l'Hôpital Rule

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Introduction

l'Hôpital rule is a technique in finding the limit of a rational function in indeterminate form in particular zero over zero and infinity over infinity. It is named after the 17th-century French mathematician Guillaume de l'Hôpital. It is also known at the Bernouilli rule as the Swiss mathematician Johann Bernoulli introduced it to him in 1694 (Albrycht, 1951).

$$\text{Lim}_{\ x\to b} \frac{f(x)}{g(x)} = \text{Lim}_{\ x\to b} \frac{f'(x)}{g'(x)}$$

The limit must be indeterminate form such as zero over zero or infinity over Infinity,
First of all, we will review key concepts before we explore l'Hôpital rule. Afterwards, we will be ready to prove the theorem and then explore when l'Hôpital rule can only be used.

Background

First principles of differentiation

In calculus you should be aware that:

$$f'(x) = \text{Lim}_{\ h\to 0} \frac{f(x+h)-f(x)}{h}$$

This is saying that the gradient of the function f(x) at x= b is equal to the tangent at the point (b,f(b)). If you choose any two points on the curve f(x), The gradient of the straight line between the two points will converge to the gradient of the curve at x = b when the two points get closer to each other.
If you wish to explore this concept please click on the link to access a page that I have created - understanding first principles!
Let b = x + h then this can be written as

$$f'(x)= \text{Lim}_{\ x-b\to 0} \frac{f(b)-f(x)}{b-x} = \text{Lim}_{\ x\to b} \frac{f(b)-f(x)}{b-x}$$

Switching x and b also leads to

$$f'(b)= \text{Lim}_{\ x\to b} \frac{f(x)-f(b)}{x-b}$$

Properties of Limits

It is important to be familiar with some key properties of limits.
Additive

$$\text{Lim}_{\ x\to a}\ f(x) + g(x) = \text{Lim}_{\ x\to a}\ f(x) + \text{Lim}_{\ x\to a}\ g(x)$$

Multiplicative

$$\text{Lim}_{\ x\to a}\ f(x)\ g(x) = \text{Lim}_{\ x\to a}\ f(x)\ \text{Lim}_{\ x\to a}\ g(x)$$

Quotient

$$\text{Lim}_{\ x\to a} \frac{f(x)}{g(x)} = \frac{\text{Lim}_{\ x\to a}\ f(x)}{\text{Lim}_{\ x\to a}\ g(x)}$$

Power

$$\text{Lim}_{x\to a} f(x)^n = [\text{Lim}_{x\to a} f(x)]^n$$

The last two properties we will be using in order to prove l'Hôpital rule.

Proof of l'Hôpital Rule

let f(b) = g(b) = 0

$$\text{Lim}_{x\to b} \frac{f(x)}{g(x)} = \text{Lim}_{x\to b} \frac{f(x)-f(b)}{g(x)-g(b)} = \text{Lim}_{x\to b} \frac{\frac{f(x)-f(b)}{(x-b)}}{\frac{g(x)-f(b)}{(x-b)}}$$

Using the quotient rule of limits

$$\text{Lim}_{x\to b} \frac{f(x)}{g(x)} = \frac{\text{Lim}_{x\to b} \frac{f(x)-f(b)}{(x-b)}}{\text{Lim}_{x\to b} \frac{g(x)-f(b)}{(x-b)}}$$

Applying the first principles of differentiation this leads to:

$$\text{Lim}_{x\to b} \frac{f(x)}{g(x)} = \frac{f'(b)}{g'(b)}$$

Example

We can find the limit of the following function by simply differentiating the numerator and denominator.

$$\text{Lim}_{x\to 0} \frac{\sin(x)}{x} = \text{Lim}_{x\to 0} \frac{\cos(x)}{1} = 1$$

Do note that this limit is in indeterminate form since we get zero over zero. Therefore, we can apply l'Hôpital rule.

l'Hôpital rule can also be explained using linear approximations (Stewart, J. 2012).
Let f(b) = g(b) = 0

$$f(x) \approx f'(b)(x-b)+f(b) \text{ and } g(x) \approx g'(b)(x-b) + g(b)$$

These expressions are derived from first principles.

$$f'(b) = \text{Lim}_{x\to b} \frac{f(x)-f(b)}{x-b} \text{ and } g'(b) = \text{Lim}_{x\to b} \frac{g(x)-g(b)}{x-b}$$

It can be seen that the approximation of f(x) and g(x) improves as x gets closer to b.

$$\text{Lim}_{x\to b} \frac{f(x)}{g(x)} = \text{Lim}_{x\to b} \frac{f'(b)(x-b)+f(b)}{g'(b)(x-b)+g(b)} = \frac{f'(b)}{g'(b)}$$

Applying l'Hôpital rule to infinity over infinity

We can use the above proof to show that l'Hôpital rule can be used when the limit is infinity over infinity.
Let $\text{Lim}_{x\to b} f(x) = \text{Lim}_{x\to b} g(x) = \infty$

$$L = \text{Lim}_{x\to b} \frac{f(x)}{g(x)} = \text{Lim}_{x\to b} \frac{1/g(x)}{1/f(x)}$$

Let us show that L can also be expressed as the following:

$$L = \text{Lim}_{x\to b} \frac{(1/g(x))'}{(1/f(x))'} = \text{Lim}_{x\to b} \frac{-(g(x))^{-2}g'(x)}{-(f(x))^{-2}f'(x)} = \text{Lim}_{x\to b} \frac{(f(x))^2 g'(x)}{(f(x))^2 f'(x)}$$

Next we will use the product and power properties of limits.

$$L = L^2 \text{ Lim}_{x \to b} \frac{g'(x)}{f'(x)}$$

Hence, we have shown that :

$$L = \text{Lim}_{x \to b} \frac{f(x)}{g(x)} = \text{Lim}_{x \to b} \frac{f'(x)}{g'(x)}$$

When l'Hôpital rule can not used?

l'Hôpital rule can only be used when the limit is in indeterminate form. Let us look at the example where we will mistakenly use l'Hopital rule because the limit of our rational function is neither zero over zero nor infinity over infinity.
Example

$$\text{Lim}_{x \to 2} \frac{x+6}{2x}$$

Substituting x= 2 into the limit will lead to the following:

$$\text{Lim}_{x \to 2} \frac{x+6}{2x} = 2$$

Applying l'Hôpital rule incorrectly by ignoring the necessary condition and immediately differentiating the numerator and denominator would give us the following answer:

$$\text{Lim}_{x \to 2} \frac{1}{2} = 0.5$$

Conclusion

l'Hôpital rule is a great tool in finding the limits of rational functions. Differentiating the numerator and the denominator can be so much easier that using other methods to find the limits such as factorising the numerator and denominator and cancelling out the common factors or expressing the the rational function as a polynomial (Taylor's expansion). However, we must not forget that l'Hôpital rule can **only** be used for limits in indeterminate form such as zero over zero or infinity over infinity.

References

Albrycht, J. (1951). L’Hopital’s rule for vector-valued functions. ^ Colloq. Math. 2: 176–177.
Stewart, J. (2012), Calculus : early transcendentals , Brooks/Cole, Cengage Learning , Belmont, Cal. .