4.1 EE 5356 FILE 3 K. R. Rao

Solutions to problems in ch. 4

262.5 lines scanned in $1/60$ sec

time taken to scan 1 line $= (1/60)/262.5$

$= 1/15750$ s

$\therefore$ horizontal scan rate $=$ 15750 Hz.

1 field scanned in $(1/60)$ seconds

$\therefore$ Vertical scan rate $=$ 60 Hz.

4.2

If $f(x,y)$ is bandlimited image, then its Fourier Transform can be expressed as

$$F(\xi_1,\xi_2) = \begin{cases} F'(\xi_1,\xi_2), & |\xi_1| \leq \xi_{x0},\ |\xi_2| \leq \xi_{y0} \\ 0, & \text{otherwise} \end{cases}$$

$$= \mathrm{Rect}\left(\frac{\xi_1}{2\xi_{x0}}, \frac{\xi_2}{2\xi_{y0}}\right) F'(\xi_1,\xi_2)$$

$$f(x,y) = \mathcal{F}^{-1}\left\{\mathrm{Rect}\left(\frac{\xi_1}{2\xi_{x0}}, \frac{\xi_2}{2\xi_{y0}}\right) F'(\xi_1,\xi_2)\right\}$$

$$= \left(\mathcal{F}^{-1}\left\{\mathrm{Rect}\left(\frac{\xi_1}{2\xi_{x0}}, \frac{\xi_2}{2\xi_{y0}}\right)\right\}\right) \circledast \left(\mathcal{F}^{-1}\left\{F'(\xi_1,\xi_2)\right\}\right)$$

$$= 4\xi_{x0}\xi_{y0}\,\mathrm{sinc}(2\xi_{x0}x, 2\xi_{y0}y) \circledast f'(x,y)$$

while $f'(x,y)$ may or may not be spacelimited, the result of convolving it with the non-space-limited sinc function will yield a nonspace-limited image. Using the principle of duality we can show that if $f(x,y)$ is

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space-limited then $F(\xi_1,\xi_2)$ can not be band limited.

4.3

$$\mathcal{F}\{\cos 2m\pi x\} = \tfrac{1}{2}[\delta(\xi_1 - m) + \delta(\xi_1 + m)]$$

$$f(x,y) = 4\cos 4\pi x \cos 6\pi x$$

$$F(\xi_1,\xi_2) = \delta(\xi_1-2,\xi_2-3) + \delta(\xi_1-2,\xi_2+3) + \delta(\xi_1+2,\xi_2-3) + \delta(\xi_1+2,\xi_2+3)$$

The Fourier transform of a sampled image is given by:

$$F_s(\xi_1,\xi_2) = \xi_{xs}\xi_{ys} \sum\sum_{k,l=-\infty}^{\infty} F(\xi_1 - k\xi_{xs}, \xi_2 - l\xi_{ys})$$

Case 1: $\Delta x = \Delta y = 0.5$

$$F_s(\xi_1,\xi_2) = 4\sum\sum_{k,l=-\infty}^{\infty}[\delta(\xi_1-2-2k, \xi_2-3-2l) + \delta(\xi_1-2-2k, \xi_2+3-2l) + \delta(\xi_1+2-2k, \xi_2-3-2l) + \delta(\xi_1+2-2k, \xi_2+3-2l)]$$

Reconstruction is through the use of an ideal LPF with cutoff frequency $(1/2\Delta x, 1/2\Delta y)$

$$H(\xi_1,\xi_2) = \begin{cases} 1/4, & -1 \le \xi_1 \le 1,\ -1 \le \xi_2 \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\tilde{F}(\xi_1,\xi_2) = H(\xi_1,\xi_2)F_s(\xi_1,\xi_2)$$
$$= \frac{4}{4}[4\delta(\xi_1,\xi_2+1) + 4\delta(\xi_1,\xi_2-1)]$$

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$\tilde{f}(x,y) = 8\cos(2\pi y)$

since sampling frequencies were not greater than twice the bandwidth of the image, aliasing effects were expected. (frequencies above half the sampling frequencies in the original image will appear as frequencies below half the sampling frequencies and the original image can not be recovered by the LPF)

Case 2: $\Delta x = \Delta y = 0.2$

$$F_s(\xi_1,\xi_2) = 25 \sum_{k,l=-\infty}^{\infty}\sum F(\xi_1 - 5k, \xi_2 - 5l)$$

$$H(\xi_1,\xi_2) = \begin{cases} 1/25, & -2.5 \le \xi_1 \le 2.5,\ -2.5 \le \xi_2 \le 2.5 \\ 0, & \text{otherwise} \end{cases}$$

$$\tilde{F}(\xi_1,\xi_2) = \frac{25}{25}\left[\delta(\xi_1-2,\xi_2+2) + \delta(\xi_1-2,\xi_2-2) + \delta(\xi_1+2,\xi_2+2) + \delta(\xi_1+2,\xi_2-2)\right]$$

$$\tilde{f}(x,y) = 4\cos(4\pi x)\cos(4\pi y)$$

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$$\xi_{xs} = \frac{1}{\Delta x} \qquad \Delta x = \frac{1}{2\xi_{x0}}$$

