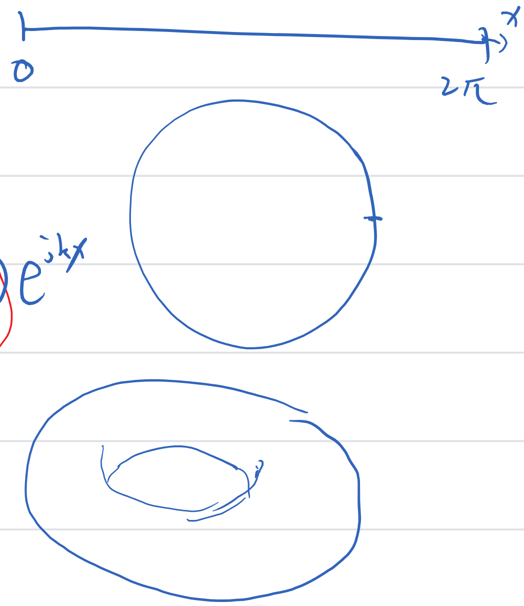
# Von Neumann Stability Analysis

Monday, September 26, 2016 8:54 AM

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}$$




$$u = \sum_{k=-\infty}^{\infty} \hat{u}_k e^{jkx}$$

$$\sum_k \frac{d\hat{u}_k}{dt} e^{jkx} = \kappa \sum_k \hat{u}_k(-k^2) e^{jkx}$$

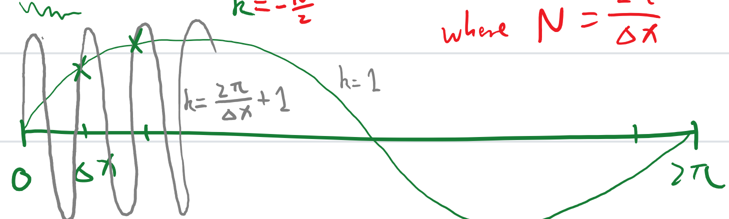
$$\frac{d\hat{u}_k}{dt} = \kappa(-k^2)\hat{u}_k$$

After discretization

$$\frac{du_i}{dt} = \kappa \frac{u_{i+1} + u_{i-1} - 2u_i}{\Delta x^2}$$

$$u_i := u(i\Delta x) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \hat{u}_k e^{jki\Delta x}$$

where $N = \frac{2\pi}{\Delta x}$




$$e^{j\left(\frac{2\pi}{\Delta x}+k\right)i\Delta x} = \underbrace{e^{j2\pi \cdot i}}_{=1} e^{jki\Delta x}$$

$$\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \frac{d\hat{u}_k}{dt} e^{jki\Delta x} = \kappa \frac{\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \hat{u}_k \left(e^{jk(i+1)\Delta x} + e^{jk(i-1)\Delta x} - 2e^{jki\Delta x}\right)}{\Delta x^2}$$

Monday, September 26, 2016 9:59 AM

$$e^{jk(i+1)\Delta x} + e^{jk(i-1)\Delta x} - 2e^{jki\Delta x}$$

$$\parallel \qquad\qquad \parallel$$

$$e^{jki\Delta x}e^{jk\Delta x} \qquad e^{jki\Delta x}e^{-jk\Delta x}$$

$$= e^{jki\Delta x}\left(e^{jk\Delta x} + e^{-jk\Delta x} - 2\right)$$

$$\boxed{\sum_k \frac{d\hat{u}_k}{dt} e^{jki\Delta x} = K\sum_k \hat{u}_k e^{jki\Delta x} \frac{e^{jk\Delta x} + e^{-jk\Delta x} - 2}{\Delta x^2}}$$

Discrete:
$$\frac{d\hat{u}_k}{dt} = K\frac{e^{jk\Delta x} + e^{-jk\Delta x} - 2}{\Delta x^2}\hat{u}_k$$

$$\hat{u}_k = e^{\,!\,t}$$

$$\frac{2\cos k\Delta x - 2}{\Delta x^2} \overset{\Delta x k \ll 1}{\approx} -k^2$$

