DATA 609 Homework 1

Dan Smilowitz

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Section 1.1, Problem 10

Each month, the annuity increased through interest earned equal to 1% of the previous month's balance and decreases through withdrawal of $1,000. Therefore, the change in monthly balance can be modeled by $\Delta a = 0.01a_n - 1000$. Substituting in $\Delta a = a_{n+1} - a_n$ and solving for a_{n+1}, the dynamical system can be modeled using

$$a_{n+1} = 1.01a_n - 1000$$
$$a_0 = 50000$$

Iterating through this dynamical system, the annuity will run out of money (i.e. $a_n = 0$) after 70 months (six years), as shown in the table and plot below.

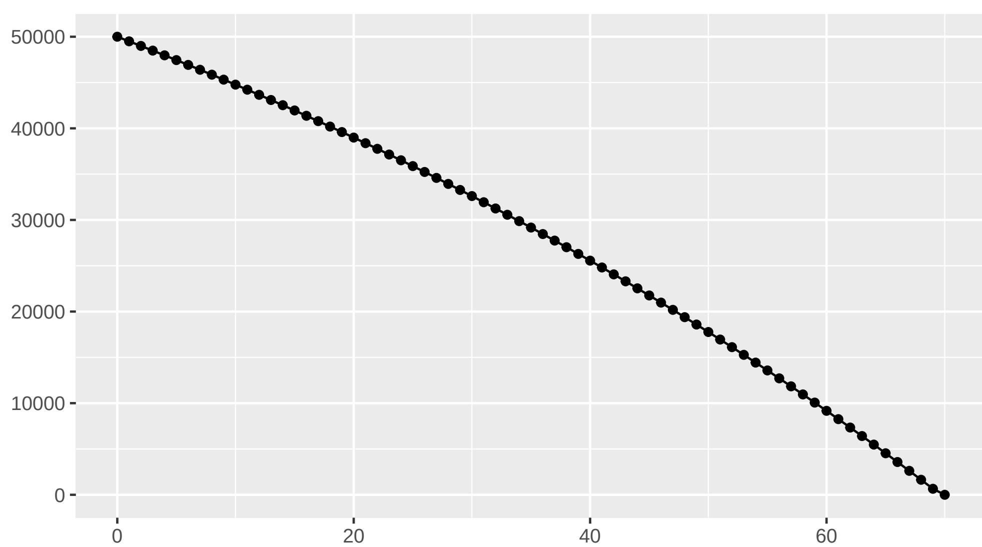
```r
# set initial value a_0
a <- 50000

# iterate until a_n <= 0
while (min(a) > 0) {
  a_n <- a[length(a)] - 1000 + 0.01 * a[length(a)]
  a <- c(a, a_n)
}

# set a = 0 to avoid negatives
a[length(a)] <- 0
```

Value of Annuity Earning 1% Interest with $1,000 withdrawals

n	a
0	50000
1	49500
2	48995
...	...
68	1638.89
69	655.28
70	0



Section 1.2, Question 9

The changes in a_n are calculated and shown below:

