Separação de Variáveis e Séries de Fourier

$$u_{tt} - c^2 u_{xx} = 0 \qquad 0 < x < l$$

$$u(t,0) = 0 = u(t,l)$$

$$u(0,x) = f(x) \; ; \quad \frac{\partial u}{\partial t}(0,x) = g(x)$$

Ideia: construir soluções particulares e depois satisfazer condições iniciais por combinação linear.

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EDO linear homogênea no $\mathbb{R}^n$

$$\frac{dx}{dt} = Ax \qquad x(0) = x_0$$

soluções da forma

$$x(t) = e^{\lambda t} v \qquad v \neq 0$$

$$\lambda e^{\lambda t} v = e^{\lambda t} A v \qquad A v = \lambda v$$

ou $\quad x(t) = T(t)\, v$

$$\frac{dT}{dt} v = T(t) A v$$

$$A v = \lambda(t) v \quad \Rightarrow \quad \lambda(t) \text{ é const. } \lambda(t) = \lambda$$

$$\text{e} \quad \frac{dT}{dt} = \lambda T \quad \Rightarrow \quad T(t) = e^{\lambda t} c$$

Existe base de autovalores?

Também p/

$$\frac{d^2x}{dt^2} = Ax \qquad x(t) = e^{\lambda t} v$$

$$Av = \lambda^2 v$$

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$$u_{tt} - c^2 u_{xx} = 0 \qquad u(t,0) = 0 = u(t,l)$$

$$+ \text{ C. I.}$$

$$u(t,x) = T(t)\varphi(x) \qquad \varphi(0) = \varphi(L) = 0$$

$$\varphi \neq 0$$

$$T''\varphi = c^2 T(t)\varphi''$$

$$\frac{T''}{c^2 T} = \frac{\varphi''}{\varphi} = -\lambda$$

$$\varphi'' + \lambda\varphi = 0 \qquad \varphi(0) = \varphi(l) = 0, \quad \varphi \neq 0.$$

$$\lambda = -\alpha^2$$

$$\varphi(x) = A e^{\alpha x} + B e^{-\alpha x}$$

$$0 = \varphi(0) = A + B = 0$$

$$\varphi(x) = A(e^{\alpha x} - e^{-\alpha x})$$

$$\varphi(l) = 0 \Rightarrow e^{2\alpha l} = 1 \Rightarrow \alpha = 0$$

$$\Rightarrow \varphi(x) = Ax + B \Rightarrow A = B = 0$$

Então $\lambda > 0 \Rightarrow \lambda = w^2$

$\varphi(x) = A \cos wx + B \operatorname{sen} wx$

$\varphi(0) = 0 \Rightarrow A = 0$

$\varphi(x) = B \operatorname{sen} wx$

$\varphi(l) = 0 \Rightarrow B \operatorname{sen} wl = 0$

$B \neq 0 \Rightarrow \operatorname{sen} wl = 0$ $wl = n\pi$

$w = \frac{n\pi}{l}$

Então

$$\frac{T''}{c^2 T} = -w^2 = -\frac{n^2\pi^2}{l^2}$$

$T'' + w^2 c^2 T = 0 \Rightarrow$

$$T_n(t) = \alpha_n \cos wct + \beta_n \operatorname{sen} wct =$$

$$= \alpha_n \cos\left(\frac{n\pi c}{l} t\right) + \beta_n \operatorname{sen}\left(\frac{n\pi c}{l} t\right)$$

Solução da eq. da onda

$$u(t,x) \overset{?}{=} \sum_{n=1}^{\infty} \left(\alpha_n \cos\left(\frac{n\pi c}{l} t\right) + \beta_n \operatorname{sen}\left(\frac{n\pi c}{l} t\right)\right) \operatorname{sen} \frac{n\pi}{l} x$$

$$f(x) = u(0,x) = \sum_{n=1}^{\infty} \alpha_n \cos\left(\frac{n\pi c}{\ell}\right) \operatorname{sen} \frac{n\pi}{\ell} x \quad ? \qquad (1)$$

$$g(x) = \frac{\partial u}{\partial t}(0,x) = \sum_{n=1}^{\infty} \frac{n\pi c}{\ell} \beta_n \operatorname{sen} \frac{n\pi x}{\ell} \qquad ? \qquad (2)$$

(1) e (2) valem p/ extensões ímpares.