$= 2\xi_{x0}$ ∴ sampling at the Nyquist rate in the x-direction

$$\xi_{ys} = \frac{1}{\Delta y}$$

$= f_l$ less than twice the bandwidth, thus

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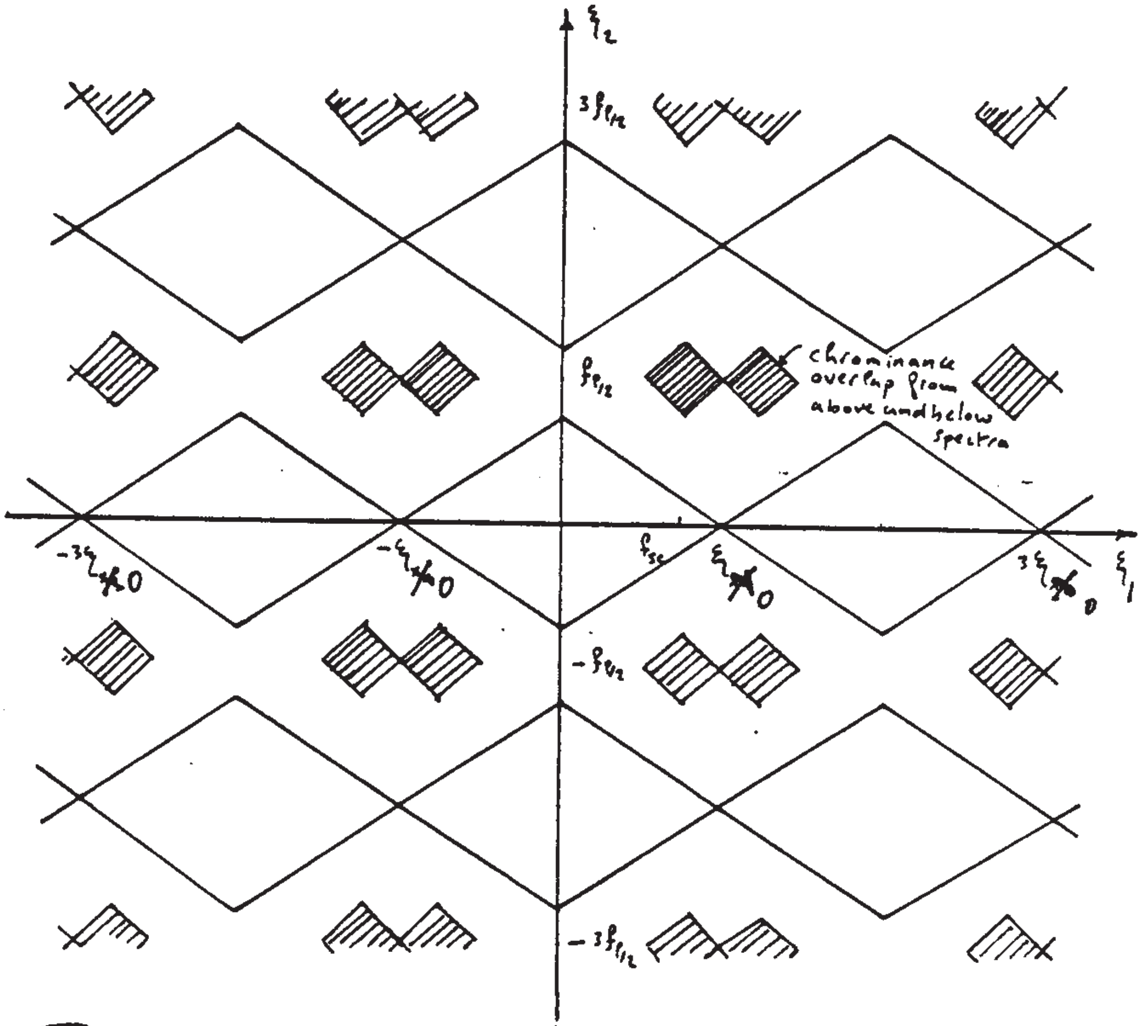
aliasing effects are expected.



4.5.

$$E\{|f(x,y)-\tilde{f}(x,y)|^2\} = E\{[f(x,y)-\tilde{f}(x,y)]f(x,y)\} - E\{[f(x,y)-\tilde{f}(x,y)]\tilde{f}(x,y)\} \quad \text{— ①}$$

$$E\{[f(x,y)-\tilde{f}(x,y)]f(x,y)\}$$
$$= E\{f(x,y)f(x,y)\} - E\{\tilde{f}(x,y)f(x,y)\}$$
$$= R(0,0) - \sum_{m,n=-\infty}^{\infty}\sum E\{f(m\Delta x, n\Delta y)f(x,y)\}\,\mathrm{sinc}(x\xi_{xs}-m)\cdot \mathrm{sinc}(y\xi_{ys}-n)$$
$$= R(0,0) - \sum_{m,n=-\infty}^{\infty}\sum R(m\Delta x - x, n\Delta y - y)\,\mathrm{sinc}(x\xi_{xs}-m)\cdot \mathrm{sinc}(y\xi_{ys}-n) \quad \text{— ②}$$

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$$E\{[f(x,y)-\hat{f}(x,y)]\hat{f}(x,y)\} =$$

$$\sum_{m,n=-\infty}^{\infty}\sum E\{[f(x,y)-\hat{f}(x,y)]f(m\Delta x, n\Delta y)\}\,\mathrm{sinc}(x\xi_{xs}-m)\cdot \mathrm{sinc}(y\xi_{ys}-n) \quad ③$$

$$E\{[f(x,y)-\hat{f}(x,y)]f(m\Delta x, n\Delta y)\} =$$

$$E\{f(x,y)f(m\Delta x,n\Delta y)\} - \sum_{k,l=-\infty}^{\infty}\sum E\{f(k\Delta x, l\Delta y)\cdot f(m\Delta x, n\Delta y)\}\,\mathrm{sinc}(x\xi_{xs}-k)\,\mathrm{sinc}(y\xi_{ys}-l)$$

$$= R(x-m\Delta x, y-n\Delta y) - \sum_{k,l=-\infty}^{\infty}\sum R(k\Delta x - m\Delta x, l\Delta y - n\Delta y)\,\sin(x\xi_{xs}-k)\,\mathrm{sinc}(y\xi_{ys}-l) \quad ④$$

$R(.,.)$ is a deterministic function and is bandlimited and can be written as

$$R(x-x_0, y-y_0) = \sum_{k,l=-\infty}^{\infty}\sum R(k\Delta x - x_0, l\Delta y - y_0)\,\mathrm{sinc}(x\xi_{xs}-k)\,\mathrm{sinc}(y\xi_{ys}-l)$$

with $x_0 = x$ and $y_0 = y$

$$R(0,0) = \sum_{k,l=-\infty}^{\infty}\sum R(k\Delta x - x, l\Delta y - y)\,\mathrm{sinc}(x\xi_{xs}-k)\,\mathrm{sinc}(y\xi_{ys}-l) \quad ⑤$$

with $x_0 = m\Delta x$ and $y_0 = n\Delta y$

$$R(x-m\Delta x, y-n\Delta y) = \sum_{k,l=-\infty}^{\infty}\sum R(k\Delta x - m\Delta x, l\Delta y - n\Delta y)\,\mathrm{sinc}(x\xi_{xs}-k)\,\mathrm{sinc}(y\xi_{ys}-l) \quad ⑥$$

Using ⑤ in ② then the right hand side of ② = 0 ⑦

using ⑥ in ④ then the right hand side of ④ = 0

∴ the right hand side of ③ = 0 ⑧

From ⑦ & ⑧ then the right hand side of ① = 0

$$\therefore E\{|f(x,y)-\hat{f}(x,y)|^2\} = 0$$

4.6

The image is band limited and sampling is done at Nyquist rate (or higher) of the image.

