Smoothed Particle Hydrodynamics(Particle Based Fluid Simulation for Interactive Applications)

Force Analysis

$$\vec{F} = \vec{F}_{pressure} + \vec{F}_{viscosity} + \vec{F}_{external}$$

Density

$$\rho(\vec{r}_i) = \sum_j m_j \frac{\rho_j}{\rho_j} W(\vec{r}_i - \vec{r}_j, h) = \sum_j m_j W_{poly6}(\vec{r}_i - \vec{r}_j, h), i \neq j$$

$$K_{poly6}(\vec{r}, h) = \frac{1}{\int_0^{2\pi} \int_0^{\pi} \int_0^{h} r^2 sin(\varphi)(h^2 - r^2)^3 d_r d_\varphi d_\theta} = \frac{315}{64\pi h^9}$$

$$W_{poly6}(\vec{r}, h) = \frac{315}{64\pi h^9} \begin{cases} (h^2 - r^2)^3, & 0 \le r \le h \\ 0, & otherwise \end{cases}$$
$$where\ r\ = |\vec{r}_i - \vec{r}_j|$$

$$\rho(\vec{r}_i) = m\frac{315}{64\pi h^9} \sum_j (h^2 - |\vec{r}_i - \vec{r}_j|^2)^3, i \neq j$$

Pressure

The density ρ varies and needs to be evaluated at every time step

$$p = k\rho$$

$$where\ p\ is\ the\ pressure\ \text{field};$$
$$k\ is\ a\ gas\ constant\ that\ depends\ on\ the\ temperature;$$
$$\rho\ is\ the\ density$$

In our simulations we use a modified version suggested by Desbrun2

$$p = k(\rho - \rho_0)$$
$$where\ \rho_0\ is\ the\ rest\ density$$

Pressure Force

$$\vec{F}_{pressue} = -\nabla p(\vec{r}_i) = -\sum_j m_j \frac{p_j}{\rho_j} \nabla W_{spiky}(\vec{r}_i - \vec{r}_j, h), i \neq j$$
$$where\ m\ is\ the\ mass\ of\ particle;$$
$$p\ is\ the\ pressure\ of\ a\ single\ particle;$$
$$\rho\ is\ the\ density;$$
$$\vec{r}\ corresponds\ to\ the\ location;$$
$$h\ corresponds\ to\ the\ core\ radius;$$
$$W(\vec{r}, h)\ the\ smoothing\ kernel;$$
$$i\ and\ j\ is\ the\ index\ of\ a\ particle$$

Because the pressures at the locations of the two particles are not equal in general, the pressure forces will not be symmetric. Different ways of symmetrization of the equation have been proposed in the literature.We suggest a very simple solution which we found to be best suited for our purposes of speed and stability. The so computed pressure force is symmetric because it uses the arithmetic mean of the pressures of interacting particles.

$$\vec{F}_{pressue} = -\nabla p(\vec{r}_i) = -\sum_j m_i \frac{p_i + p_j}{2\rho_j} \nabla W_{spiky}(\vec{r}_i - \vec{r}_j, h), i \neq j$$

$$W_{spiky}(\vec{r}, h) = \frac{15}{\pi h^6} \begin{cases} (h - r)^3, & 0 \le r \le h \\ 0, & otherwise \end{cases}$$

$$\nabla W_{spiky}(\vec{r}, h) = \frac{15}{\pi h^6} \nabla (h - r)^3 = -\frac{45}{\pi h^6} (h - r)^2 \frac{\vec{r}}{r}$$

$$\vec{a}_i^{pressure} = -\frac{\nabla p(\vec{r}_i)}{\rho_i} = m \frac{45}{\pi h^6} \sum_j (\frac{p_i + p_j}{2\rho_i \rho_j} (h - r)^2 \frac{\vec{r}_i - \vec{r}_j}{r}), i \neq j$$
$$where\ r\ = |\vec{r}_i - \vec{r}_j|$$

Viscosity

$$\vec{F}_{viscosity} = \mu \nabla^2 \vec{v}(\vec{r}_a) = \mu \sum_j m_j \frac{\vec{v}_j}{\rho_j} \nabla^2 W(\vec{r}_i - \vec{r}_j, h), i \neq j$$
$$where\ \mu\ is\ the\ viscosity\ of\ the\ fluid$$

because the velocity field varies from particle to particle. Since viscosity forces are only dependent on velocity differences and not on absolute velocities, there is a natural way to symmetrize the viscosity forces by using velocity differences

$$\vec{F}_{viscosity} = \mu \nabla^2 \vec{v}(\vec{r}_a) = \mu \sum_j m_j \frac{\vec{v}_j - \vec{v}_i}{\rho_j} \nabla^2 W_{viscosity}(\vec{r}_i - \vec{r}_j, h), i \neq j$$

A possible interpretation of the equation is to look at the neighbors of particle i from i′s own moving frame of reference.

