DATA 609 Assignment 9: Game Theory

Dan Smilowitz

Section 10.1, Problem 1

Part a

			Colin	
		C1		**C2**
	R1	10	$\Longleftrightarrow$	10
Rose		$\Uparrow$		$\Uparrow$
	R2	10	$\Longrightarrow$	10

The game has a pure Nash equilibrium with a value of 10. Strategy R1 maximizes Rose's value regardless of Colin's strategy. While the value can be achieved at more than one combination of strategies, it is still a Nash equilibrium since neither party can benefit by departing from that strategy (i.e. either (R1, C1) or (R2, C2)).

Part c

			Pitcher	
		Fastball		**Knuckleball**
	Guesses fastball	0.400	$\Longrightarrow$	0.100
Batter		$\Uparrow$		$\Downarrow$
	Guesses knuckleball	0.300	$\Longrightarrow$	0.250

The game has a pure Nash equilibrium with a value of 0.250. Pitcher strategy *Knuckleball* minimizes the score regardless of batter strategy; given this pitcher strategy, batter strategy *Guess knuckleball* maximizes the score.

Section 10.2, Problem 2a

Referring to x as the portion of times that Rose plays strategy R1 and $1 - x$ the portion of the time that Rose plays strategy R2, her goal is to maximize the payoff P. If Colin plays purely strategy C1, the expected value of P is $10x + 5(1 - x)$; if he plays purely strategy C3, the expected value of P is $10x$. Thus, since x is a probability, the linear program for Rose is

$$\text{Maximize } P$$

Subject to

$$\begin{gathered} P \leq 10x + 5(1 - x) \\ P \leq 10x \\ x \geq 0 \\ x \leq 1 \end{gathered}$$

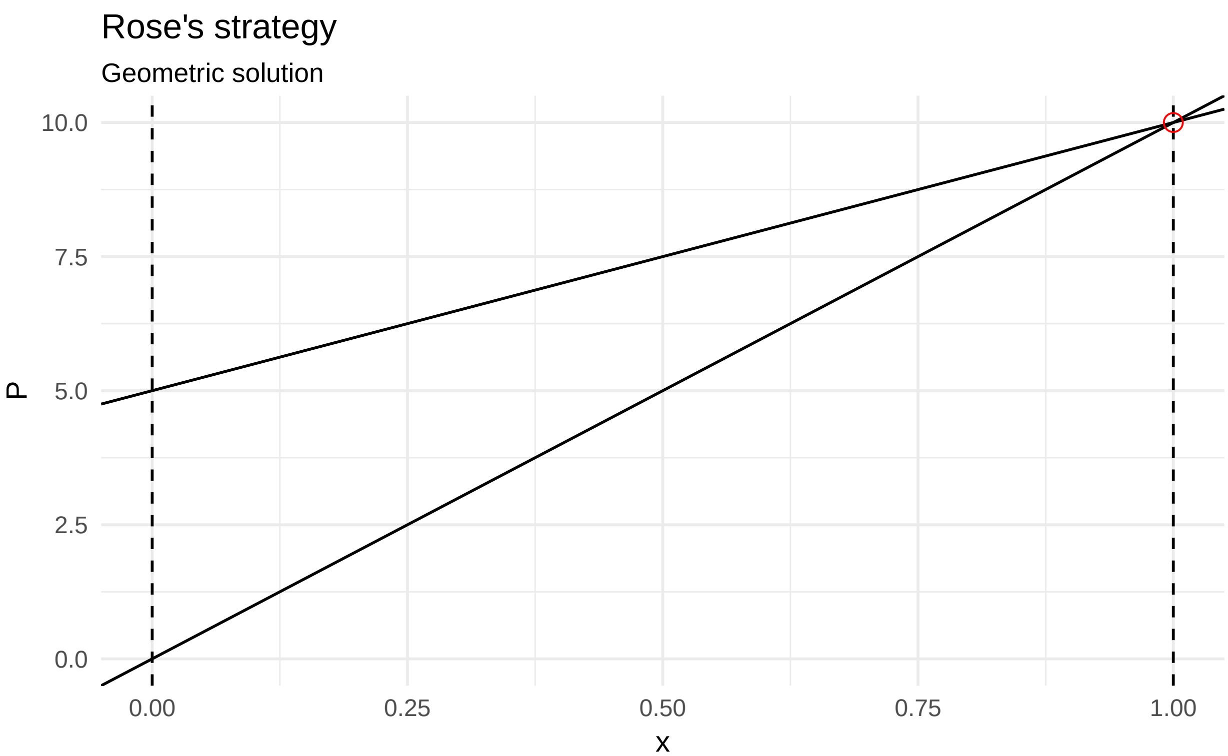
If y represents the portion of the time that Colin plays strategy C1, then the expected value of P is $10y + 10(1 - y) = 10$ if Rose plays purely strategy R1 and $5y$ if Rose plays purely strategy R2. This means that, for Colin, the linear program is

