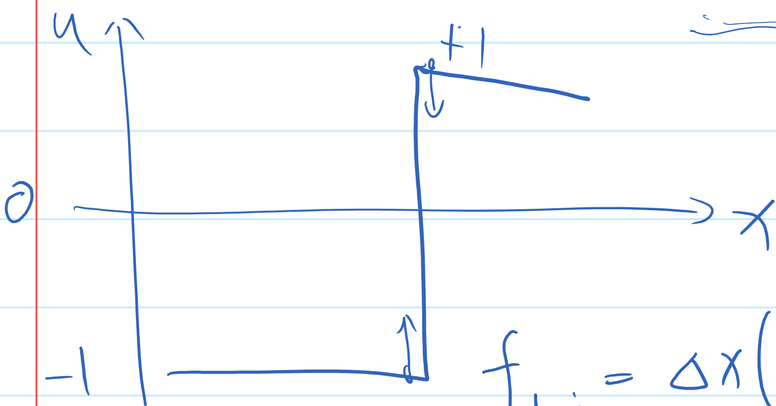


$$f_{i+\frac{1}{2}} = \frac{f(u_i)+f(u_{i+1})}{2} + \underbrace{\frac{\Delta x \left|\frac{df}{du}\right|_{max}}{2}}\frac{u_{i+1}-u_i}{\Delta x}$$

$$\frac{du_i}{dt} = \frac{f_{i-\frac{1}{2}} - f_{i+\frac{1}{2}}}{\Delta x} = \frac{f(u_{i+1}) - f(u_{i-1})}{2\Delta x} + \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}$$

$$\frac{\left|\frac{df}{du}\right| \Delta x}{2}$$



$$f_{dissip} = \Delta x\left(|u| + \sqrt{gh}\right)\frac{u_{i+1}-u_i}{\Delta x}$$

Systems of conservation laws -- JST scheme

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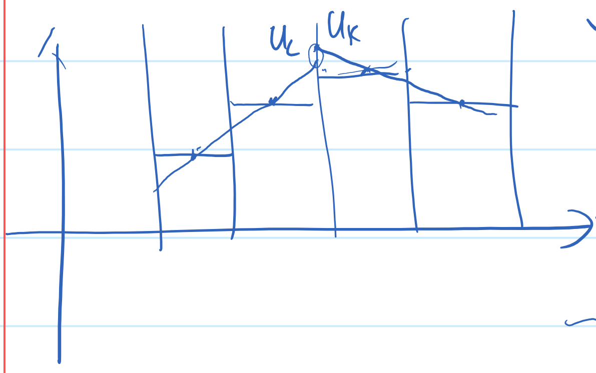

$$f = \frac{f(u_R)+f(u_L)}{2} + f_{dissip}$$

$$\phi(r) = 1$$

$$f_{dissip} = \Delta x \left|\frac{df}{du}\right| \frac{u_R - u_L}{\Delta x}$$

$$= \left|\frac{df}{du}\right| (u_R - u_L)$$

$$= \left|\frac{df}{du}\right| \left(u_{i+1} - \frac{\Delta x}{2}\frac{u_{i+2}-u_{i+1}}{\Delta x} - \left(u_i + \frac{\Delta x}{2}\frac{u_i - u_{i-1}}{\Delta x}\right)\right)$$

$$= \left|\frac{df}{du}\right| \left(-\frac{1}{2}u_{i+2} + \frac{3}{2}u_{i+1} - \frac{3}{2}u_i + \frac{1}{2}u_{i-1}\right)$$

$$= \left|\frac{df}{du}\right| \left(-\frac{1}{2}(u_{i+2} - 2u_{i+1} + u_i) + \frac{1}{2}(u_{i+1} - 2u_i + u_{i-1})\right)$$

$$= -\frac{1}{2}\Delta x^3 \left|\frac{df}{du}\right| \frac{\partial^3 u}{\partial x^3}$$

$$\frac{du_i}{dt} = \frac{-f_{i+\frac{1}{2}} + f_{i-\frac{1}{2}}}{\Delta x} = \frac{-f_{i+\frac{1}{2},c} + f_{i-\frac{1}{2},c}}{\Delta x} + \frac{-f_{i+\frac{1}{2},dissip} + f_{i-\frac{1}{2},dissip}}{\Delta x}$$

1. Construct limiter $\phi = \begin{cases} 0 & \text{nonsmooth} \\ 1 & \text{smooth} \end{cases}$
2. $\Delta x \frac{\partial u}{\partial x} \cdot \left|\frac{df}{du}\right| \rightarrow \phi$ $\quad \times (1-\phi)$
3. $\Delta x^3 \frac{\partial^3 u}{\partial x^3} \left|\frac{df}{du}\right|$ $\quad \times 1$

Systems of conservation laws -- JST scheme

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Finite element -- Linear space and basis

