File # 2 ch.4 Image Sampling and Quantization

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EE 5356

For digital (computer) processing of images

1) sampling on a discrete grid
2) quantizing each sample (pel or pixel) into a finite # of levels (bits)

1) + 2) gives a digitized image

See Fig. 4.1 Sampling, quantization & display of images

Image Scanning & TV standards (NTSC, PAL, SECAM)

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Image display/Recording

CRT display, Halftone

NTSC : 525 scan lines / frame
30 frames/sec
(2 interlaced fields/frame) | 60 fields/sec

Horizontal & vertical blanking intervals

Composite signal

$$u(t) = Y(t) + I(t)\cos(2\pi f_{sc} t + \phi) + Q(t)\sin(2\pi f_{sc} t + \phi)$$

$Y(t)$ = Luminance, $I(t)$, $Q(t)$ = chrominance color

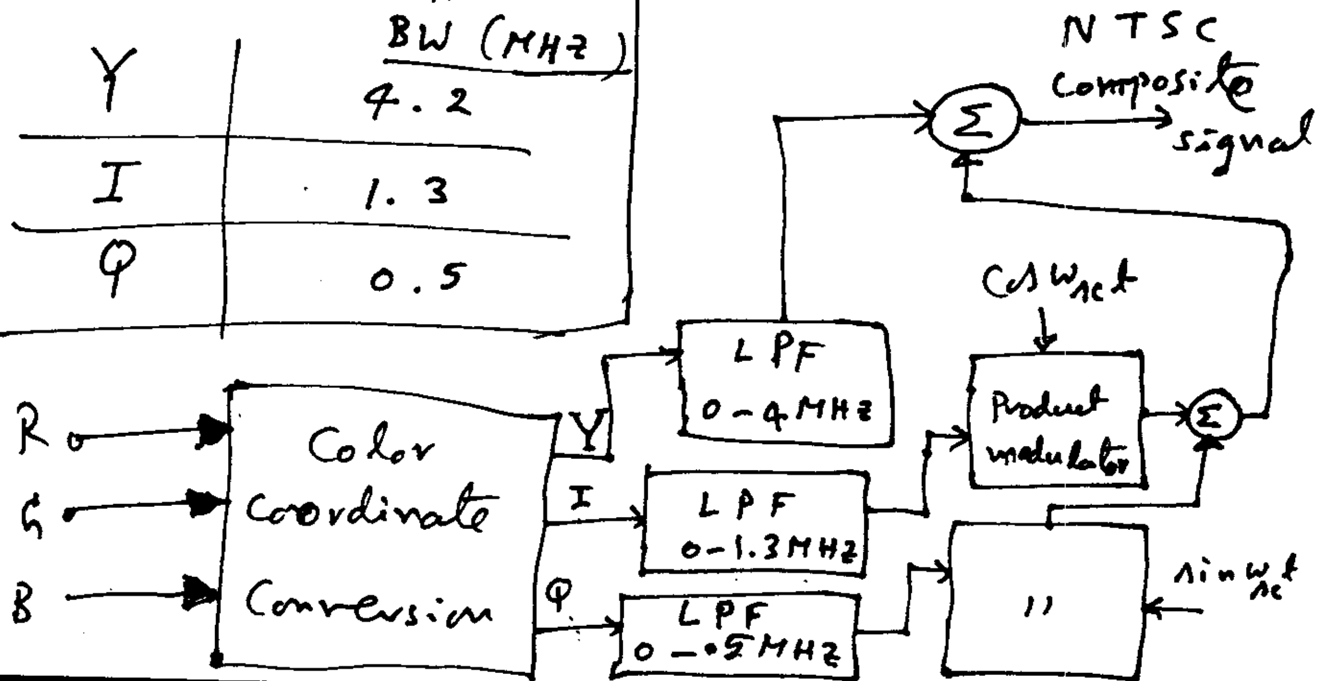
$f_{sc} = 3.58\ MHz = \frac{455}{2} f_l$

f_{sc} = color subcarrier frequency.

f_l = horizontal scanning frequency

= 15.75 KHz

	BW (MHz)
Y	4.2
I	1.3
Q	0.5



4.2 2-D Sampling Theory

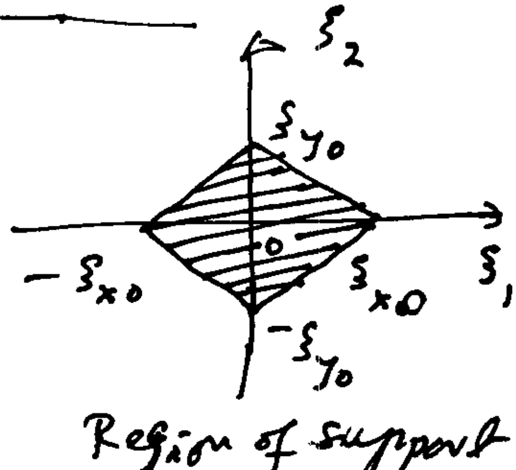
Bandlimited images

$f(x,y)$ is bandlimited when (See Fig. 4.6)

$F(\xi_1, \xi_2) = 0$

for $|\xi_1| > \xi_{x0}$

& $|\xi_2| > \xi_{y0}$



Region of support

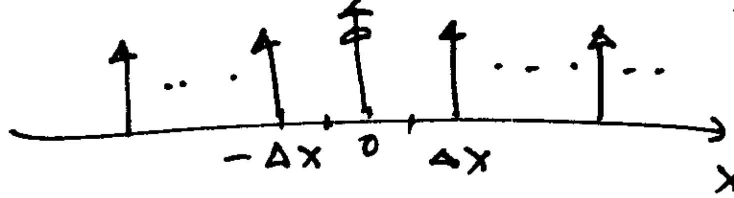
$F(\xi_1, \xi_2)$ = 2-D FT of $f(x,y)$

ξ_{x0}, ξ_{y0} are bandwidths along x & y

For circularly symmetric spectra, $\xi_{x0} = \xi_{y0} = \xi_0$ is the BW.

F.T. of a periodic impulse train. (period $= \Delta x$)

$$\text{Comb}(x : \Delta x) = \sum_{m=-\infty}^{\infty} \delta(x - m\Delta x)$$



Define $$F(\xi_1) = \int_{-\infty}^{\infty} f(x)\, e^{-j2\pi\xi_1 x}\, dx$$

& $$f(x) = \int_{-\infty}^{\infty} F(\xi_1)\, e^{j2\pi\xi_1 x}\, d\xi_1$$

Fourier series expansion of the impulse train

$f(x) = f(x + m\Delta x)$

$$f(x) = \sum_{k=-\infty}^{\infty} a_k\, e^{j2\pi k x/\Delta x} \quad \Big| \quad \frac{1}{\Delta x} = \xi_{xs}$$

$$a_k = \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} f(x)\, e^{-j(2\pi k/\Delta x)\,x}\, dx = \text{F.S. Coefft}$$

$$a_k = \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\Delta x/2} \delta(x)\, e^{-j2\pi k x/\Delta x}\, dx$$

$$= \frac{1}{\Delta x} = \xi_{xs} = \textbf{Sampling rate}$$

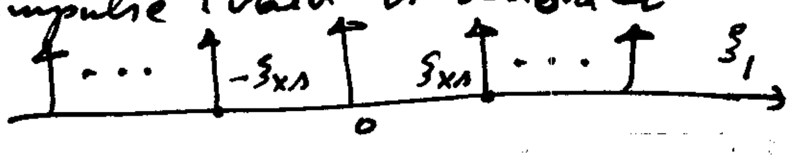
$$\therefore F(\xi_1) = \int_{-\infty}^{\infty} \underbrace{\sum_{k=-\infty}^{\infty} \frac{1}{\Delta x}\left[e^{j2\pi k x \xi_{xs}}\right]}_{f(x)} e^{-j2\pi\xi_1 x}\, dx$$

$$= \frac{1}{\Delta x}\sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j2\pi x(\xi_1 - k\xi_{xs})}\, dx$$

(see next page)

$$= \frac{1}{\Delta x}\sum_{k=-\infty}^{\infty} \delta(\xi_1 - k\xi_{xs}) \quad \boxed{\frac{1}{\Delta x} = \xi_{xs}}$$

F.T. of a periodic impulse train is another impulse train.



$$\mathcal{F}[\delta(x)] = \int_{-\infty}^{\infty} \delta(x)\, e^{-j2\pi\xi_1 x}\, dx = 1$$

$$\therefore \mathcal{F}^{-1}[1] = \int_{-\infty}^{\infty} e^{j2\pi\xi_1 x}\, d\xi_1 = \delta(x)$$