$$\text{Minimize } P$$

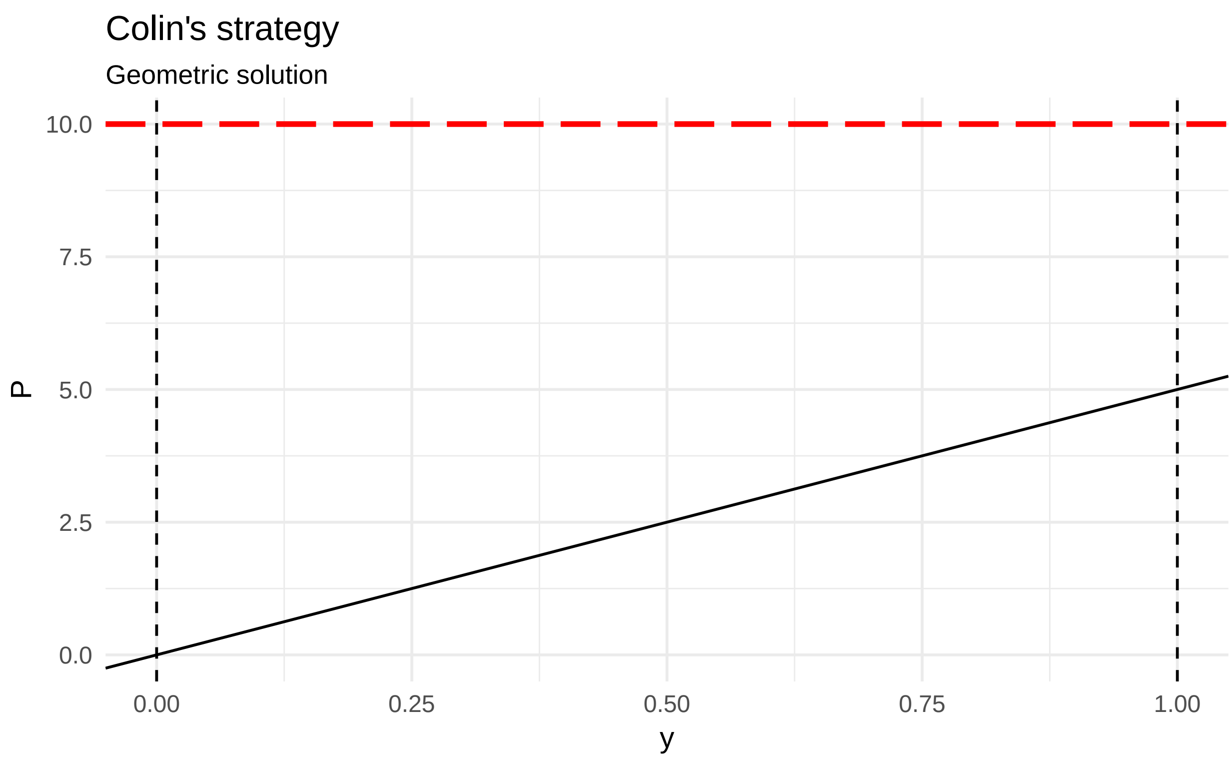
Subject to

$$P \geq 10$$
$$P \geq 5y$$
$$x \geq 0$$
$$x \leq 1$$

Geometric Solution



From this graph, any solution along or below the line $P = 10x$ is a feasible solution in the range $0 \leq x \leq 1$. The maximized value of P is $P = 10$ at $x = 1$ – if Rose plays strategy R1 100% of the time, she is guaranteed a maximum payoff of 10.



This graph shows that there is no optimal strategy for Colin – regardless of his mix of strategies, the furthest he can minimize the payoff is to $P = 10$. Due to this, Colin should likely play a strategy of pure C2, as this places him in the best position to take advantage of suboptimal play by Rose.

Algebraic Solution

For Rose, the intersection points of the above-stated constraints are shown below:

x	P	Feasible
0	0	Y
1	**10**	**Y**
0	5	N
1	**10**	**Y**

For Colin, the values are below:

x	P	Feasible
0	**10**	**Y**
1	**10**	**Y**
0	0	N
1	5	N

As in the geometric solution, the best strategy for Rose is $x = 1$ i.e. always playing strategy R1, and Colin's strategy does not matter.