Analytical:
$$\frac{d\hat{u}_k}{dt} = K(-k^2)\,\hat{u}_k$$

$$\begin{pmatrix} e^{jk\Delta x} \\ + \; e^{-jk\Delta x} \\ - \; 2 \end{pmatrix} \begin{matrix} = \cos k\Delta x + i\sin k\Delta x \\ = \cos k\Delta x - i\sin k\Delta x \\ = 2\cos k\Delta x - 2 \end{matrix}$$

# Space-time discretization, CFL Condition

Monday, September 26, 2016 9:01 AM

Forward Euler is stable if

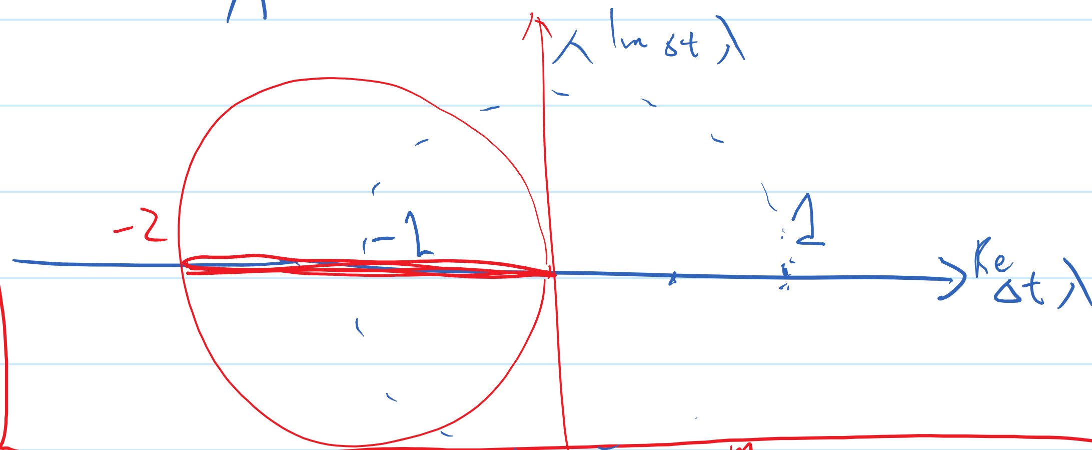
$$\frac{x^{k+1}-x^k}{\Delta t} = \lambda x^k$$

$x(t)$ not space

$k$: time step not space

$$\frac{x^{k+1}}{\Delta t} = \frac{x^k}{\Delta t} + \lambda x^k = \left(\frac{1}{\Delta t}+\lambda\right)x^k$$

$$x^{k+1} = (1+\underbrace{\Delta t\,\lambda}_{})\,x^k$$




$$\frac{u_i^m - u_i^m}{\Delta t} = K\,\frac{u_{i+1}^m + u_{i-1}^m - 2u_i^m}{\Delta x^2}$$

$$u_i^m = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \hat{u}_k^m\, e^{jki\Delta x}$$

$$\frac{\hat{u}_k^{m+1} - \hat{u}_k^m}{\Delta t} = \left(K\,\frac{2\cos k\Delta x - 2}{\Delta x^2}\right)\hat{u}_k^m$$

