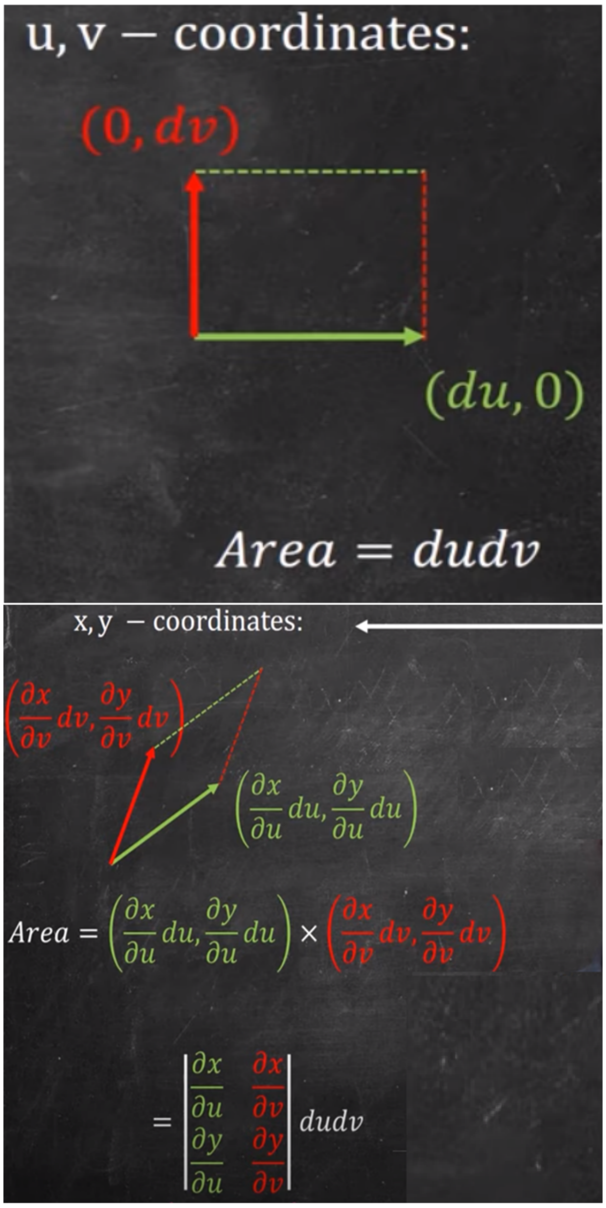
Change of Variables

Background

Area of parallelogram = base x perpendicular height $= |b||a|sin\theta = |a \times b|$

Jacobian



$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

For x= g(u,v) and y = h(u,v)

$$\iint_R f(x,y)dxdy = \iint_G f(g(u,v), h(u,v))J(u,v)dudv$$

u(x,y) and v(x,y)

$\Delta u \approx u_x \Delta x + u_y \Delta y$

$\Delta v \approx v_x \Delta x + v_y \Delta y$

$\begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} \approx \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$

The scaling factor for the area is the determinant of the matrix.

$< \Delta x, 0 > \rightarrow < \Delta u, \Delta v > \approx < U_x \Delta x, V_x \Delta x >$

$< 0, \Delta y > \rightarrow < \Delta u, \Delta v > \approx < U_y \Delta y, V_y \Delta y >$

area' = det () dxdy

Alternatively,

x(u,v) and y(u,v)

$\Delta x \approx x_u \Delta u + x_v \Delta v$

$\Delta y \approx y_u \Delta u + y_v \Delta v$

$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \approx \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix}$

The scaling factor for the area is the determinant of the matrix.