Section 10.3, Problem 3

Investor's Game

For the investor, the variables of interest are

- P = Payoff
- x_A = Portion of the time to play alternative A
- x_B = Portion of the time to play alternative B
- x_C = Portion of the time to play alternative C

The linear program is then

$$\text{Maximize } P$$

Subject to

$$\begin{aligned}
3000x_A + 1000x_B + 4500x_C - P &\geq 0 \quad \textit{Economy } 1 \\
4500x_A + 9000x_B + 2000x_C - P &\geq 0 \quad \textit{Economy } 2 \\
6000x_A + 2000x_B + 3500x_C - P &\geq 0 \quad \textit{Economy } 3 \\
x_A, x_B, x_C &\geq 0 \\
x_A, x_B, x_C &\leq 1 \\
x_A + x_B + x_C &= 1 \\
P &\geq 0
\end{aligned}$$

This linear program can be solved using the `lpSolve` R package:

```r
# LHS of constraints in matrix form
inv_mat <- matrix(c(3000, 1000, 4500, -1,
                    4500, 9000, 4000, -1,
                    6000, 2000, 3500, -1,
                    1, 0, 0, 0,
                    0, 1, 0, 0,
                    0, 0, 1, 0,
                    1, 0, 0, 0,
                    0, 1, 0, 0,
                    0, 0, 1, 0,
                    1, 1, 1, 0,
                    0, 0, 0, 1),
                  ncol = 4, byrow = TRUE,
                  dimnames = list(NULL, c('xA', 'xB', 'xC', 'P')))
# objective as vector
inv_obj <- c(0, 0, 0, 1)
# direction & RHS of constraints as vectors
inv_dir <- c(rep('>=', 3), rep('>=', 3), rep('<=', 3), "==", ">=")
inv_rhs <- c(rep(0, 3), rep(0, 3), rep(1, 3), 1, 0)
# solve system
library(lpSolve)
inv_strat <- lp('max', inv_obj, inv_mat, inv_dir, inv_rhs)
```

The optimal strategy for the investor is $x_A = 0.25$, $x_B = 0$, $x_C = 0.75$, which yields an optimal payoff of $P = 4125$.

Economy's Game

For the economy, the variables of interest are

- P = Payoff
- y_1 = Portion of the time to play condition 1
- y_2 = Portion of the time to play condition 2
- y_3 = Portion of the time to play condition 3

The linear program is

$$\text{Minimize } P$$

Subject to

$$\begin{aligned} 3000y_1 + 4500y_2 + 6000y_3 - P &\leq 0 \quad \textit{Alternative A} \\ 1000y_1 + 9000y_2 + 2000y_3 - P &\leq 0 \quad \textit{Alternative B} \\ 4500y_1 + 4000y_2 + 3500y_3 - P &\leq 0 \quad \textit{Alternative C} \\ y_1, y_2, y_3 &\geq 0 \\ y_1, y_2, y_3 &\leq 1 \\ y_1 + y_2 + y_3 &= 1 \\ P &\geq 0 \end{aligned}$$

The economy's program is solved in the same way as the investor's:

```r
# LHS of constraints in matrix form
eco_mat <- matrix(c(3000, 4500, 6000, -1,
                    1000, 9000, 2000, -1,
                    4500, 4000, 3500, -1,
                    1, 0, 0, 0,
                    0, 1, 0, 0,
                    0, 0, 1, 0,
                    1, 0, 0, 0,
                    0, 1, 0, 0,
                    0, 0, 1, 0,
                    1, 1, 1, 0,
                    0, 0, 0, 1),
                  ncol = 4, byrow = TRUE,
                  dimnames = list(NULL, c('y1', 'y2', 'y3', 'P')))
# objective as vector
eco_obj <- c(0, 0, 0, 1)
# direction & RHS of constraints as vectors
eco_dir <- c(rep('<=', 3), rep('>=', 3), rep('<=', 3), "==", ">=")
eco_rhs <- c(rep(0, 3), rep(0, 3), rep(1, 3), 1, 0)
# solve system
eco_strat <- lp('min', eco_obj, eco_mat, eco_dir, eco_rhs)
```

The optimal strategy for the economy is $y_1 = 0.625$, $y_2 = 0$, $y_3 = 0.375$, which yields an optimal payoff of $P = 4125$.

Section 10.4, Problem 1

The movement diagram from Section 10.1 is replicated below, with row minima and column maxima added:

	C1	**Colin**	**C2**	**Row min**
R1	10	$\Longleftrightarrow$	10	**10**
Rose	$\Uparrow$		$\Uparrow$	
R2	5	$\Longrightarrow$	0	0
Col max	**10**		**10**	

Two pure strategy solutions exist – Rose playing R1 and Colin playing either C1 or C2. In both cases, the value of the game is 10.

