Finite element -- Linear basis and projection

Monday, October 17, 2016 8:47 AM

$f(x)$ known in $[a, b]$

Define a finite-dimensional space

$$X = \left\{ \underline{g(x) = \sum_{i=1}^{n} g_i \phi_i(x)}, \ \forall g_1, \dots, g_n \right\}$$

here $\{\phi_i, i=1, \dots, n\}$ is a basis

Best approximation of f in X

$$\min_{g \in X} \|f - g\|_2^{2} = \int_a^b (f-g)^2 \, dx$$

$$\min_{g_i, i=1,\dots,n} \|f - g\|_2^2 = \int_a^b \left(f - \sum g_i \phi_i\right)^2 dx$$

$$\Downarrow \text{ variational analysis}$$

$$\int_a^b \left(f - \sum g_i \phi_i\right) \cdot \phi_j \, dx = 0$$

$$\Downarrow$$

$$A g = b$$

Finite element -- projection of 1D Poisson's equation

Monday, October 17, 2016 8:47 AM

$$\frac{\partial^2 u}{\partial x^2} + f = 0 \qquad u(0) = u(1) = 0$$

Approach 1: Let $\forall g(x)$ in $[0,1]$ be smooth.

$$\int_0^1 g(x)\cdot\left(\frac{\partial^2 u}{\partial x^2} + f\right)dx = 0 \leftarrow$$

Restrict $\quad u = \sum_{i=1}^{n} u_i \phi_i(x)$

$$g(x) = \phi_j(x) \qquad j = 1, \ldots, n$$

$$\int_0^1 \phi_j(x)\cdot\left(\sum_{i=1}^{n} u_i \frac{\partial^2 \phi_i}{\partial x^2} + f\right)dx = 0$$

$$0 = \sum_{i=1}^{n} u_i\left(\int_0^1 \phi_j(x)\frac{\partial^2 \phi_i}{\partial x^2}\,dx\right) + \left(\int_0^1 \phi_j\cdot f\,dx\right)$$

$$A_{ij} = \int_0^1 \phi_j \frac{\partial^2 \phi_i}{\partial x^2}\,dx \qquad b_j = \int_0^1 \phi_j f\,dx$$

$$Au + b = 0$$

Finite element -- projection of 1D Poisson's equation

Monday, October 17, 2016 8:47 AM

Integration by parts

$$\int_0^1 g(x)\cdot\left(\frac{\partial^2 u}{\partial x^2}+f\right)dx = 0$$

$$\frac{d}{dx}\left(g\frac{\partial u}{\partial x}\right) = \frac{\partial g}{\partial x}\cdot\frac{\partial u}{\partial x} + g\cdot\frac{\partial^2 u}{\partial x^2}$$

$$\int_0^1 \frac{d}{dx}\left(g\frac{\partial u}{\partial x}\right)dx = g\cdot\frac{\partial u}{\partial x}\Big|_0^1$$

$$= \int_0^1 \frac{\partial g}{\partial x}\cdot\frac{\partial u}{\partial x}\,dx + \int_0^1 g\frac{\partial^2 u}{\partial x^2}\,dx$$

$$\int_0^1 g\,\cancel{x}\,f\,dx + g\cdot\frac{\partial u}{\partial x}\Big|_0^1 - \int_0^1 \frac{\partial g}{\partial x}\cdot\frac{\partial u}{\partial x} = 0$$