$\therefore$ The output signal component will be the same as the input with σ_f^2 power.

a. Noise power at the Sensor output $= \iint_{-\xi_f}^{\xi_f} (\eta/4)\, d\xi_1\, d\xi_2$

$$\iint_{-2\xi_f}^{2\xi_f} (\eta/4)\, d\xi_1\, d\xi_2 = 4\xi_f^2 \eta \qquad \boxed{(SNR)_g = \frac{\sigma_f^2}{4\eta\xi_f^2}}$$

$$= \eta \xi_f^2$$

$$(SNR)_g = \sigma_f^2/\eta\xi_f^2$$

b. without prefiltering aliasing occurs for the noise component. There will be 4 overlapping components of the spectra at each point and the noise spectral density becomes $(4 \times \eta/4)$

Output noise spectrum $= \begin{cases} \eta, & -\xi_f < \xi_1, \xi_2 < \xi_f \\ 0, & \text{otherwise} \end{cases}$

Noise power in reconstructed image

$$= \iint_{-\xi_f}^{\xi_f} \eta \, d\xi_1\, d\xi_2$$

$$= 4\eta\xi_f^2$$

$$(SNR)_g = \sigma_f^2/4\eta\xi_f^2$$

with prefiltering, no aliasing

noise power spectrum $\begin{cases} \eta/4, & -\xi_f < \xi_1, \xi_2 < \xi_f \\ 0, & \text{otherwise} \end{cases}$

noise power in reconstructed image $= \iint_{-\xi_f}^{\xi_f} (\eta/4)\, d\xi_1\, d\xi_2$

$$= \eta\xi_f^2$$

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$$(SNR)_g = \sigma_f^2 / \eta \xi_f^2$$

$\Rightarrow$ Without prefiltering but sampling at the Nyquist rate of the noise we end up with the same SNR as with prefiltering $\sigma_f^2/\eta \xi_f^2$

comparing the above schemes we realize that the last two cases (prefiltering or sampling at Nyquist rate of the noise) yield the same SNR but in practice the noise bandwidth is much larger than that of the image and will require a very high sampling rate. Then, prefiltering is the recommended way.

(4.7)

$$\int_{-\infty}^{\infty} \text{sinc}(x\xi_{xs} - m)\, \text{sinc}(x\xi_{xs} - m')\, dx = \frac{1}{\xi_{xs}} \delta(m-m') \;\forall m, m'$$

$\therefore$ the set of functions $\sin(x\xi_{xs} - m)$ are orthogonal. To minimize σ_{ls}^2 we should have $\dfrac{\partial \sigma_{ls}^2}{\partial a(m,n)} = 0$.

$$\frac{\partial \sigma_{ls}^2}{\partial a(m,n)} = -2\iint_{-\infty}^{\infty} \Big[f(x,y) - \sum_{k,l}\sum a(k,l)\, \phi_k(x)\, \psi_l(y) \Big] \phi_m(x)\, \psi_n(y)\, dx\,dy$$

$= 0$ to find condition for minimum

$$\iint_{-\infty}^{\infty} f(x,y)\, \phi_m(x)\, \psi_n(y)\, dx\,dy = \sum_{k,l}\sum \iint_{-\infty}^{\infty} a(k,l)\, \phi_k(x)\, \psi_l(y)\, \phi_m(x)\, \psi_n(y)\, dx\,dy$$

$$= \sum_{k,l}\sum a(k,l) \Big[\int_{-\infty}^{\infty} \phi_k(x)\, \phi_m(x)\, dx\Big]\Big[\int_{-\infty}^{\infty} \psi_l(y)\, \psi_n(y)\, dy\Big]$$

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$$= \frac{1}{\xi_{xs}\xi_{ys}} \sum_{k,l}\sum a(k,l)\,\delta(k-m)\,\delta(l-m)$$

$$= \frac{1}{\xi_{xs}\xi_{ys}}\, a(m,n)$$

Hence to minimize σ_{is}^2 $a(m,n)$ must be chosen such that:

$$a(m,n) = \xi_{xs}\xi_{ys} \iint_{-\infty}^{\infty} f(x,y)\,\phi_m(x)\,\phi_n(y)\,dx\,dy$$

Using the inner product property of the Fourier transform

$$a(m,n) = \xi_{xs}\xi_{ys} \iint_{-\infty}^{\infty} F(\xi_1,\xi_2)\left[\mathcal{F}\{\operatorname{sinc}(x\xi_{xs}-m)\operatorname{sinc}(y\xi_{ys}-n)\}\right]^* d\xi_1\, d\xi_2$$

$$\mathcal{F}\{\operatorname{sinc}(x\xi_{xs}-m)\operatorname{sinc}(y\xi_{ys}-n)\} = \frac{1}{\xi_{xs}\xi_{ys}} e^{-j2\pi(\xi_1 m\Delta x+\xi_2 n\Delta y)} \operatorname{rect}(\xi_1/\xi_{xs})\operatorname{rect}(\xi_2/\xi_{ys})$$

$f(x,y)$ is bandlimited, then $F(\xi_1,\xi_2) = 0$ for $|\xi_1| > \xi_{xs}$, $|\xi_2| > \xi_{ys}$

$$a(m,n) = \int_{-\xi_{xs}/2}^{\xi_{xs}/2}\int_{-\xi_{ys}/2}^{\xi_{ys}/2} F(\xi_1,\xi_2)\, e^{j2\pi(\xi_1 m\Delta x+\xi_2 n\Delta y)}\, d\xi_1\, d\xi_2$$

$$(\Delta x = 1/\xi_{xs},\ \Delta y = 1/\xi_{ys})$$

The above integral is the inverse Fourier transform of $F(\xi_1,\xi_2)$ computed at $x = m\Delta x$, $y = n\Delta y$, i.e. $f(m\Delta x, n\Delta y)$ (this would not have been true if $f(x,y)$ was not band-

(4.7)

limited)

$\therefore a(m,n) = f(m\Delta x, n\Delta y)$

with this choice and using (4.15) we can see that σ_{ls}^2 becomes zero.