Similarly

$$\mathcal{F}[\delta(x-x_0)] = \int_{-\infty}^{\infty} \delta(x-x_0)\, e^{-j2\pi\xi_1 x}\, dx$$

$$= e^{-j2\pi\xi_1 x_0}$$

$$\therefore \mathcal{F}^{-1}\left[e^{-j2\pi\xi_1 x_0}\right] = \int_{-\infty}^{\infty} e^{j2\pi\xi_1 (x-x_0)}\, d\xi_1$$

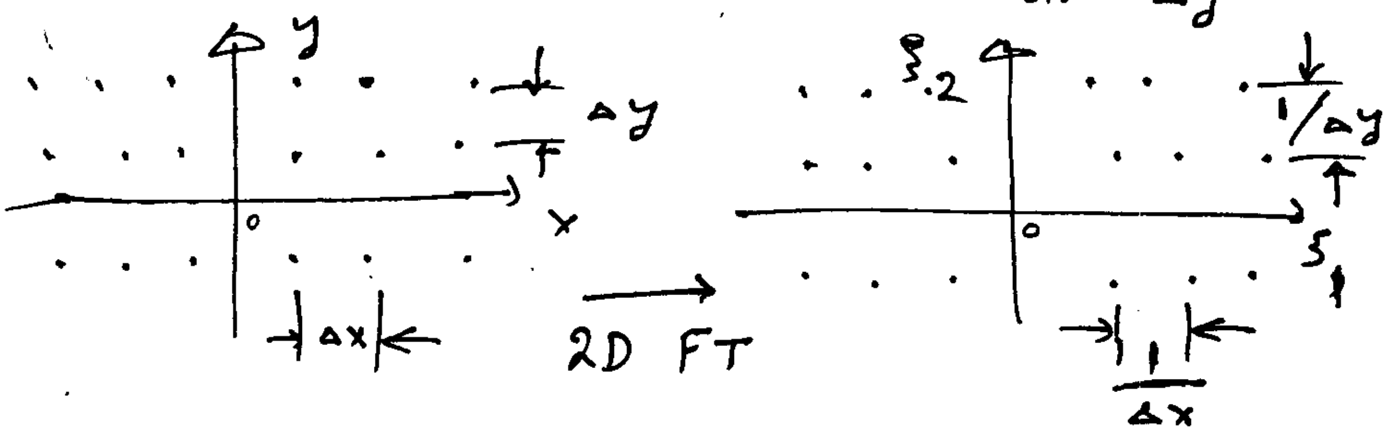
$$= \delta(x-x_0)$$

Similarly 2-D FT of an 2-D impulse train (periodic both along x & y) is a 2-D impulse train (" " " ξ_1 & ξ_2) 8

sampling interval in (x,y) is Δx & Δy

" " " (ξ_1, ξ_2) is $\frac{1}{\Delta x}$ & $\frac{1}{\Delta y}$

where $\xi_{xs} = \frac{1}{\Delta x}$ & $\xi_{ys} = \frac{1}{\Delta y}$



FT of a sampled function, is periodic repetition of FT of the original function (before sampling).

2-D infinite array of Dirac delta fns.

$$\mathrm{Comb}(x, y; \Delta x, \Delta y) = \sum_{m,n=-\infty}^{\infty}\sum \delta(x - m\Delta x, y - n\Delta y) \quad (4.5)$$

Sampled image $f_s(x, y)$ (uniformly sampled along both x & y, sampling intervals Δx & Δy)

$$f_s(x, y) = f(x, y)\ \mathrm{Comb}(x, y; \Delta x, \Delta y)$$

$$= \left[\sum_{m,n=-\infty}^{\infty}\sum \delta(x - m\Delta x, y - n\Delta y)\right] f(x, y)$$

$$f_s(x, y) = \sum_{m,n=-\infty}^{\infty}\sum f(m\Delta x, n\Delta y)\ \delta(x - m\Delta x, y - n\Delta y) \quad (4.6)$$

F.T. of $\mathrm{Comb}(x, y; \Delta x, \Delta y)$ is another Comb fn., with spacing $\left(\frac{1}{\Delta x}, \frac{1}{\Delta y}\right)$ in 2D-freq., domain (see pages 5 & 6 for proof in the 1-D case)

$$\therefore \mathrm{Comb}(\xi_1, \xi_2) = \mathcal{F}\left[\mathrm{Comb}(x, y; \Delta x, \Delta y)\right]$$

$$= \xi_{xs}\,\xi_{ys} \sum_{k,l=-\infty}^{\infty}\sum \delta(\xi_1 - k\xi_{xs}, \xi_2 - l\xi_{ys})$$

$$= \xi_{xs}\,\xi_{ys}\ \mathrm{Comb}\left(\xi_1, \xi_2; \frac{1}{\Delta x}, \frac{1}{\Delta y}\right) \quad (4.7)$$

where, $\xi_{xs} = (1/\Delta x)$, $\xi_{ys} = (1/\Delta y)$

FT of a 2-D impulse train is another impulse train (2-D). The uniformly (2-D) sampled image

$$f_s(x, y) = f(x, y)\ \mathrm{comb}(x, y\,;\ \Delta x, \Delta y) \qquad (4.6)$$

$\therefore$ 2-D FT $[\, f_s(x, y)\,]$, (See Fig. 4.7b)

$$= F(\xi_1, \xi_2) \circledast \mathrm{COMB}(\xi_1, \xi_2)$$

$$= \xi_{xs}\, \xi_{ys} \sum_{k,l=-\infty}^{\infty}\sum F(\xi_1, \xi_2) \circledast \delta(\xi_1 - k\xi_{xs},\ \xi_2 - l\xi_{ys}) \qquad (4.8)$$

$= (\xi_{xs}\xi_{ys})\,\big(F(\xi_1, \xi_2)$ repeated uniformly along ξ_1 & ξ_2 with periods ξ_{xs} & ξ_{ys} respectively.

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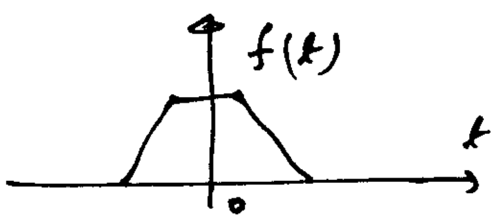
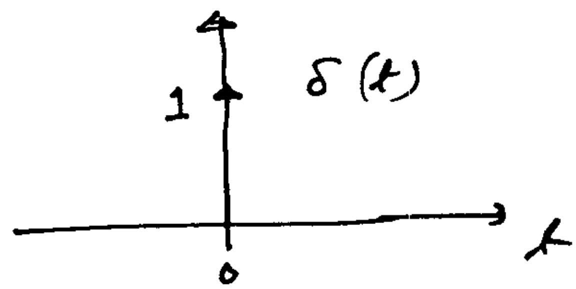
Convolution of an impulse and a function $f(t)$

$f(t) \circledast \delta(t)$

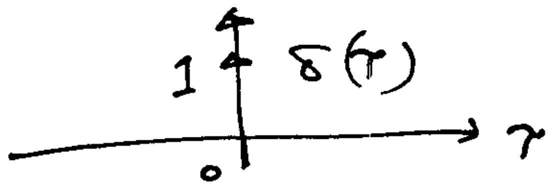


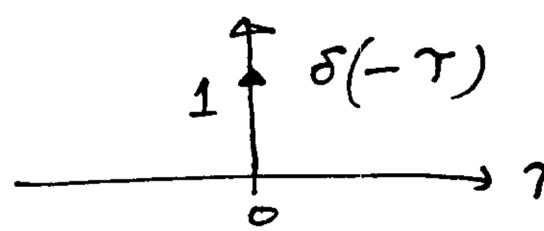


$$\int_{-\infty}^{\infty} f(\tau)\ \delta(t-\tau)\, d\tau$$

$$= f(t)$$

$$= f(t) \circledast \delta(t)$$





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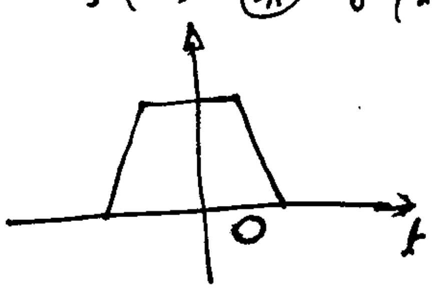
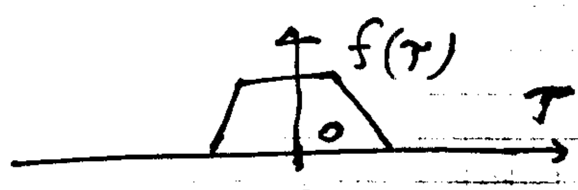
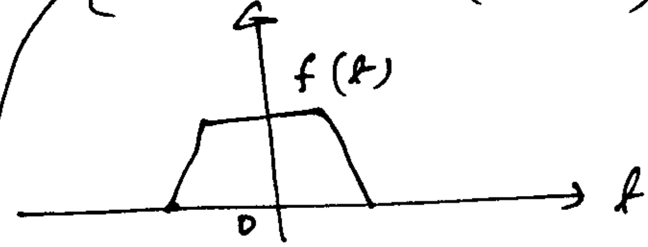
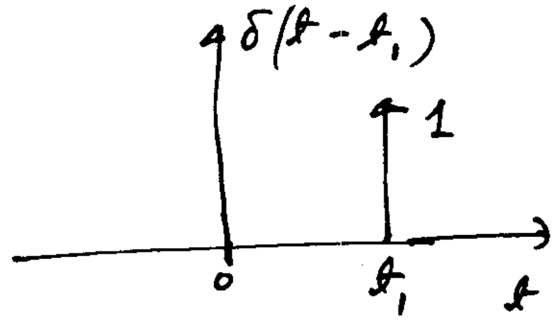