$< \Delta u, 0 > \rightarrow < \Delta x, \Delta y > \approx < X_u \Delta u, Y_u \Delta u >$

$< 0, \Delta v > \rightarrow < \Delta x, \Delta v > \approx < X_v \Delta v, Y_v \Delta v >$

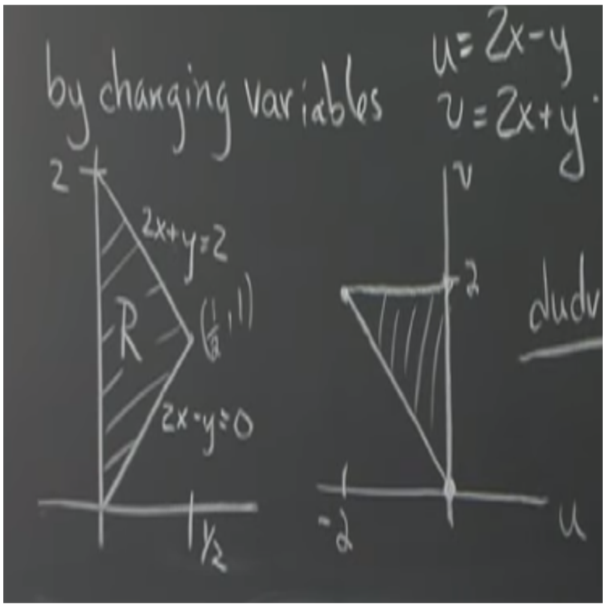
area' = det () dudv

Example

Given region defined by 2x-y=0 , 2x+y=0 and the y axis and x axis. Compute $\iint_R (4x^2 - y^2)^4 dxdy$

u = 2x- y

v=2x + y

u-v = -2y 

$\iint_R (4x^2-y^2)^4 dxdy = \int_0^2 \int_{u=-v}^{u=0} ((uv)^4)\frac{1}{4} dudv$

Example

Region defined by y = x-1 , y=x-2 and y=1, y = 2

u = x- y

v=u

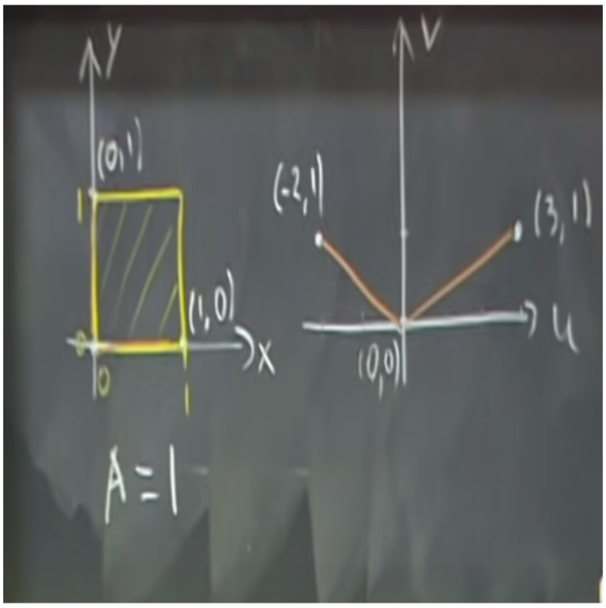
$\iint_R f(x,y)dxdy = \int_1^2 \int_1^2 f(g(u,v),h(u,v))(1)dudv$

Example

Find the scaling factor (dxdy vs dudv)

u = 3x - 2y

v = x + y

dA=dxdy dA' = dudv 

$\iint dxdy = \iint \frac{1}{5}dudv$

Example

Find the area of a circle

$\iint_R dxdy = \int_{-r}^{r} \int_{\theta=0}^{\theta=2\pi} rdrd\theta = \pi r^2$

Example

Find the area of an ellipse with semi-axes a,b

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
Change the Variables

$\frac{x}{a} = u$

$\frac{y}{b} = v$

$\iint_R dxdy = \iint_{u^2+v^2<1} dudv = ab \iint_{u^2+v^2<1} dudv$

ab (area of unit circle $= ab\pi$

Example

Find the volume of the region under the curve $z = 9 - x^2 - y^2$
$\int_{x=-3}^{x=3} \int_{y=-\sqrt{9-x^2}}^{\sqrt{9-x^2}} 9 - x^2 - y^2 dxdy$
$\int_{r=0}^{r=3} \int_{\theta=0}^{\theta=2\pi} (9 - r^2) rdrd\theta = \frac{81\pi}{2}$

Example

Compute $\int_0^1 \int_0^1 x^2 ydxdy$ using a change of variables

u=x

v=xy
$\int_0^1 \int_0^1 x^2 ydxdy = \int_0^1 \int_v^1 vdudv$