Finite element -- Error in output functional

$$\boxed{a(u, v) = l(v)}$$ $u \in X_u$

$v = X_h$

$$a(u^h, v^h) = l(v^h)$$

What we are interested in $l^0(u)$ is a linear functional of the solution

Error estimate $\sim$ $\underline{l^0(u^h) - l^0(u)}$

$$= l^0(u^h - u)$$

Adjoint equation

$$a^*(\hat{u}, v) = l^0(v) \quad \text{for all } v \in X_u$$

whose solution $\hat{u} \in X_v$.

$$l^0(u^h - u) = a^*(\hat{u}, u^h - u)$$

a^* & a are adjoint in that

$$a(u, v) = a^*(v, u) \quad \text{for all } u \in X_u, \ v \in X_v$$

$$\underline{l^0(u^h - u)} = a(u^h - u, \hat{u})$$

$$= a(u^h, \hat{u}) - a(u, \hat{u})$$

$$= a(u^h, \hat{u}) - l(\hat{u})$$

$$= a(u^h, v) - l(v) \quad \text{for } v \neq \hat{u}$$

Finite element -- Error in output functional estimated by the adjoint

$$l^o(u^h - u) = a(u^h, \hat{u}) - l(\hat{u})$$

$$\text{if} \simeq a(u^h, \hat{u}^h) - l(\hat{u}^h)$$

$$\hat{u}^h \in X_v^h$$

normally, $\hat{u}$ is approximated with higher order solution

Finite element in 2D -- weak form

$$\nabla\cdot\nabla u + f = 0$$

$f = 1$ $\qquad$ $u = 0$ at $\partial\Omega$

$$\int_\Omega v(\nabla\cdot\nabla u + f) = 0 \quad \forall v$$

$$= -\int_\Omega \nabla v\cdot\nabla u\,dx + \int_\Omega v\cdot f\,dx$$

$\underline{a(u,v)} = \int_\Omega \nabla u\cdot\nabla v\,dx$ $\qquad$ $\underline{l(v)} = \int_\Omega v f\,dx$

$\le \left(\int_\Omega v^2\right)^{\frac{1}{2}}\left(\int_\Omega f^2\right)^{\frac{1}{2}}$

$u, v \in H^1$

$$a(u,v) \le \left(\int_\Omega \nabla u\cdot\nabla u\right)^{\frac{1}{2}}\left(\int_\Omega \nabla v\cdot\nabla v\right)^{\frac{1}{2}}$$

Finite element in 2D -- weak form

Finite element in 2D -- basis functions

$$u^h = \sum_{i=1}^{n} u_i \phi_i$$

$$a(u^h, \phi_j) = l(\phi_j) \quad \text{for } j = 1, \dots, n$$

$$= \sum_{i=1}^{n} u_i \underbrace{a(\phi_i, \phi_j)}_{A_{ij}} = \underbrace{l(\phi_j)}_{b_j}$$

$$A \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = b$$

$$A_{ij} = a(\phi_i, \phi_j) = \int_\Omega \nabla \phi_i \cdot \nabla \phi_j \, dx$$

$$= \sum_{k \in \text{elements}} \int_{\Omega_k} \nabla \phi_i \cdot \nabla \phi_j \, dx$$

$$= \underline{\sum_{k \in \text{elem}} \nabla \phi_i \Big|_k \cdot \nabla \phi_j \Big|_k \cdot a_k}$$

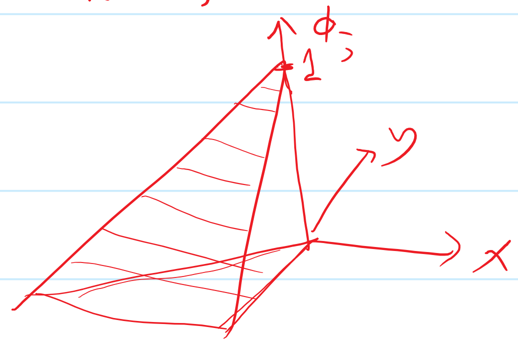
$$\nabla \phi_i \cdot (x_i - x_j) = \phi_i(x_i) - \phi_i(x_j) = 1$$

$$\begin{pmatrix} x_1 - x_2 & y_1 - y_2 \\ x_1 - x_3 & y_1 - y_3 \end{pmatrix} \begin{pmatrix} \nabla_x \phi_1 \\ \nabla_y \phi_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{unless } i = j$$

