$u(t,x)$

(1) $$\frac{\partial^2 u(t,x)}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad t \in \mathbb{R}$$

$x \in \mathbb{R}$ (corda infinita)

$u(t,x)$ de classe C^2 é solução de (1)

$$\Longleftrightarrow \quad u(t,x) = F(x+ct) + G(x-ct)$$

Dem.

$\Longleftarrow$ óbvio

$\Longrightarrow$ $\quad x+ct = \xi; \; x-ct = \eta$

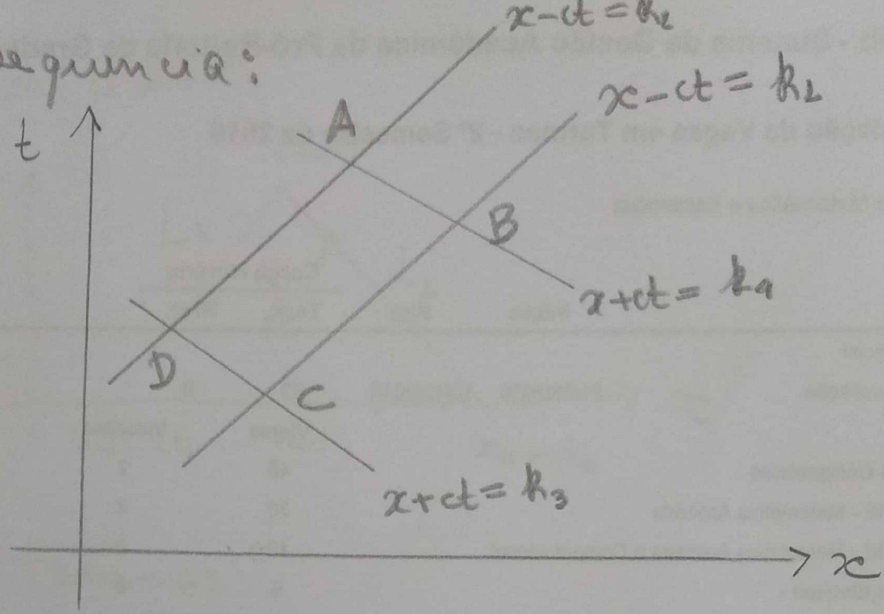
$$x = \frac{\xi+\eta}{2} \qquad t = \frac{\xi-\eta}{2c}$$

$$u(t,x) = u\left(\frac{\xi-\eta}{2c}, \frac{\xi+\eta}{2}\right) = v(\xi,\eta)$$

$u(t,x)$ satisfaz (1) $\Longleftrightarrow$

$$\frac{\partial^2}{\partial \xi \partial \eta} v(\xi,\eta) = 0 \Longleftrightarrow v(\xi,\eta) = f_1(\xi) + f_2(\eta)$$

Consequência:



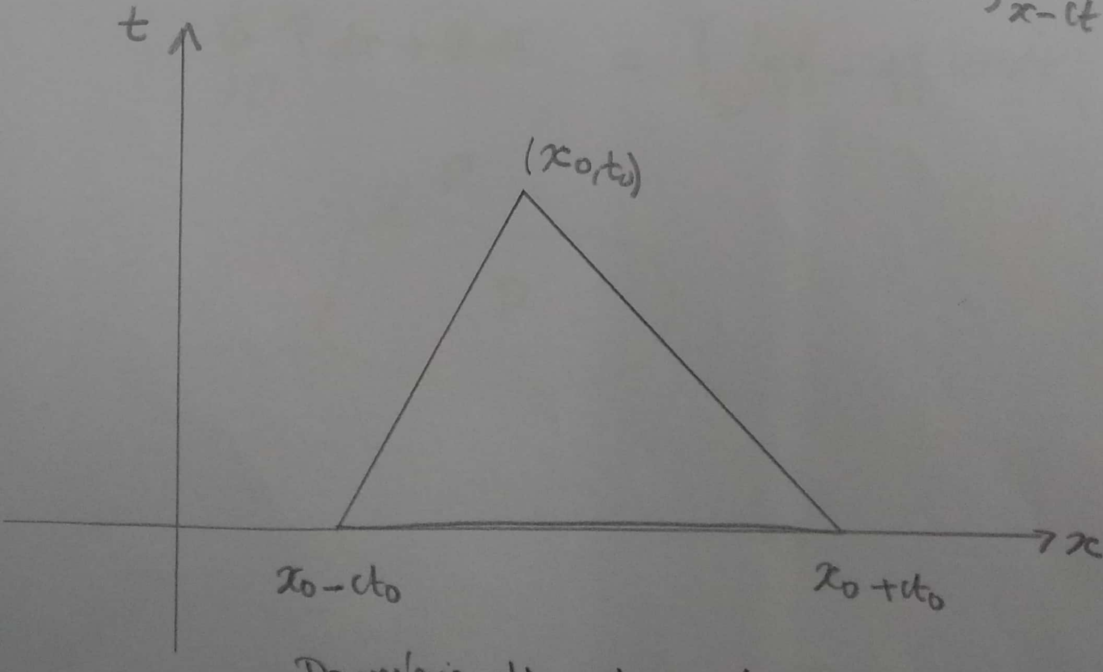
$$u(A)+u(C)=u(B)+u(D)$$

—— // ——

condição inicial:

$u(0,x)=f(x)$ $\qquad \frac{\partial u}{\partial t}(0,x)=g(x)$

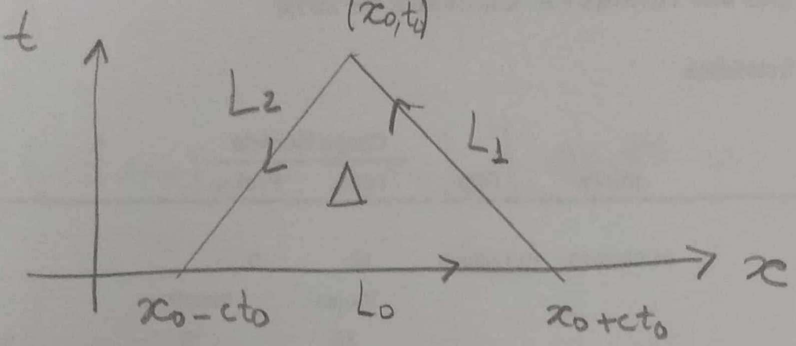
$$\Rightarrow \quad u(t,x)=\frac{f(x+ct)+f(x-ct)}{2}+\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$



Domínio de dependência

$[x_0-ct_0,\, x_0+ct_0]\times\{0\}$

Unicidade



Teorema:

$u_{tt} - c^2 u_{xx} = 0$

$$u(0,x) = 0 = \frac{\partial u}{\partial t}(0,x)$$

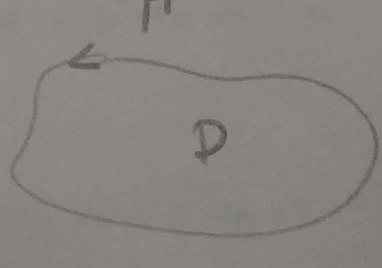
$$x_0 - ct_0 \le x \le x_0 + ct_0$$

Então $u(t,x) = 0$ p/ $(t,x) \in \Delta$

<u>Dem</u>

Fórmula de Green no plano:

$$\oint_\Gamma P\,dx + Q\,dt = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial t}\right) dx\,dt$$