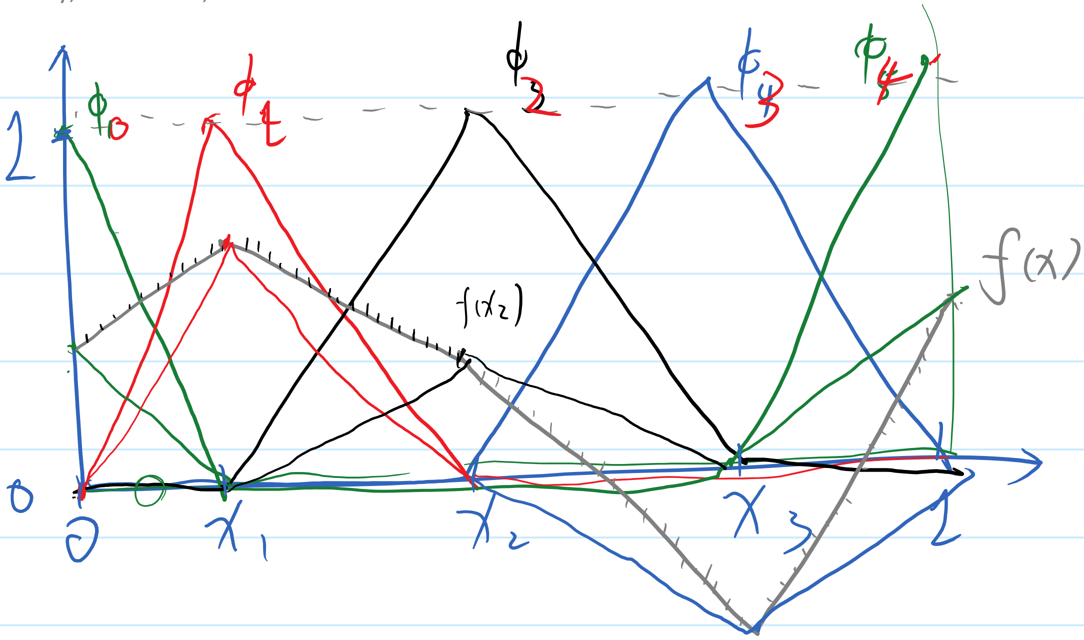
$$g \longrightarrow \phi_j$$

$$u \longrightarrow \sum_{i=1}^{n} u_i\phi_i \qquad \Rightarrow\quad Au+b=0$$

$$A_{ij} = -\int_0^1 \frac{\partial \phi_j}{\partial x}\cdot\frac{\partial \phi_i}{\partial x}\,dx + \phi_j\frac{\partial \phi_i}{\partial x}\Big|_0^1$$

Finite element -- basis functions for piecewise linear continuous space

Monday, October 17, 2016 8:47 AM



$$f(x) = f(0)\,\phi_0(x) + f(x_1)\,\phi_1(x) + f(x_2)\,\phi_2(x) + f(x_3)\,\phi_3(x) + f(x_4)\,\phi_4(x)$$

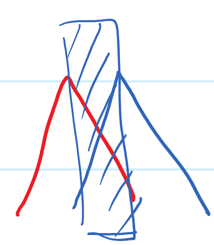
If we know $u(0) = u(1) = 0$

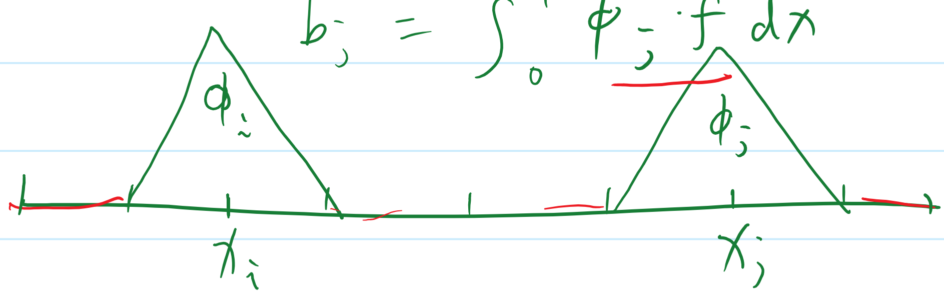
$$u(x) = \sum_{i=1}^{n-1} u_i \phi_i(x)$$ after removing two basis functions.

for $j = 1, \dots, n-1$, $\phi_j \frac{\partial \phi_i}{\partial x}\Big|_0^1 = 0$.

So, $$A_{ij} = \int_0^1 \frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x}\,dx$$ nonzero if $|i-j| \le 1$

$$b_j = \int_0^1 \phi_j \cdot f\,dx$$





Finite element -- linear functional and bilinear form

Monday, October 17, 2016 8:47 AM

$$A_{ii} = \int_0^1 \left(\frac{\partial \phi_i}{\partial x}\right)^2 dx = \int_{x_{i-1}}^{x_i} \left(\frac{\partial \phi_i}{\partial x}\right)^2 dx + \int_{x_i}^{x_{i+1}} \left(\frac{\partial \phi_i}{\partial x}\right)^2 dx$$

$$\left(\frac{1}{x_i - x_{i-1}}\right)^2 \qquad \left(\frac{+1}{x_{i+1} - x_i}\right)^2$$

$$= \frac{1}{x_i - x_{i-1}} + \frac{1}{x_{i+1} - x_i}$$

$$A_{i\,i+1} = \int_0^1 \left(\frac{\partial \phi_i}{\partial x}\right)\left(\frac{\partial \phi_{i+1}}{\partial x}\right) dx = \int_{x_i}^{x_{i+1}} \left(\frac{\partial \phi_i}{\partial x}\right)\left(\frac{\partial \phi_{i+1}}{\partial x}\right) dx$$

$$-\frac{1}{x_{i+1} - x_i} \qquad \frac{1}{x_{i+1} - x_i}$$

$$= -\int_{x_i}^{x_{i+1}} \frac{1}{(x_{i+1} - x_i)^2} dx$$

$$= -\frac{1}{x_{i+1} - x_i}$$