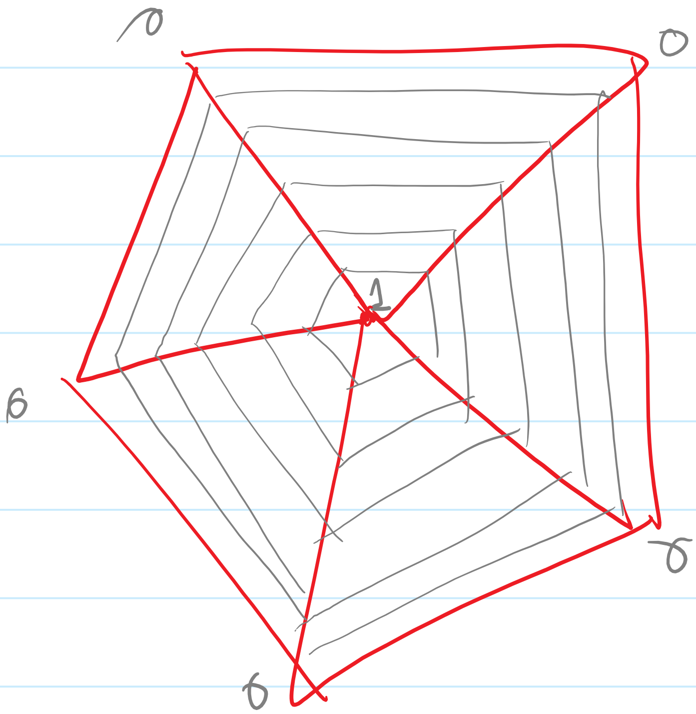
Finite element in 2D -- matrix form

$$b_j = \int_\Omega \phi_j \cdot \underbrace{f}_{\equiv 1} \, dx = \int_\Omega \phi_j \, dx$$

$$= \sum_{k \in elems} \int_\Omega \phi_j \Big|_k dx = \sum_{k \in elems} \frac{a_k}{3}$$

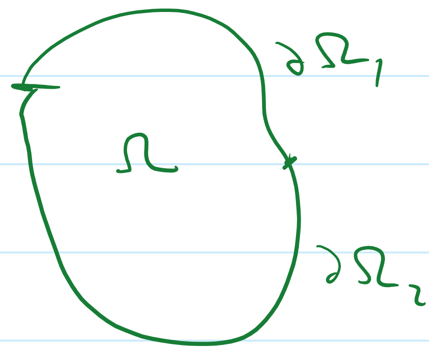


Finite element in 2D -- matrix form



Finite element in 2D -- matrix form

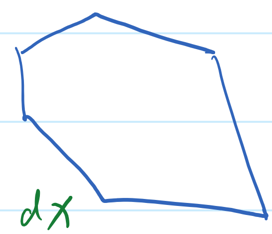
$$\nabla\cdot\nabla u + f = 0$$



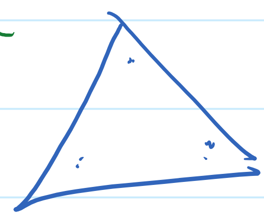
$$u\big|_{\partial\Omega_1} = 0$$

$$\vec{n}\cdot\nabla u\big|_{\partial\Omega_2} = 0$$

$$\int_\Omega v\cdot(\nabla\cdot\nabla u + f)\,dx = 0$$



$$= \int_{\partial\Omega} \underbrace{v}_{0|_{\partial\Omega_1}}\, \underbrace{\vec{n}\cdot\nabla u}_{0|_{\partial\Omega_2}}\Big|_{\partial\Omega} ds - \int_\Omega \nabla v\cdot\nabla u\,dx + \underline{\int v f\,dx}$$



$$\| $$

$$\sum w_i\, v(x_i)\, f(x_i)$$

$$\vec{w} = \underline{v\nabla u}$$

Gauss theorem

$$\int_\Omega \nabla\cdot\vec{w}\,dx = \int_{\partial\Omega} \vec{n}\cdot\vec{w}\,ds$$

$$\int_\Omega \nabla\cdot(v\nabla u) = \int_\Omega \underbrace{\nabla v\cdot\nabla u}_{RHS} + \underbrace{v\nabla\cdot\nabla u}_{LHS} = \underbrace{\int_{\partial\Omega} v\,\vec{n}\cdot\nabla u\,ds}_{RHS}$$

$$\int_\Omega v\nabla\cdot\nabla u\,dx = \int_{\partial\Omega} v\,\vec{n}\cdot\nabla u\,ds - \int_\Omega \nabla v\cdot\nabla u\,dx$$