(4.8)

a. $E\{a_{m,n} a_{m',n'}\} = E\{\iint_{-L}^{L} f(x,y)\phi_{m,n}(x,y)\,dx\,dy$

$$\iint_{-L}^{L} f(x',y')\phi_{m',n'}(x',y')\,dx'\,dy'\}$$

$$= \iint_{-L}^{L}\iint_{-L}^{L} E\{f(x,y)f(x',y')\}\phi_{m,n}(x,y)\phi_{m',n'}(x',y')\,dx\,dy\,dx'\,dy'$$

$$= \iint_{-L}^{L}\left[\iint_{-L}^{L} R(x,x';y,y')\phi_{m',n'}(x',y')\,dx'\,dy'\right]\phi_{m,n}(x,y)\,dx\,dy$$

$$= \lambda_{m',n'}\iint_{-L}^{L}\phi_{m',n'}(x,y)\,\phi_{m,n}(x,y)\,dx\,dy$$

$$= \lambda_{m',n'}\,\delta(m'-m, n'-n)$$

Hence $\{a_{m,n}\}$ are orthogonal random variables

b. From the completeness property of $\{\phi_{m,n}(x,y)\}$ we have $E\{|f(x,y) - \sum\sum_{m,n=0}^{\infty} a_{m,n}\phi_{m,n}(x,y)|^2\} = 0$

expanding this and using part a. we get

$$R(x,x';y,y') = \sum\sum_{m,n=0}^{\infty}\lambda_{m,n}\phi_{m,n}^2(x,y)$$

mean square error taking MN terms:

$$\sigma_{M,N}^2 \triangleq \iint_{-L}^{L} E\{|f(x,y) - \breve{f}(x,y)|^2\}\,dx\,dy$$

4.8

$$\sigma^2_{M,N} = \iint_{-L}^{L} E\left\{\left[f(x,y) - \hat{f}_{M,N}(x,y)\right] f^*(x,y)\right\} dx\,dy$$

$$f(x,y) = f^*(x,y) \qquad \text{(real)}$$

$$f(x,y) = \sum\sum_{m,n=0}^{\infty} a_{m,n}\,\phi_{m,n}(x,y)$$

$$\hat{f}_{M,N}(x,y) = \sum_{m,n=0}^{M-1}\sum^{N-1} a_{m,n}\,\phi_{m,n}(x,y)$$

$$\sigma^2_{M,N} = \iint_{-L}^{L} \Big[\sum\sum_{m,n=0}^{\infty} \sum\sum_{k,l=0}^{\infty} E\{a_{m,n}a_{k,l}\}\,\phi_{m,n}(x,y)\,\phi_{k,l}(x,y)$$

$$- \sum_{m,n=0}^{M-1}\sum^{N-1} \sum\sum_{k,l=0}^{\infty} E\{a_{m,n}a_{k,l}\}\,\phi_{m,n}(x,y)\,\phi_{k,l}(x,y)\Big] dx\,dy$$

substituting from part a.

$$\sigma^2_{M,N} = \sum\sum_{m,n=0}^{\infty} \lambda_{m,n} \iint_{-L}^{L} \phi^2_{m,n}(x,y)\,dx\,dy$$

$$- \sum_{m,n=0}^{M-1}\sum^{N-1} \lambda_{m,n} \iint_{-L}^{L} \phi^2_{m,n}(x,y)\,dx\,dy$$

From the orthogonality property of $\phi_{m,n}(x,y)$ the integral $= 1$

$$\sigma^2_{M,N} = \sum\sum_{m,n=0}^{\infty} \lambda_{m,n} - \sum_{m,n=0}^{m-1}\sum^{N-1} \lambda_{m,n}$$

To minimize, maximize the second term by choosing the largest eigenvalues

$$\sigma^2_{M,N} = \sum_{m=M}^{\infty}\sum_{n=N}^{\infty} \lambda_{m,n}$$

4.8

$$\sigma^2_{M,N} = \iint_{-L}^{L} E\left\{\left| f(x,y) - \sum_{m,n=0}^{M-1\,N-1}\sum a_{m,n}\,\phi_{m,n}(x,y)\right|^2\right\} dx\,dy$$

$$= \iint_{-L}^{L} R(x,x;y,y)\,dx\,dy - \sum_{m,n=0}^{M-1\,N-1}\sum \lambda_{m,n}$$

Since the terms in the summation are positive ($\lambda_{m,n} = E\{|a_{m,n}|^2\} \geq 0$), it is obvious that $\sigma^2_{M,N}$ is minimized when:

$\{\lambda_{m,n}\ \ 0 \leq m \leq M-1,\ 0 \leq n \leq N-1\}$ are chosen as large as possible. i.e. MN largest eigen-values.

From the expression for $R(x,x;y,y)$, we get

$$\iint_{-L}^{L} R(x,x;y,y)\,dx\,dy = \sum_{m,n=0}^{\infty}\sum \lambda_{m,n} \iint_{-L}^{L} \phi^2_{m,n}(x,y)\,dx\,dy$$

$$= \sum_{m,n=0}^{\infty}\sum \lambda_{m,n}$$

Then $\sigma^2_{M,N}$ becomes

$$\sigma^2_{M,N} = \sum_{m,n=0}^{\infty}\sum \lambda_{m,n} - \sum_{m,n=0}^{M-1\,N-1}\sum \lambda_{m,n}$$

$$= \sum_{m=M}^{\infty}\sum_{n=N}^{\infty} \lambda_{m,n}$$

4.9

$g(x,y)$ is the sum of 2 comb functions.
The Fourier transform of a comb function

(4.9)

(with spacing $\Delta x, \Delta y$) is another comb function (with spacing $\frac{1}{\Delta x}, \frac{1}{\Delta y}$)

$$G(\xi_1, \xi_2) = \left(\tfrac{1}{2}\cdot\tfrac{1}{2}\right) \sum_{k,l}\sum \delta\left(\xi_1 - \tfrac{k}{2}, \xi_2 - \tfrac{l}{2}\right)$$

$$+ e^{-j2\pi(k/2 + l/2)} \left(\tfrac{1}{2}\cdot\tfrac{1}{2}\right) \sum_{k,l}\sum \delta\left(\xi_1 - \tfrac{k}{2}, \xi_2 - \tfrac{l}{2}\right)$$

$$= \tfrac{1}{4}\Big[1 + \underbrace{e^{-j\pi(k+l)}}_{\substack{= -1 \quad k+l = \text{odd} \\ = 1 \quad k+l = \text{even}}}\Big] \sum_{k,l}\sum \delta\left(\xi_1 - \tfrac{k}{2}, \xi_2 - \tfrac{l}{2}\right)$$

$$= \tfrac{1}{2} \sum_{k,l = \text{even}}\sum \delta(\xi_1 - k\xi_0, \xi_2 - l\xi_0), \quad \xi_0 \triangleq \tfrac{1}{2}$$