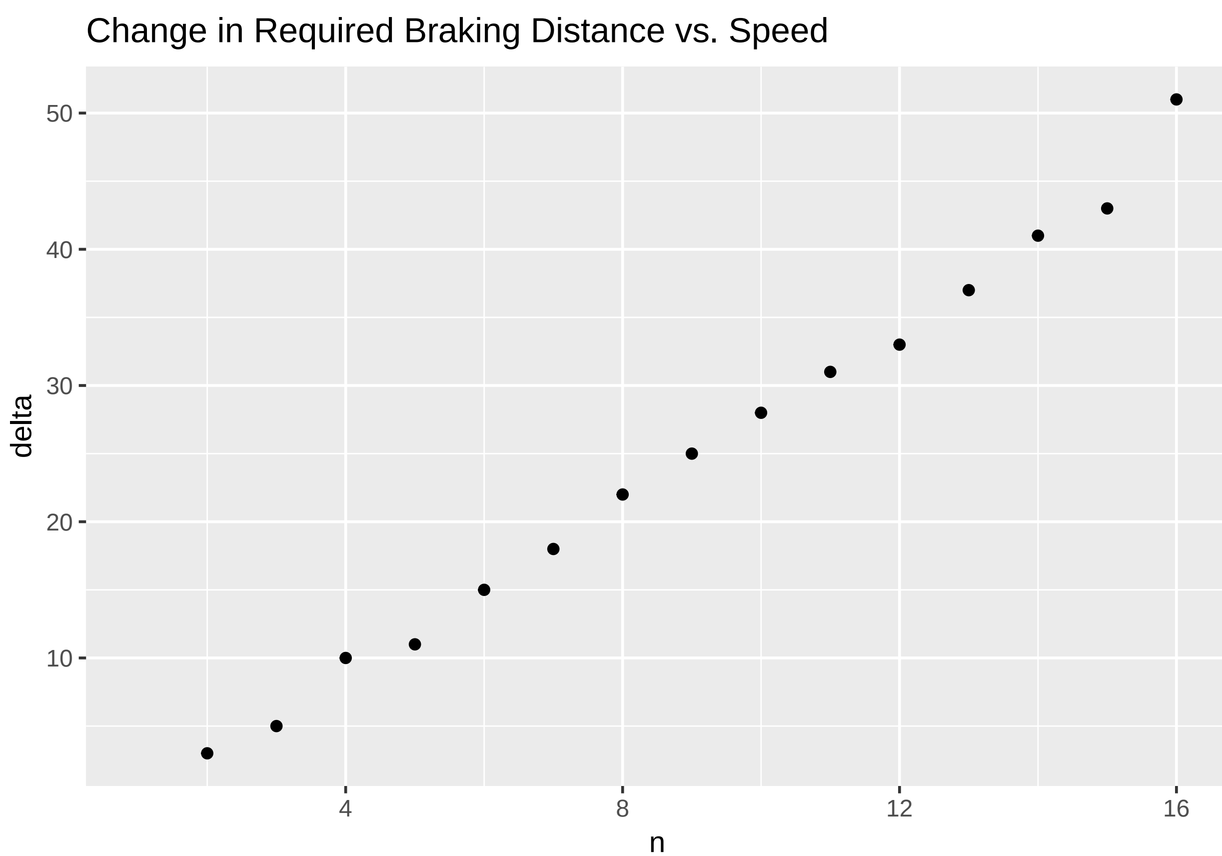
```
# read in data
txt <- textConnection('1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
3 6 11 21 32 47 65 87 112 140 171 204 241 282 325 376')
dat <- strsplit(readLines(txt), ' ')

library(readr)
df_brake <- data.frame(
  matrix(parse_number(unlist(dat)), ncol = 2, dimnames = list(NULL, c('n', 'a')))
  )

# calculate delta
library(dplyr)
df_brake$delta <- df_brake$a - lag(df_brake$a)
```

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a	3	6	11	21	32	47	65	87	112	140	171	204	241	282	325	376