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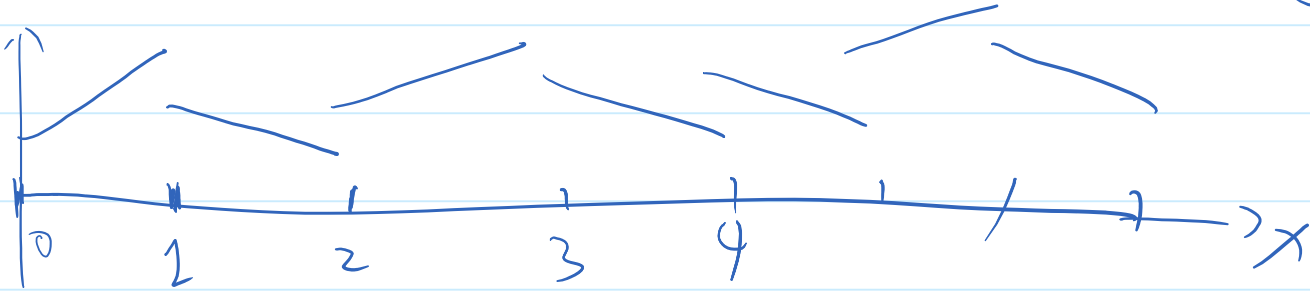
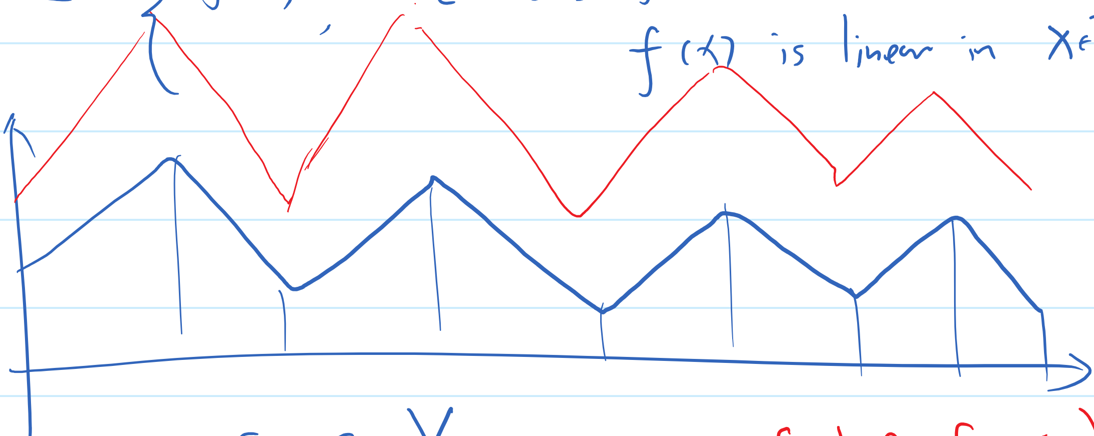
① $X = \{ f(x),\ x\in[0,1],\ \text{s.t. for all } i,\ f(x) \text{ is linear in } x\in[x_{i-1}, x_i] \}$



② $X = \{ f(x),\ x\in[0,1],\ f \text{ is continuous, and } \forall i,\ f(x) \text{ is linear in } x \in [x_{i-1},x_i]\}$



Linearity: $\forall f_1, f_2 \in X$ $\quad a_1 f_1 + a_2 f_2 \in X$ for $\forall a_1$ and a_2

Basis: $\{x_1, x_2, \dots x_n\}$ is a basis of X iff

① $\forall x\in X$, $x = \sum_{i=1}^n a_i x_i$

② $\forall x \in X$, $(a_i, i=1,\dots,n)$ s.t. $x=\sum a_i x_i$ is unique.

$\min_{x\in X} dist(f,x) \Longleftrightarrow \min_{a\in\mathbb{R}^n} dist(f, \sum_{i=1}^n a_i x_i)$

Finite element -- projection

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$$\min_{g \in X} \text{dist}(f, g) = \left(\int_a^b (f-g)^2 dx \right)^{\frac{1}{2}}$$

$$g^* - f \perp X := \forall g' \in X, \int_a^b (g^* - f) \cdot g' \,\cancel{dx=0}$$

$g' = g - g^*$

$$\int_a^b (f-g)^2 dx = \int_a^b (f - g^* - g')^2 dx$$

$$= \int_a^b \left((f-g^*)^2 - \underbrace{2g'(f-g^*)}_{=0} + \underbrace{g'^2}_{>0} \right) dx$$

if $\int_a^b g'(f-g^*)dx = 0$

then

$$\int_a^b (f-g)^2 dx = \int_a^b (f-g^*)^2 dx + \int_a^b g'^2 dx$$

i.e. g^* minimizes $\int_a^b (f-g)^2 dx$ among all $g \in X$

Optimization $\Longrightarrow$ $\int_a^b g'(f-g^*) dx = 0$ for all $g' \in X$

express $g^* = \sum_{i=1}^n a_i \phi_i$ where $\{\phi_i\}$ is a basis for X

if $\boxed{\int_{a_n}^b \phi_j \left(f - \sum_{i=1}^n a_i \phi_i\right) dx = 0}$ $j = 1, \dots, n$

then $\forall g' = \sum b_i \phi_i$ $\sum b_i \int_a^b \phi_i (f - \sum a_i \phi_i) dx = 0$

then $\forall g' = \sum_{j=1}^{n} b_j \phi_j, \quad \sum_{j=1}^{n} b_j \int_a^b \phi_j \left(f - \sum a_i \phi_i\right) dx = 0$

Finite element -- projection as a linear system

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$$\int_a^b \phi_j \left(f - \sum_{i=1}^{n} a_i \phi_i \right) dx = 0$$

$$\underline{\int_a^b \phi_j f \, dx} - \sum_{i=1}^{n} a_i \underline{\int_a^b \phi_j \phi_i \, dx} = 0$$

If we construct $\phi_1 \quad \phi_2 \quad \ldots$

define $\vec{b} = \begin{pmatrix} \int_a^b \phi_1 f \, dx \\ \int_a^b \phi_2 f \, dx \\ \vdots \\ \int_a^b \phi_n f \, dx \end{pmatrix}$

$$\bar{\bar{A}} = \begin{pmatrix} \int \phi_1 \phi_1 \, dx & \int \phi_1 \phi_2 \, dx & & \int \phi_1 \phi_n \, dx \\ \vdots & & & \\ \int \phi_n \phi_1 \, dx & & & \int \phi_n \phi_n \, dx \end{pmatrix}$$

$$\bar{\bar{A}} a = b$$