(4.10)

a. $$L_k^q(x) \triangleq \prod_{\substack{m=k_0 \\ m \neq k}}^{k_1} \left(\frac{x-m}{k-m}\right) \quad k_0 \le k \le k_1$$

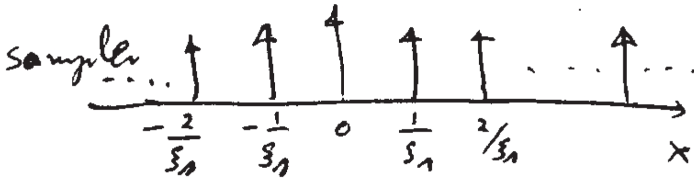
put $n = k - m$ and take Lim as $q \to \infty$

$\therefore k_0 \to -\infty, k_1 \to \infty$

$$\lim_{q\to\infty} L_k^q(x) = \prod_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left(\frac{x-k+n}{n}\right)$$

$$= \prod_{n=1}^{\infty} \left(\frac{x-k+n}{n}\right)\left(\frac{x-k-n}{-n}\right)$$

$$= \prod_{n=1}^{\infty} \left[1 - \frac{(x-k)^2}{n^2}\right]$$

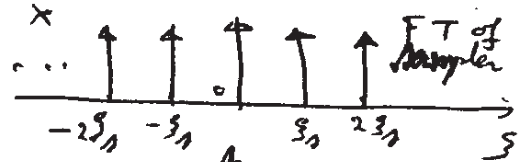


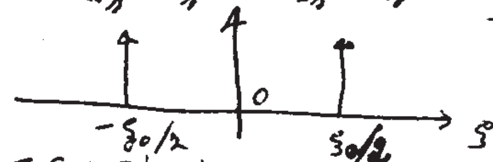
$$\sum_{n=-\infty}^{\infty} \delta(x - n/\xi_1)$$

4.11

a. $f(x) = 2\cos(\pi \xi_0 x)$

$$F(\xi) = \delta(\xi - \xi_0/2) + \delta(\xi + \xi_0/2)$$





The Fourier transform of the sampled signal $F_s(\xi)$ will be given by

$$F_s(\xi) = \xi_s \sum_{k=-\infty}^{\infty} F(\xi - k\xi_s)$$, periodic impulse train

The reconstructed image, $\hat{F}(\xi) = H(\xi) F_s(\xi)$ where $H(\xi)$ is the freq. response of the reconstruction filter

$$\hat{F}(\xi) = \xi_s \Big[H(\xi_0/2)\big(\delta(\xi - \xi_0/2) + \delta(\xi + \xi_0/2)\big) + H(\xi_s - \xi_0/2)\big(\delta(\xi - \xi_s + \xi_0/2) + \delta(\xi + \xi_s - \xi_0/2)\big)\Big]$$

$$= a\left[\delta(\xi - \xi_0/2) + \delta(\xi + \xi_0/2)\right] + b\left[\delta(\xi - \xi_s + \xi_0/2) + \delta(\xi + \xi_s - \xi_0/2)\right]$$

where $a \triangleq \xi_s H(\xi_0/2) = \xi_s H(-\xi_0/2)$

$b \triangleq \xi_s H(\xi_s - \xi_0/2) = \xi_s H(-\xi_s + \xi_0/2)$

$$\hat{f}(x) = 2a\cos(\pi \xi_0 x) + 2b\cos(2\pi(\xi_s - \xi_0/2)x)$$

$$= 2a\cos(\pi\xi_0 x) + 2b(\cos 2\pi \xi_s x \cos \pi \xi_0 x + \sin 2\pi \xi_s x \sin \pi \xi_0 x)$$

4.11

$$= 2\,(a + b\cos 2\pi \xi_s x)\cos(\pi \xi_0 x) + 2b \sin 2\pi \xi_s x \sin \pi \xi_0 x$$

b. if $\xi_0 \ll \xi_s$ then $b \simeq H(\xi_s)\,\xi_s$
$= 0$

$\hat{f}(x) \simeq 2a \cos \pi \xi_0 x$

NO Moire effect.

if $\xi_0 = 0$, $b = H(\xi_s)\,\xi_s$
$= 0$

$\hat{f}(x) = 2a$
$= \text{constant}$

4.12

$$f(x,y) = 4(\cos 4\pi x)(\cos 4\pi y)$$

$$F(\xi_1, \xi_2) = [\delta(\xi_1 - 2) + \delta(\xi_1 + 2)][\delta(\xi_2 - 2) + \delta(\xi_2 + 2)]$$

$$F_s(\xi_1, \xi_2) = 25 \sum_{k,l=-\infty}^{\infty}\sum F(\xi_1 - 5k, \xi_2 - 5l)$$

Reconstruction filter

$$H(\xi_1, \xi_2) = \begin{cases} \mathcal{F}\{\text{rect}(\frac{x}{0.2}, \frac{y}{0.2})\}, & -5 \le \xi_1, \xi_2 \le 5 \\ 0 & \text{, otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{25}\,\text{sinc}(0.2\xi_1)\,\text{sinc}(0.2\xi_2), & -5 \le \xi_1, \xi_2 \le 5 \\ 0 & \text{, otherwise} \end{cases}$$

Reconstructed image spectrum

4.12

$$\hat{F}(\xi_1,\xi_2) = F_s(\xi_1,\xi_2)\, H(\xi_1,\xi_2)$$

since both $F(\xi_1,\xi_2)$ and $H(\xi_1,\xi_2)$ are separable, then $\hat{F}(\xi_1,\xi_2)$ will also be separable

$$\hat{F}(\xi_1,\xi_2) = G(\xi_1)G(\xi_2)$$

where $G(\xi) = \text{sinc}(0.2\times 2)[\delta(\xi-2)+\delta(\xi+2)] + \text{sinc}(0.2\times 3)[\delta(\xi-3)+\delta(\xi+3)]$

$$\hat{f}(x,y) = 4\,\text{sinc}^2(0.4)\cos 4\pi x \cos 4\pi y + 4\,\text{sinc}^2(0.6)\cos 6\pi x \cos 6\pi y + 4\,\text{sinc}(0.4)\,\text{sinc}(0.6)[\cos 4\pi x \cdot \cos 6\pi y + \cos 6\pi x \cos 4\pi y]$$

