Finite element -- Review and demo

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$\frac{1}{2}x(1-x)$

$\frac{1}{2}x(1-x) + (1-x)$

$$\frac{\partial^2 u}{\partial x^2} + f = 0 \qquad u(0) = u(1) = 0$$

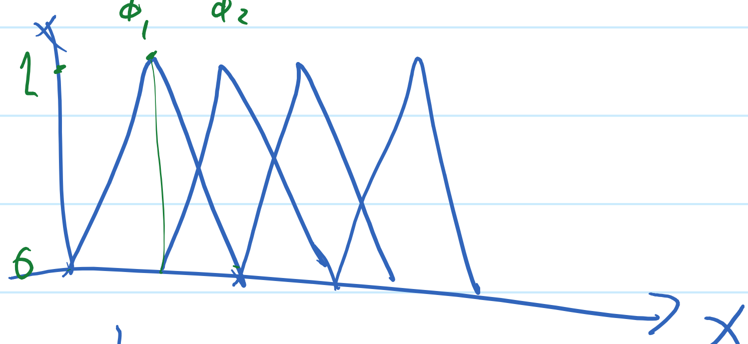
$$\int_a^b v\left(\frac{\partial^2 u}{\partial x^2} + f\right) dx = 0 \qquad \forall\, v$$

$$-\int_a^b \frac{\partial v}{\partial x}\frac{\partial u}{\partial x}\, dx + \int_a^b v \cdot f\, dx = 0 \qquad \text{IBP}$$

$$u = \sum_{i=1}^{n} u_i \phi_i(x) + u_0 \phi_0(x) \qquad v = \phi_j(x)$$

$$-\int_a^b u_0 \frac{\partial \phi_0}{\partial x}\cdot\frac{\partial \phi_j}{\partial x} - \sum u_i \underbrace{\int_a^b \frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x}\, dx}_{A_{ij}} + \underbrace{\int_a^b f \cdot \phi_j(x)\, dx}_{b_j} = 0$$

$$Au = b$$



$$A_{ii} = \frac{1}{x_i - x_{i-1}} + \frac{1}{x_{i+1} - x_i}$$

if $f = 1$, $\qquad b_j = \dfrac{x_{j+1} - x_{j-1}}{2}$

if $f = \sin \pi x \qquad b_j = \int^{x_j} \dfrac{x - x_{j-1}}{x_j - x_{j-1}} \cdot \sin \pi x\, dx$

1) $J = \sin \pi x$

$$b_j = \int_{x_{j-1}}^{x_j} \frac{x - x_{j-1}}{x_j - x_{j-1}} \cdot \sin \pi x \, dx + \int_{x_j}^{x_{j+1}} \frac{-x + x_{j+1}}{x_{j+1} - x_j} \sin \pi x \, dx$$

Finite element -- Sobolev space

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$$L^2_\Omega = \left\{ f : \int_\Omega f^2 \, dx < \infty \right\}$$

$$f, g \in L^2_\Omega$$

$$\int (f+g)^2 \, dx = \int \underline{f^2} + 2fg + \underline{g^2} \, dx < \infty$$

$$\int f g \, dx \le \frac{1}{2} \int \underline{f^2} + \underline{g^2} \, dx$$

s.o, $f, g \in L^2_\Omega$

then $af + bg \in L^2_\Omega$

$$H^1_\Omega = \left\{ f : \int_\Omega f^2 + \underline{\|\nabla f\|^2} \, dx < \infty \right\}$$

$$\Updownarrow$$

$$f \in L^2_\Omega \;\; \& \;\; \nabla f \in L^2_\Omega$$

discontinuous function

$f = \sqrt{x} \quad x \in [0,1]$

in L^2 but not H^1

$$H^{(2)} = \left\{ f : \int_\Omega f^2 + \|\nabla f\|^2 + \|\nabla \cdot \nabla f^2\|^2 \, dx < \infty \right\}$$

Finite element -- Bilinear form and Linear functional

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$$\boxed{-\int_a^b \frac{\partial v}{\partial x}\frac{\partial u}{\partial x}\,dx + \int_a^b v\cdot f\,dx = 0}$$

<u>weak</u> form of $\frac{\partial^2 u}{\partial x^2} + f = 0$ is

with $u(0)=u(1)=0$

Find $u \in H^1_{[a,b],0} = \{ f \in H^1_{[a,b]},\ \text{s.t } f(0)=f(1)=0 \}$

s.t for all $v \in H^1_{[a,b],0}$

$$\boxed{-\int_a^b \quad + \int_a^b \quad = 0}$$ is true

It holds even if f is distribution

e.g. $f(x) = \delta(x-0.5)$ in $[0,1]$.

$$\int_a^b f\,v\,dx = v(0.5)$$

$$\int_a^b \frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial x}\,dx = \underline{v(0.5)}$$

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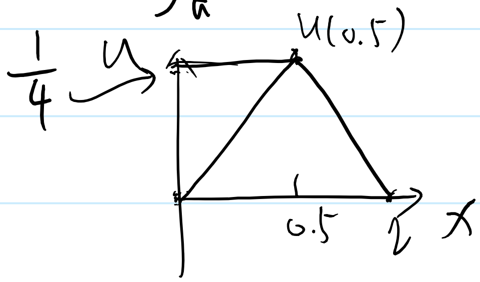
u has to be liner in $[0, 0.5]$ & $[0.5, 1]$

