Conservation law $\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$

Characteristics and boundary conditions -- scalar PDEs

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$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = 0 \qquad f(u) = Uu$$

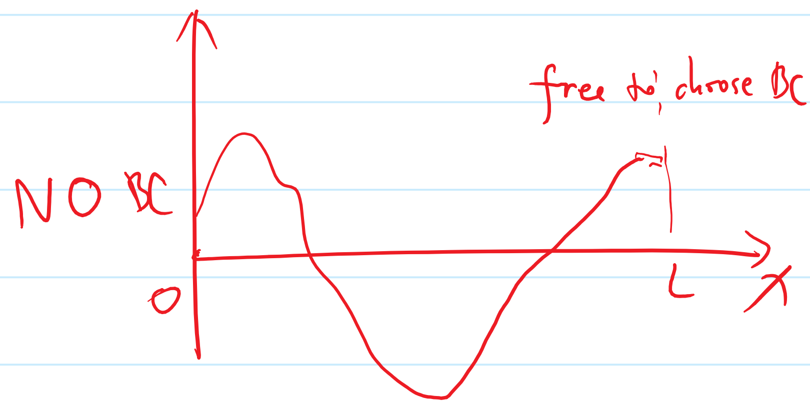
characteristics : $\frac{df}{du} = U$

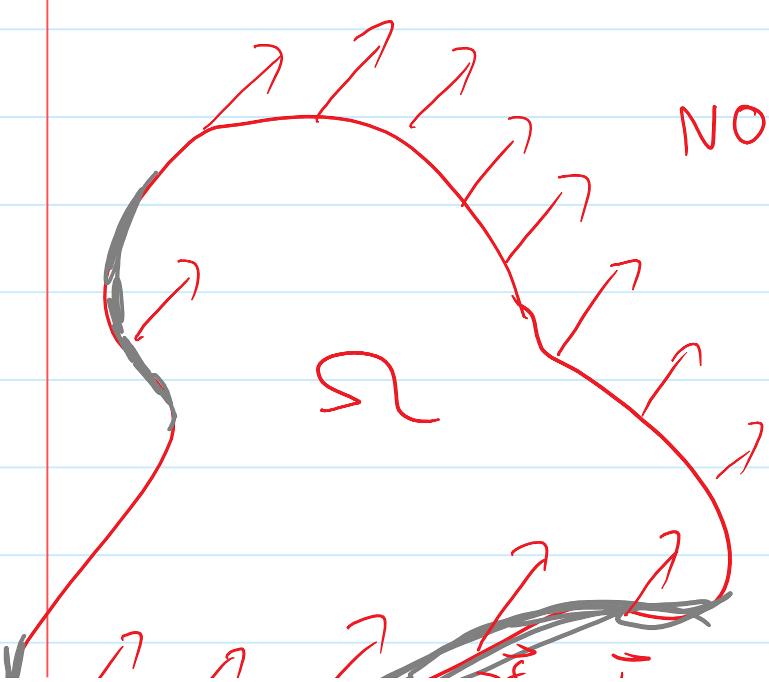
speed of discontinuity

$$\frac{\Delta f}{\Delta u} = \frac{f_L - f_R}{u_L - u_R} = U$$

1st case: $U > 0$ NO BC at L

2nd case $U < 0$

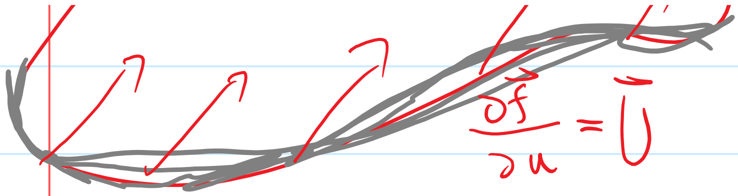




$$\frac{\partial u}{\partial t} + U_x\frac{\partial u}{\partial x} + U_y\frac{\partial u}{\partial y} = 0$$

$$\vec{U} = (U_x, U_y)$$

$$\frac{\partial u}{\partial t} + \vec{U}\cdot\nabla u = 0$$



$$\frac{\partial \vec{f}}{\partial u} = \vec{U}$$

$$\frac{\partial u}{\partial t} + \vec{U} \cdot \nabla u = 0$$

$$\vec{f} = (U_x u, U_y u) = \vec{U} u$$

Characteristics and boundary conditions -- system of PDEs

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$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial t} = \dfrac{\partial u}{\partial x} \end{cases}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$$

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix}.$$

eigenvalues + eigenvectors

$$1\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$-1\begin{pmatrix} +1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} +1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\overbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}$$

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix}$$

$$\frac{\partial}{\partial t}\begin{pmatrix} \frac{u+v}{2} \\ \frac{u-v}{2} \end{pmatrix} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} \frac{u+v}{2} \\ \frac{u-v}{2} \end{pmatrix}$$