$\forall k = -\frac{N}{2}, \dots, \frac{N}{2}-1$

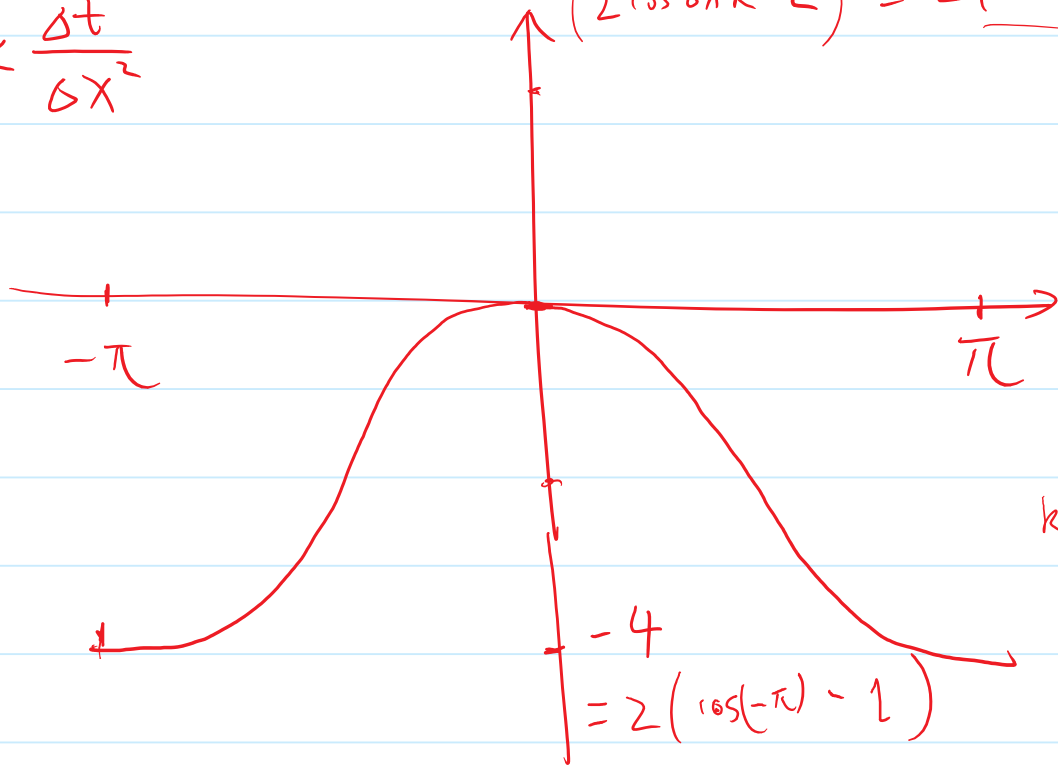
$$\left|1 + \Delta t\, K\,\frac{2\cos k\Delta x - 2}{\Delta x^2}\right| < 1$$

Monday, September 26, 2016 10:23 AM

$$\left| 1 + \Delta t \, k \frac{(2\cos \Delta x k - 2)}{\Delta x^2} \right| < 1$$

$$C = k \frac{\Delta t}{\Delta x^2}$$

$$(2\cos \Delta x k - 2) = 2(\cos - 1)$$




$$\Delta x = \frac{2\pi}{N}$$

$$k = -\frac{N}{2} \dots \frac{N}{2} - 1$$

$$-1 < 1 + C \cdot [-4, 0] < 1 \iff 0 < C < \frac{1}{2}$$

CFL Condition

$$k \cdot \frac{\Delta t}{\Delta x^2} < \frac{1}{2}$$

# Linear Advection Equation

Monday, September 26, 2016 10:33 AM

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u_i}{\partial t} + U\,\frac{u_{i+1} - u_{i-1}}{2\Delta x} = 0$$

$$u_i = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \hat{u}_k\, e^{jki\Delta x}$$

