Implicit Function Theorem

$F(x,y) \epsilon C^1$ in a neighbourhood of (x_0, y_0)

$F(x_0, y_0) = 0$

$\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$

df = $\frac{\partial f(x_0, y_0)}{\partial y} dy + \frac{\partial f}{\partial x} dx$

If these conditions are met then there is an explicit function y=f(x)

Background

Function

Function is one-to-one or many to one. One-to-one function is one x values corresponds to one y values.

Projection of $\vec{a}$ onto $\vec{b}$

Projection = $|a|cos(\theta) = \frac{a.b}{|b|}$

Resolved vector of $\vec{a}$ onto $\vec{b}$

$\text{Resolved} = \text{Projection} \times \hat{b}$

$|\text{resolved}| = |\text{Projection}|$

Chain Rule

w = w(x,y,z), x = x(t), y= y(t), z= z(t)

$\frac{dw}{dt} = w_x \frac{dx}{dt} + w_y \frac{dy}{dt} + w_z \frac{dx}{dt}$

$\frac{dw}{dt} = \nabla w . \frac{d\vec{F}}{dt}$

Gradient is perpendicular to the level curves.

It points towards higher values of w.

Example

$w = a_1 x + a_2 y + a_3 z,$

$\nabla w =< a_1, a_2, a_3 >$

Level surface $c = a_1 x + a_2 y + a_3 z$

This is a plane

The normal to the plane is the vector \lt a_1, a_2, a_3 \gt \)

This is the same as the gradient.

Example

$w = x^2 + y^2$,

$\nabla w = < 2x, 2y, >$

Level curve $c = x^2 + y^2$

This is a circle where $\frac{dy}{dx} = \frac{-x}{y} =$

The normal to the tangent is the vector $< x, y, >$

This is the same direction as the gradient.

Proof

$\vec{r} = \vec{r}(t)$ stays on the level surface $w = c$.

velocity vector is going to be tangential to the curve and also tangential to level surface (curve is inside the surface).

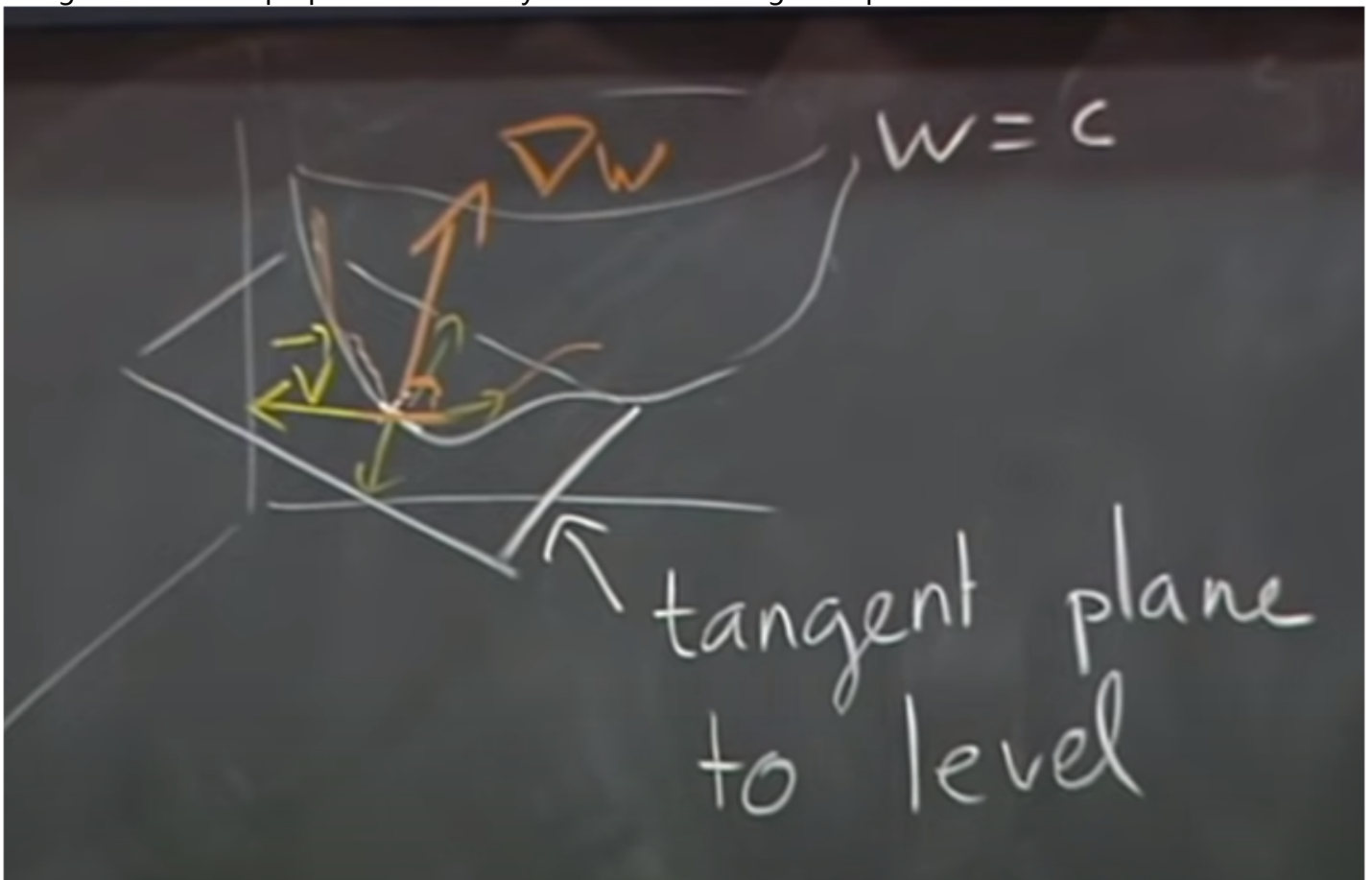$\vec{v} = \frac{d\vec{r}}{dt}$ is tangential to the level surface $w = c$.

