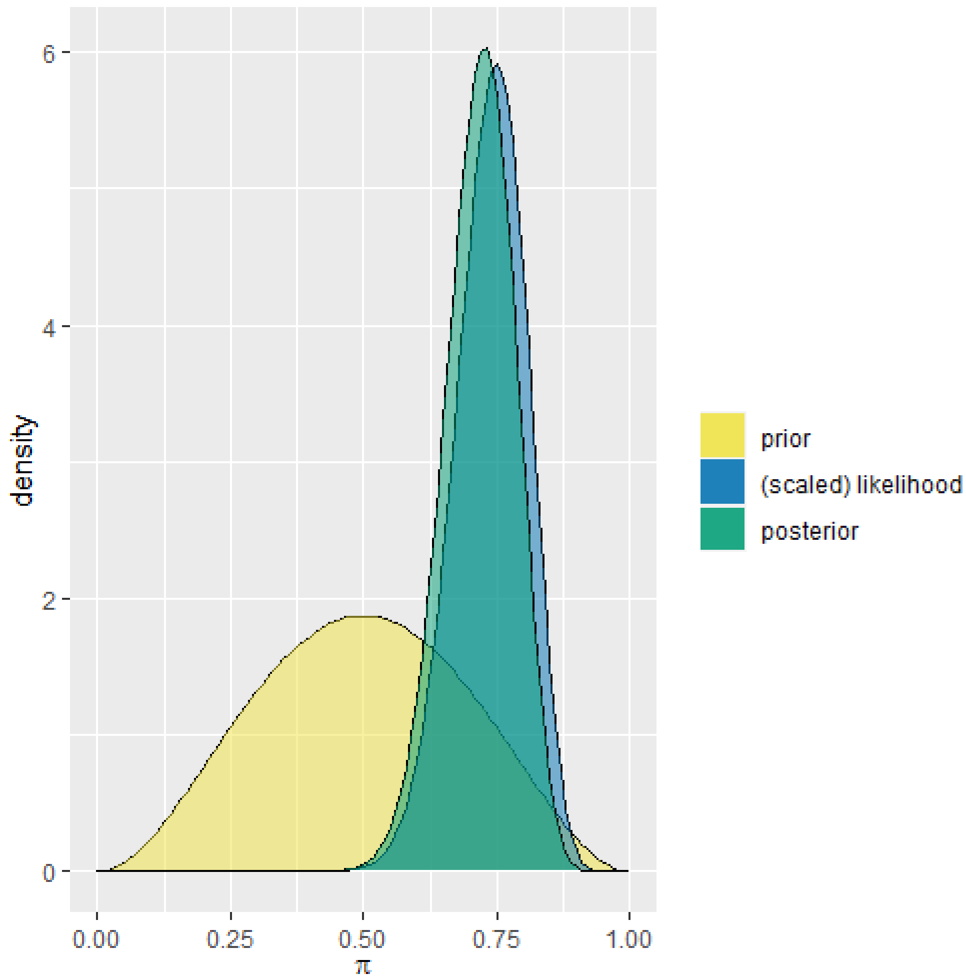
Complete the following tasks. Show your work where applicable. Some answers have been provided. Assemble your work into one PDF document and upload the PDF back into our CatCourses page.

1. (*Bayes Rules!* exercise 3.18)

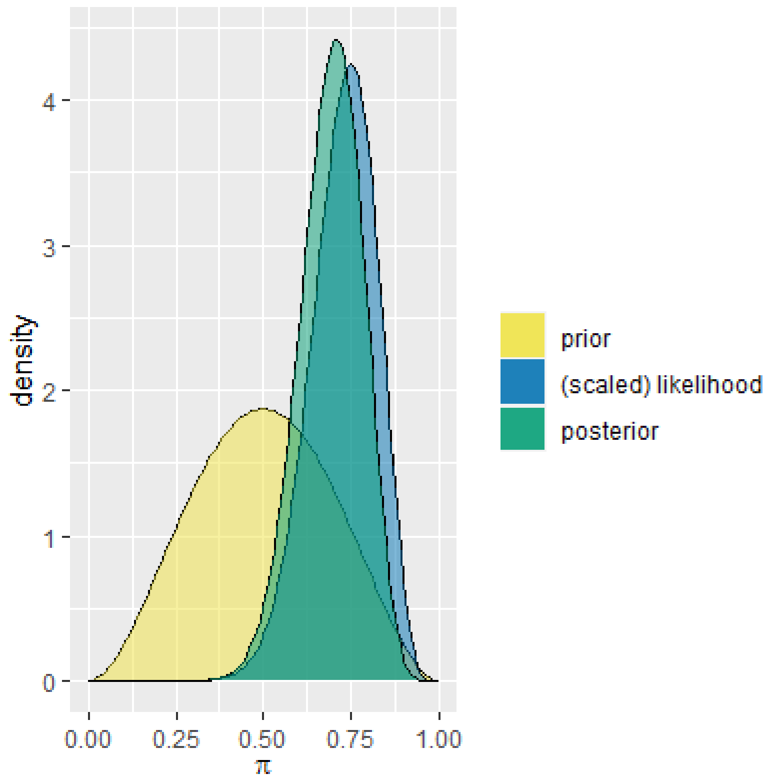
   (a) Patrick has a Beta(3,3) prior for π, the probability that someone in their town attended a protest in June 2020. In their survey of 40 residents, 30 attended a protest. Summarize Patrick's analysis using `summarize_beta_binomial()` and `plot_beta_binomial()`.

```
> library("bayesrules")
> library("tidyverse")
> summarize_beta_binomial(3,3,30,40) |>
    mutate_if(is.numeric, round, digits = 4)
      model alpha beta   mean   mode    var     sd
1     prior     3    3 0.5000 0.5000 0.0357 0.1890
2 posterior    33   13 0.7174 0.7273 0.0043 0.0657
> plot_beta_binomial(3,3,30,40)
```