$$e^{jki\Delta x}\left(e^{jk\Delta x} - e^{-jk\Delta x}\right)$$

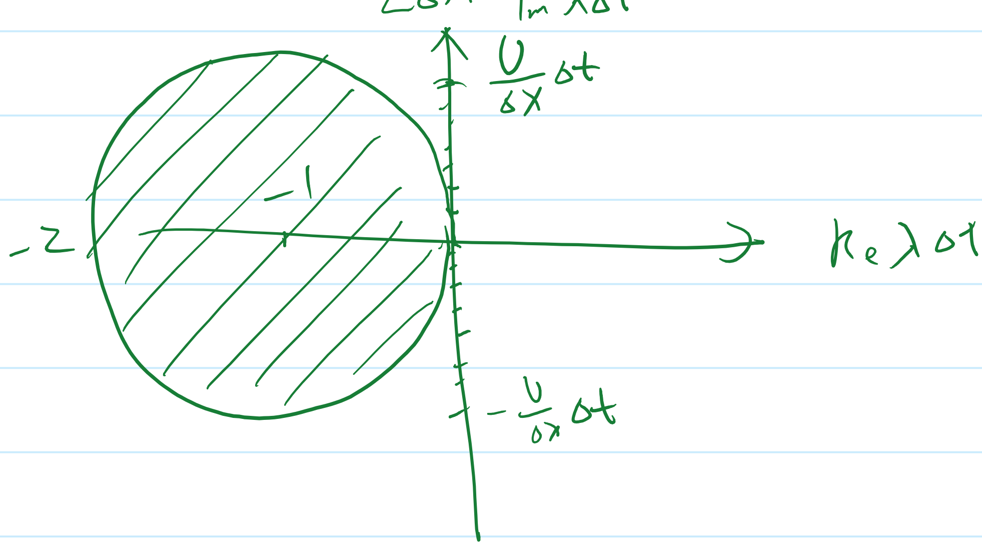
$$\parallel$$

$$\sum_k \frac{d\hat{u}_k}{dt} e^{jki\Delta x} + U\,\frac{\sum_k \hat{u}_k\left(e^{jk(i+1)\Delta x} - e^{jk(i-1)\Delta x}\right)}{2\Delta x} = 0$$

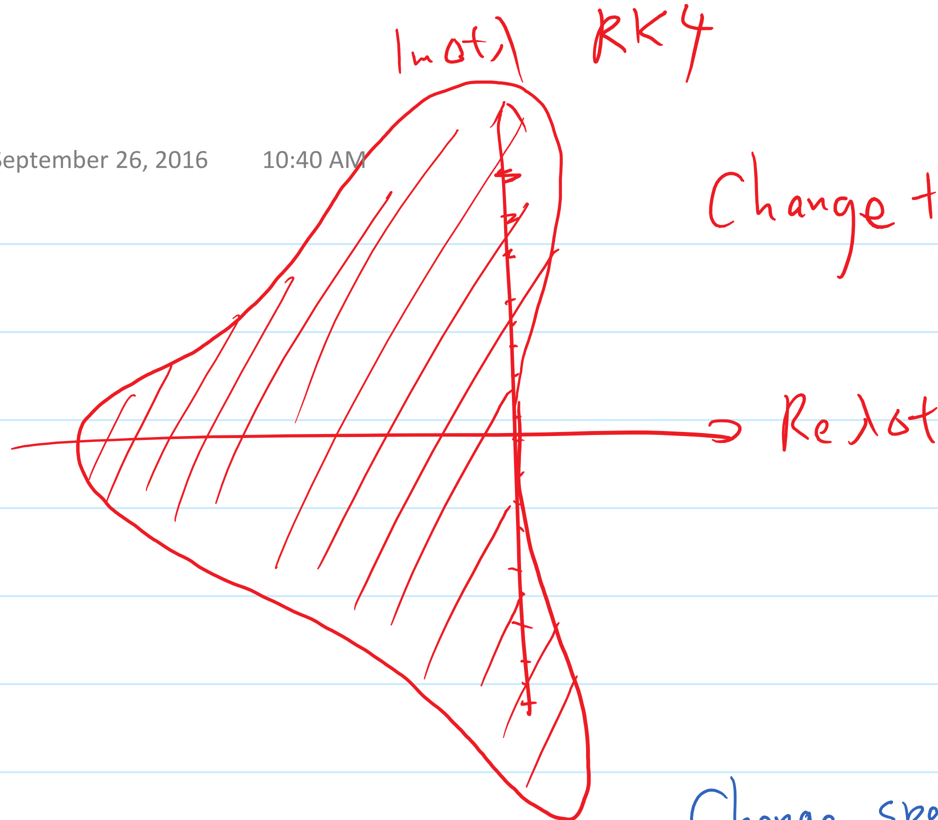
$$\frac{d\hat{u}_k}{dt} + U\,\hat{u}_k\,\frac{e^{jk\Delta x} - e^{-jk\Delta x}}{2\Delta x} = 0$$

$$\frac{d\hat{u}_k}{dt} = \underbrace{-U\,\frac{e^{jk\Delta x} - e^{-jk\Delta x}}{2\Delta x}}_{\parallel}\,\hat{u}_k$$

$$-U\,\frac{2j\sin k\Delta x}{2\Delta x} = -Uj\,\frac{\sin k\Delta x}{\Delta x}$$



