DATA 609 Assignment 6: Linear Programming

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Section 7.1, Problem 2

The rancher must decide how to meet the minimum weekly nutritional requirements for an average-sized horse in such a way as to minimize cost.

The decision variables are as follows:

- x_h: bales of hay purchased
- x_o: sacks of oats purchased
- x_f: feeding blocks purchased
- x_p: sacks of high-protein concetrate purchased

The function that must be minimized is the cost:

$$c(X) = 1.80x_h + 3.50x_o + 0.40x_f + 1.00x_p$$

There are three constraints related to the nutritional requirements:

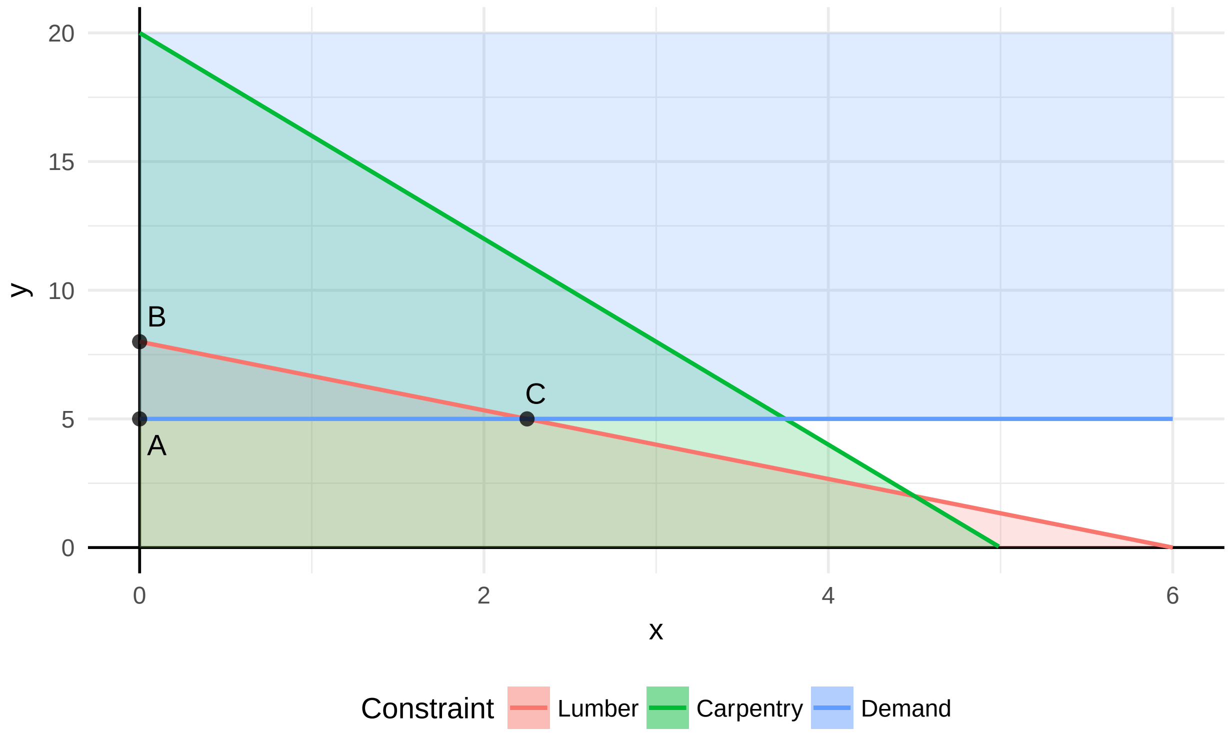
$$\begin{aligned}
0.5x_h + 1.0x_o + 2.0x_f + 6.0x_p &\geq 40.0 \quad &\text{(Protein)} \\
2.0x_h + 4.0x_o + 0.5x_f + 1.0x_p &\geq 20.0 \quad &\text{(Carbohydrates)} \\
5.0x_h + 2.0x_o + 1.0x_f + 2.5x_p &\geq 45.0 \quad &\text{(Roughage)}
\end{aligned}$$

$$\text{where} \quad x_h, x_o, x_f, x_p \geq 0$$

The assumptions for a linear program are met:

- There is a unique objective function $c(X)$
- The objective and constraint functions contain only variables of order 1
- The objective and constraint functions contain no products of variables
- The coefficients in the objective and constraint functions are constant
- It can be reasonably assumed that the decision variables can assume fractional values

