Modified Equation, Dissipation and Dispersion Error

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original equation

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} = 0 \qquad U > 0$$

$$\frac{d u_i}{dt} + U \frac{u_i - u_{i-1}}{\Delta x} = 0$$

$$u_{i-1} = \cancel{u_i} - \Delta x\, u_i' + \frac{\Delta x^2}{2} u_i'' - \frac{\Delta x^3}{6} u_i''' + O(\Delta x^4)$$

modified equation

$$\frac{d u_i}{dt} + U \frac{-\left(-\Delta x\, u_i' + \frac{\Delta x^2}{2} u_i'' - \frac{\Delta x^3}{6} u_i'''\right)}{\Delta x} + O(\Delta x^3) = 0$$

$$\underbrace{\frac{\partial u_i}{\partial t} + U \frac{\partial u_i}{\partial x} - \frac{\Delta x}{2} U \frac{\partial^2 u_i}{\partial x^2}}_{\text{diffusion / dissipation}} + \underbrace{\left(\frac{\Delta x^2}{6} U \frac{\partial^3 u_i}{\partial x^3}\right)}_{\text{dispersion}} = O(\Delta x^3)$$

diffusion / dissipation

dispersion

$$\frac{d u_i}{dt} + U \frac{u_{i+1} - u_{i-1}}{2\Delta x} = 0$$

$$u_{i+1} = u_i + \Delta x\, u_i' + \frac{\Delta x^2}{2} u_i'' + \frac{\Delta x^3}{6} u_i''' + O(\Delta x^4)$$

$$u_{i-1} = u_i - \Delta x\, u_i' + \frac{\Delta x^2}{2} u_i'' - \frac{\Delta x^3}{6} u_i''' + O(\Delta x^4)$$

$$\frac{u_{i+1} - u_{i-1}}{2\Delta x} = u_i' + \frac{\Delta x^2}{6} u_i''' + O(\Delta x^3)$$

$O(\Delta x^4)$ Because u_i'''' is canceled

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + U \frac{\Delta x^2}{6} \frac{\partial^3 u}{\partial x^3} = O(\Delta x^4)$$

Solving System of Equations

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$$\begin{cases} \frac{\partial u}{\partial t} + U_{11}\frac{\partial u}{\partial x} + U_{12}\frac{\partial v}{\partial x} = 0 \\ \frac{\partial v}{\partial t} + U_{21}\frac{\partial u}{\partial x} + U_{22}\frac{\partial v}{\partial x} = 0 \end{cases}$$

$$\begin{cases} \frac{du_i}{dt} + U_{11}\frac{u_{i+1}-u_{i-1}}{2\Delta x} + U_{12}\frac{v_{i+1}-v_{i-1}}{2\Delta x} = 0 \\ - - - - - - = 0 \end{cases}$$

$$u_i = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \hat{u}_k e^{jki\Delta x} \qquad v_i = \sum_k \hat{v}_k e^{jki\Delta x}$$

$$\begin{cases} \frac{d\hat{u}_k}{dt} + U_{11}\hat{u}_k \frac{e^{jk\Delta x}-e^{-jk\Delta x}}{2\Delta x} + U_{12}\hat{v}_k \frac{e^{jk\Delta x}-e^{-jk\Delta x}}{2\Delta x} \\ \frac{d\hat{v}_k}{dt} + U_{21}\hat{u}_k \frac{e^{jk\Delta x}-e^{-jk\Delta x}}{2\Delta x} + U_{12}\hat{v}_k \frac{e^{jk\Delta x}-e^{-jk\Delta x}}{2\Delta x} \end{cases}$$

$$\frac{d}{dt}\begin{pmatrix}\hat{u}_k \\ \hat{v}_k\end{pmatrix} + C_k \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22}\end{pmatrix}\begin{pmatrix}\hat{u}_k \\ \hat{v}_k\end{pmatrix} = 0$$

$$C_k = \frac{e^{jk\Delta x} - e^{-jk\Delta x}}{2\Delta x}$$

$$\frac{d}{dt}\begin{pmatrix}\hat{u}_k \\ \hat{v}_k\end{pmatrix} = \underset{\sim}{A_k}\begin{pmatrix}\hat{u}_k \\ \hat{v}_k\end{pmatrix} \qquad A_k = V_k \Lambda_k V_k^{-1}$$

$$V_k^{-1}\frac{d}{dt}\begin{pmatrix}\hat{u}_k \\ \hat{v}_k\end{pmatrix} = \Lambda_k V_k^{-1}\begin{pmatrix}\hat{u}_k \\ \hat{v}_k\end{pmatrix} \qquad \begin{pmatrix}\check{u}_k \\ \check{v}_k\end{pmatrix} := V_k^{-1}\begin{pmatrix}\hat{u}_k \\ \hat{v}_k\end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix}\check{u}_k \\ \check{v}_k\end{pmatrix} = \begin{pmatrix}\lambda_{k1} & 0 \\ 0 & \lambda_{k2}\end{pmatrix}\begin{pmatrix}\check{u}_k \\ \check{v}_k\end{pmatrix}$$

Solving System of Equations

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$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

(ϕ, ψ)

$$\phi := \frac{\partial u}{\partial t}$$

$$\psi := \frac{\partial u}{\partial x}$$

$$\frac{\partial \psi}{\partial t} = \frac{\partial^2 u}{\partial x \partial t} = \frac{\partial \phi}{\partial x}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial \phi}{\partial t} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial \psi}{\partial x}$$

$$\begin{cases} \dfrac{\partial \psi}{\partial t} = \dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial \phi}{\partial t} = \dfrac{\partial \psi}{\partial x} \end{cases} \qquad \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$\lambda_{k1} = \frac{e^{jk\delta x} - e^{-jk\delta x}}{2\delta x}$$

$$\lambda_{k2} = \frac{e^{jk\delta x} - e^{jk\delta x}}{2\delta x}$$