delta	NA	3	5	10	11	15	18	22	25	28	31	33	37	41	43	51

The values, plotted against n, show a nearly linear relationship:



Using the lm function in R, the best linear fit between values of Δa_n and a_n which passes through the origin has a slope of 0.1526. This means that $\Delta a_n = 0.1526$, so the difference equation model is given by

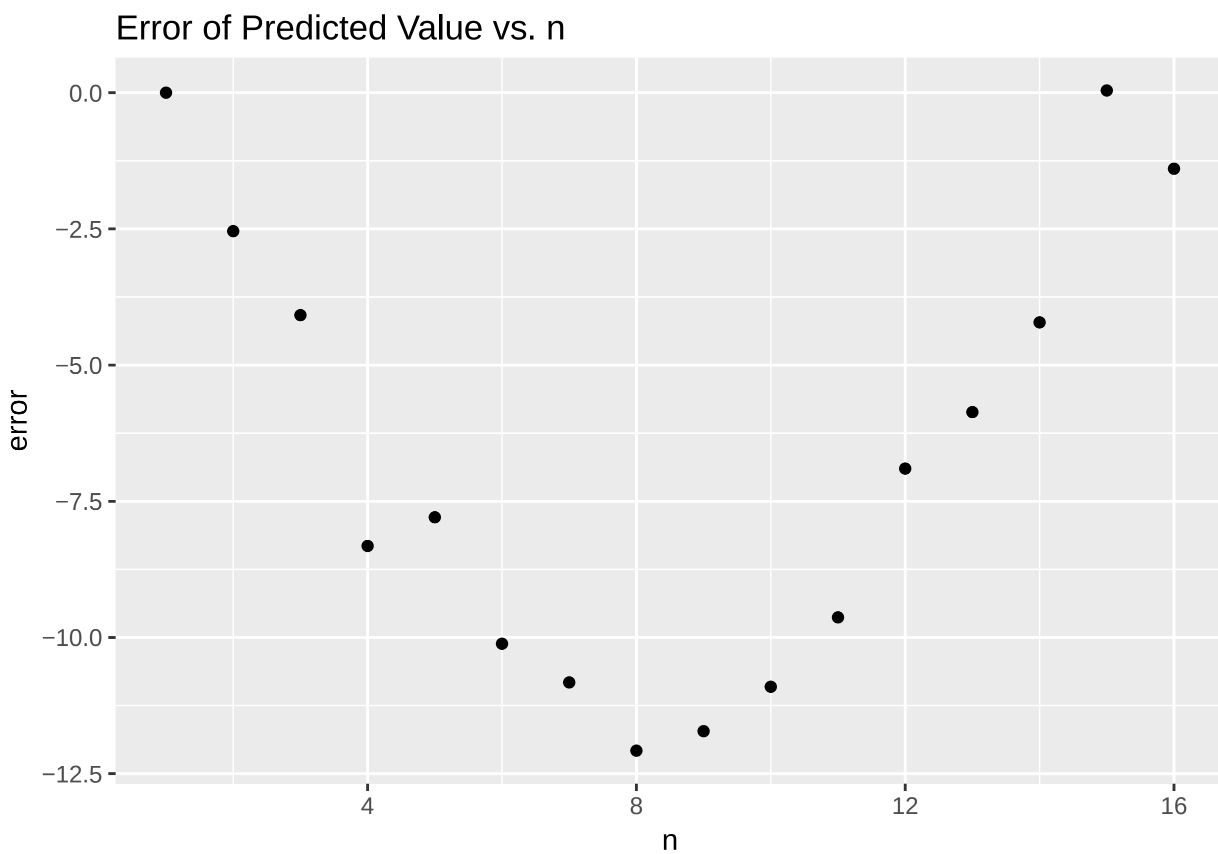
$$
\begin{aligned}
a_{n+1} &= a_n + 0.1526 a_n = 1.1526 a_n \\
a_1 &= 3
\end{aligned}
$$