By the chain rule $\frac{dw}{dt} = \nabla w \cdot \frac{d\vec{r}}{dt}$

$\frac{dw}{dt} = \nabla w \cdot \vec{v}$

since w = c, therefore $\frac{dw}{dt} = 0$

Hence the velocity as the gradient are perpendicular to eachother

The gradient is also perpendicular to any vector on the tangential plane.

Finding Tangential plane to a surface

Level surface $x^2 + y^2 - z^2 = 4$ at (2,1,1)

$\nabla w =< 2x, 2y, -2z >$

Normal to tangential plane \lt 4, 2,-2\gt \)

Tangential plane: $4x + 2y - 2z = 8$

Alternative method

\(dw = 2xdx + 2ydy - 2zdz

at (2,1,1) $dw = 4dx + 2dy - 2dz$

at (2,1,1) $\Delta W \approx 4\Delta x + 2\Delta y - 2\Delta z$

We stay on the level surface $\Delta W = 0$

$4(x - 2) + 2(y - 1) - 2(z - 1) = 0$

Gradient

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$$

level curves

z = 2x + y

All these level curves will be lines e.g.

0 = 2x + y

1 = 2x + y

$z = x^2 + y^2$ e.g $1 = x^2 + y^2$

Level Surface

A surface S in the R^3 is called a level surface of f(x,y,z) if the value of f on every point S is some fixed constant. For example every body in a class room is the level surface of 37 degrees celsius- providing students do not have fever.

Example

$f(x, y, z) = x^2 + y^2 - z$

$0 = x^2 + y^2 - z$ - circular parabola.

$1 = x^2 + y^2 - z$ - circular parabola where the vertex is at -1.

Gradient is the direction of steepest ascent

Example 1

$$x^2 + y^2 = 1$$

$$F(x, y) = x^2 + y^2 - 1 = 0$$

df $= (2y)dy + (2x)dx$

$$\frac{dy}{dx} = \frac{-x}{y}$$

Example 2

$$F(x, y) = y^5 + y^3 + y + x = 0$$

$$F_y(x, y) = 5y^4 + 3y^2 + 1 > 0$$

This function is strictly increasing

exactly one root

df $= (5y^4 + 3y^2 + 1)dy + (x)dx$

$$\frac{dy}{dx} = \frac{-1}{(5y^4 + 3y^2 + 1)}$$

Generalisation- n + 1 coordinates

$$\vec{x} = (x_1, x_2, \cdots, x_n)$$

$F(\vec{x}, y)\epsilon C^1$ in a neighbourhood of $N_0(\vec{x}_0, y_0)$

$$F(N_0) = 0$$

$$\frac{\partial f}{\partial y}(N_0) \neq 0$$

If these conditions are met then there is an explicit function $y = f(\vec{x})$

$$\frac{\partial f}{\partial x_i} = -\frac{\frac{\partial F}{\partial x_i}}{\frac{\partial F}{\partial y}} \ (i = 1, 2, 3, \cdots, k)$$