⑨

$$0 = \iint_\Delta (u_{tt} - c^2 u_{xx})\,dx\,dy =$$

$$\iint_\Delta \left[(-c^2 u_x)_x - (-u_t)_t\right] dx\,dt =$$

$$\oint_{\partial\Delta} (-u_t\,dx - c^2 u_x\,dt)$$

L_0: $\quad dt = 0$

L_1: $\quad dx = -c\,dt \qquad x + ct = k_1 = x_0 + ct_0$

L_2: $\quad dx = ct \qquad x - ct = k_2 = x_0 - ct_0$

$$\int_{L_0} -u_t\,dx - c^2 u_x\,dt = \int_{L_0} -u_t\,dx$$

$= 0$ pois $u_t(0,x) = 0$

$$\int_{L_1} (-u_t\,dx - c^2 u_x\,dt) =$$

$$\int_{L_1} c\,u_t\,dt - c^2 u_x\,dt =$$

$$c\int_{L_1} u_t\,dt - c\,u_x\,dt =$$

$L_1 \quad x = x_0 + ct_0 - ct$

$0 \leq t \leq t_0$

Como

$$\frac{d}{dt} u(t, x_0 + ct_0 - ct) =$$

$$u_t(t, x_0 + ct_0 - ct) - c\, u_x(t, x_0 + ct_0 - ct)$$

temos

$$c \int_{L_1} (u_t - c u_x)\, dt =$$

$$c \quad u(t, x_0 + ct_0 - ct)\Big]_0^{t_0} =$$

$$c\, u(t_0, x_0) - c\, \underset{0}{\underset{\parallel}{u(0, x_0 + ct_0)}} =$$

$$c\, u(t_0, x_0)$$

Idem

$$\int_{L_2} -u_t\, dx - c^2 u_x\, dt = c\, u(t_0, x_0)$$

$$\Rightarrow \quad u(t_0, x_0) = 0$$

Variando (t_0, x_0) temos o resultado.

Generalizando, se

$$u(0,x) = f(x) \quad e \quad \frac{\partial u}{\partial t}(0,x) = g(x)$$

$$\Rightarrow$$

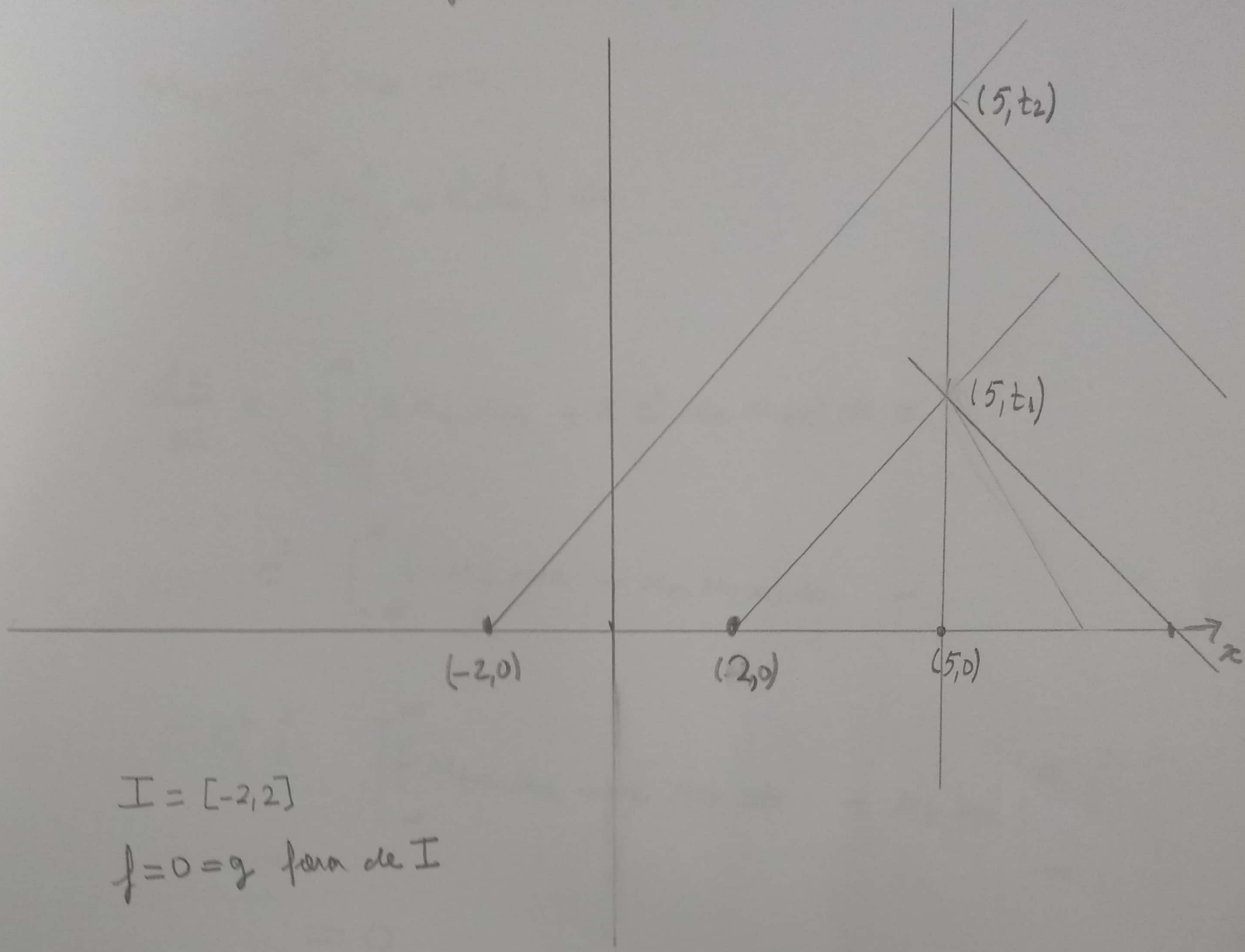
$$u(t_0,x_0) = \frac{f(x_0+ct_0)+f(x_0-ct_0)}{2} + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(s)\,ds$$