(b) Harold has the same prior as Patrick, but lives in a different town. In their survey, 15 out of 20 people attended a protest. Summarize Harold's analysis using `summarize_beta_binomial()` and `plot_beta_binomial()`

```
> summarize_beta_binomial(3,3,15,20) |>
    mutate_if(is.numeric, round, digits = 4)
      model alpha beta   mean   mode    var     sd
1     prior     3    3 0.5000 0.5000 0.0357 0.1890
2 posterior    18    8 0.6923 0.7083 0.0079 0.0888
> plot_beta_binomial(3,3,15,20)
```



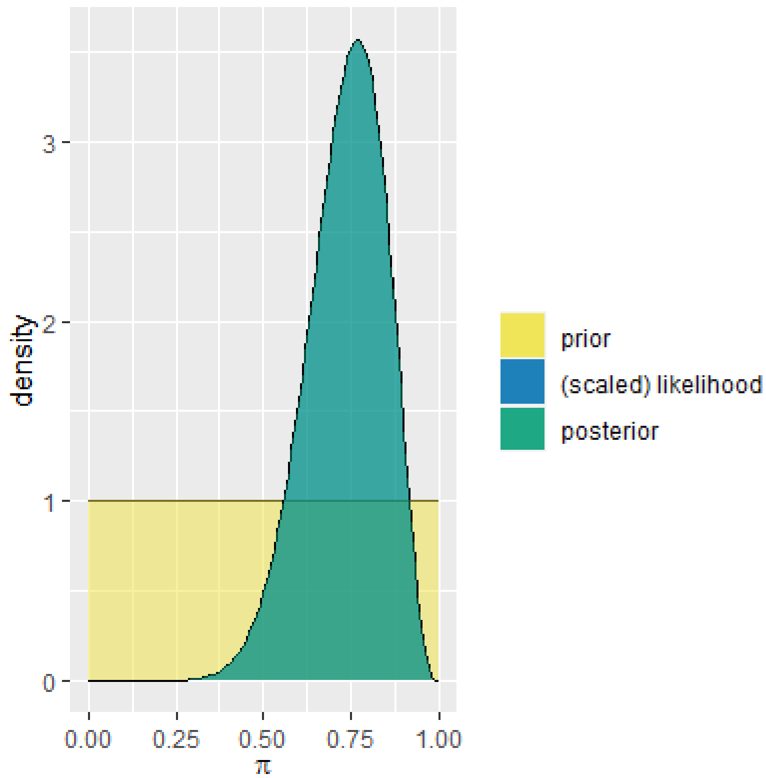
(c) How do Patrick and Harold's posterior models compare? Briefly explain what causes these similarities and differences.

While both Patrick and Harold observed a 75 percent participation rate in the protests, since Patrick's data included more people, the posterior distribution there favored the new data more compared to the prior distribution.

2. (*Bayes Rules!* exercise 4.11) **(Different data, uninformative prior)** In each situation below we have the same prior on the probability of a success, $\pi \sim \text{Beta}(1, 1)$, but different data. Identify the corresponding posterior model and utilize `plot_beta_binomial()` to sketch the prior pdf, likelihood function, and posterior pdf.