Section 7.2, Problem 6



The feasible area is the triangle outlined by the intersection of the lumber constraint, the demand constraint, and the non-negativity constraint of x. These three points (A, B, and C) are located at $(0, 5)$, $(0, 8)$, and $(2.25, 5)$. The maximum value must lie at one of these extreme points, so the objective function $10x + 35y$ is calculated for each:

	x	y	$f(x, y)$
A	0	5	175
B	0	8	280
C	2.25	5	197.5

So the function is maximized at point B:

$$f(0, 8) = 280$$

Section 7.3, Problem 6

The constraints, with x & y re-assigned symbols x_1 & x_2, and slack variables added can be shown as

$$\begin{aligned} 8x_1 + 6x_2 + y_1 &= 48 && \text{(lumber)} \\ 4x_1 + x_2 + y_2 &= 20 && \text{(carpentry)} \\ x_2 - y_3 &= 5 && \text{(demand)} \end{aligned}$$

$$x_1, x_2, y_1, y_2, y_3 \geq 0$$

The coefficient for y_3 is negative since the sign in its inequality is the opposite of the other constraints.

Due to the demand constraint ($x_2 \geq 5$), it is known that any points considered with $x_2 = 0$ will be infeasible. As such, to find extreme points, the variables $x_1, y_1, y_2,$ & y_3 are set to zero in pairs to find feasible extreme points.

$\boldsymbol{x_1, y_1 = 0}$:

$$\begin{aligned} 6x_2 &= 48 \\ x_2 + y_2 &= 20 \\ x_2 - y_3 &= 5 \end{aligned}$$

This has the solution $x_2 = 8; y_2 = 12; y_3 = 3$. This is feasible since y_2 & y_3 are non-negative. This corresponds to the point $(0, 8)$, represented by point B in the previous problem.

$\boldsymbol{x_1, y_2 = 0}$:

$$\begin{aligned} 6x_2 + y_1 &= 48 \\ x_2 &= 20 \\ x_2 - y_3 &= 5 \end{aligned}$$

This has the solution $x_2 = 20; y_1 = -36; y_3 = -15$. This is not feasible since y_1 & y_3 are negative.

$\boldsymbol{x_1, y_3 = 0}$:

$$\begin{aligned} 6x_2 + y_1 &= 48 \\ x_2 + y_2 &= 20 \\ x_2 &= 5 \end{aligned}$$

This has the solution $x_2 = 5; y_1 = 18; y_2 = 15$. This is feasible since y_1 & y_3 are non-negative. This corresponds to the point $(0, 5)$, represented by the point A in the previous problem.

$\boldsymbol{y_1, y_2 = 0}$:

$$\begin{aligned} 8x_1 + 6x_2 &= 48 \\ 4x_1 + x_2 &= 20 \\ x_2 - y_3 &= 5 \end{aligned}$$

This has the solution $x_2 = 2; x_1 = 4.5; y_3 = -3$. This is not feasible since y_3 is negative.

$\boldsymbol{y_1, y_3 = 0}$:

$$\begin{aligned} 8x_1 + 6x_2 &= 48 \\ 4x_1 + x_2 + y_2 &= 20 \\ x_2 &= 5 \end{aligned}$$

This has the solution $x_2 = 5; x_1 = 2.25; y_2 = 6$. This is feasible since y_2 is non-negative. This corresponds to the point $(2.25, 5)$, represented by the point C in the previous problem.

$\boldsymbol{y_2, y_3 = 0}$:

$$\begin{aligned} 8x_1 + 6x_2 + y_1 &= 48 \\ 4x_1 + x_2 &= 20 \\ x_2 &= 5 \end{aligned}$$

This has the solution $x_2 = 5; x_1 = 3.75; y_1 = -36$. This is not feasible since y_1 & y_3 are negative.

x_1	x_2	$f(x_1, x_2)$
0	5	175
0	8	280
2.25	5	197.5

As above, the function is maximized:

$$f(0, 8) = 280$$

Section 7.4, Problem 6

The problem can be shown in tableau format:

$$
\begin{aligned}
8x_1 + 6x_2 + y_1 &= 48 \\
4x_1 + x_2 + y_2 &= 20 \\
-x_2 + y_3 &= -5 \\
-10x_1 - 35x_2 + z &= 0
\end{aligned}
$$

Tableau 0

x_1	x_2	y_1	y_2	y_3	z	RHS
8	6	1	0	0	0	48
4	1	0	1	0	0	20
0	-1	0	0	1	0	-5
-10	-35	0	0	0	1	0

The entering value is x_2, corresponding to -35 in the bottom row.

x_1	x_2	y_1	y_2	y_3	z	RHS	Ratio
8	6	1	0	0	0	48	8
4	1	0	1	0	0	20	20
0	-1	0	0	1	0	-5	5
-10	-35	0	0	0	1	0	*

The exiting variable is y_3, corresponding to the ratio of 5 in the third row.

