DATA 609 Assignment 11: Differential Systems

Dan Smilowitz

Section 12.1, Problem 1

$$\frac{dx}{dt} = \frac{d}{dt}\left(-e^t\right) = -e^t = -y$$
$$\frac{dy}{dt} = \frac{d}{dt}\left(e^t\right) = e^t = -x$$

Section 12.1, Problem 6

The only point at which $\frac{dx}{dt} = \frac{dy}{dt} = 0$ is $(2, 1)$.

Integrating the equations,

$$x = \int \left(\frac{dx}{dt}\right) dt = -\int (y-1)\, dt = -t\,(y-1)$$

As $t \to \infty$, $x(t) \to \infty$ if $y_0 < 1$ or $x(t) \to -\infty$ if $y_0 > 1$.

$$y = \int \left(\frac{dy}{dt}\right) dt = \int (x-2)\, dt = t\,(x-2)$$

As $t \to \infty$, $y(t) \to \infty$ if $x_0 > 2$ or $y(t) \to -\infty$ if $x_0 < 2$.

Since $(x, y) \to (\pm\infty, \pm\infty)$ for $(x_0, y_0) \notin (2, 1)$, the point is unstable.

Section 12.2, Problem 7

Part a

$$\frac{dt}{dx} = \left(\frac{dx}{dt}\right)^{-1} = \frac{1}{(a-by)x}$$
$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{(m-nx)y}{(a-by)x}$$

Part b

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\frac{m-nx}{a-by} \longrightarrow \frac{ady}{y} - bdy = \frac{mdx}{x} - ndx$$

$$\int \frac{ady}{y} - \int bdy = \int \frac{mdx}{x} - \int ndx \longrightarrow a\ln y - by + K_1 = m\ln x - nx + K_2$$

$$y^a e^{-by} = Kx^m e^{-nx}$$

Part c

$$\frac{d}{dy}f(y) = ay^{a-1} \times e^{-by} + (-be^{-by}) \times y^a = y^{a-1}e^{-by}\,(a - by)$$

Setting $f' = 0$ to find the maximum:

$$y^{a-1}e^{-by}\,(a - by) = 0 \longrightarrow a - by = 0 \longrightarrow y = \frac{a}{b}$$

At $y = \frac{a}{b}$,

$$f(\frac{a}{b}) = \left(\frac{a}{b}\right)^a e^{-a} = \left(\frac{a}{eb}\right)^a$$

$$\frac{d}{dx}g(x) = mx^{m-1} \times e^{-nx} + (-ne^{-ny}) \times x^m = x^{m-1}e^{-nx}\,(m - nx)$$

Setting $g' = 0$ to find the maximum:

$$x^{m-1}e^{-ny}\,(m - nx) = 0 \longrightarrow m - nx = 0 \longrightarrow x = \frac{m}{n}$$

At $x = \frac{m}{n}$,

$$g(\frac{m}{n}) = \left(\frac{m}{n}\right)^m e^{-m} = \left(\frac{m}{en}\right)^m$$

Part d

The constant K from part b is equivalent to

$$K = \frac{e^{K_1}}{e^{K_2}}$$

Where K_1 and K_2 are the integration constants from the left and right sides, respectively. As $(x, y) \to (m/n, a/b)$, it is clear from the provided equations that $dx/dt, dy/dt \to 0$.

Using this information, the constants can be solved for:

$$a \ln y - by + K_1 = 0 \longrightarrow e^{K_1} = \frac{y^a}{e^{by}}$$

$$m \ln x - ny + K_2 = 0 \longrightarrow e^{K_2} = \frac{x^m}{e^{nx}}$$

Thus, as $(x, y) \to (m/n, a/b)$,

$$\lim \left[\left(\frac{y^a}{e^{by}}\right)\left(\frac{e^{nx}}{x^b}\right)\right] = \frac{e^{K_1}}{e^{K_2}} = K$$

Part f

For $y_0 > a/b$, $f(y_0) < M_y$. This implies

$$\frac{M_y}{M_x}\left(\frac{x^m}{e^{nx}}\right) = \frac{y_0^a}{e^{by_0}} < M_y \longrightarrow \frac{x^m}{e^{nx}} < M_x$$

Per the plot of $g(x)$, there is a single value $x > m/n$ satisfying this condition. Thus, there exists a unique trajectory approaching $(m/n, a/b)$ from above.