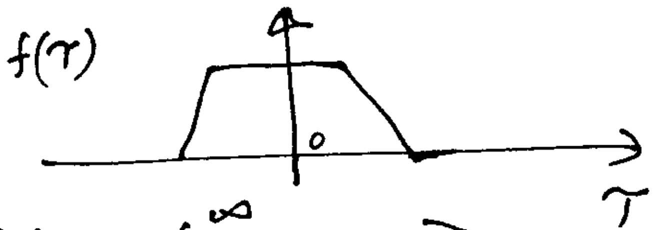


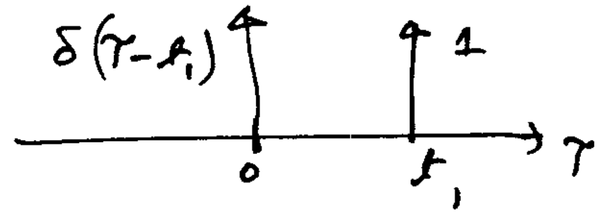
Convolution of a delayed impulse and $f(t)$

$$\left[f(t) \circledast \delta(t-t_1)\right]$$

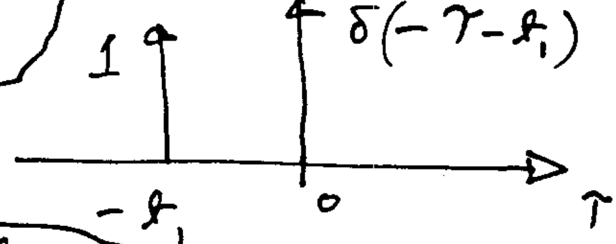








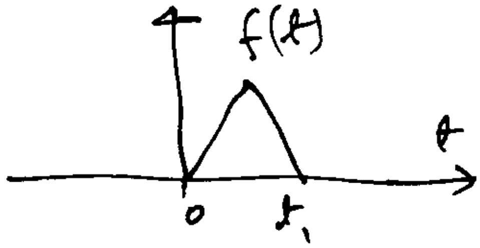
$$= \int_{-\infty}^{\infty} f(\tau)\, \delta(t-\tau-t_1)\, d\tau$$

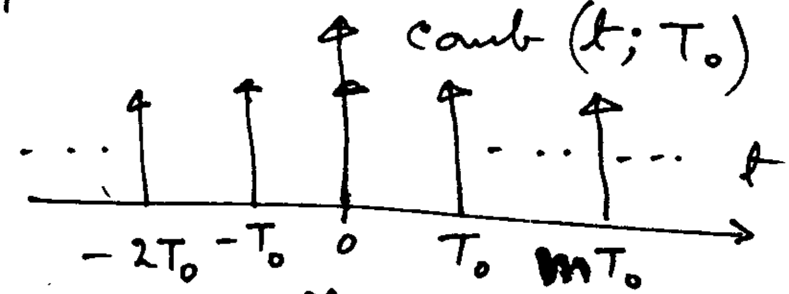


$$= f(t-t_1)$$

$= f(t)$ shifted to the right by t_1.

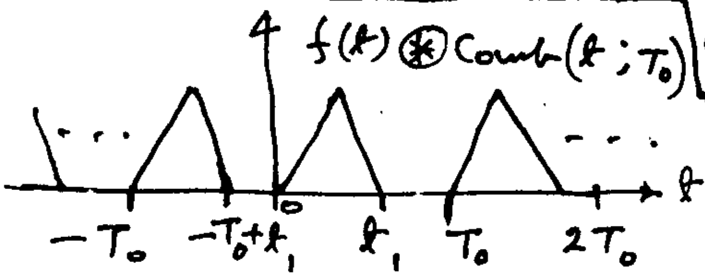
convolution of a periodic impulse train with $f(t)$





$$\text{Comb}(t;T_0) = \sum_{m=-\infty}^{\infty} \delta(t - mT_0)$$

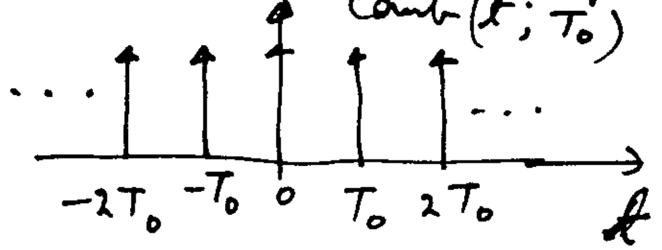
$f(t) \circledast \text{Comb}(t;T_0)$ is $f(t)$ repeated periodically with period T_0. There is no overlap provided $T_0 > t_1$



This property is valid in any domain. i.e., time or frequency domain.

Reconstruction of the original image from the sampled image (See Fig. 4.7)

sampling $f(t)$ uniformly implies multiplying $f(t)$ with an impulse train with period T_0



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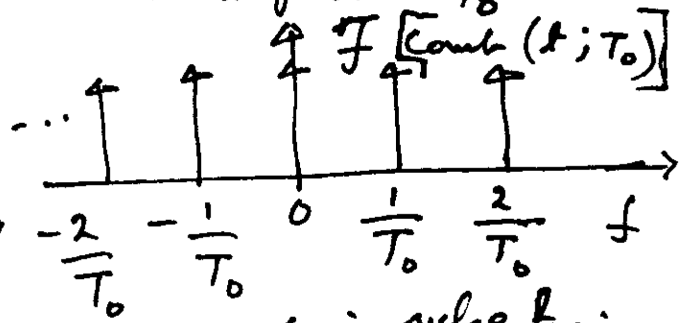
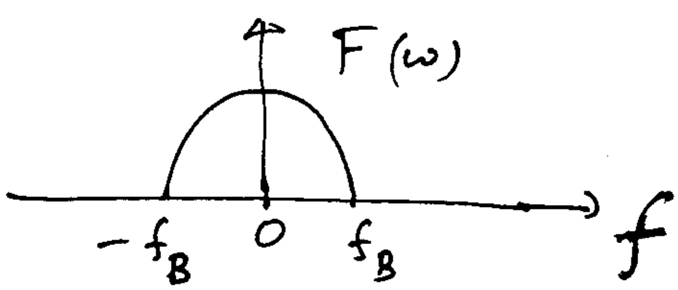


FT of impulse train.

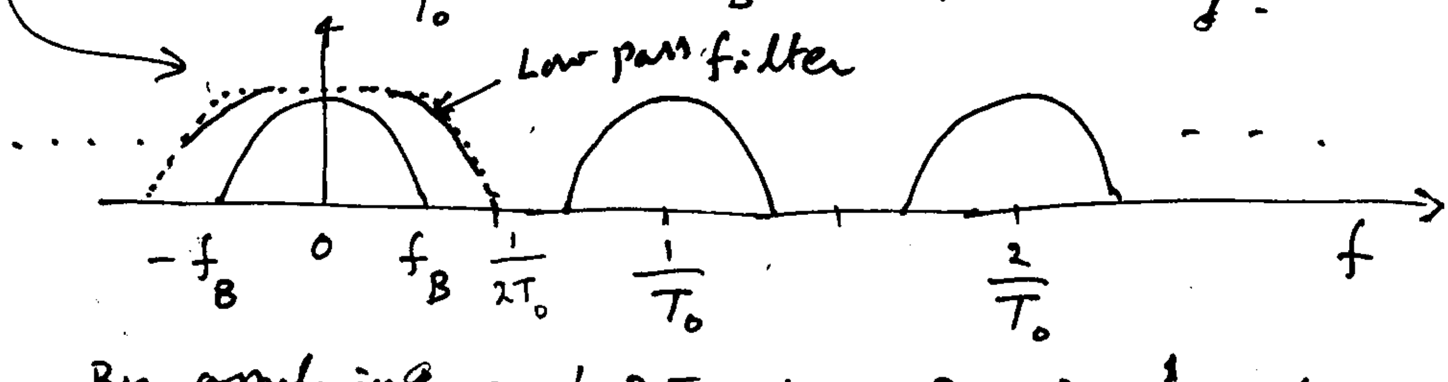


$F(\omega) = \mathcal{F}[f(t)]$ is band limited. Bandwidth $= f_B$.