Example 3

$F(x, y, z) = 3x^2y - yz^2 - 4xz - 7 = 0$ We can show that near (-1,1,2) we can write y = f(x,z)
$F(-1, 1, 2) = 0$

$$\frac{\partial f}{\partial x} = 6xy - 4z$$

$$\frac{\partial f}{\partial y} = 3x^2 - z^2$$

$$\frac{\partial f}{\partial z} = 2zy$$

$$\frac{\partial f}{\partial x}(-1, 2) = -\frac{6xy - 4z}{3x^2 - z^2}(-1, 1, 2) = -14$$

We can find y explicitly without the theorem.

$$y = f(x, z) = \frac{4xz + 7}{3x^2 - z^2}$$

Using the quotient rule

$$\frac{\partial f}{\partial x}(-1, 2) = -14$$

Example 4

$$F(x, y, z) = x^2 y - yz^2 - 4xz - 7 = 0$$

In this example, we can write z = f(x,y) explicitly by the quadratic formula

$$z = {4x \pm \sqrt{(-4x)^2 - 4(-y)(-7 + 3x^2 y)} \over 6x^2 y }$$

The theorem fails

$$\frac{\partial F}{\partial z} = -2zy - 4x$$

$$\frac{\partial F}{\partial z}(N_0) = 0$$

Example 5

$$F(x, y) = (x - y)^3$$

$$F(x, y) = 0 \quad therefore \quad y = x$$ There is an explicit function at any point. However at (0,0)

$$\frac{\partial f}{\partial x}(0, 0) = 0$$

$$\frac{\partial f}{\partial y}(0, 0) = 0$$

The theorem does not apply

Proof

$$\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$$

Case 1

$$\frac{\partial F}{\partial y} > 0$$

At a neighbourhood of (x_0, y_0)

$F(x_0, y)$ is strictly increasing in terms of y.

$$F(x_0, y_0) = 0$$

There exists a y_1 such that $F(x_0, y_1) > 0$

There exists a y_2 such that $F(x_0, y_2) < 0$

For every x near x_0

$$F(x, y_1) > 0$$

$$F(x, y_2) < 0$$

For such an x near x_0, since

$\frac{\partial f}{\partial y} > 0$, F(x,y) is increasing (as an increasing function of y)

Therefore there exists a unique y such that $F(x, y) = 0$

This proves that y=f(x) exists and is unique proof of the formula for f'(x)

F(x,f(x)) = 0

By the chain rule

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} f'(x) = 0$$

$$f'(x) = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

The gradient is perpendicular to level surfaces

Suppose we have a function $g(x, y, z) \epsilon C^1$ at $M_0(x_0, y_0, z_0)$

$$g(M_0) = g(x_0, y_0, z_0) = C_0$$

Denote by S the level surface $g(x, y, z) = c_0$

Assume that $\nabla g(x_0, y_0, z_0) \neq 0$

Say for example that $\frac{\partial g}{\partial z} \neq 0$

$$F(x, y, z) = g(x, y, z) - c_0$$

$$F(M_0) = 0, F \epsilon C^1$$

$$\frac{\partial F}{\partial z}(M_0) \neq 0$$

By the implicit function theorem

$$f \epsilon C^1, \backslash (z_0 = f(x_0, y_0)$$

$F(x, y, f(x, y)) = 0$ in a neighbourhood

$g(x, y, f(x, y)) = c_0$ near M_0

Hence near M_0 the level surface S is the graph of f(x,y).

Example 6

$$x^2 + y^2 + z^2 = sin(xy)$$

$$F(x, y, z) = x^2 + y^2 + z^2 - sin(zy)$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_y}$$

$$\frac{\partial z}{\partial x} = -\frac{2x}{2z - y cos(zy)}$$

$$\frac{\partial z}{\partial y} = -\frac{F_x}{F_z}$$

$$\frac{\partial z}{\partial y} = -\frac{2y - z cos(zy)}{2z - y cos(zy)}$$