Scalar conservation law -- Total variation

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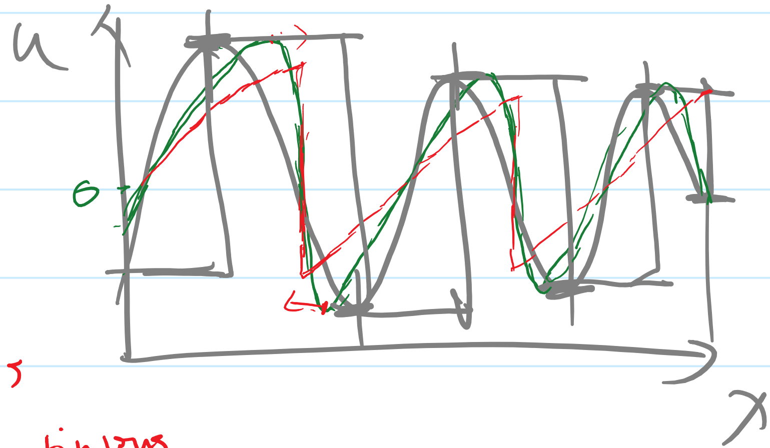
$$TV(t) = \int_0^L \left| \frac{\partial u}{\partial x} \right| dx$$



$\frac{dTV}{dt} = 0$ if u is smooth

$\frac{dTV}{dt} \le 0$ if u is discontinuous

$$\frac{dTV}{dt} \le 0$$

Finite Volume -- cell average, numerical flux

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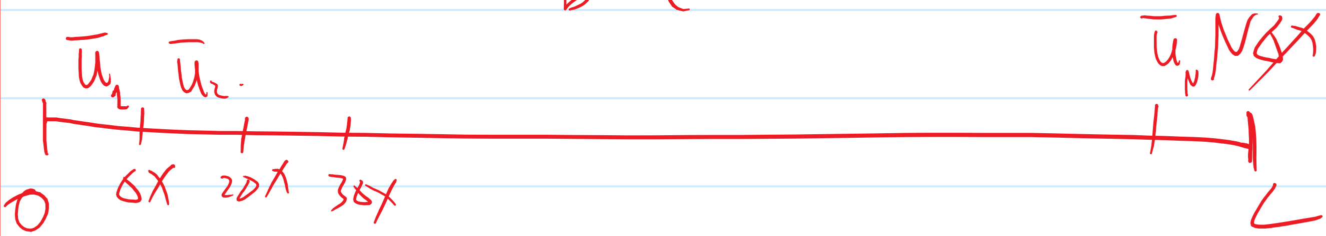
$$\frac{\frac{d}{dt}\int_a^b u\,dx + f(u(b)) - f(u(a))}{b-a} = 0$$

$$\frac{d}{dt}\bar{u} + \frac{f_R - f_L}{b-a} = 0$$



$$\frac{d\bar{u}_i}{dt} + \frac{f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}}{\Delta x} = 0$$ EXACT

approximation

$$f_{i+\frac{1}{2}} = f(\underbrace{u(x_{i+\frac{1}{2}})}_{\text{interpolate}})$$

Central flux scheme

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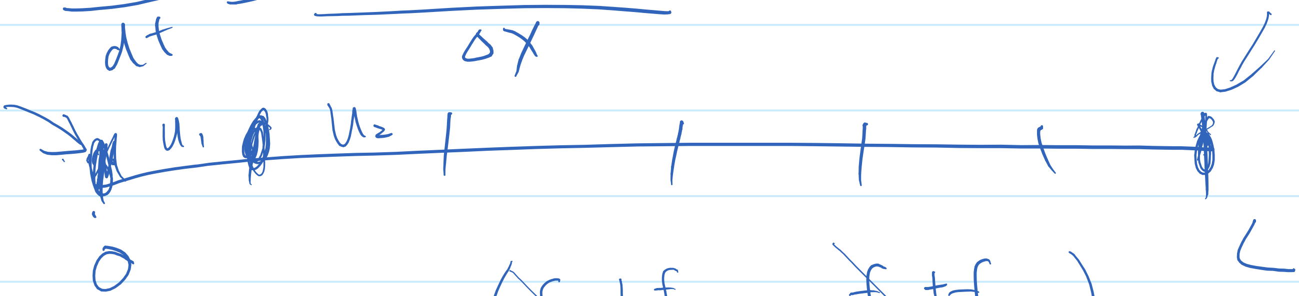
$$f_{i+\frac{1}{2}} = \frac{f_i + f_{i+1}}{2} \quad \text{where } f_i = f(\bar{u}_i)$$

$$\frac{d\bar{u}_i}{dt} = \frac{f_{i-\frac{1}{2}} - f_{i+\frac{1}{2}}}{\Delta x}$$



$$\frac{d\bar{u}_i}{dt} = \frac{1}{\Delta x}\left(\frac{\cancel{f_i} + f_{i-1}}{\cancel{2}} - \frac{\cancel{f_i} + f_{i+1}}{\cancel{2}}\right)$$

