Convergence of Jacobi method for Poisson's equation

$$Du^{k+1} = -(L+U)u^k + b$$

$$Du = -(L+U)u + b$$

$$e^k := u^k - u$$

$$e^{k+1} = \underbrace{-D^{-1}(L+U)}_{\text{Jacobi iteration matrix}} e^k$$

$$-D^{-1}(L+U) = V \Lambda V^{-1}$$

$$e^k = V \Lambda^k V^{-1} e^0$$

$$\lambda_1 = 1 - \varepsilon$$

$e^k \sim \frac{1}{2} e^0$ require $(1-\varepsilon)^k \sim \frac{1}{2}$

$$1-\varepsilon \sim e^{-\varepsilon}$$

$$(1-\varepsilon)^k \sim e^{-\varepsilon k} \qquad k \sim \frac{1}{\varepsilon} \log 2$$

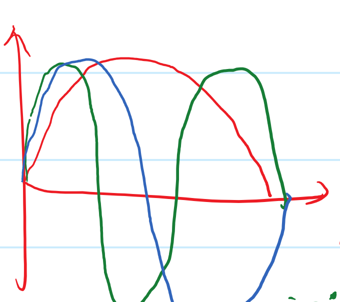
$$A = \begin{pmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{pmatrix} \frac{1}{\Delta x^2}$$

$$D = -2 I \frac{1}{\Delta x^2}$$

$$L+U = \begin{pmatrix} 0 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{pmatrix} \frac{1}{\Delta x^2}$$

$$-D^{-1}(L+U) = \begin{pmatrix} 0 & \frac{1}{2} & & \\ \frac{1}{2} & \ddots & \ddots & \\ & \ddots & \ddots & \frac{1}{2} \\ & & \frac{1}{2} & 0 \end{pmatrix}$$

V_{ij}



i

$$V_{ij} = \sin \frac{ij\pi}{(n+1)}$$

$$\lambda_j = \left(\sin \frac{(i-1)j\pi}{n+1} + \sin \frac{(i+1)j\pi}{n+1} \right) / 2 \Big/ \sin \frac{ij\pi}{n+1} = \frac{e^{-i\frac{j\pi}{n+1}} + e^{i\frac{j\pi}{n+1}}}{2} = \cos \frac{j\pi}{n+1}$$

$$\lambda_j = \left(\sin\frac{(i-1)j\pi}{n+1} + \sin\frac{(i+1)j\pi}{n+1}\right)\Big/2 \Big/ \sin\frac{ij\pi}{n+1} = \frac{e^{-\frac{j\pi}{n+1}i} + e^{\frac{j\pi}{n+1}i}}{2} = \cos\frac{j\pi}{n+1}$$

Gershgorin circle theorem and the convergence of Jacobi method

Diagonal dominance

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots \\ a_{21} & a_{22} & \cdots & \cdots \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix}$$

if $\quad |a_{kk}| > \sum_{i \neq k} |a_{ki}|$

Every λ_k must be in $\left\{ \lambda : \underline{|\lambda - a_{kk}| \le \sum_{i \neq k} |a_{ki}|} \right\}$

$\underline{A V_j = \lambda_j V_j}$ $\quad$ Pick k s.t.

V_{kj} has greatest magnitude

$\underline{|V_{kj}|} \ge |V_{ij}| \quad \forall i \neq k$

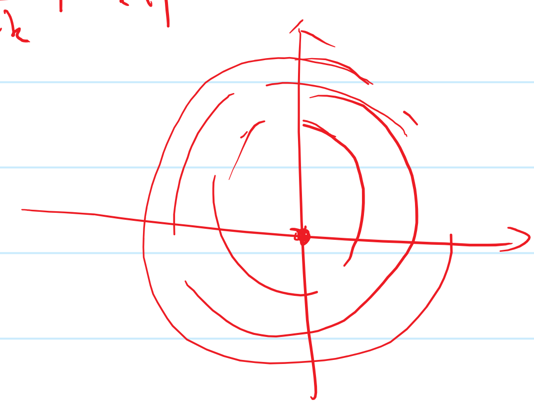
$$\sum a_{ki} V_{ij} = \lambda_j V_{kj}$$

$$a_{kk} V_{kj} - \lambda_j V_{kj} = -\sum_{i \neq k} a_{ki} V_{ij}$$

$$|a_{kk} - \lambda_j| \le \sum_{i \neq k} |a_{ki}| \frac{|V_{ij}|}{|V_{kj}|} \le 1$$

$$\le \sum_{i \neq k} |a_{ki}|$$

$$-D^{-1}(L+U) = -\begin{pmatrix} 0 & \frac{|a_{12}|}{|a_{11}|} & \frac{|a_{13}|}{a_{11}} & \cdots & \cdots \\ & \ddots & & & \\ & & \ddots & & \\ & & & & 0 \end{pmatrix}$$