FT of $\left([f(t)] \, [\text{comb}(t\,;T_0)] \right)$ is

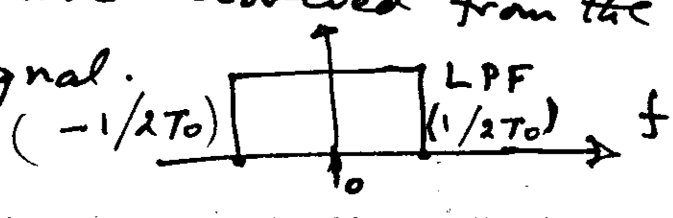
$F(\omega)$ repeated periodically with period $\frac{1}{T_0}$.

when $\frac{1}{T_0} > 2 f_B$. No aliasing.



By applying a LPF whose BW is $\frac{1}{2T_0}$, the original signal $f(t)$ can be recovered from the uniformly sampled signal.

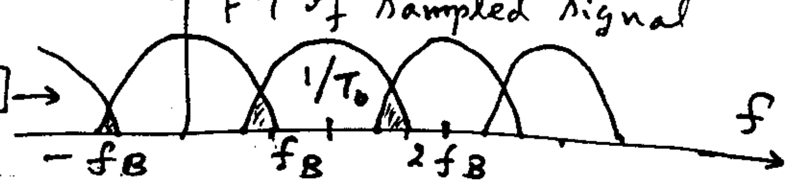
(See Fig. 4.7)



NYQUIST Rate aliasing & foldover frequencies 17

when $\frac{1}{T_o} = 2f_B$ then $\frac{1}{T_o}$, the sampling rate (T_o = sampling interval) is called the Nyquist rate. To recover the original signal $f(t)$ with BW = f_B Hz, from the sampled signal $f(n)$ the sampling rate (# of samples/sec) must be at least $= 2f_B$. When $\frac{1}{T_o} < 2f_B$, the periodic repetitions of $F(\omega)$ overlap. This is called aliasing.

Freq. spectrum of the sampled signal when $\frac{1}{T_o} < 2f_B$



Sampling theorem 18

$f(x, y)$ bandlimited image, ξ_{x0} = BW along x
ξ_{y0} = " " y

Sampled image $f(m\Delta x, n\Delta y)$

where Δx = sampling interval along x
Δy = " " y

$\frac{1}{\Delta x} = \xi_{xs}$ = sampling rate along x
$\frac{1}{\Delta y} = \xi_{ys}$ = " " " y

$f(x,y)$ can be recovered from $f(m\Delta x, n\Delta y)$

provided $\xi_{xs} \geq 2\xi_{x0}$ and $\xi_{ys} \geq 2\xi_{y0}$

$$f(x,y) = \sum_{m,n=-\infty}^{\infty}\sum f(m\Delta x, n\Delta y)\, \text{sinc}[(x\xi_{xs} - m)] \cdot \text{sinc}[(y\xi_{ys} - n)], \quad (4.16)$$

$\boxed{\text{See } (4.13) \text{ and } (4.14)}$

For proof of (4.16) see M. Schwartz "Info Transmission ..." pages 99-104 McGraw-Hill Book Co.

$$f(x,y) = \mathcal{F}^{-1}\left[F_s(\xi_1, \xi_2)\, H(\xi_1, \xi_2)\right]$$

where $H(\xi_1, \xi_2)$ is the ideal 2-D LPF.

Its region of support is the rectangle in the 2-D frequency domain

$$\left[-\frac{1}{2}\xi_{xs}, \frac{1}{2}\xi_{xs}\right] \times \left[-\frac{1}{2}\xi_{ys}, \frac{1}{2}\xi_{ys}\right] \quad (4.13)$$

$$\mathcal{F}^{-1}\left[H(\xi_1, \xi_2)\right] = h(x,y) = \left[\text{sinc}(x\xi_{xs})\right]\left[\text{sinc}(y\xi_{ys})\right], \quad (4.14)$$

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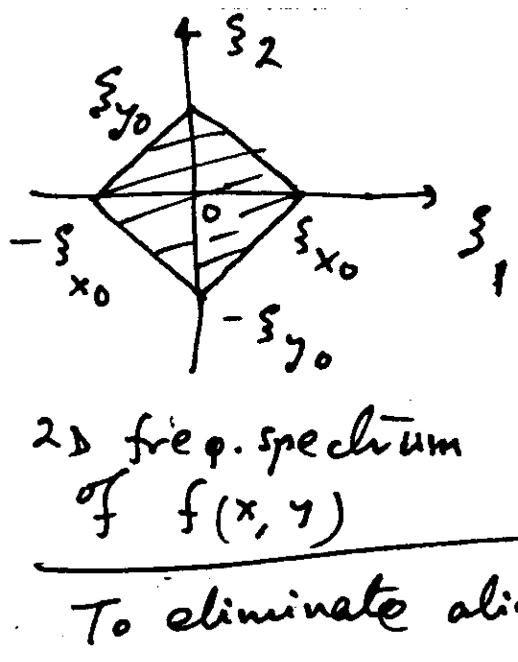


2D freq. spectrum of $f(x,y)$

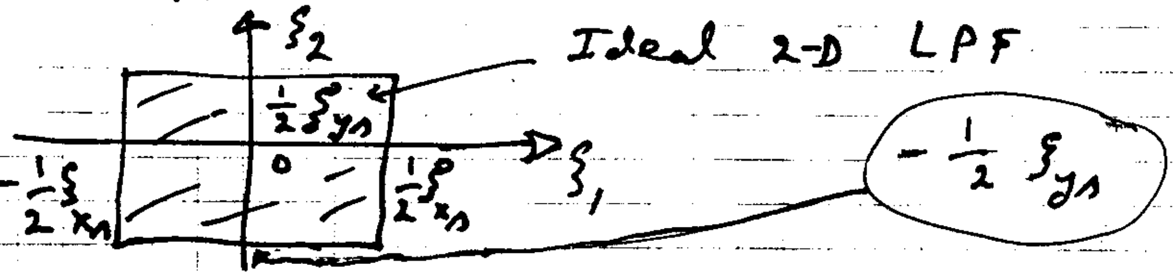
ξ_{x0} = BW along ξ_1

ξ_{y0} = " " ξ_2

$2\xi_{x0}$, $2\xi_{y0}$ are Nyquist sampling rates

To eliminate aliasing, sampling rates should be

$$\xi_{xs} \geq 2\xi_{x0} \quad \text{and} \quad \xi_{ys} \geq 2\xi_{y0}$$



EX.4.1 / P.89 Image $f(x,y) = 2\cos 2\pi(3x+4y)$ (20)

sampling intervals $\Delta x = \Delta y = 0.2$

$$f(x,y) = \cancel{2}\left[e^{j2\pi(3x+4y)} + e^{-j2\pi(3x+4y)}\right]$$

$$F(\xi_1,\xi_2) = \iint_{-\infty}^{\infty}\left[e^{j2\pi(3x+4y)} \cancel{2} + e^{-j2\pi(3x+4y)}\right] e^{-j2\pi(\xi_1 x+\xi_2 y)}\,dx\,dy$$

$$= \iint_{-\infty}^{\infty}\left[e^{-j2\pi[x(\xi_1-3)+y(\xi_2-4)]} + e^{-j2\pi[x(\xi_1+3)+y(\xi_2+4)]}\right] dx\,dy$$

$$= \left(\int_{-\infty}^{\infty} e^{-j2\pi x(\xi_1-3)}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-j2\pi y(\xi_2-4)}\,dy\right)$$

$$+ \left(\int_{-\infty}^{\infty} e^{-j2\pi x(\xi_1+3)}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-j2\pi y(\xi_2+4)}\right) dy$$ 21

$$= \left[\delta(\xi_1-3)\,\delta(\xi_2-4) + \delta(\xi_1+3)\,\delta(\xi_2+4)\right]$$

$$= \left[\delta(\xi_1-3,\ \xi_2-4) + \delta(\xi_1+3,\ \xi_2+4)\right]$$

$F(\xi_1,\xi_2)$ is zero for $|\xi_1| > 3$ & $|\xi_2| > 4$

$\therefore\ \xi_{x0} = 3$ and $\xi_{y0} = 4$, (cycles/meter)

sampling rate (in the spatial domain): $\xi_{xs} = \frac{1}{\Delta x} = \frac{1}{0.2} = 5$ samples/meter