$$W_{viscosity}(\vec{r}, h) = \frac{15}{2\pi h^3} \begin{cases} -\frac{r^3}{2h^3} + \frac{r^2}{h^2} + \frac{h}{2r} - 1, & 0 \leq r \leq h \\ 0, & otherwise \end{cases}$$

$$\nabla^2 W_{viscosity}(\vec{r}, h) = \frac{45}{\pi h^6}(h - r)$$
$$where\ r\ = |\vec{r}_i - \vec{r}_j|$$

$$\vec{a}_i^{viscosity} = \frac{\vec{F}_i^{viscosity}}{\rho_i} = m\mu \frac{45}{\pi h^6} \sum_j \frac{\vec{v}_j - \vec{u}_i}{\rho_i \rho_j}(h - |\vec{r}_i - \vec{r}_j|), i \neq j$$

Gravity

$$\vec{F}_{gravity} = \rho * g$$

$$where\ \rho\ is\ the\ mass\ of\ a\ single\ particle;$$
$$g\ is\ an\ external\ force\ density\ \text{field}$$

Surface Tension

The surface of the fluid can be found by using an additional field quantity which is 1 at particle locations and 0 everywhere else. This field is called color field in the literature. For the smoothed color field we get

$$C_S(r) = \sum_j m_j \frac{1}{\rho_j} W_{poly6}(\vec{r}_i - \vec{r}_j, h), i \neq j$$

$$\triangledown W_{poly6} = -\frac{945}{32\pi h^9} \begin{cases} (h^2 - r^2)^2(\vec{r}_i - \vec{r}_j), & 0 \leq r \leq h \\ 0, & otherwise \end{cases}$$

$$\triangledown^2 W_{poly6} = \frac{945}{8\pi h^9} \begin{cases} (h^2 - r^2)^2[r^2 - \frac{3}{4}(r^2 - h^2)], & 0 \leq r \leq h \\ 0, & otherwise \end{cases}$$

The gradient field of the smoothed color field yields the surface normal field pointing into the fluid and the divergence of n measures the curvature of the surface

$$\kappa = \frac{-\triangledown^2 C_S}{|\vec{n}|}$$

$$\vec{n} = \triangledown C_S$$

$$\triangledown C_S(\vec{r}_i) = -m\frac{945}{32\pi h^9} \sum_j \frac{1}{\rho_j}(h^2 - r^2)^2(\vec{r}_i - \vec{r}_j), i \neq j$$

$$\nabla^2 C_s(\vec{r}_i) = m\frac{945}{8\pi h^9}\sum_j \frac{1}{\rho_j}(h^2 - r^2)[r^2 - \frac{3}{4}(h^2 - r^2)], i \neq j$$

$$where\, r\ = |\vec{r}_i - \vec{r}_j|$$

The minus is necessary to get positive curvature for convex fluid volumes. Putting it all together, we get for the surface traction:

$$t^{surface} = \sigma\kappa\frac{\vec{n}}{|\vec{n}|}$$
$$where\, \sigma\, is\, tension\, coeffcient$$

$$f^{surface} = -\sigma\nabla^2 C_S\frac{\vec{n}}{|\vec{n}|}$$

Evaluating n/|n| at locations where |n| is small causes numerical problems. We only evaluate the force if |n| exceeds a certain threshold.

$$\vec{a}_i^{surface\, tension} = \frac{\vec{F}^{surface\, tension}}{\rho_i} = -\sigma\nabla^2 C_s\frac{\nabla C_s}{\rho_i|\nabla C_s|}$$
$$= -\sigma\frac{945}{8\pi h^9}\sum_j \frac{m_j}{\rho_j}(h^2 - r^2)[r^2 - \frac{3}{4}(h^2 - r^2)]\frac{\nabla C_s}{\rho_i|\nabla C_s|}, i \neq j$$

