Pset 2

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1 Stable Matching

1.1 Hard example for stable matching

(a '+' in the rejected list means that man was rejected in that stage)

Stage #	Men's Proposals	Women Rejections & String
1	**A**: 1 **B**: 2 **C**: 3 **D**: 1	**1**: **String**: D **Rejected List**: +A **2**: **String**: B **Rejected List**: **3**: **String**: C **Rejected List**: **4**: **String**: **Rejected List**:
2	**A**: 2 **B**: 2 **C**: 3 **D**: 1	**1**: **String**: D **Rejected List**: A **2**: **String**: A **Rejected List**: +B **3**: **String**: C **Rejected List**: **4**: **String**: **Rejected List**:
3	**A**: 2 **B**: 3 **C**: 3 **D**: 1	**1**: **String**: D **Rejected List**: A **2**: **String**: A **Rejected List**: B **3**: **String**: B **Rejected List**: +C **4**: **String**: **Rejected List**:

4	**A**: 2 **B**: 3 **C**: 1 **D**: 1	**1**: **String**: C **Rejected List**: A, +D **2**: **String**: A **Rejected List**: B **3**: **String**: B **Rejected List**: C **4**: **String**: **Rejected List**:
5	**A**: 2 **B**: 3 **C**: 1 **D**: 2	**1**: **String**: C **Rejected List**: A, D **2**: **String**: D **Rejected List**: B, +A **3**: **String**: B **Rejected List**: C **4**: **String**: **Rejected List**:
6	**A**: 3 **B**: 3 **C**: 1 **D**: 2	**1**: **String**: C **Rejected List**: A, D **2**: **String**: D **Rejected List**: B, A **3**: **String**: A **Rejected List**: C, +B **4**: **String**: **Rejected List**:

7	**A**: 3 **B**: 1 **C**: 1 **D**: 2	**1**: **String**: B **Rejected List**: A, D, +C **2**: **String**: D **Rejected List**: B, A **3**: **String**: A **Rejected List**: C, B **4**: **String**: **Rejected List**:
8	**A**: 3 **B**: 1 **C**: 2 **D**: 2	**1**: **String**: B **Rejected List**: A, D, C **2**: **String**: C **Rejected List**: B, A, +D **3**: **String**: A **Rejected List**: C, B **4**: **String**: **Rejected List**:
9	**A**: 3 **B**: 1 **C**: 2 **D**: 3	**1**: **String**: B **Rejected List**: A, D, C **2**: **String**: C **Rejected List**: B, A, D **3**: **String**: D **Rejected List**: C, B, +A **4**: **String**: **Rejected List**:

10	**A**: 4 **B**: 1 **C**: 2 **D**: 3	**1**: **String**: B **Rejected List**: A, D, C **2**: **String**: C **Rejected List**: B, A, D **3**: **String**: D **Rejected List**: C, B, A **4**: **String**: A **Rejected List**:

1.2 A tight bound on the number of proposals

We first show that there is at most one man who proposes to the last woman on his list.

Proof. Suppose there is more than 1 man who proposes to the last woman on his list. Let those men be M_1 and M_2 with least preferred women W_1 and W_2 respectively and M_1 attempts to propose before M_2 in execution of the algorithm.

We know that if M_1 and M_2 are proposing to the last woman on their preference lists, every other woman has rejected them and has another man on her "string". Therefore to end with a perfect matching, M_1 must be matched with W_1 and M_2 must be matched with W_2. There are now two scenarios regarding these women:

1. $W_1 == W_2$. If the two women are the same we run into a contradiction as both M_1 and M_2 cannot be matched with the same person in a perfect stable matching.

2. $W_1 \neq W_2$. We know every women except W_1 has rejected M_1. That means every woman except W_1 has a man on her string at the time M_1 is looking to propose to W_1. By the same reasoning, we can say that every women except W_2 has rejected M_2 and every women except W_2 has a man on her string. However this is a contradiction of our previous statement that all women except W_1 have a man on her string.

That means there can only be at most 1 man who proposes to n women. The rest of the men can only propose to at max $n - 1$ women so the algorithm is bounded at $n(n - 1) + 1$ proposals. $\square$

Example 1.1 starts off with 4 proposals in stage 1 and increments the number of unique proposals by one for each stage except the last totalling in 13 different proposals which is the worst case for $n = 4$. This takes 10 days to finish.