$\xi_{ys} = \frac{1}{\Delta y} = \frac{1}{0.2} = 5$ "

Nyquist rates are $2 \times 3 = 6$, samples/meter along x

and $2 \times 4 = 8$ " " " y

(there is aliasing in the freq. domain.)
sampling rate is below the Nyquist rate,

2-D sampling function (2-D periodic impulse train)

$$\sum\sum_{m,n=-\infty}^{\infty} \delta(x-0.2m,\ y-0.2n)$$

$\Delta x = \Delta y = 0.2$ meters

2-D FT of sampling fn.,

$$\mathcal{F}\left[\sum\sum_{m,n=-\infty}^{\infty} \delta(x-0.2m,\ y-0.2n)\right]$$

$$= \xi_{xs}\,\xi_{ys}\left[\sum\sum_{k,l=-\infty}^{\infty} \delta(\xi_1-5k,\ \xi_2-5l)\right]$$

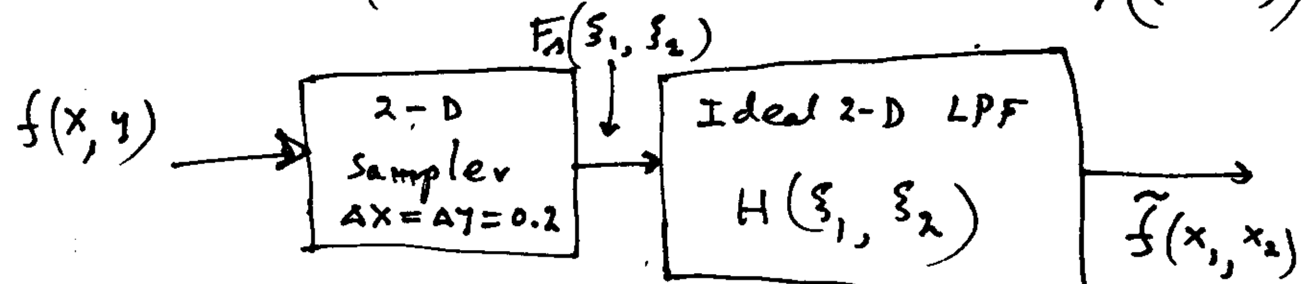
periodic Impulse train in the 2-D frequency domain

where $\xi_{xs} = \frac{1}{\Delta x} = 5$ cycles/meter

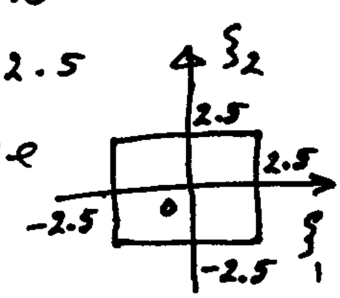
$\xi_{ys} = \frac{1}{\Delta y} =$ " " "

$$\therefore F_s(\xi_1,\xi_2) = 25 \sum\sum_{k,l=-\infty}^{\infty}\Big[\delta(\xi_1-3-k5,\ \xi_2-4-5l)$$

$$+\ \delta(\xi_1+3-5k,\ \xi_2+4-5l)\Big],\ \text{(see (4.8))}$$



$$H(\xi_1,\xi_2) = \begin{cases} \frac{1}{25} & |\xi_1| \le 2.5,\ |\xi_2| \le 2.5 \\ 0 & \text{otherwise} \end{cases}$$



$$\tilde{F}(\xi_1,\xi_2) = F_s(\xi_1,\xi_2)\ H(\xi_1,\xi_2)$$

$$\tilde{F}(\xi_1, \xi_2) = \left[\delta(\xi_1 - 2, \xi_2 - 1) + \delta(\xi_1 + 2, \xi_2 + 1)\right]$$

$$\tilde{f}(x, y) = 2\cos[2\pi(2x + y)]$$

$$f(x, y) = 2\cos[2\pi(3x + 4y)]$$

Any freq., component in the original image that is above $(\xi_{xs}/2, \xi_{ys}/2)$ by $(\Delta\xi_x, \Delta\xi_y)$ is reproduced as a freq., component at $(\xi_{xs}/2 - \Delta\xi_x, \xi_{ys}/2 - \Delta\xi_y)$

Here $(\xi_{xs}/2) = (\xi_{ys}/2) = 2.5$, $\Delta\xi_x = 0.5$, $\Delta\xi_y = 1.5$

signal is reproduced at freqs., $(2, 1)$ cycles/meter

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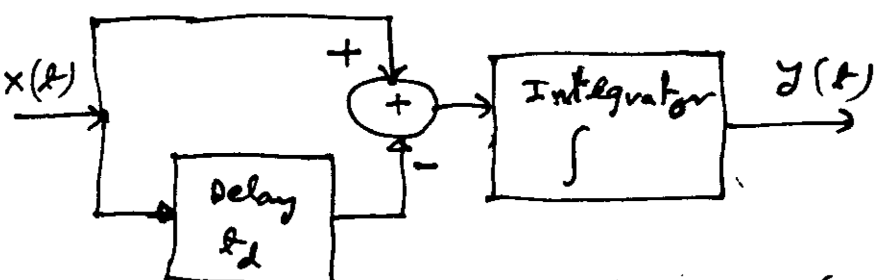
ZOH Zero-order hold



delay by t_d] $e^{-j2\pi f t_d}$

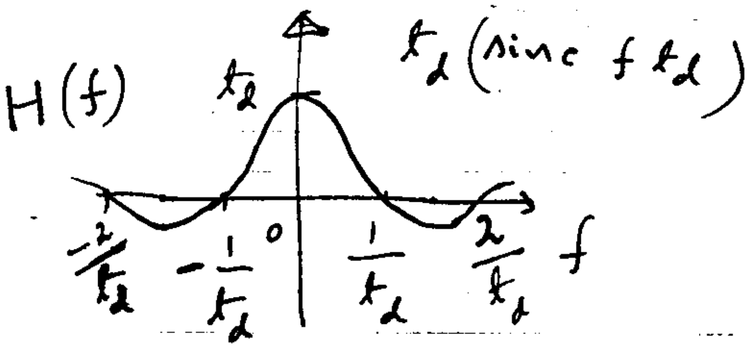
$\int \longrightarrow \dfrac{1}{j2\pi f}$

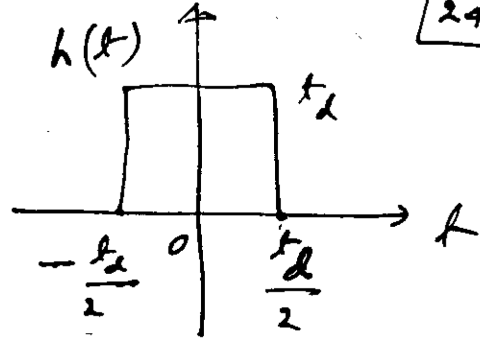
Transfer fn., $\dfrac{Y(f)}{X(f)} = \left(\dfrac{1 - e^{-j2\pi f t_d}}{j2\pi f}\right)$

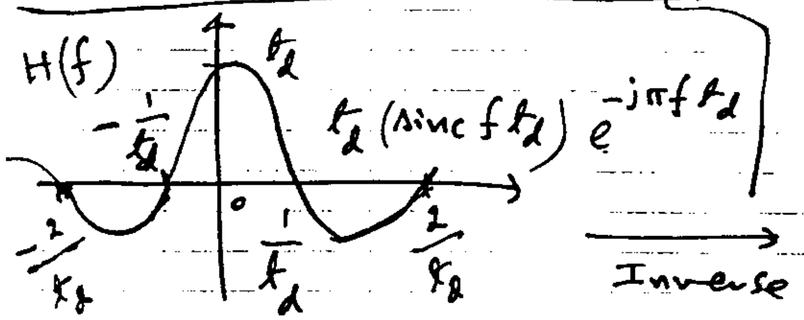
$$= \left(\frac{e^{j\pi f t_d} - e^{-j\pi f t_d}}{2j}\right)\left(\frac{e^{-j\pi f t_d}}{\pi f}\right)$$