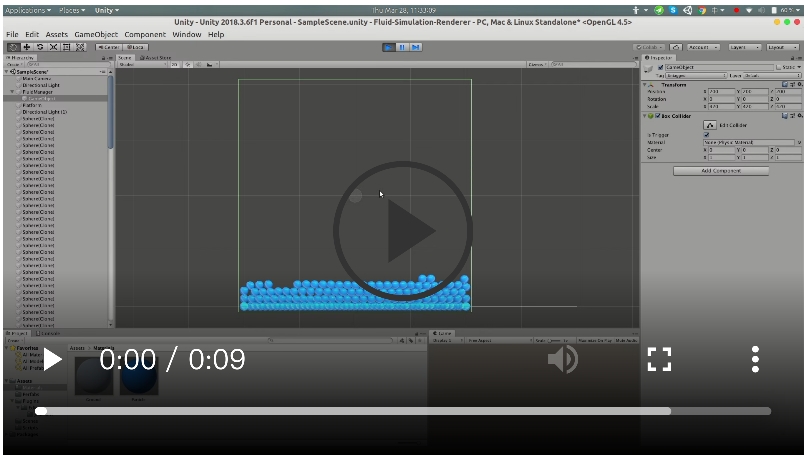
Particle Acceleration

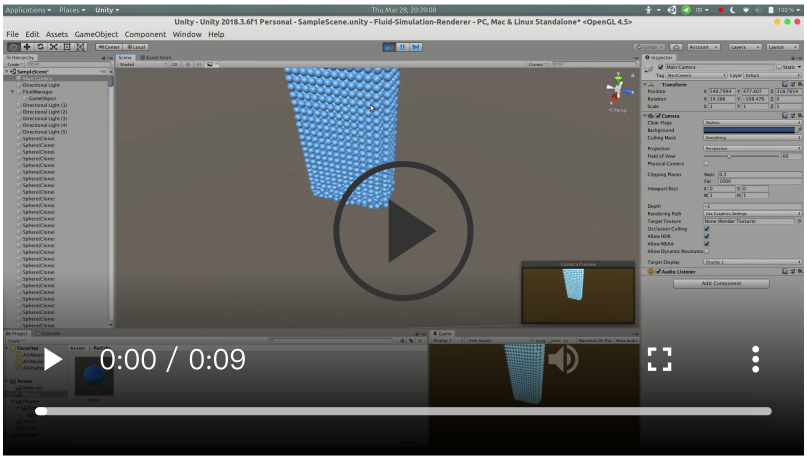
$$\vec{a}(\vec{r}_i) = \vec{g} + m\frac{45}{\pi h^6}\sum_j(\frac{p_i + p_j}{2\rho_i\rho_j}(h - r)^2\frac{\vec{r}_i - \vec{r}_j}{r})$$
$$+m\mu\frac{45}{\pi h^6}\sum_j\frac{\vec{u}_j - \vec{u}_i}{\rho_i\rho_j}(h - r) - \sigma\nabla^2 C_s\frac{\nabla C_s}{\rho_i|\nabla C_s|}, i \neq j$$
$$where\, r\ =\ |\vec{r}_i - \vec{r}_j|$$

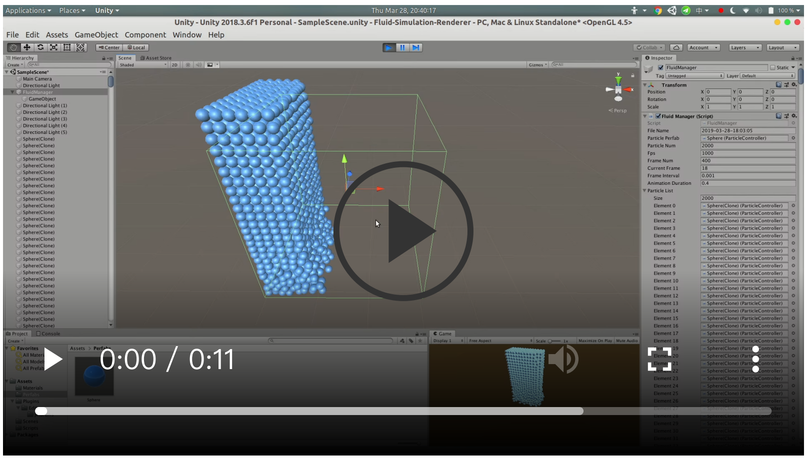
External Forces

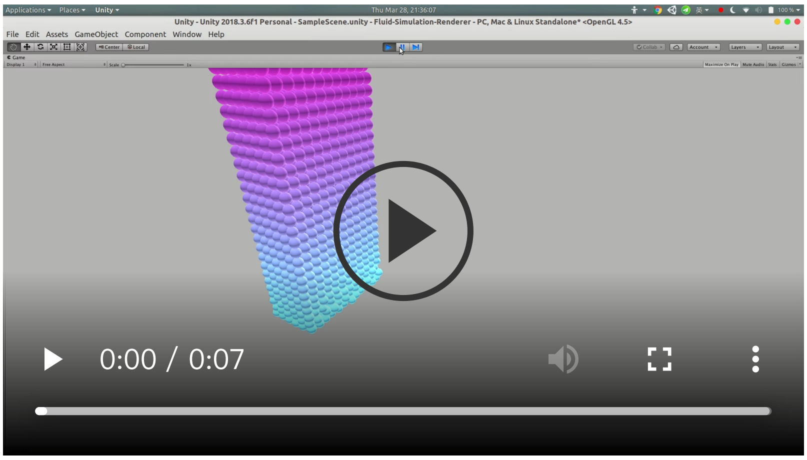
Our simulator supports external forces such as gravity, collision forces and forces caused by user interaction. These forces are applied directly to the particles without the use of SPH. When particles collide with solid objects such as the glass in our examples, we simply push them out of the object and reflect the velocity component that is perpendicular to the object's surface.

Results





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