$(\text{sinc}(0.4) = 0.757 \quad \text{sinc}(0.6) = 0.505)$

If the input image is constant gray

$$f(x,y) = c$$

$$F(\xi_1,\xi_2) = c\,\delta(\xi_1,\xi_2)$$

$$\hat{F}(\xi_1,\xi_2) = \text{sinc}^2(0.0)\,c\,\delta(\xi_1,\xi_2)$$

$\hat{f}(x,y) = c$, flat field

4.13

$$\hat{r}_k = \frac{\int_{\hat{t}_k}^{\hat{t}_{k+1}} v P_v(v)\,dv}{\int_{\hat{t}_k}^{\hat{t}_{k+1}} P_v(v)\,dv}$$

4.13

$$v = \sigma u + \mu \qquad p_v(v) = p_u(u)\left|\frac{du}{dv}\right|$$

$$dv = \sigma\, du$$

$$P_v(v) = \frac{1}{\sigma} P_u(u)$$

performing the substitution will also change the integral limits such that the new limits,

$$t_k = (\hat{t}_k - \mu)/\sigma$$

$$\hat{r}_k = \frac{\int_{t_k}^{t_{k+1}} (\sigma u + \mu) \cdot \frac{1}{\sigma} P_u(u)\, \sigma\, du}{\int_{t_k}^{t_{k+1}} \frac{1}{\sigma} P_u(u)\, \sigma\, du}$$

$$= \sigma \frac{\int_{t_k}^{t_{k+1}} u\, P_u(u)\, du}{\int_{t_k}^{t_{k+1}} P_u(u)\, du} + \mu$$

$$= \sigma r_k + \mu$$

Also note that $\hat{t}_k = \left(\frac{\hat{r}_k + \hat{r}_{k+1}}{2}\right)$

$$= \frac{\sigma r_k + \mu + \sigma r_{k+1} + \mu}{2}$$

$$= \sigma\left(\frac{r_k + r_{k+1}}{2}\right) + \mu$$

$$= \sigma t_k + \mu$$ (same result obtained from the limit change)

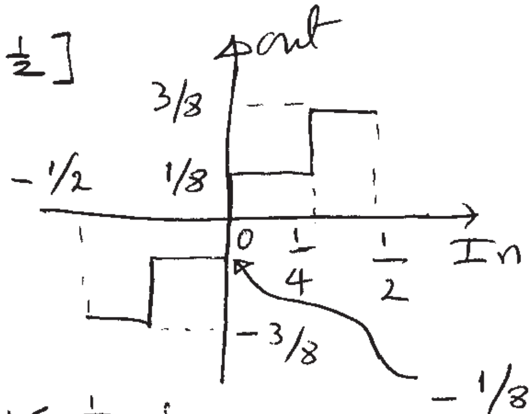
4.14

a. $F(u) = \int_0^u (1-x)\,dx \quad u>0$

$= u - \frac{u^2}{2}$

$$w = F(u) = \begin{cases} u - \frac{u^2}{2}, & 0 \le u \le 1 \\ u + \frac{u^2}{2}, & -1 \le u \le 0 \end{cases}$$

w uniformly distributed $[-\frac{1}{2}, \frac{1}{2}]$



$L = 4 \quad q = \frac{1}{4}$

$t_w = -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \cancel{\frac{1}{8}} \ \frac{1}{2}$

$r_w = -\frac{3}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8}$

$$u' = F^{-1}(w) = \begin{cases} 1 - \sqrt{1-2w}, & 0 \le w \le \frac{1}{2} \\ -1 + \sqrt{1+2w}, & -\frac{1}{2} \le w \le 0 \end{cases}$$

which gives

$t_u = -1, -0.293, 0, 0.293, 1$

$r_u = -0.5, -0.134, 0.134, 0.5$

Mean square error $= E[(u-u')^2]$

$$= \sum_{i=1}^{4} \int_{t_i}^{t_{i+1}} (x - r_i)^2 P_u(x)\,dx$$

$$= \int_{-1}^{-0.293} (x+0.5)^2 (1+x)\,dx$$

$$+ \int_{-0.293}^{0} (x+0.134)^2 (1+x)\,dx + \int_0^{0.293} (x-0.134)^2 (1-x)\,dx$$

$$+ \int_{0.293}^{1} (x-0.5)^2 (1-x)\,dx$$

$$= 0.0178$$

4.14

b. $$f(x) = a \frac{\int_0^x (1-y)^{1/3} dy}{\int_0^1 (1-y)^{1/3} dy}$$

$$= \begin{cases} a\left[1-(1-x)^{4/3}\right] & x \geq 0 \\ -a\left[1-(1+x)^{4/3}\right] & x \leq 0 \end{cases} \quad (a:\text{arbitrary})$$

$w = f(u)$ is a random variable distributed over $[-a, a]$ and is quantized uniformly.

$t_w = -a, \ -\frac{a}{2}, \ 0, \ \frac{a}{2}, \ a$

$r_w = -\frac{3a}{4}, \ -\frac{a}{4}, \ \frac{a}{4}, \ \frac{3a}{4}$

$$u' = f^{-1}(w) = \begin{cases} 1-\left(1-\frac{w}{a}\right)^{3/4} & , 0 \leq w \leq a \\ -1+\left(1+\frac{w}{a}\right)^{3/4} & -a \leq w \leq 0 \end{cases}$$

pick $a = \frac{1}{2}$ so that we have the same distribution range of $[-\frac{1}{2}, \frac{1}{2}]$ as part a., then

$t_u = -1, \ -0.405, \ 0, \ 0.405, \ 1$

$r_u = -0.646, \ -0.194, \ 0.194, \ 0.646$

Mean square error $= E\{(u-u')^2]$

$$= \sum_{i=1}^{4} \int_{t_i}^{t_{i+1}} (x-r_i)^2 P_u(x)\, dx$$

$= 0.0163 <$ Mean square error in part a.

$\therefore$ compander in part a is suboptimal compared to this one.