De maneira geral, se

$f: \mathbb{R} \to \mathbb{R}$ é periódica de período 2ℓ, queremos saber se é possível escrever:

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{\ell} + B_n \operatorname{sen} \frac{n\pi x}{\ell}\right)$$

Relações de ortogonalidade

$$\underset{f_0}{1}, \ \underset{f_1}{\cos\frac{\pi x}{\ell}}, \ \underset{f_2}{\cos\frac{2\pi x}{\ell}}, \ \dots \ \underset{f_n}{\cos\frac{n\pi x}{\ell}} \dots$$

$$\underset{g_1}{\operatorname{sen}\frac{\pi x}{\ell}} \ \ \underset{g_2}{\operatorname{sen}\frac{2\pi x}{\ell}}, \ \dots \ \underset{g_n}{\operatorname{sen}\frac{n\pi x}{\ell}}$$

Produto escalar real

$f, g : [-l, l] \to \mathbb{R}$

$$\langle f, g \rangle = \int_{-l}^{l} f(x) g(x) \, dx$$

O.K. se $f, g \in L^2([-l, l], \mathbb{R})$

ou $f \in L^1([-l, l], \mathbb{R}) \quad g \in L^\infty([-l, l], \mathbb{R})$

ou $f \in L^p, \; g \in L^{p'}$

Então

$\langle f_n, f_m \rangle = 0$ se $n \neq m, \quad n, m = 0, 1, 2, \ldots$

$\langle g_n, g_m \rangle = 0$ se $n \neq m, \quad n, m = 1, 2, \ldots$

$\langle f_n, g_m \rangle = 0 \quad \forall n, m.$

Por exemplo,

$$\operatorname{sen} x \operatorname{sen} y = \frac{1}{2} [\cos(x-y) - \cos(x+y)]$$

$$\int_{-l}^{l} \cos \frac{k\pi x}{l} \, dx = \begin{cases} 2l & \text{se } k = 0 \\ 0 & \text{se } k \neq 0 \text{ inteiro} \end{cases}$$

$$\int_{-l}^{l} \cos^2 \frac{n\pi x}{l}\, dx = l = \int_{-l}^{l} \operatorname{sen}^2 \frac{n\pi x}{l}\, dx \qquad n \neq 0$$

$$\left(\cos^2 x = \frac{1+\cos 2x}{2} \right)$$

$$\int_{-l}^{l} 1^2\, dx = 2l$$

Coeficientes de Fourier

$$\vec{v} = a\vec{\imath} + b\vec{\jmath} + c\vec{k}$$

$$a = \langle \vec{v}, \vec{\imath} \rangle$$

Supondo

$$f = \frac{A_0}{2} f_0 + A_1 f_1 + B_1 g_1 + \dots A_n f_n + B_n g_n + \dots$$

$$\int_{-l}^{l} f(x) \cdot f_0(x)\, dx = \langle f, f_0 \rangle = \frac{A_0}{2} \underbrace{\langle f_0, f_0 \rangle}_{2l} =$$

$$= A_0 l$$

Portanto

$$A_0 = \frac{1}{l} \int_{-l}^{l} f(x)\, dx$$

$$\int_{-l}^{l} f(x)\, f_1(x)\, dx = \langle f, f_1 \rangle = A_1 \langle f_1, f_1 \rangle = A_1 l$$

$$A_1 = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{\pi x}{l}\, dx$$

Em geral, (7)

$$A_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx = \frac{1}{\ell}\langle f, f_n\rangle \qquad n = 0, 1, 2, \ldots$$

$$B_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\,\mathrm{sen}\,\frac{n\pi x}{\ell}\,dx = \frac{1}{\ell}\langle f, g_n\rangle \qquad n = 1, 2, \ldots$$

Os coeficientes estão bem definidos

se $\int_{-\ell}^{\ell} |f(x)|\,dx < \infty$.

Definindo

$$s_n(x) = \frac{A_0}{2} + \sum_{k=1}^{n}\left(A_k\cos\frac{k\pi x}{\ell} + B_k\,\mathrm{sen}\,\frac{k\pi x}{\ell}\right)$$

é verdade que $s_n \to f$? Em que sentido?