Se $u_{tt} - c^2 u_{xx} = h(t,x)$

$$u(t_0,x_0) = \cdots \quad + \frac{1}{2c}\iint_\Delta h(s,y)\,ds\,dy$$

$$\iint_\Delta h(s,y)\,ds\,dy = \frac{1}{2c}\int_0^{t_0}\left[\int_{x_0-c(t_0-s)}^{x_0+c(t_0-s)} h(s,y)\,dy\right]ds$$

Domínio de dependência



$I = [-2, 2]$

$f = 0 = g$ fora de I

$u(t,5) = 0$ se $0 \leq t \leq t_1$

$t_1 = \dfrac{5-2}{c} = \dfrac{3}{c}$

$u(t,0) = \dfrac{1}{2c}\displaystyle\int_{-2}^{2} g(s)\,ds \quad t_2 \leq t$

$t_2 = \dfrac{5-(-2)}{c} = 7/c$

Unicidade via energia

$$u_{tt} - c^2 u_{xx} = 0$$

$$E(t) = \int_{-\infty}^{\infty} (u_t^2 + c^2 u_x^2)\, dx$$

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} (2 u_t u_{tt} + 2 c^2 u_x u_{tx})\, dx =$$

$$c^2 \int_{-\infty}^{\infty} (u_t u_{xx} + u_x u_{tx})\, dx =$$

$$c^2 \left[\int_{-\infty}^{\infty} (-u_{tx} u_x + u_x u_{tx})\, dx + u_t u_x \Big|_{-\infty}^{\infty} \right.$$

$$= 0$$

Portanto se $E(0) = 0 \Rightarrow E(t) = 0 \Rightarrow$

u é constante $\Rightarrow$ $u \equiv 0$

Regularidade da solução.

$f \in C^2,\ g \in C^1 \Rightarrow u$ é C^2

e não mais que isso.