$$= \frac{1}{2\Delta x}\left(f_{i-1} - f_{i+1}\right)$$

$$= \frac{1}{2\Delta x}\left(\frac{\bar{u}_{i-1}^2}{2} - \frac{\bar{u}_{i+1}^2}{2}\right)$$

$$\bar{u}_i^2 \frac{d\bar{u}_i}{dt} = \frac{1}{2\Delta x}\left(\frac{\bar{u}_{i-1}^2 \bar{u}_i^2}{2} - \frac{\bar{u}_i^2 \bar{u}_{i+1}^2}{2}\right)$$

$$\frac{1}{3}\frac{d\bar{u}_i^3}{dt} = \frac{\bar{u}_{i-1}^2 \bar{u}_i^2}{4\Delta x} - \frac{\bar{u}_i^2 \bar{u}_{i+1}^2}{4\Delta x}$$

$$\frac{d}{dt}\sum_{i=i_0}^{i_1} \frac{\bar{u}_i^3}{3} = \frac{\bar{u}_{i_0-1}^2 \bar{u}_{i_0}^2}{4\Delta x} - \frac{\bar{u}_{i_1}^2 \bar{u}_{i_1+1}^2}{4\Delta x}$$

Periodic domain

$$\frac{d}{dt}\sum_{\text{all indices}} \frac{\bar{u}_i^3}{3} = 0$$

Central flux scheme -- conservation of higher order statistics

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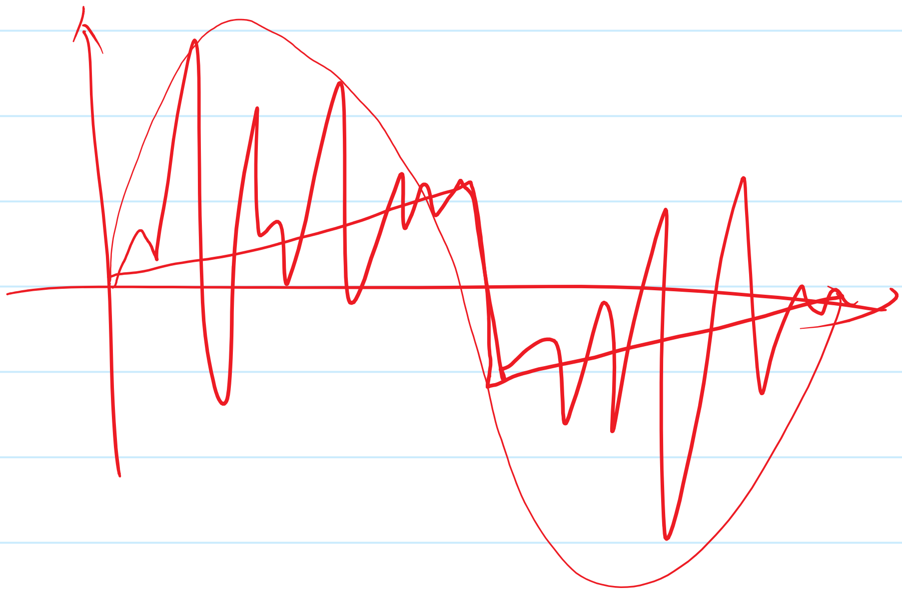

$$f_{i+\frac{1}{2}} = \lambda \frac{f(\bar{u}_i) + f(\bar{u}_{i+1})}{2} + (1-\lambda) f\left(\frac{\bar{u}_i + \bar{u}_{i+1}}{2}\right)$$

$$\bar{u}_i \frac{d\bar{u}_i}{dt} = \frac{1}{\Delta x}\left(f_{i-\frac{1}{2}} - f_{i+\frac{1}{2}}\right)\bar{u}_i$$

$$\frac{1}{2}\frac{d\bar{u}_i^2}{dt} = \frac{1}{\Delta x}\Bigg(\frac{\lambda}{2}\left(\frac{\bar{u}_i^2 + \bar{u}_{i-1}^2}{2}\bar{u}_i - \frac{\bar{u}_{i+1}^2 + \bar{u}_i^2}{2}\bar{u}_i\right)$$

$$+ (1-\lambda)\left(\frac{1}{2}\left(\frac{\bar{u}_i + \bar{u}_{i+1}}{2}\right)^2 \bar{u}_i - \frac{1}{2}\left(\frac{\bar{u}_i + \bar{u}_{i+1}}{2}\right)^2 \bar{u}_i\right)\Bigg)$$

$$\frac{\bar{u}_i^2 + 2\bar{u}_i\bar{u}_{i+1} + \bar{u}_{i+1}^2}{8}\bar{u}_i$$

$$\lambda \frac{\cancel{\bar{u}_i^3} + \bar{u}_{i-1}^2 \bar{u}_i}{4} + (1-\lambda)\frac{\cancel{\bar{u}_i^3} + 2\bar{u}_i^2\bar{u}_{i-1}}{8}$$

$$+ \bar{u}_{i-1}^2 \bar{u}_i$$