Noções de convergência

$f_n, f : I = [a,b] \to \mathbb{R}$

$f_n \to f$ em <u>média quadrática</u>:

$$\int_a^b |f_n - f|^2 dx \to 0 \quad (\text{Norma } L^2)$$

$f_n \to f$ <u>uniformemente</u>

$\forall \varepsilon > 0, \exists N(\varepsilon) : n \geq N(\varepsilon) \Rightarrow$

$$|f_n(x) - f(x)| < \varepsilon, \forall x \in I \quad (L_\infty)$$

$f_n \to f$ <u>quase sempre</u>:

$\{x : f_n(x)$ não converge p/ $f(x)\}$ tem medida zero.

Seja $f: [-\ell, \ell] \to \mathbb{R}$ tal que

$$\int_{-\ell}^{\ell} f^2(x)\,dx < \infty \implies \int_{-\ell}^{\ell} |f(x)| \cdot 1\,dx \leq$$

$$\left(\int_{-\ell}^{\ell} f^2(x)\,dx\right)^{1/2} (2\ell)^{1/2} < \infty.$$

$\implies$ A_n e B_n estão bem definidos.

Teorema.

$$\lim_{n \to \infty} \int_{-\ell}^{\ell} (S_n(x) - f(x))^2\,dx = 0$$

Dem. Teoria sobre espaços de Hilbert e teorema de aproximação de Weierstrass.

Corolário. Identidade de Parseval (Pitágoras)

$$\frac{1}{\ell} \int_{-\ell}^{\ell} f^2(x)\,dx = \frac{A_0^2}{2} + \sum_{k=1}^{\infty} (A_k^2 + B_k^2)$$

Demonstração

$$\int_{-l}^{l} (f(x) - s_m(x))^2 dx =$$

$$\int_{-l}^{l} \left(f(x) - \frac{A_0}{2} - \sum_{k=1}^{m} A_k \cos \frac{k\pi x}{l} + B_k \operatorname{sen} \frac{k\pi x}{l}\right)^2 dx$$

$$= \int_{-l}^{l} f^2(x)\, dx + \frac{A_0^2}{2} l + \sum_{k=1}^{m} A_k^2 l + B_k^2 l$$

$$-2\left[\int_{-l}^{l} f(x) \left(\frac{A_0}{2} + \sum_{k=1}^{m} A_k \cos \frac{k\pi x}{l} + B_k \operatorname{sen} \frac{k\pi x}{l} \right) dx \right.$$

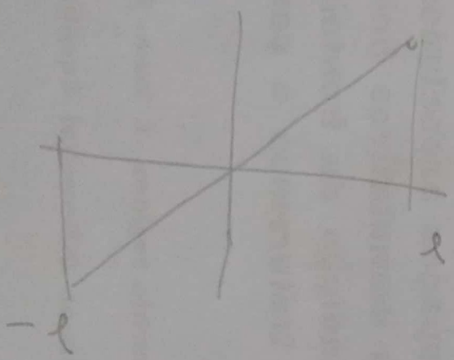
$$= \int_{-l}^{l} f^2(x)\, dx + \frac{A_0^2}{2} l + \sum_{k=1}^{m} (A_k^2 + B_k^2) l$$

$$-2\left[\frac{A_0}{2} l A_0 + \sum_{k=1}^{m} A_k l A_k + B_k l B_k \right] =$$

$$\int_{-l}^{l} f^2(x) - \frac{A_0^2 l}{2} - l \sum_{k=1}^{m} (A_k^2 + B_k^2) \xrightarrow[m \to +\infty]{} 0 \text{ qd}$$

Aplicações de Parseval.

1) Soma de séries



$f(x) = x$ ímpar

$A_n = 0$

$$B_n = \frac{1}{\ell} \int_{-\ell}^{\ell} x \operatorname{sen} \frac{n\pi x}{\ell}\, dx =$$

$$= (-1)^{n+1} \frac{2\ell}{n\pi}$$

Parseval:

$$\sum_{n=1}^{\infty} \frac{4\ell^2}{n^2\pi^2} = \frac{1}{\ell} \int_{-\ell}^{\ell} x^2 dx = \frac{2}{\ell} \frac{\ell^3}{3}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Problema isoperimétrico (Dido).

De todas as curvas (C^1) no plano que tem um comprimento dado L qual engloba a maior área?

A menor é zero.

Usando Parseval:

$$A \leq \frac{L^2}{4\pi}.$$

Além disso, a igualdade ocorre $\Longleftrightarrow$ círculo.