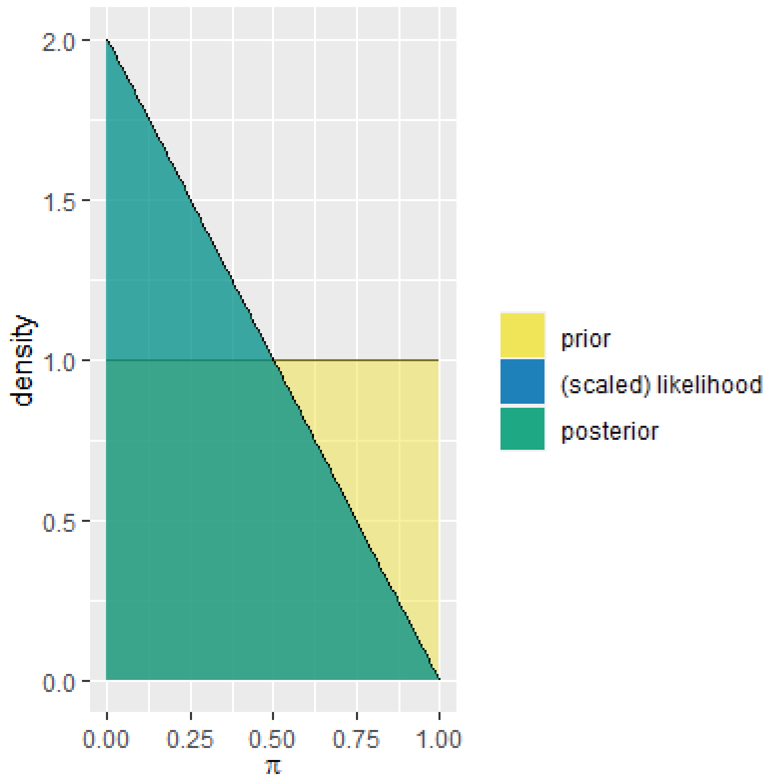
(a) $Y = 10$ in $n = 13$ trials

```
> summarize_beta_binomial(1,1,10,13) |> mutate_if(is.numeric, round, digits =
      model alpha beta   mean   mode    var     sd
1     prior     1    1 0.5000    NaN 0.0833 0.2887
2 posterior    11    4 0.7333 0.7692 0.0122 0.1106
> plot_beta_binomial(1,1,10,13)
```



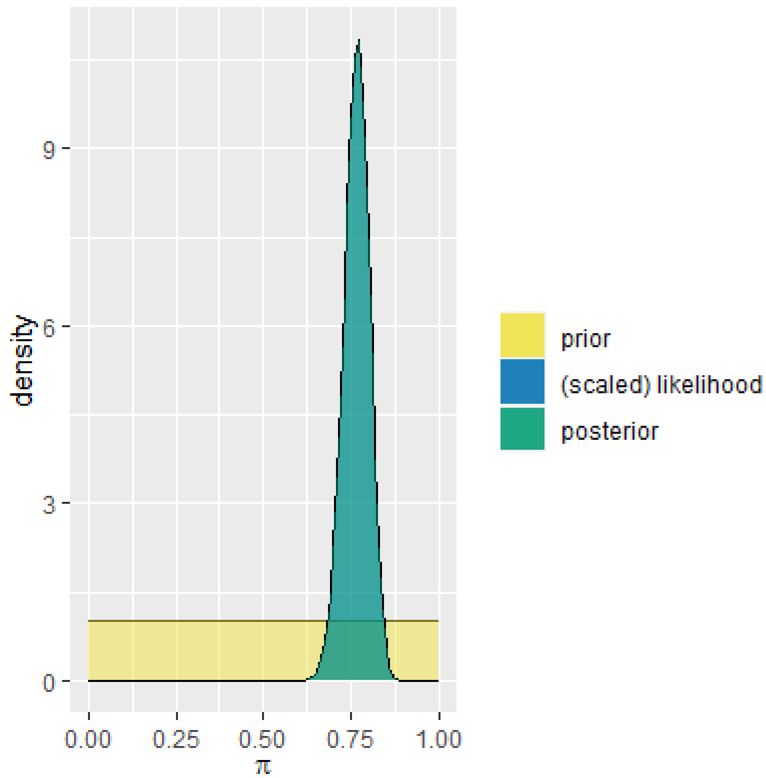
(b) $Y = 0$ in $n = 1$ trials

```
> summarize_beta_binomial(1,1,0,1) |> mutate_if(is.numeric, round, digits = 4
      model alpha beta   mean mode    var     sd
1     prior     1    1 0.5000  NaN 0.0833 0.2887
2 posterior     1    2 0.3333    0 0.0556 0.2357
> plot_beta_binomial(1,1,0,1)
```