Section 12.3, Problem 1

Critical Point

From above, the derivative is given by

$$f'(y) = y^{a-1}e^{-by}\,(a - by)$$

Setting $f' = 0$ to find the maximum:

$$y^{a-1}e^{-by}\,(a - by) = 0 \longrightarrow a - by = 0 \longrightarrow y = \frac{a}{b}$$

Since $y = \frac{a}{b}$ is the only root of the equation $f'(y) = 0$, it is the only critical point of $f(y)$.

The second derivative is given by

$$f''(y) = a(a-1)y^{a-2} \times e^{-by} + ay^{a-1} \times -be^{-by} + b^2e^{-by} \times y^a + ay^{a-1} \times -be^{-by}$$

$$f''(y) = y^{a-2}e^{-by}\left(a^2 - 2aby - a + b^2y^2\right)$$

At the critical point, this is

$$f''(\frac{a}{b}) = \left(\frac{a}{b}\right)^{a-2} e^{-a}\left(a^2 - 2a^2 - a + a^2\right) = -\frac{a^{a-1}}{e^a\;b^{a-2}}$$

This second derivative, indicating that the slope is decreasing. Thus, the critical point is a maximum.

Limit

The limit of the equation $\lim_{y\to\infty} f(y)$ does not provide a useful limit. Using L'Hopital's rule to take the derivative of the numerator and denominator, however, provides useful insight:

$$\lim_{y\to\infty} \frac{y^a}{e^{by}} = \lim_{y\to\infty} \frac{\frac{d^n}{dy^n}y^a}{\frac{d^n}{dy^n}e^{by}} = \lim_{y\to\infty} \frac{\prod_i^n \{a - i + 1\}y^{a-n}}{b^n e^{by}}$$

From this, it is clear that the denominator will grow much larger than the numerator; thus