Monday, September 26, 2016 10:40 AM




Change time discretization

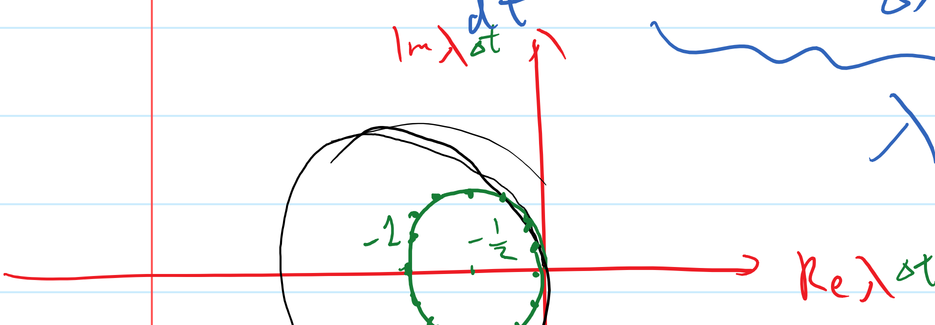
Change spatial Discretization

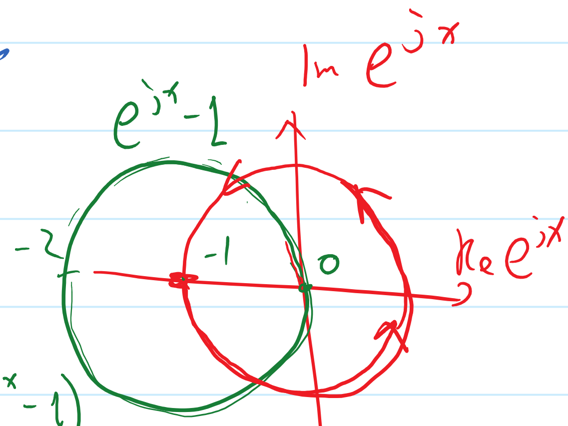
$$\frac{du_i}{dt} + U\frac{u_{i+1} - u_i}{\Delta x} = 0$$

$$u_i = \sum_k \hat{u}_k e^{jki\Delta x}$$

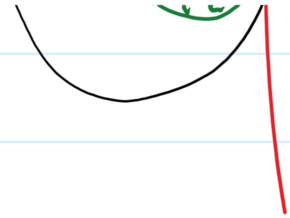
$$\frac{d\hat{u}_k}{dt} + U\hat{u}_k\frac{e^{jk\Delta x} - 1}{\Delta x} = 0$$

$$\frac{d\hat{u}_k}{dt} = \underbrace{-U\frac{(e^{jk\Delta x} - 1)}{\Delta x}}_{\lambda}\hat{u}_k$$