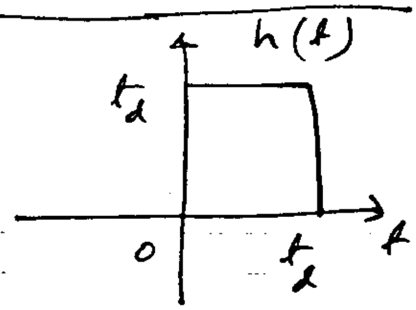
$$= t_d \left(\frac{\sin \pi f t_d}{\pi f t_d}\right) e^{-j\pi f t_d}$$

$$\frac{Y(f)}{X(f)} = t_d\,(\mathrm{sinc}\, f t_d)\, e^{-j\pi f t_d} = H(f)$$



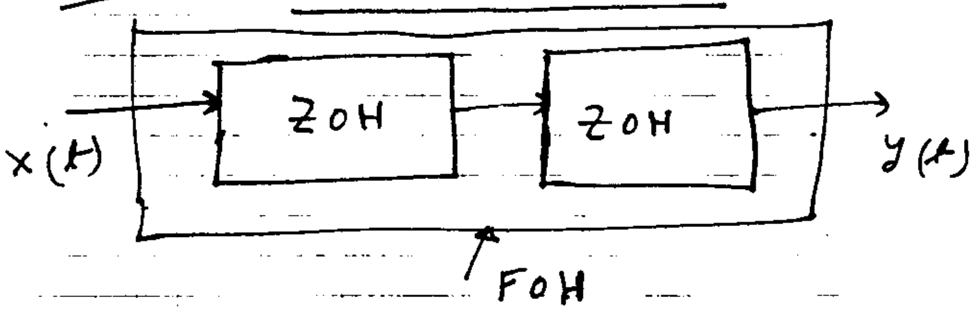




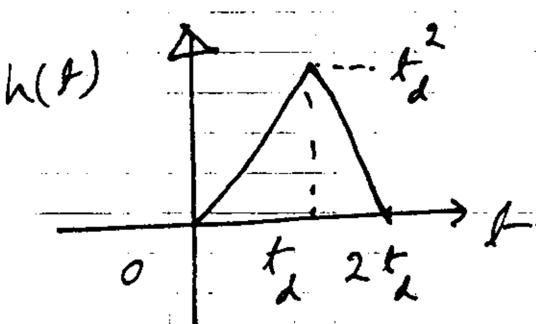


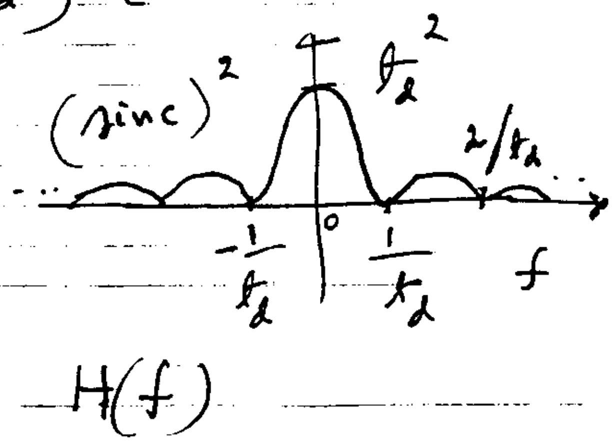
(See Fig. 4.13 / p. 96) (ZOH)

FOH First order hold 24c



$$H(f) = t_d^{\,2}\,(\text{sinc}\, f t_d)^2\, e^{-j2\pi f t_d}$$





4.3 Extensions of sampling Theory

$f(x,y)$ stationary random field is bandlimited when its PSD

$$S(\xi_1, \xi_2) = 0, \quad \text{for } |\xi_1| > \xi_{x0} \ \& \ |\xi_2| > \xi_{y0} \quad (4.17)$$

power spectral density

Sampling theorem for random fields

Given $f(x,y)$: stationary bandlimited random field, then

$$\tilde{f}(x,y) = \sum\sum_{m,n=-\infty}^{\infty} f(m\Delta x, n\Delta y) \, \mathrm{sinc}(x\xi_{xs} - m) \, \mathrm{sinc}(y\xi_{ys} - n) \quad (4.18)$$

where

$$\mathrm{sinc}(x\xi_{xs} - m) = \left[\frac{\sin \pi (x\xi_{xs} - m)}{\pi (x\xi_{xs} - m)}\right]$$

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converges to $f(x,y)$ in the mean square sense i.e.,

$$E\left[\left|f(x,y) - \tilde{f}(x,y)\right|^2\right] = 0, \quad (4.19)$$

where $\xi_{xs} = \frac{1}{\Delta x}$ & $\xi_{ys} = \frac{1}{\Delta y}$, spatial sampling rate

and $\xi_{xs} \geq 2\xi_{x0}$ and $\xi_{ys} \geq 2\xi_{y0}$

$\Delta x, \Delta y$ are spatial sampling intervals

$2\xi_{x0}$ and $2\xi_{y0}$ are Nyquist sampling rates

(27)

$S_s(\xi_1, \xi_2) \to$ PSD of the sampled image $f_s(x, y)$

(27)

$f_s(x, y) = f(x, y)$ sampled uniformly along x & y with sampling intervals Δx & Δy.

$S(\xi_1, \xi_2) =$ PSD of $f(x, y)$. Then

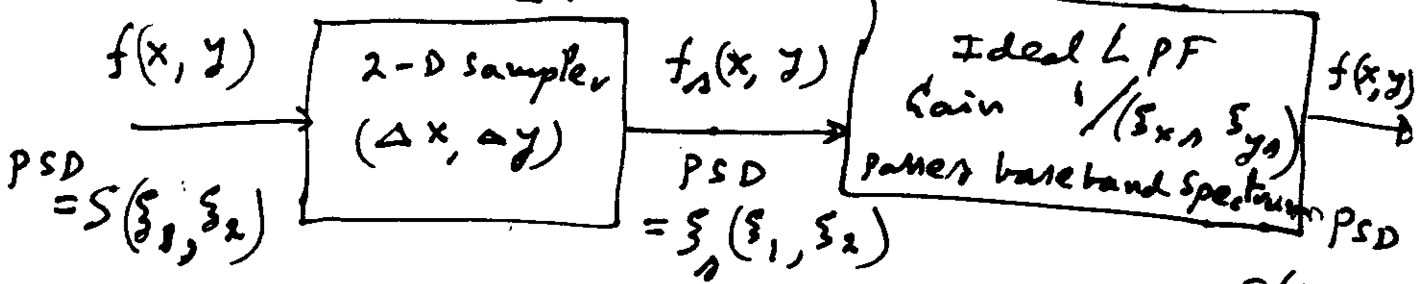
$$S_s(\xi_1, \xi_2) = \xi_{xs}\,\xi_{ys} \sum_{k,l=-\infty}^{\infty}\sum S(\xi_1 - k\xi_{xs}, \xi_2 - l\xi_{ys}), \quad (4.20)$$

$= S(\xi_1, \xi_2)$ repeated periodically along ξ_1 & ξ_2 with sampling intervals ξ_{xs} & ξ_{ys}

where $\xi_{xs} = \dfrac{1}{\Delta x}$ & $\xi_{ys} = \dfrac{1}{\Delta y}$.

proof of (4.20) is similar to that shown for (4.8). (28)

$\xi_{xs} = \dfrac{1}{\Delta x} \geq 2\xi_{x0}$ and $\xi_{ys} = \dfrac{1}{\Delta y} \geq 2\xi_{y0}$



$S(\xi_1, \xi_2)$ is bandlimited to ξ_{x0} & ξ_{y0}.