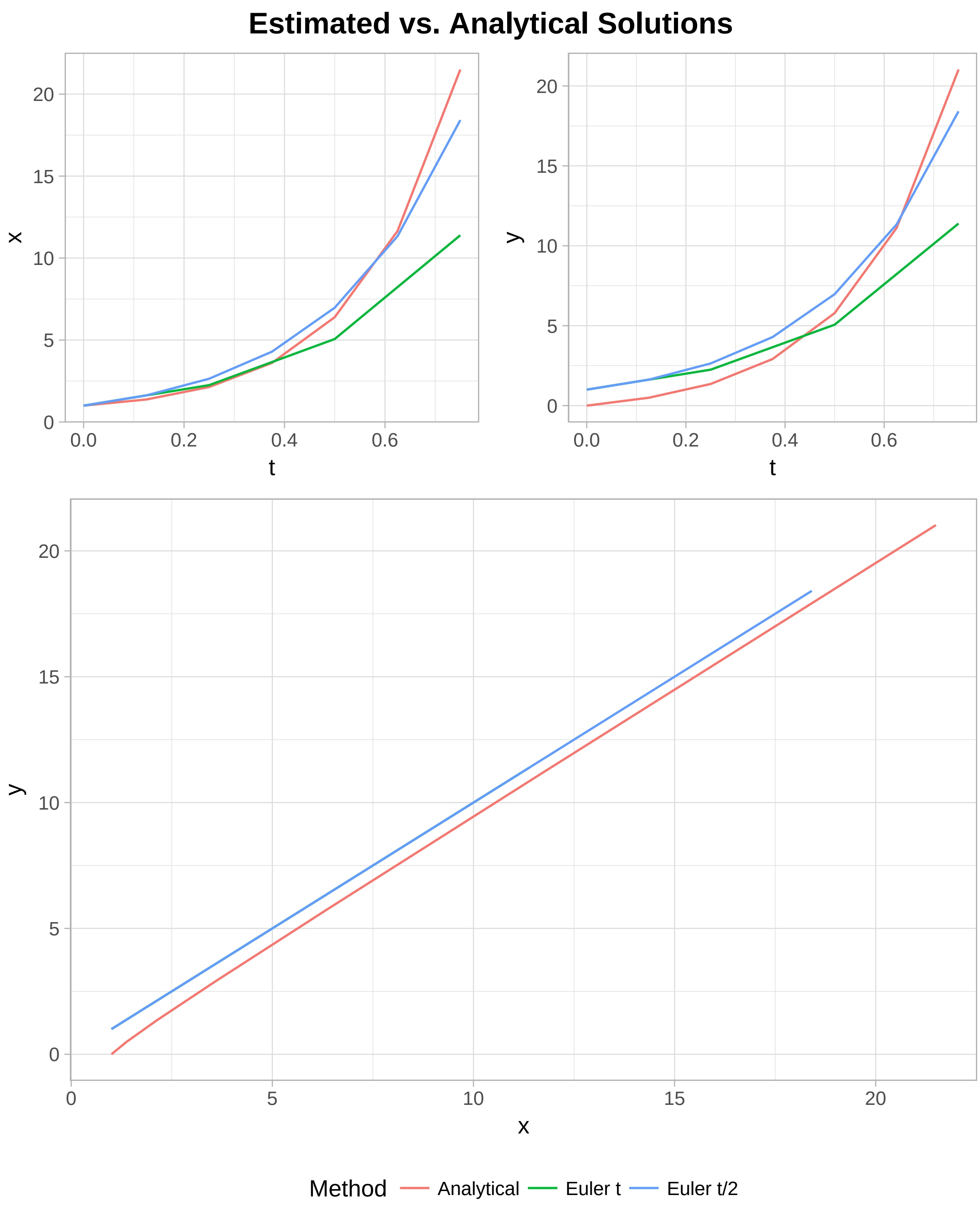
$$\lim_{y\to\infty} f(y) = 0$$

Section 12.5, Problem 1

```
# function to implement Euler's method
euler <- function(dxdt, dydt, t0, x0, y0, delta_t, n) {
  # initial conditions
  t <- t0
  x <- x0
  y <- y0
  # run loop to get approximations
  for (i in 1:n) {
    t_i <- t[length(t)] + delta_t
    x_i <- x[length(x)] + f(x[length(x)], y[length(y)]) * delta_t
    y_i <- y[length(y)] + g(x[length(x)], y[length(y)]) * delta_t
    t <- c(t, t_i)
    x <- c(x, x_i)
    y <- c(y, y_i)
  }
  # return estimate
  return(data.frame(t, x, y))
}

# functions for dx/dt and dy/dt
f <- function(x, y) {return(2 * x + 3 * y)}
g <- function(x, y) {return(3 * x + 2 * y)}

# get results for delta_t and 0.5*delta_t
est_a <- euler(f, g, 0, 1, 1, 1/4, 3)
est_b <- euler(f, g, 0, 1, 1, 1/8, 6)

# calculate analytical solutions
soln <- data.frame(t = est_b$t,
                   x = 0.5 * exp(-est_b$t) + 0.5 * exp(5 * est_b$t),
                   y = -0.5 * exp(-est_b$t) + 0.5 * exp(5 * est_b$t))
```

The results of the above estimates (Euler's method using Δt, Euler's method using $\Delta t/2$, and the analytical solution) are presented below in tabular format and graphically on the following page.

t	$x_{E(\Delta t)}$	$y_{E(\Delta t)}$	$x_{E(\Delta t/2)}$	$y_{E(\Delta t/2)}$	x_A	y_A
0	1	1	1	1	1	0
0.125	NA	NA	1.625	1.625	1.375	0.4929
0.25	2.25	2.25	2.641	2.641	2.135	1.356
0.375	NA	NA	4.291	4.291	3.604	2.917
0.5	5.062	5.062	6.973	6.973	6.395	5.788
0.625	NA	NA	11.33	11.33	11.65	11.11
0.75	11.39	11.39	18.41	18.41	21.5	21.02



Estimated vs. Analytical Solutions
x
t
y
t
y
x
Method
Analytical
Euler t
Euler t/2