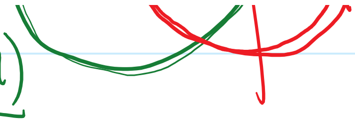



$$\lambda\Delta t = \left(-\frac{U\Delta t}{\Delta x}\right)(e^{jx} - 1)$$

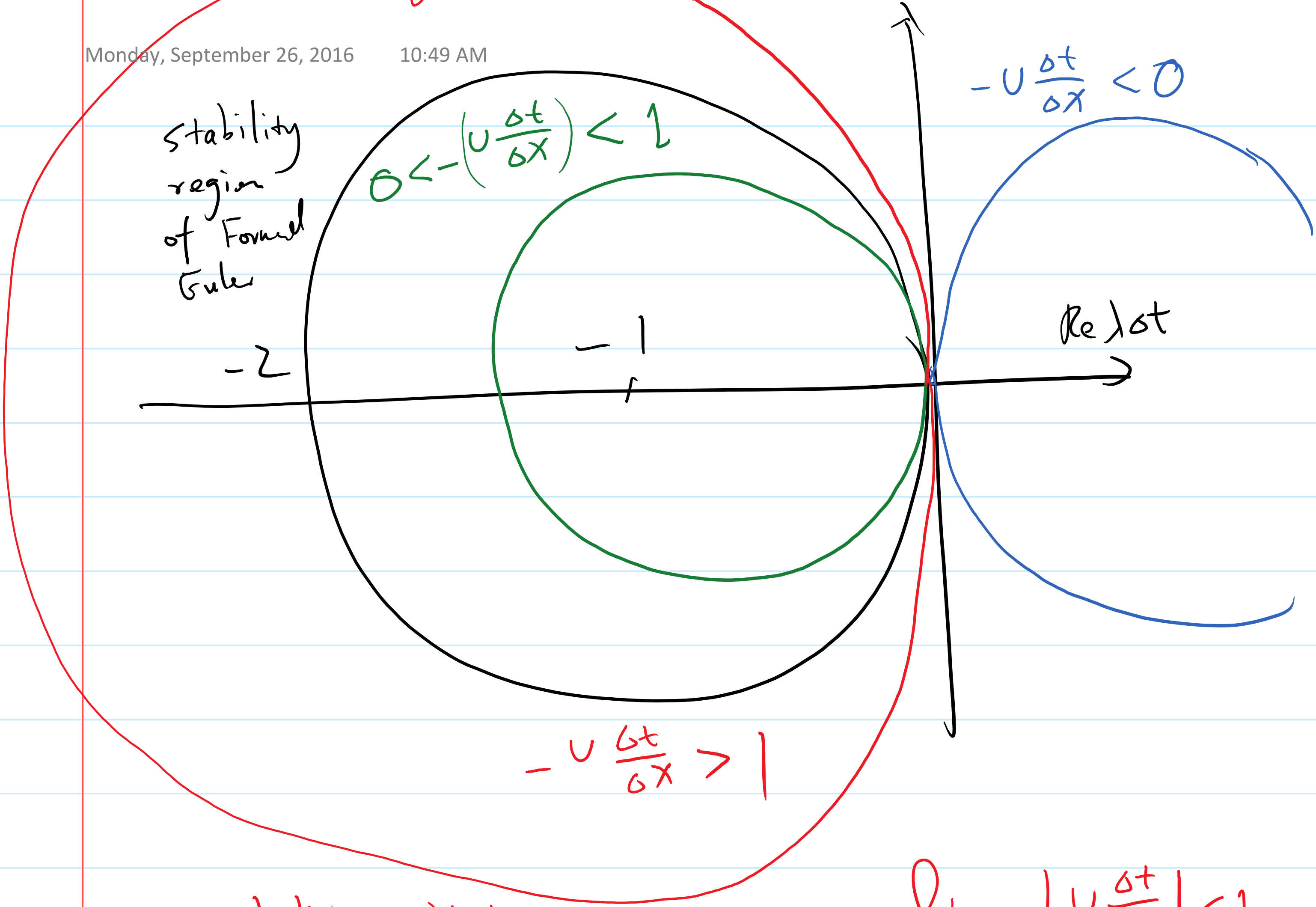


$$\lambda \Delta t = \left(-\frac{U \Delta t}{\Delta x}\right) \underbrace{\left(e^{ix} - 1\right)}$$



if $-\frac{U \Delta t}{\Delta x} = 0.5$

$$\frac{du_i}{dt} + U\frac{u_{i+1}-u_i}{\Delta x} = 0$$




Stability criterion: $U < 0$ & $\underbrace{\left|U\frac{\Delta t}{\Delta x}\right|}_{\text{CFL number}} < 1$

If $U > 0$

$$\frac{du_i}{dt} + U\frac{u_i - u_{i-1}}{\Delta x} = 0$$

$U > 0$ & $\left|U\frac{\Delta t}{\Delta x}\right| < 1$

# Upwinding