Section 10.5, Problem 3

As shown by the movement diagram below, a pure strategy exists – Rose playing R2 and Colin playing either C1 or C2

			Colin	
		C1		**C2**
	R1	0.5	$\Longrightarrow$	0.3
Rose		$\Downarrow$		$\Downarrow$
	R2	0.6	$\Longleftarrow$	1

Due to this, neither the equating expected values or method of oddments will return useful solutions; however, they are still conducted for demonstration purposes.

Equating Expected Values

For Rose, the expected value under each of Colin's strategies are

$$E(C1) = 0.5x + 0.6(1-x)$$
$$E(C2) = 0.3x + 1(1-x)$$

where x is the portion of the time Rose uses strategy R1. Setting these equal to one another and solving,

$$0.5x + 0.6(1-x) = 0.3x + 1(1-x) \longrightarrow x = \frac{2}{3}; 1-x = \frac{1}{3}$$

The value of the game is

$$E(C1) = 0.5x + 0.6(1-x) = \frac{8}{15} \approx 0.5333$$

For Colin, the expected value under each of Rose's strategies are

$$E(R1) = 0.5y + 0.3(1-y)$$
$$E(R2) = 0.6y + 1(1-y)$$

where y is the portion of the time Colin uses strategy C1. Setting these equal to one another and solving,

$$0.5y + 0.3(1-y) = 0.6y + 1(1-y) \longrightarrow y = \frac{7}{6}; 1-y = -\frac{1}{6}$$

Clearly this is a violation of the condition $0 \leq y \leq 0$ that applies since y is a probability. Nonetheless, the value of the game is calculated:

$$E(R1) = 0.5y + 0.3(1-y) = \frac{7}{12} - \frac{3}{60} = \frac{8}{15} \approx 0.5333$$

Method of Oddments

As above, the solution does not produce useful results:

	C1	**C2**	$[\Delta]$
R1	0.5	0.3	0.2
R2	0.6	1	0.4
$[\Delta]$	0.1	0.7	$0.6 \neq 0.8$

Section 10.6, Problem 2

$$\begin{array}{cccccc} & & & \textbf{Colin} & \\ & & \textbf{C1} & & \textbf{C2} \\ & \textbf{R1} & (1,2) & \Longrightarrow & (3,1) \\ \textbf{Rose} & & \Downarrow & & \Downarrow \\ & \textbf{R2} & (2,4) & \Longleftarrow & (4,3) \end{array}$$

There is a stable Nash equilibrium at $(2,4)$ – neither player can utilaterally improve from this position.

Rose would rather Colin play C2, as it increases her potential payoff. To do this, she can issue a threat:

> *If Colin plays C1, Rose will play R1*

This meets the critera for a threat:

- it is contingent on Colin's action
- it harms Rose (lowers her payoff from 2 playing R2 to 1 playing R1)
- it is harmful to Colin (lowers his payoff from R under R2 to 2 under R1)

The game then becomes

$$\begin{array}{cccccc} & & & \textbf{Colin} & \\ & & \textbf{C1} & & \textbf{C2} \\ & \textbf{R1} & (1,2) & \Longrightarrow & (3,1) \\ \textbf{Rose} & & & & \Downarrow \\ & \textbf{R2} & \textit{Eliminated} & & (4,3) \end{array}$$

Thus Colin will choose strategy C2, and Rose will choose strategy R2, maximizing her payoff at $(4,3)$.

Section 10.7, Problem 3

To get the table of payoffs, the probabilities must be matched in a 3-by-3 grid and multiplied by the associated payoff per each outcome and summing the two numbers:

	IL	**IM**	**IC**
DL	$(3, -5)$	$(3, -10)$	$(3, -10)$
DM	$(10, -5)$	$(8, -6)$	$(8, -10)$
DC	$(10, -5)$	$(10, -6)$	$(10, -10)$

Summing these values and completing the movement diagram yields

$$
\begin{array}{cccccc}
 & \textbf{IL} & & \textbf{IM} & & \textbf{IC} \\
\textbf{DL} & -2 & \Rightarrow & -7 & \Leftrightarrow & -7 \\
 & \Downarrow & & \Downarrow & & \Downarrow \\
\textbf{DM} & 5 & & 2 & & -2 \\
 & \Updownarrow & & \Downarrow & & \Downarrow \\
\textbf{DC} & 5 & \Rightarrow & 4 & \Rightarrow & 0
\end{array}
$$

There is a nash equilibrium at (DC, IC) – here the game has a value of 0 and neither player can unilaterally improve.