Using this equation to estimate values of a_n for the provided values of n, the errors in predicted values are related to n as follows:

```r
# get slope of line through origin
m <- coefficients((lm(delta ~ 0 + a, df_brake)))[[1]]

# get predicted values
df_brake$pred <- lag(df_brake$a) * (m + 1)
df_brake$pred[1] <- df_brake$a[1] # plug in a_1 for pred_1

# calculate errors
df_brake$error <- df_brake$pred - df_brake$a
```



This distribution is clearly non-random – the errors are nearly all negative, and first increase in magnitude through $n = 8$ then decrease in magnitude. This suggests that the linear model may not be appropriate, as it fails to capture changes in Δa_n at intermediate values of n.

Section 1.3, Question 13

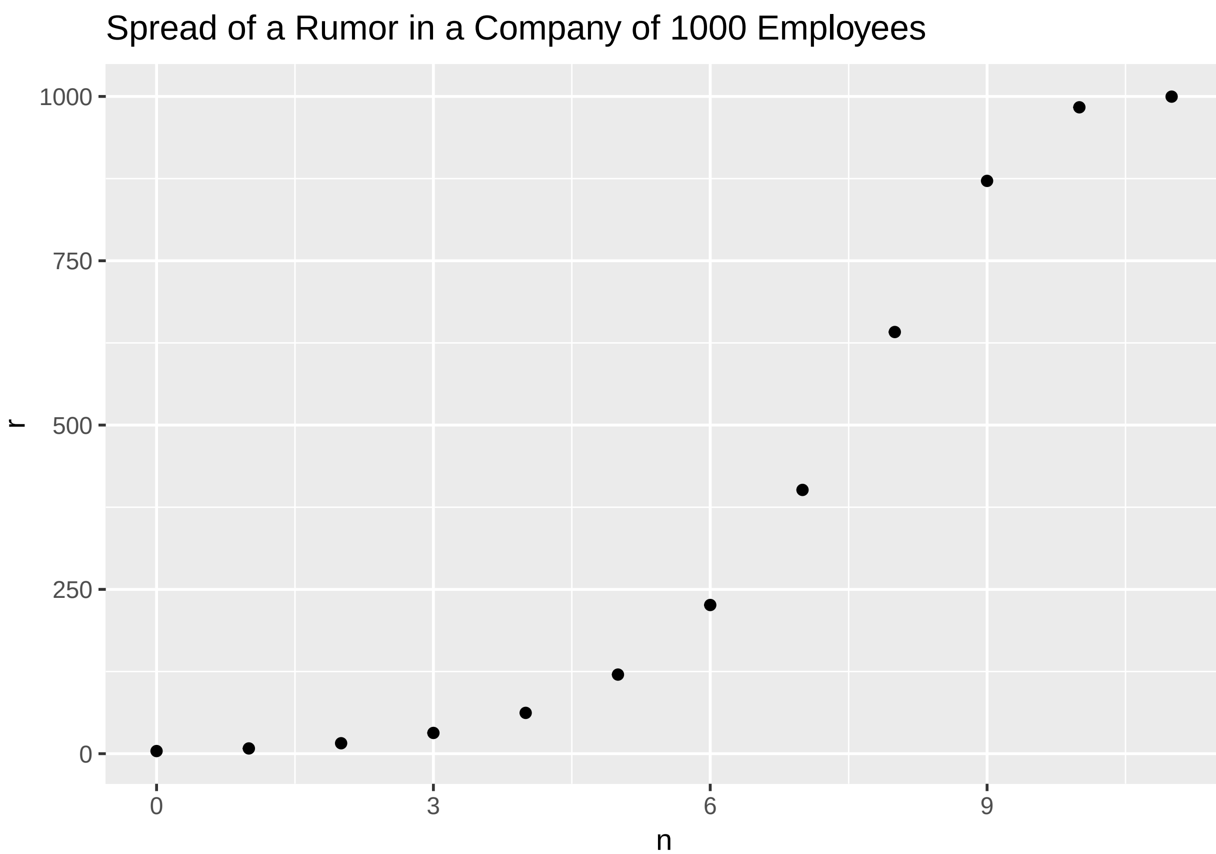
Iterating through the system

$$r_{n+1} = r_n + 0.001 r_n \times (1000 - r_n)$$
$$r_0 = 4$$

all 1,000 employees will have heard the rumor after 10 days (assuming that fractional values are not significant – if they are, the rumor will reach all employees within 14 days). The spread of the rumor is shown below.

```
# set initial conditions
r <- 4
n <- 0
k <- 0.001

# iterate until r = 1000
while(round(max(r), 0) < 1000) {
  n <- c(n, max(n) + 1)
  r <- c(r, max(r) + k * max(r) * (1000 - max(r)))
}
```



Section 1.4, Question 6

For the equilibrium conditions (P, Q), the system becomes

$$\begin{aligned} P &= P - 0.1(Q - 500) \\ 0 &= 0.1Q + 50 \\ Q &= 500 \end{aligned} \qquad\qquad \begin{aligned} Q &= Q + 0.2(P - 100) \\ 0 &= 0.2P + 20 \\ P &= 100 \end{aligned}$$

So the equilibrium conditions are $(P, Q) = (100, 500)$

These values represent the price (100) at which quantity will remain the same as in the previous period and the quantity (500) at which the price will remain the same as in the previous period. The constants of -0.1 and 0.2 represent the strength and direction of the interaction between price and quantity – the signs indicate that increased quantity will decrease the price, and that increased price will increase the quantity. This makes sense intuitively – suppliers will be more interested in producing the product if the price is higher, and retailers will be likely to lower prices if there is a surplus of the product available.

```
# set initial conditions
P0 <- c(100, 200, 100, 100)
Q0 <- c(500, 500, 600, 400)

# create data frame container
df_econ <- data.frame(n = numeric(), Variable = character(),
                      Value = numeric(), Model = character())

# model long-term behavior
library(tidyr)
for (i in 1:length(P0)) {
  n <- 0
  P <- P0[i]
  Q <- Q0[i]
  for (j in 1:500) {
    Pn <- P[length(P)]
    Qn <- Q[length(Q)]
    Pn1 <- Pn - 0.1 * (Qn - 500)
    Qn1 <- Qn + 0.2 * (Pn - 100)
    P <- c(P, Pn1)
    Q <- c(Q, Qn1)
    n <- c(n, j)
  }
  # gather into tidy data frame and join to container
  tmp_df <- data.frame(n, P, Q) %>% gather(Variable, Value, -n)
  tmp_df$Model <- rep(paste('Case', LETTERS[i]), length(n))
  df_econ <- rbind(df_econ, tmp_df)
}
```

In Case A, where the starting conditions are equal to the equilibrium conditions, the values remain at these conditions (this follows from the definition of equilibrium conditions). For the three other cases, the values oscillate around zero while growing without bound – this makes sense due to the difference in signs between the coeffcients for P_n and Q_n. Case B shows the greatest increase in values over the 500 periods modeled – this is due to the fact that the magnitude of the coefficient for the reaction of Q_n to P_n is higher than the coefficient for the reaction of P_n to Q_n. Cases C and D show the same approximate variability in P and Q, but exhibit a half-period lag relative to one another – this is due to the difference in sign between the starting conditions and equilibrium conditions in these cases.