The third row is divided by the coefficient for x_2, and x_2 is eliminated from the remaining rows:

Tableau 1

x_1	x_2	y_1	y_2	y_3	z	RHS
8	0	1	0	6	0	18
4	0	0	1	1	0	15
0	1	0	0	-1	0	5
-10	0	0	0	-35	1	175

The next entering variable is y_3, corresponding -35 in the last row.

x_1	x_2	y_1	y_2	y_3	z	RHS	Ratio
8	0	1	0	6	0	18	3
4	0	0	1	1	0	15	15
0	1	0	0	-1	0	5	-5
-10	0	0	0	-35	1	175	*

The exiting variable is y_1, corresponding to the ratio of 3 in the first row. The first row is divided by the coefficient for y_3, and y_3 is eliminated from all other rows:

Tableau 2

x_1	x_2	y_1	y_2	y_3	z	RHS
1.333	0	0.1667	0	1	0	3
2.667	0	-0.1667	1	0	0	12
1.333	1	0.1667	0	0	0	8
36.67	0	5.833	0	0	1	280

There are no negative coefficients in the last row, so the solution provided is optimal. The two indepdent variables x_1 and y_1 have a value of 0. Substituting these values into the first constraint gives $x_2 = 8$. Thus, this solution corresponds to the same solutions as above:

$$z(0, 8) = 280$$

Section 7.5, Problem 1

The extreme point $(12, 15)$ remains optimal if the slope of the objective function in the x_1x_2-plane lies between $-\frac{2}{3}$ (lumber constraint) and $-\frac{5}{4}$ (labor constraint). The slope of the objective function $z = c_1x_1 + c_2x_2$ is given by $-\frac{c_1}{c_2}$. Given $c_1 = 25$, the sensitiviity of the solution to changing c_2 can be solved using

$$\begin{aligned}
-\frac{5}{4} \leq -\frac{25}{c_2} \leq -\frac{2}{3} \\
-\frac{4}{5} \geq -\frac{c_2}{25} \geq -\frac{3}{2} \\
20 \leq x_2 \leq 37.5
\end{aligned}$$

For value of the profit per bookcase between \$20 and \$37.50, the carpenter should continue to produce 12 tables and 15 bookcases. If the value falls below \$20, the carpenter should produce only tables; if the value rises above \$37.50, he should produce only bookcases. If the value falls exactly at either of these points, producing exclusively one item or the mix of 12 tables and 15 bookcases will yield the same optimal profit. Interestingly, the sensitivity of the constant c_2 is the same as that of c_1 – this is due to the linear additive nature of the relationship between x_1 and x_2 in the objective function z.

Section 7.6, Problem 3

A function is created to calculate the value of c that minimizes the total absolute deviation of a model $y = cx^{power}$. The function takes data (x and y), the power, the range of interest (a, b), and tolerance t as inputs. After applying the golden section method, it returns the best estimate c_{opt} and the minimum deviation $f(c_{opt})$.

```
# golden ratio
r <- 0.618
# create function to calculate minimum value
golden_section <- function(x, y, power, a, b, t) {
  # set up initial c1 & c2
  c1 <- a + (1 - r) * (b - a)
  c2 <- a + r * (b - a)
  # determine current interval
  ival <- min(b - c1, c2 - a)
  # create function f(c)
  f <- function(c) {
    sum(abs(y - x^power * c))
  }
  # loop until within tolerance
  while (ival > t) {
    ival <- min(b - c1, c2 - a)
    # calculate values at endpoints
    f1 <- f(c1)
    f2 <- f(c2)
    # adjust a & b based on f(x1) & f(x2)
    if (f1 > f2) {
      a <- c1
      c1 <- c2
      c2 <- a + r * (b - a)
    } else {
      b <- c2
      c2 <- c1
      c1 <- a + (1 - r) * (b - a)
    }
  }
  # record and return best values
  c_opt <- (a + b) / 2
  return(list(c = c_opt, f = do.call(f, list(c_opt))))
}
```

The data is loaded in, and the function is applied for powers 1-3 using a tolerance $t = 0.01$. The range is set to be $(0, 10)$ to capture the highest possible slope between any two points.

	c_{opt}	$f(c_{opt})$
$y = cx$	7.074	199.5
$y = cx^2$	0.2284	216.3
$y = cx^3$	0.003668	454.2