Gauss-Seidel method

$$Au = b \qquad A = L + D + U$$

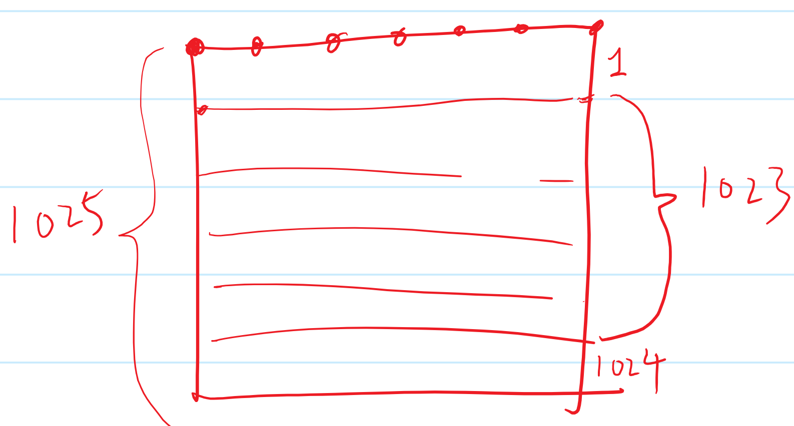
$$Lu + Du + Uu = b \qquad (L+D)\,u^{k+1} = b - Uu^{k}$$

$$\begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ a_{31} & a_{32} & a_{33} & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} u_1^{k+1} \\ u_2^{k+1} \\ \vdots \\ u_n^{k+1} \end{pmatrix} = \begin{pmatrix} b_1 - (Uu^k)_1 \\ b_2 - (Uu^k)_2 \\ \vdots \\ b_n - (Uu^k)_n \end{pmatrix}$$

$$\frac{u_{i-1}^{(k+1)} - 2u_i^{(k+1)} + u_{i+1}^{(k)}}{\Delta x^2} = b_i$$

$$u_i^{(k+1)} = -\frac{\Delta x^2}{2} b_i + \frac{u_{i-1}^{(k+1)} + u_{i+1}^{(k)}}{2}$$

$$u_{ij}^{(k+1)} = -\frac{\Delta x^2}{4} b_{ij} + \frac{u_{i-1\,j}^{(k+1)} + u_{i+1\,j}^{(k)} + u_{i\,j-1}^{(k+1)} + u_{i\,j+1}^{(k)}}{4}$$

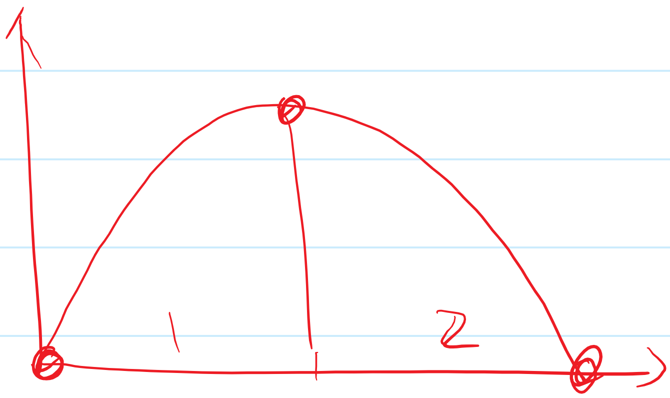


$$\lambda = \cos\left(\frac{\pi}{n}\right) \approx 1 - \frac{1}{2}\left(\frac{\pi}{n}\right)^2$$

$$N = \log / (1-\lambda) = \frac{\log 2}{\frac{1}{2}\left(\frac{\pi}{N}\right)^2} \sim N^2$$

$$\approx 1 - \frac{1}{2}\left(\frac{\pi}{n}\right)^2 \qquad N = \ldots/(1-\lambda) = \frac{\ldots}{\frac{1}{2}\left(\frac{\pi}{N}\right)^2} \sim N$$

The idea of multigrid



$N=2$. $\cos\frac{\pi}{N} = \cos\frac{\pi}{2} = 0$

$\cos\frac{2\pi}{N}$

Grid

finer

coarser

$r^{(0)} = b - Ax^{(0)}$

solve δx : s.t. $A\delta x^{(0)} = r^{(0)}$

$Ax^{(0)} = b - r^{(0)}$

$A(x^{(0)} + \delta x^{(0)}) = b$

interpolate $r^{(0)}$ to coarse grid

$\delta x^{(0)} \approx A^{-1} r^{(0)}$

interpolate $\delta x^{(0)}$ to fine grid

$x^{(1)} = x^{(0)} + \delta x^{(0)}$

iteration on fine grid

algorithm

1. Computing residual
2. Interpolating residual to coarser grid
3. Iterative solver on both coarse and fine grid
4. Interpolating solution to finer grid