Reconstruction (recovery) of bandlimited stationary random signal from its sampled data.

$$f_s(x,y),\ \text{PSD} = S_s(\xi_1,\xi_2) \longrightarrow \boxed{\text{Ideal LPF gain } \left[1/(\xi_{xs}\,\xi_{ys})\right]} \longrightarrow \tilde{S}(\xi_1,\xi_2),\ \hat{f}(x,y)$$

$$\tilde{S}(\xi_1,\xi_2) = \left[\sum_{k,l=-\infty}^{\infty} S(\xi_1 - k\xi_{xs},\, \xi_2 - l\xi_{ys})\right] \cdot W(\xi_1,\xi_2) \tag{4.21}$$

where

$$W(\xi_1,\xi_2) = \begin{cases} 1 & (\xi_1,\xi_2) \in \mathcal{R} \text{ (region of support)} \\ 0 & \text{otherwise} \end{cases}$$

When $\xi_{xs} \geq 2\xi_{x0}$ & $\xi_{ys} \geq 2\xi_{y0}$, $\tilde{S}(\xi_1,\xi_2) = S(\xi_1,\xi_2)$

30

otherwise the aliasing power

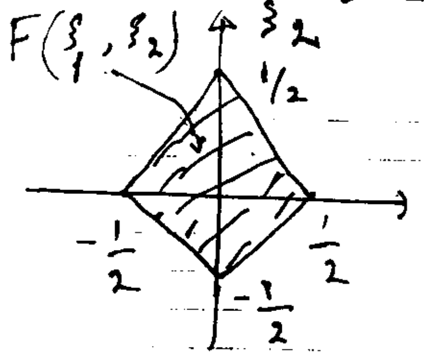
= in the tails of PSD outside $\mathcal{R}$, i.e.,

$$\sigma_a^2 = \iint_{(\xi_1,\xi_2)\notin\mathcal{R}} S(\xi_1,\xi_2)\,d\xi_1\,d\xi_2$$

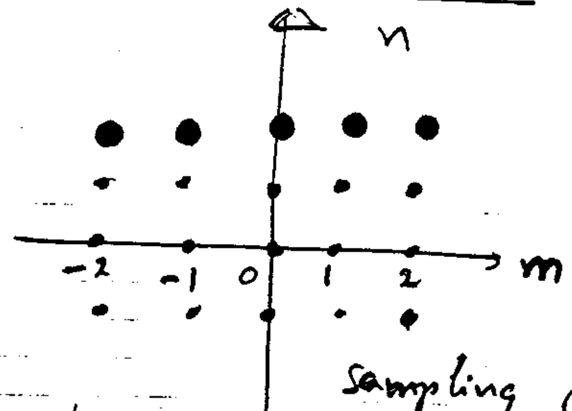
$$= \iint_{-\infty}^{\infty} \left[1 - W(\xi_1,\xi_2)\right] S(\xi_1,\xi_2)\,d\xi_1\,d\xi_2$$

$$= \left[\iint_{-\infty}^{\infty} S(\xi_1,\xi_2)\,d\xi_1\,d\xi_2 - \iint_{-\infty}^{\infty} W(\xi_1,\xi_2)\,S(\xi_1,\xi_2)\,d\xi_1\,d\xi_2\right] \tag{4.23}$$

Nonrectangular grid sampling and interlacing.



Baseband spectrum



Sampling G_1

Rectangular grid

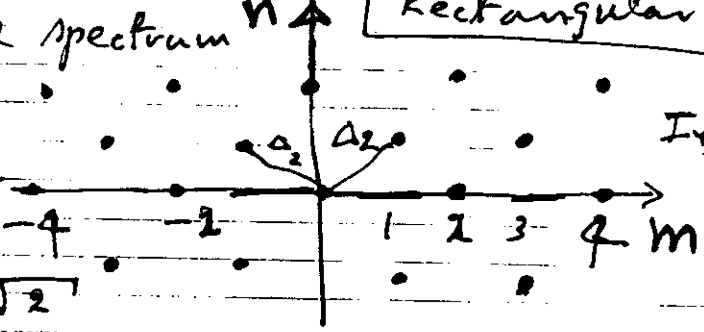


Fig 4.9

Interlaced grid G_2

$\Delta_2 = \sqrt{2}$

prob. 4.9

Interlaced sampling grid G_2

$$g(x,y) = \sum_{m,n}\sum \delta(x-2m,\ y-2n)$$

$$+ \sum_{m,n}\sum \delta(x-2m-1,\ y-2n-1)$$

2D-FT is

$$G(\xi_1, \xi_2) = \frac{1}{4}\sum_{k,l}\sum \delta\left(\xi_1 - \frac{k}{2},\ \xi_2 - \frac{l}{2}\right)$$

$$+\frac{1}{4}e^{-j\pi(k+l)}\left[\sum_{k,l}\sum \delta\left(\xi_1 - \frac{k}{2},\ \xi_2 - \frac{l}{2}\right)\right]$$

$$= \frac{1}{2}\sum_{k+l=\text{even}}\sum \delta(\xi_1 - k\xi_0,\ \xi_2 - l\xi_0)$$

Since $e^{-j\pi(k+l)} = 1$ $k+l$ is even

$k+l$ is odd $\longrightarrow -1$

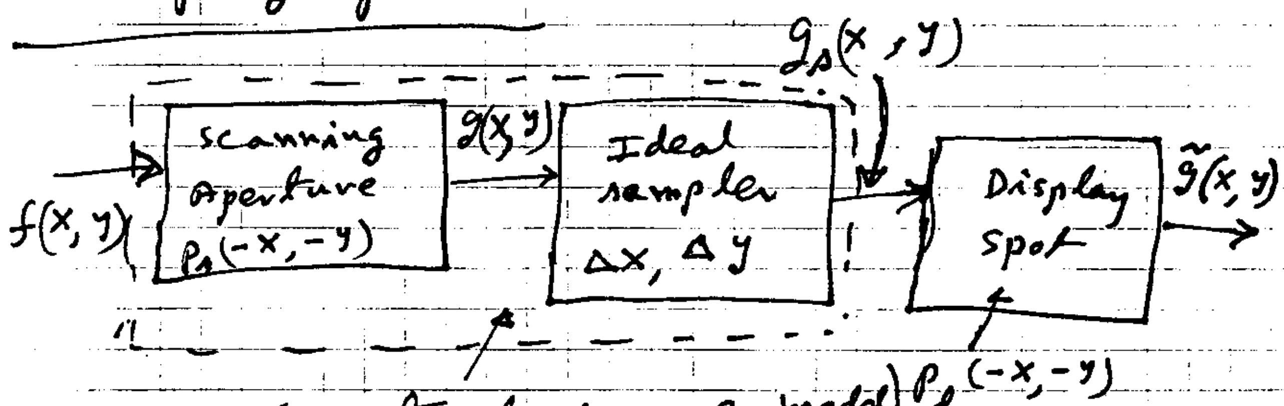
$$\xi_0 = 1/2$$

Section (4.4)
Practical limitations in sampling and reconstruction

practical scanners have finite aperture. This is similar to low pass filtering the input image before sampling. High frequency terms are dropped out resulting in image blurring (resolution loss).

Also there are no ideal LPF for recovering the original signal.

Sampling aperture



(practical scanner model)

Fig 4.10 Practical sampling and reconstruction

For the ideal case $p_s(x, y) = p_d(x, y)$
$= \delta(x, y)$

$p_s(x, y)$ = light distribution in the aperture

$$g(x,y) = p_s(x,y) \star f(x,y)$$

correlation (4.25)

$$= \iint_A p_s(x',y')\, f(x+x', y+y')\, dx'dy'$$

Aperture shape

Let $x + x' = p$ and $y + y' = q$

$x' = p - x$ $\quad$ $y' = q - y$

$$g(x,y) = \iint_A p_s(p-x, q-y)\, f(p,q)\, dp\, dq$$

$$= \iint_A p_s\big(-(x-p), -(y-q)\big)\, f(p,q)\, dp\, dq$$

$$\boxed{g(x,y) = p_s(-x,-y) \circledast f(x,y)} \quad (4.25)$$

2D convolution

$$g_s(x,y) = \big[comb(x,y;\Delta x,\Delta y)\big]\, g(x,y) \quad (4.26)$$

2-D Infinite impulse train

$$\big[Comb(x,y;\Delta x,\Delta y)\big] = \sum_{m,n=-\infty}^{\infty}\sum \delta(x-m\Delta x, y-n\Delta y)$$

Light distribution in the aperture is symmetric WRT 180° rotation, i.e.,

$$p_s(x,y) = p_s(-x,-y)$$

For an $(L \times L)$ square aperture with uniform distribution

$$g(x,y) = \int_{x-\frac{L}{2}}^{x+L/2}\int_{y-\frac{L}{2}}^{y+L/2} f(x',y')\, dx'dy' \quad (4.27)$$

= Integral of the image over the scanner aperture at (x,y)

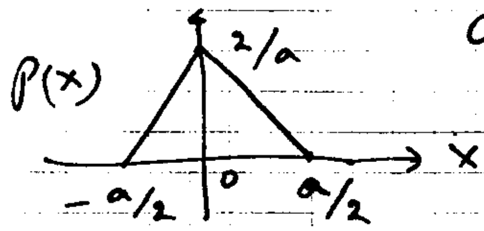
EX 4.2 CCD Camera 256×256 photodetectors of size $(a \times a)$

Spacing $\Delta x = \Delta y = a \leq \Delta$ (see Fig. 4.3)