4.15

a. $P_u(u) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{\left(\frac{-u^2}{2\sigma^2}\right)}$ Gaussian Distribution

$P_u(u) = P_u(-u)$

using (4.53) with $a=1$

$$f(x) = \frac{\int_0^x e^{-u^2/6\sigma^2}\,du}{\int_0^\infty e^{-u^2/6\sigma^2}\,du}$$

$$= y = \boxed{\frac{2}{\sigma\sqrt{6\pi}} \int_0^x e^{-u^2/6\sigma^2}\,du \quad , \; x \geq 0}$$

$$= 2\,\mathrm{erf}\,(x/\sigma\sqrt{6})$$

(4.53) $$f(x) = a\,\frac{\int_0^x [P_u(u)]^{1/3}\,du}{\int_0^{t_{L+1}} [P_u(u)]^{1/3}\,du} \quad (x \geq 0)$$

(error function) where $\mathrm{erf}(x) \triangleq \frac{1}{\sqrt{\pi}} \int_0^x e^{-y^2}\,dy$

$$g(y) = f^{-1}(y) = x$$

$$= \sigma\sqrt{6}\;\mathrm{erf}^{-1}(y/2)\;, \quad y \geq 0$$

4.15

b. $$P_u(u) = \begin{cases} \frac{u}{\sigma^2} e^{-u^2/2\sigma^2}, & u \geq 0 \\ 0 & u < 0 \end{cases}$$ Rayleigh distribution

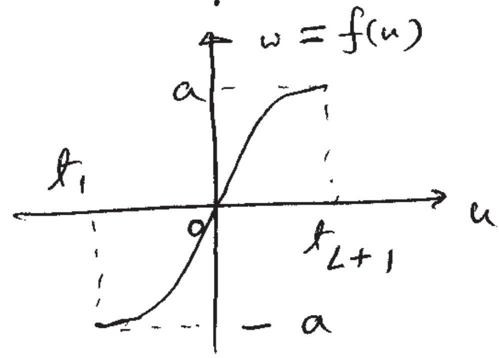
using (4.52) with $a=1$

$$f(x) = 2\,\frac{\int_0^x u^{1/3} e^{-u^2/6\sigma^2}\,du}{\int_0^\infty u^{1/3} e^{-u^2/6\sigma^2}\,du} - 1$$

$$f(x) = 2a\,\frac{\int_{t_1}^x [P_u(u)\,du]^{1/3}\,du}{\int_{t_1}^{t_{L+1}} [P_u(u)\,du]^{1/3}\,du} - a$$

$$= y = c \int_0^x u^{1/3} e^{-u^2/6\sigma^2}\,du - 1$$

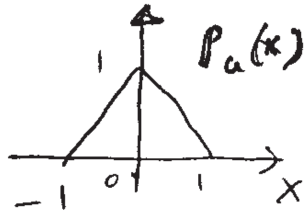
where $c = 2 \Big/ \int_0^\infty u^{1/3} e^{-u^2/6\sigma^2}\,du$



Compressor

4.16

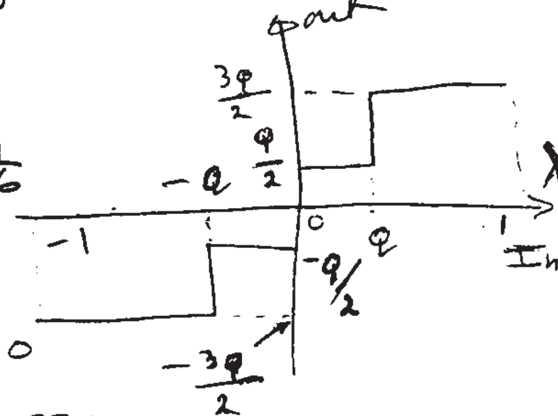
$$p_u(x) = \begin{cases} 1-|x| & , -1 \le x \le 1 \\ 0 & , \text{otherwise} \end{cases}$$



a. Optimum uniform quantizer. $L=4$

Mean square error, $\mathcal{E} = 2\int_0^q (x-q/2)^2(1-x)\,dx + 2\int_q^1 (x-3/2\,q)^2(1-x)\,dx$

$$= \frac{2}{3}q^4 - 2q^3 + \frac{9}{4}q^2 - q + \frac{1}{6}$$



To minimize $\mathcal{E}$ set $\frac{d\mathcal{E}}{dq} = 0$

$$\therefore \frac{8}{3}q^3 - 6q^2 + \frac{9}{2}q - 1 = 0$$

which gives $q = 0.3894$

(other 2 roots are complex)

$\mathcal{E} = 0.0157$ at the above value of q

Decision and reconstruction levels are

t: $-1, -q, 0, q, 1$

r: $-\frac{3}{2}q, -\frac{q}{2}, \frac{q}{2}, \frac{3}{2}q$

output probabilities $P_k = \int_{t_k}^{t_{k+1}} p_u(x)\,dx$

$P_1 = P_4 = \int_{0.39}^{1} (1-x)\,dx = 0.1864$

$P_2 = P_3 = \int_0^{0.39} (1-x)\,dx = 0.3136$

Entropy $= \sum_{i=1}^{4} P_i \log_2 P_i$

$= 1.95$ bits/sample

4.16

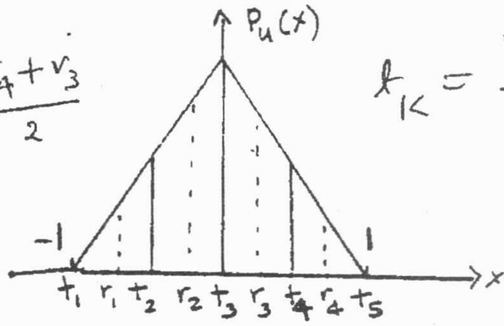
b. Lloyd - Max quantizer, $L=4$

Decision levels t_1, t_3 & t_5 are known immediately to be $-1, 0, 1$ respectively.

To find t_2 and t_4 we consider the following

⇒ decision level lies in the middle of immediate reconstruction levels.