Eq. de Laplace: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

soluções C^2 são C^∞.

corda vibrante na semi reta $x>0$

$$u_{tt} - c^2 u_{xx} = 0 \qquad x>0$$

$$u(0,x) = f(x), \qquad \frac{\partial u}{\partial t}(0,x) = g(x)$$

Dirichlet $\qquad u(t,0)=0$

Neumann $\qquad \frac{\partial u}{\partial x}(t,0) = 0$

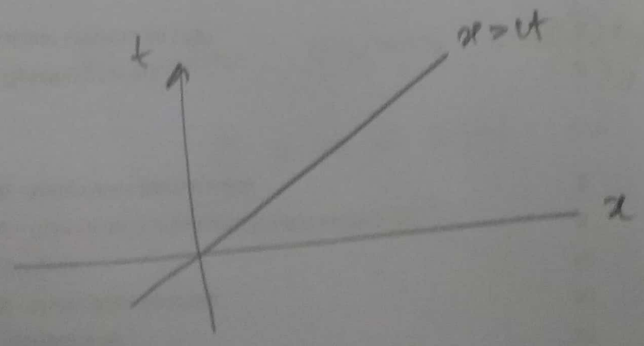
Dirichlet: $\quad f(0) = 0 = g(0)$

$\tilde{f}(x) \quad \tilde{g}(x)$ extensões ímpar.

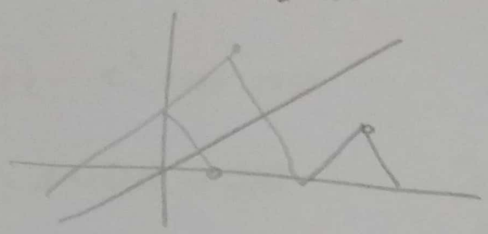
$$\tilde{u}(t,x) = \frac{\tilde{f}(x+ct) + \tilde{f}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

$u(t,x)$ restrição de $\tilde{u}(t,x)$ p/ $x \geq 0$

Em termos das condições iniciais



$x \geq 0, \quad t \geq 0 \quad$ e $\quad x - ct \geq 0$



$$u(t,x) = \frac{1}{2}\left[f(x+ct) + f(x-ct) \right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

$$x - ct < 0 \Rightarrow ct - x > 0$$

$$u(t,x) = \frac{1}{2}\left[f(x+ct) - f(ct-x) \right] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(s)\,ds$$

Idem para $t \leq 0$

As formulas acima valem p/

$$0 \leq x \leq c|t| \quad \text{ou}$$

$$c|t| \leq x$$

Condição de Neumann: extensão par

Eq. da corda vibrante num intervalo finito

$$u_{tt} - c^2 u_{xx} = 0 \qquad 0 < x < l$$

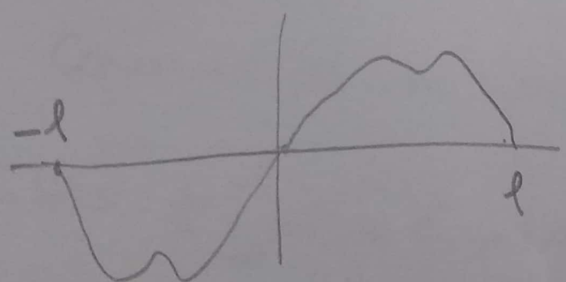
$$u(0,x) = f(x) \qquad \frac{\partial u}{\partial t}(0,x) = g(x)$$