Impulse response of detectors

$$P_s(x, y) = p(x)\, p(y)$$

$$p(x) = \begin{cases} \frac{2}{a}\left(1 - \frac{2|x|}{a}\right) & |x| \leq a/2 \\ 0 & \text{otherwise} \end{cases}$$



See Fig. 4.10

Image $f(x, y) = 2\left[\cos 2\pi\left(\frac{x}{4a} + \frac{y}{8a}\right)\right]$ (30h

$(a = \Delta = \Delta x = \Delta y$ (photodetector size is $a \times a$))

FT gives $G(\xi_1, \xi_2) = \left[\operatorname{sinc}^2\left(\frac{a\xi_1}{2}\right)\operatorname{sinc}^2\left(\frac{a\xi_2}{2}\right)\right] F(\xi_1, \xi_2)$

$$F(\xi_1, \xi_2) = \left[\delta\left(\xi_1 - \frac{1}{4a}, \xi_2 - \frac{1}{8a}\right) + \delta\left(\xi_1 + \frac{1}{4a}, \xi_2 + \frac{1}{8a}\right)\right]$$

$$G(\xi_1, \xi_2) = \operatorname{sinc}^2\left(\frac{1}{8}\right)\operatorname{sinc}^2\left(\frac{1}{16}\right) F(\xi_1, \xi_2)$$

$$= 0.94\, F(\xi_1, \xi_2)$$

30i

$2\cos[2\pi(\frac{x}{4a} + y/8a)]$

$$= \left[e^{j2\pi\left(\frac{x}{4a} + \frac{y}{8a}\right)} + e^{-j2\pi\left(\frac{x}{4a} + \frac{y}{8a}\right)}\right]$$

FT gives the delta fns. in the frequency domain

scanner output signal

$$g_s(x, y) = g(x, y)\, w(x, y) \sum\sum_{m,n=-\infty}^{\infty} \delta(x - ma, y - na)$$

$w(x, y)$ rectangular window $(-\frac{L}{2}, \frac{L}{2})$ (Square aperture)

$L = 256\Delta$ $\Delta = a$

$$G(\xi_1, \xi_2) \simeq 0.94 \left[\delta\left(\xi_1 - \frac{1}{4a}, \xi_2 - \frac{1}{8a}\right) + \delta\left(\xi_1 + \frac{1}{4a}, \xi_2 + \frac{1}{8a}\right)\right]$$ $\boxed{30j}$

$$W = L^2 \operatorname{sinc}(\xi_1 L) \operatorname{sinc}(\xi_2 L)$$

$$\tilde{G}(\xi_1, \xi_2) = G(\xi_1, \xi_2) \circledast W(\xi_1, \xi_2)$$ convolution

$$= \int\int_{-\infty}^{\infty} G(\psi_1, \psi_2)\, W(\xi_1 - \psi_1, \xi_2 - \psi_2)\, d\psi_1\, d\psi_2$$

$$= 0.94 L^2 \int\int_{-\infty}^{\infty} \left[\delta\left(\psi_1 - \frac{1}{4a}, \psi_2 - \frac{1}{8a}\right) + \delta\left(\psi_1 + \frac{1}{4a}, \psi_2 + \frac{1}{8a}\right)\right]$$

$$\cdot \operatorname{sinc}((\xi_1 - \psi_1)L) \operatorname{sinc}((\xi_2 - \psi_2)L)\, d\psi_1\, d\psi_2$$

spectrum of scanned image is

$$G_s(\xi_1, \xi_2)$$

30K

(spacing $a = \Delta$, $L = 256\Delta$)

$$= \xi_s^2 \sum\sum_{m,n=-\infty}^{\infty} \tilde{G}(\xi_1 - m\xi_s, \xi_2 - n\xi_s), \quad \xi_s = \frac{1}{\Delta}$$

$$\tilde{G}(\xi_1, \xi_2) = G(\xi_1, \xi_2) \circledast W(\xi_1, \xi_2)$$

$$W(\xi_1, \xi_2) = L^2 \operatorname{sinc}(\xi_1 L) \operatorname{sinc} \xi_2 L$$

$$\tilde{G}(\xi_1, \xi_2) = (256)^2 a^2 \times 0.94 \left[\operatorname{sinc}(256 a\xi_1 - 64)\right.$$

$$\left.\operatorname{sinc}(256 a\xi_2 - 32) + \operatorname{sinc}(256 a\xi_1 + 64) \operatorname{sinc}(256 a\xi_1 + 32)\right]$$

$$\tilde{G}(\xi_1, \xi_2) = \left[\operatorname{sinc}\left(\xi_1 - \frac{1}{4a}\right)L\right.$$

30l

$$\left.\operatorname{sinc}\left(\xi_2 - \frac{1}{8a}\right)L\right] + \left[\operatorname{sinc}\left(\xi_1 + \frac{1}{4a}\right)L\right.$$

$$\left.+ \operatorname{sinc}\left(\xi_2 + \frac{1}{8a}\right)L\right], \quad L = 256a$$

$$= 61400\, a^2 \left[\operatorname{sinc}(256 a\xi_1 - 64)\right.$$

$$\operatorname{sinc}(256 a\xi_2 - 32)$$

$$\left.+ \operatorname{sinc}(256 a\xi_1 + 64) \operatorname{sinc}(256 a\xi_2 + 32)\right]$$

(see $\frac{\text{Fig. 4.15}}{\text{p. 98}}$) $\frac{\text{Array Scanner}}{\text{frequency response}}$

Lagrange Interpolation

Polynomial interpolation.

Lagrange polynomial of order $(q-1)$

$$L_K^q(x) = \prod_{\substack{m=K_0 \\ m \neq K}}^{K_1} \left(\frac{x-m}{K-m} \right) \qquad (4.28)$$

$$\left(K_0 \le K \le K_1 , \quad q = 2, 3 \right)$$

$$L_K^1(x) = 1, \quad \forall K \text{ (for all } K)$$

$$K_0 = -\left(\frac{q-1}{2}\right), \quad K_1 = \left(\frac{q-1}{2}\right), \quad K \text{ odd}$$

$$K_0 = -\left(\frac{q-2}{2}\right), \quad K_1 = \frac{q}{2}, \quad K \text{ even}$$

Sampling interval $= \Delta$

$\hat{f}(x) = \hat{f}(m\Delta + \alpha\Delta)$ = Interpolated function between the samples

$$\hat{f}(m\Delta + \alpha\Delta) = \sum_{K=K_0}^{K_1} L_K^q(\alpha) f(m\Delta + K\Delta) \qquad (4.29)$$

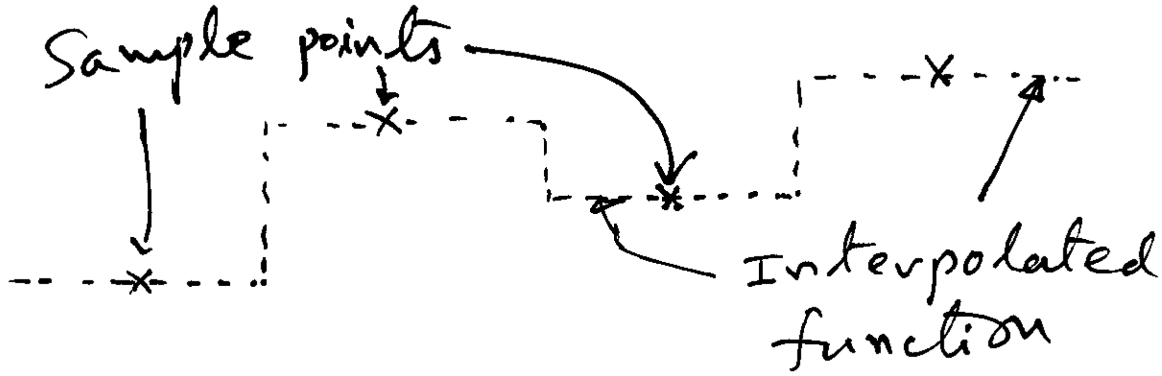
$-\frac{1}{2} \le \alpha < \frac{1}{2}$ $\quad$ q odd $\quad$ 30m

$0 \le \alpha < 1$ $\quad$ q even

For $q = 1, 2, 3$

$$q = 1 \Rightarrow \hat{f}(m\Delta + \alpha\Delta) = f(m\Delta)$$

$-\frac{1}{2} \le \alpha < \frac{1}{2}$, $\quad$ ZOH $\quad$ (4.30)



FOH $\quad$ First order hold

$$q = 2 \Rightarrow \hat{f}(m\Delta + \alpha\Delta)$$

$$= (1-\alpha) f(m\Delta) + \alpha f(\overline{m+1}\,\Delta)$$

Linear interpolation $\quad$ $0 \le \alpha < 1$