Finite element for time-dependent equations

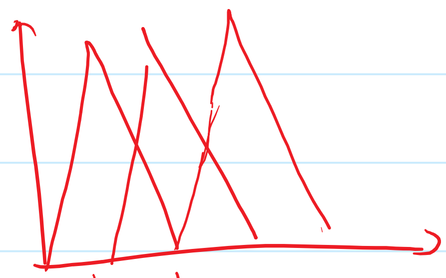
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad u(a) = u(b) = 0$$

$$\langle v, \frac{\partial u}{\partial t} \rangle = \langle v, \frac{\partial^2 u}{\partial x^2} \rangle \qquad \text{for all } v \in X_v$$

$$\int_a^b v \frac{\partial u}{\partial t}\, dx = \underline{\int_a^b v \frac{\partial^2 u}{\partial x^2}\, dx} = -\int_a^b \frac{\partial v}{\partial x} \frac{\partial u}{\partial x}\, dx$$

$$\frac{d}{dt}\left(\int_a^b u v\, dx\right) = -\int_a^b \frac{\partial v}{\partial x} \frac{\partial u}{\partial x}\, dx$$

Now restrict $u = \sum_{i=1}^{n} u_i \phi_i$



$u_i(t)$

$$\frac{d}{dt}\int_a^b \sum u_i \phi_i\, \phi_j\, dx = -\int_a^b \frac{\partial \phi_j}{\partial x} \sum u_i \frac{\partial \phi_i}{dx}\, dx$$

$$\sum \frac{du_i}{dt} \underbrace{\int_a^b \phi_i \phi_j\, dx}_{M_{ji}} = -\sum_{i=1}^{n} u_i \underbrace{\int_a^b \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial x}\, dx}_{A_{ji}}$$

$$\underbrace{M}_{\text{mass matrix}} \frac{d\vec{u}}{dt} = -\underbrace{A}_{\text{stiffness matrix}} \vec{u}$$

Iterative method for solving steady-state equations

$$\frac{\partial^2 u}{\partial x^2} = f \qquad u(0) = u(1) = 0$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - f \qquad u(0) = u(1) = 0$$

$$\frac{d\vec{u}}{dt} = A\vec{u} - \vec{f}$$

$$\frac{u^{(k+1)} - u^{(k)}}{\Delta t} = A u^{(k)} - \vec{f}$$

Jacobi iteration method

$$Au = b \qquad A = D + L + U$$

$$Du + Lu + Uu = b \qquad r^{(k)} = \underbrace{b - Au^{(k)}}_{\text{criterion for convergence}}$$

$$Du^{(k+1)} = b - Lu^{(k)} - Uu^{(k)}$$

$u^{(0)}$: initial guess

Define $e^{(k)} = u^{(k)} - u$

$$Du^{(k+1)} - Du = \left[b - Lu^{(k)} - Uu^{(k)}\right] - \left[b - Lu - Uu\right]$$

$$De^{(k+1)} = -(L+U)e^{(k)}$$

$$e^{(k+1)} = \underbrace{-D^{-1}(L+U)}_{\text{Iteration matrix}} e^{(k)}$$

$$\text{Iteration matrix} = V\Lambda V^{-1}$$

$$e^{(0)} = \sum a_i v_i = V\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

$$e^{(1)} = V\Lambda V^{-1} e^{(0)}$$

$$= V\Lambda \cancel{V^{-1} V} (a) = V\Lambda (a)$$

$$e^{(2)} = V\Lambda \cancel{V^{-1} V} \Lambda (a) = V\Lambda^2 (a)$$

$$e^{(k)} = V\Lambda^k (a)$$

Jacobi iteration method (finite difference and finite element examples)

$$\frac{\partial^2 u}{\partial x^2} = f$$

$$\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{\delta x^2} = f_i\right)$$

U D L b

$$-\frac{2}{\delta x^2}\, u_i^{(k+1)} = f_i - \frac{u_{i+1}^{(k)}}{\Delta x^2} - \frac{u_{i-1}^{(k)}}{\Delta x^2}$$

$$u_0 = u_{n+1} = c$$

$$u_i^{(k+1)} = -\frac{\Delta x^2}{2} \cdot f_i + \frac{u_{i+1}^{(k)} + u_{i-1}^{(k)}}{2}$$

$$\frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}}{\Delta x^2} = f_{ij}$$

$\Delta x = \Delta y$

D.

$$u_{ij}^{(k+1)} = -\frac{\Delta x^2}{4} f_{ij} + \frac{u_{i+1,j}^{(k)} + u_{i-1,j}^{(k)} + u_{i,j-1}^{(k)} + u_{i,j+1}^{(k)}}{4}$$