⇒ reconstruction level is the center of mass of the density between the immediate transition levels



$$t_K = \frac{r_K + r_{K-1}}{2}$$

i.e. $t_4 - r_3 = r_4 - t_4$, $t_4 = \dfrac{r_4 + r_3}{2}$

$$r_3 = \left(\frac{3 - 2t_4}{6 - 3t_4}\right) t_4$$

$$r_4 = \frac{1}{3}(1 + 2t_4)$$

$\therefore t_2 = -0.382$, $t_4 = 0.382$

$$r_K = \frac{\int_{t_K}^{t_{K+1}} u \, p_u(u)\, du}{\int_{t_K}^{t_{K+1}} p_u(u)\, du}$$

$r_1 = -0.588$ $r_2 = -0.176$ $r_3 = 0.176$ $r_4 = 0.588$

Mean square error, $\mathcal{E} = 0.0157$

probabilities $P_1 = P_4 = 0.191$

$P_2 = P_3 = 0.304$

Entropy = 1.96 bits/sample

$$r_3 = \frac{\int_0^{t_4} u \, p_u(u)\, du}{\int_0^{t_4} p_u(u)\, du}$$

$$r_4 = \frac{\int_{t_4}^{1} u \, p_u(u)\, du}{\int_{t_4}^{1} p_u(u)\, du}$$

c. $\mathcal{E}$/Entropy for optimum uniform quantizer = 0.805%

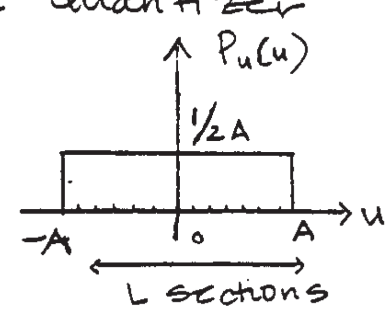
$\mathcal{E}$/Entropy for Lloyd-Max quantizer = 0.786 %

Performances are close because the number of quantizer levels is small.

4.17

For the optimum Mean Square Quantizer

$$\mathcal{E} = \frac{1}{12L^2}\left[\int_{-A}^{A}\left[p_u(u)\right]^{1/3}du\right]^3$$



$$= \frac{1}{12L^2}\cdot\frac{1}{2A}\left[\left[u\right]_{-A}^{A}\right]^3$$

$$= \frac{A^2}{3L^2}$$

Variance of u: $\sigma^2 = \frac{1}{3}A^2$

$\therefore\ A^2 = 3\sigma^2$

$L = 2^{n_s}$

$$\therefore\ \mathcal{E} = \frac{3\sigma^2}{3(2^{n_s})^2} = \sigma^2 2^{-2n_s} \qquad n_s \geq 0$$

which is equal to the Shannon lower bound for Gaussian densities (4.60)

4.18

a. $$H(q) = -\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)e^{\left(-x^2/2\sigma^2\right)}\log_2\left[\frac{1}{\sqrt{2\pi}\,\sigma}e^{-x^2/2\sigma^2}\right]dx$$

$$= -\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\,\sigma}e^{-x^2/2\sigma^2}\left[\log_2\frac{1}{\sqrt{2\pi}\,\sigma} - \frac{x^2}{2\sigma^2}\log_2 e\right]dx$$

$$= \log_2(\sqrt{2\pi}\,\sigma)\underbrace{\int_{-\infty}^{\infty}\frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi}\,\sigma}dx}_{=1} + \frac{\log e}{2\sigma^2}\underbrace{\int_{-\infty}^{\infty}\frac{x^2}{\sqrt{2\pi}\,\sigma}e^{-x^2/2\sigma^2}dx}_{=\sigma^2}$$

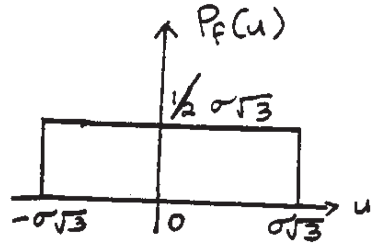
$$= \frac{1}{2}\log_2(2\pi\sigma^2) + \frac{1}{2}\log_2 e$$

$$= \frac{1}{2}\log_2(2\pi e\sigma^2)$$

(4.18)

For a uniform random variable with variance σ^2:

$$H(f) = -\int_{-\sigma\sqrt{3}}^{\sigma\sqrt{3}} \frac{1}{2\sigma\sqrt{3}} \log_2\left(\frac{1}{2\sigma\sqrt{3}}\right) dx$$



$$= \frac{\log_2(2\sigma\sqrt{3})}{2\sigma\sqrt{3}} \quad x\Big]_{-\sigma\sqrt{3}}^{\sigma\sqrt{3}}$$

$$= \frac{1}{2}\log_2(12\sigma^2)$$

b. we want to find a probability density function P(x) which maximizes

$$H = -\int_{-\infty}^{\infty} p(x)\log_2 p(x)\,dx$$

subject to the constraints

$$\int_{-\infty}^{\infty} p(x)\,dx = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} x^2 p(x)\,dx = \sigma^2$$

Using Lagrange multipliers technique, the necessary condition for maximizing H is

$$\frac{-\partial}{\partial p}(p\log_2 p) + \frac{\partial}{\partial p}(\lambda p) + \frac{\partial}{\partial p}(\mu x^2 p) = 0$$

$$-(\log_2 p + \log_2 e) + \lambda + \mu x^2 = 0$$

which gives $p(x) = e^{\lambda \ln 2 - 1} e^{\mu x^2 \ln 2}$

substituting the above in constraints yields

$$e^{\lambda \ln 2 - 1} = \sqrt{\left(\frac{-\mu \ln}{\pi}\right)}$$

$$\mu = \frac{-1}{2\sigma^2 \ln 2}$$

$$e^{\lambda \ln 2 - 1} = \frac{1}{\sqrt{2\pi}\,\sigma}$$

4.18

Hence $P(x) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right) e^{[-x^2/2\sigma^2]}$

which is a Gaussian density with variance σ^2. If x is any other random variable with variance σ^2 but with different distribution then

$$H(x) \leq H(g)$$

$$\therefore Q_x \leq Q_g \Rightarrow \alpha_x \leq 1$$

c. $$n_{min}(x) = \tfrac{1}{2}\log_2 (Q_x/D)$$

$$n_f = \tfrac{1}{2}\log_2 (\sigma^2/D)$$

$$Q_f = 6\sigma^2/\pi e$$

$$n_{min}(f) = \tfrac{1}{2}\log_2 (6\sigma^2/\pi e D)$$

$$= \tfrac{1}{2}\log_2 (\sigma^2/D) + \tfrac{1}{2}\log_2 (6/\pi e)$$

$$\simeq n_f - 0.25$$

$$\therefore n_f \simeq n_{min}(f) + \tfrac{1}{4}$$