Dirichlet dos dois lados

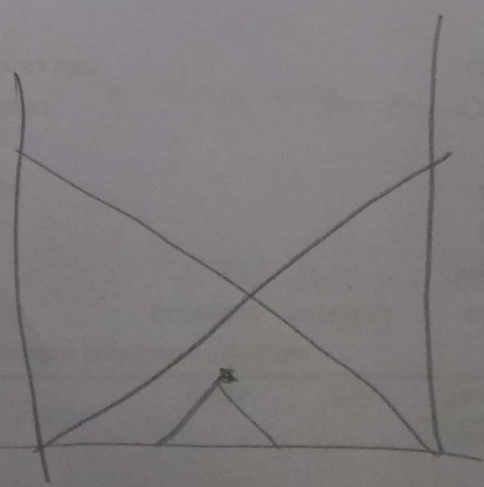
$$u(t,0) = 0 = u(t,l) = 0 \qquad \forall t$$

$$f(0) = g(0) = 0 \qquad f(l) = g(l) = 0$$

Extensão ímpar para $[-l, l]$ e periódica de período $2l$

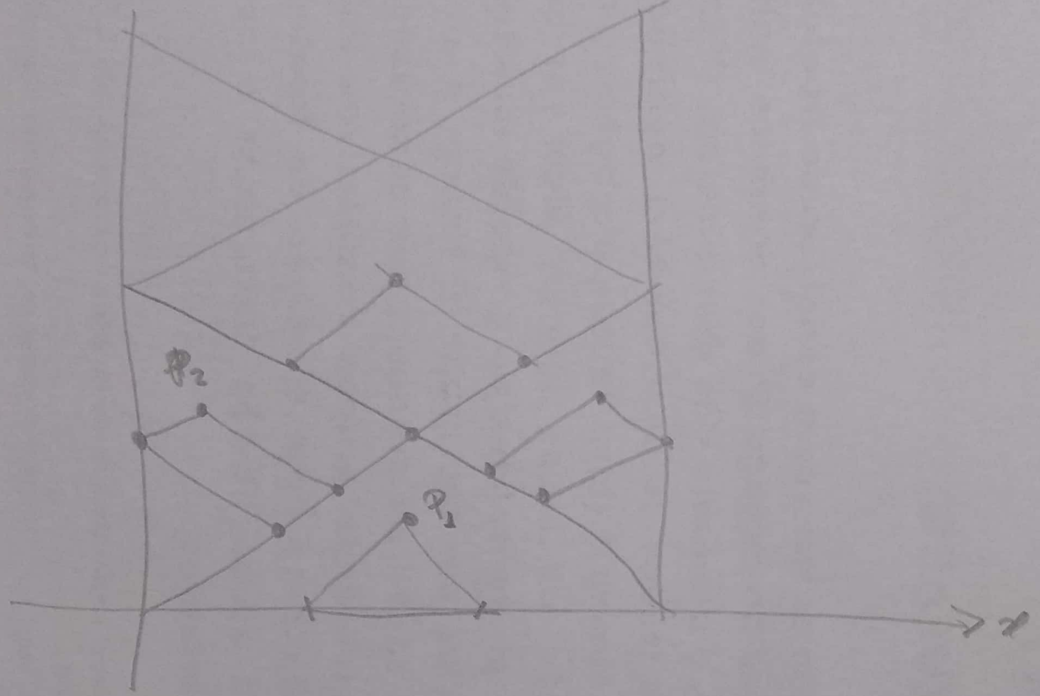


Uso $\check{f}(x)$ e $\check{g}(x)$, e restrinjo a $[0, l]$.



Outra estratégia:

$$u(A)+u(C)=u(B)+u(D)$$



Conservação da energia:

$$E(t)=\frac{1}{2}\int_0^l (u_t^2+c^2u_x^2)\,dx$$

$$\frac{dE}{dt}=\int_0^l (u_t u_{tt}+c^2u_x u_{tx})\,dx=$$

$$c^2\int_0^l (u_t u_{xx}+u_x u_{tx})\,dx=$$

$$c^2\int_0^l \frac{\partial}{\partial x}(u_t u_x)\,dx = c^2[u_t u_x]\Big|_0^l$$

$$u(t,0)=0 \Rightarrow u_t(t,0)=0$$

Neumann O.K.

Se $\alpha u_t \Rightarrow \frac{dE}{dt}=-\alpha\int_0^l u_t^2\,dx$