2 Combining Stable Matchings

(a) *Proof.* Let us assume that two women, w_1 and w_2 end up picking the same man, m_x. That means m_x has to be a matching with one woman in one set and the other woman in the other. Let's assume the following elements are in S:

$$(w_1, m_x) \in S$$
$$(w_2, m_S) \in S$$

and that the following elements are in set T:

$$(w_1, m_T) \in T$$
$$(w_2, m_x) \in T$$

for some arbitrary men m_S and m_T.

Since we assume both w_1 and w_2 pick m_x when forming W, we know the following about w_1 and w_2's preference lists:

$$w_1 : m_x > m_T$$
$$w_2 : m_x > m_S$$

Since there can be no ties in a preference list, we know m_x prefers w_1 over w_2 or w_2 over w_1.

(a) For the $w_1 > w_2$ case, we see there is an instability in T as w_1 and m_x form a rogue couple.

(b) For the $w_2 > w_1$ case, we see there is an instability in S as w_2 and m_x form a rogue couple.

Both cases contradict our assumption that T and S are stable so no two women can pick the same man. □

(b) *Proof.* Let us assume that W is not stable and we have an unstable matching:

$$\alpha = (w_1, m_1)$$
$$\beta = (w_2, m_2)$$

such that in terms of w_1's ranking:

$$m_2 > m_1$$

and in terms of m_2's ranking:

$$w_1 > w_2$$

By virtue of how W is formed, we know that $\alpha, \beta \in S \cup T$. More specifically, $\alpha \in S \implies \beta \in T$ and $\beta \in S \implies \alpha \in T$ as these unstable pairs cannot belong to the same set as S and T are stable.

Let's arbitrarily assume $(w_1, m_1) \in S$ and $(w_2, m_2) \in T$. We know that w_1 is paired with a man in T whom she ranks lower than m_1 as the $(w_1, m_1) \in S$ pair beats that pair in T. Let's call the T pairing $(w_1, m_T) \in T$

So w_1's ranking is as follows:

$$m_2 > m_1 > m_T$$

However we know that m_2's ranking is $w_1 > w_2$. So:

$$(w_1, m_T) \in T$$
$$(w_2, m_2) \in T$$

is an unstable matching in T which contradicts our assumption that T is stable. Therefore W must be a stable matching. $\square$

(c) *Proof.* Let's define a stable matching, Z as the way of combining S and T by forcing each man to pick the woman he prefers the least amongst the two.

Let's assume that $Z \neq W$. Therefore for any man m_1, we have:

$$(m_1, w_w) \in W$$
$$(m_1, w_z) \in Z$$
$$w_w \neq w_z$$

Let's arbitrarily say that the $(m_1, w_z) \in S$ and define the m_1 matching in T as $(m_1, w_T) \in T$. Since the (m_1, w_z) pair "won" over the other in creation of Z, we know that m_1 prefers w_T over w_z:

$$m_1 : w_T > w_z$$

We can apply this same logic with the $(m_1, w_w) \in W$ pair. Since $(m_1, w_w) \in S \cup T$ we also know $(m_s, w_w) \in S \cup T$ for some man m_s. We earlier assumed that $(m_1, w_z) \in S$ so that means $m_s \neq m_1$ must be paired with w_w in S: $(m_s, w_w) \in S$ and $(m_1, w_w) \in T$.

We earlier named the m_1 pair in T as (m_1, w_T) so that means $w_T = w_w$.

We also know that w_w prefers m_1 over m_s as (m_1, w_w) "won" over the (m_s, w_w) pair in S in the creation of W:

$$w_w : m_1 > m_s$$

However this forms an instability in S as:

$$(m_1, w_z) \in S$$
$$(m_s, w_w) \in S$$

but $(m_1, w_T = w_w)$ form a rogue couple. This contradicts our assumption that S is stable so $Z = W$. $\square$

(d) To create the best of both worlds for men we can create M in the same fashion as creating W in part $(a.)$ but letting men pick their preferred choice instead of women. Each man is given the name of a woman he is matched with in set S and set T. Of the two names, the man picks the woman whom he prefers more. We create M as the set of matchings where each man is paired with the woman he picks as above. It is the "best of both worlds" as each man picks the woman whom he likes greater given two stable options and the matching is stable according to $(b.)$