$a=0$

$b=1$

$$\int_a^b \frac{\partial v}{\partial x}\frac{\partial u}{\partial x}\,dx = \frac{\partial u}{\partial x}\Big|_L \int_0^{0.5} \frac{\partial v}{\partial x}\,dx + \frac{\partial u}{\partial x}\Big|_R \int_{0.5}^1 \frac{\partial v}{\partial x}\,dx$$



$$= 2\,u(0.5)\cdot v(0.5)$$

$$-2\,u(0.5)\cdot(-v(0.5))$$

$$= 4\cdot u(0.5)\cdot v(0.5) = v(0.5)$$

Finite element -- The weak form

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$u \in X_u$

Bilinear form $a(u, v)$ $v \in X_v$

example $-\int_a^b \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x}\, dx = a(u, v)$

$$a(c_1 u_1 + c_2 u_2, v) = c_1 a(u_1, v) + c_2 a(u_2, v)$$

$$a(u, c_1 v_1 + c_2 v_2) = c_1 a(u, v_1) + c_2 a(u, v_2)$$

Linear functional $l(v)$. $v \in X_v$

$$\int_a^b f \cdot v\, dx = l(v)$$

$$v(0.5) = l(v)$$

$$l(c_1 v_1 + c_2 v_2) = c_1 l(v_1) + c_2 l(v_2)$$

Weak form: find $u \in X_u$. s.t

$$a(u, v) + l(v) = 0 \quad \text{for any } v \in X_v$$

Special case is $X_u = X_v$. Galerkin

Finite element -- The weak form

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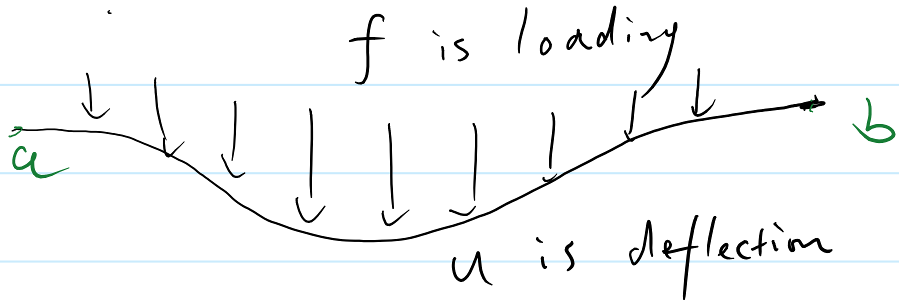
$$\frac{\partial^4 u}{\partial x^4} + f = 0$$



$$\int_a^b v \cdot \left(\frac{\partial^4 u}{\partial x^4} + f \right) dx = 0 \qquad 0 = u = \frac{\partial u}{\partial x} \text{ at } a, b.$$

$X_u = H^2_{[a,b]\,0}$ satisfy these b.c.s

$$\int_a^b v \frac{\partial^4 u}{\partial x^4} dx = - \int \frac{\partial v}{\partial x} \cdot \frac{\partial^3 u}{\partial x^3} dx$$

$$= \int \frac{\partial^2 v}{\partial x^2} \frac{\partial^2 u}{\partial x^2} dx =: \underline{a(u,v)}$$

$$l(v) = \int_a^b v \cdot f \, dx$$

Finite element -- Natural boundary condition

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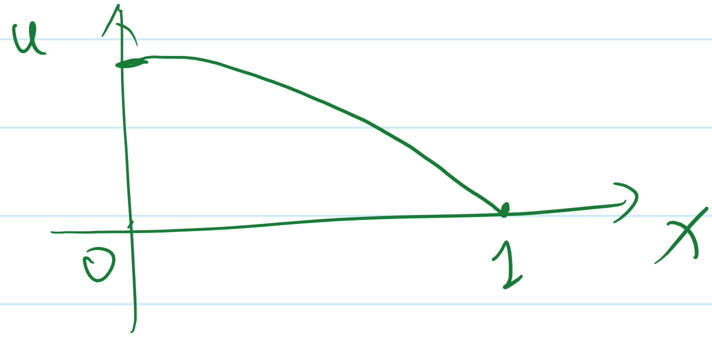
$$\frac{\partial u}{\partial x}\Big|_0 = 0 \qquad u(1)=0$$

Neumann on the left — Dirichlet on the right



$$\frac{\partial^2 u}{\partial x^2} + f = 0$$

$$\rightarrow X_u = \{ f \in H^1_{[0,1]},\ f(1)=0 \}$$

$$\rightarrow X_v = X_u$$

$$\int_a^b v \frac{\partial^2 u}{\partial x^2} + \int_a^b f \cdot v = \cancel{0} - v(0)$$

$$\downarrow$$

$$\underbrace{-v \frac{\partial u}{\partial x}\Big|_0 - \int_a^b \frac{\partial v}{\partial x}\frac{\partial u}{\partial x}\,dx + \int_a^b f \cdot v = \cancel{0} - v(0)}$$

$$\downarrow$$

for u to satisfy the weak form for $\forall\, v$

which can be anything at $x=0$,

$$\frac{\partial u}{\partial x}\Big|_0 = 0$$