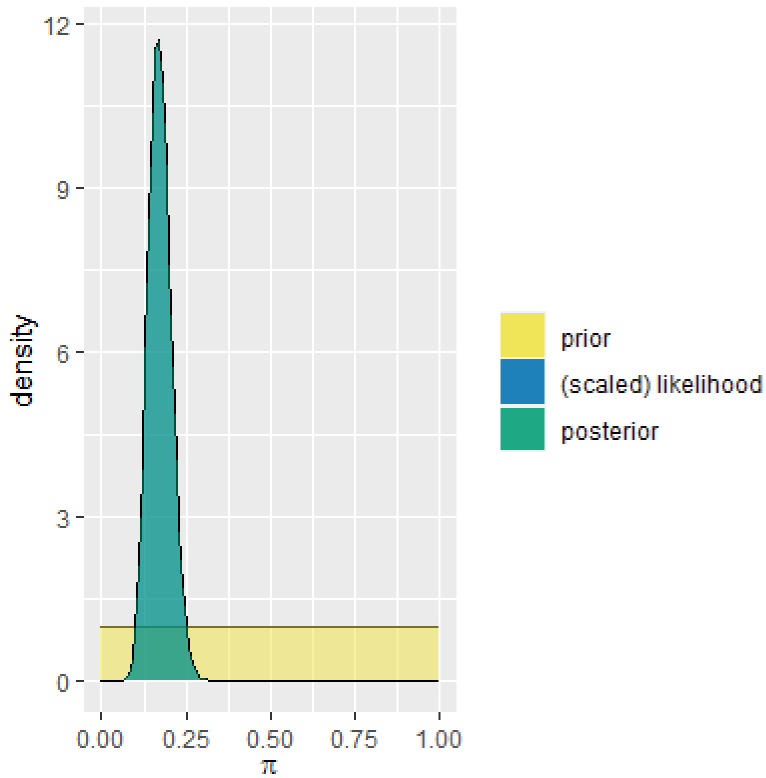
(c) $Y = 100$ in $n = 130$ trials

```
> summarize_beta_binomial(1,1,100,130) |> mutate_if(is.numeric, round, digits
      model alpha beta   mean   mode    var     sd
1     prior     1    1 0.5000    NaN 0.0833 0.2887
2 posterior   101   31 0.7652 0.7692 0.0014 0.0368
> plot_beta_binomial(1,1,100,130)
```



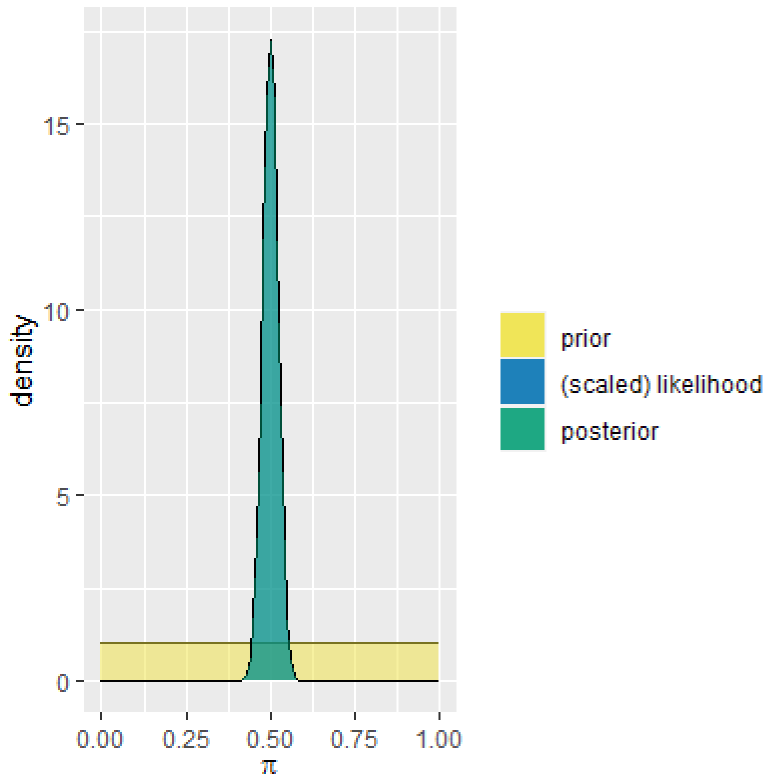
(d) $Y = 20$ in $n = 120$ trials

```
> summarize_beta_binomial(1,1,20,120) |> mutate_if(is.numeric, round, digits
      model alpha beta   mean   mode    var     sd
1     prior     1    1 0.5000    NaN 0.0833 0.2887
2 posterior    21  101 0.1721 0.1667 0.0012 0.0340
> plot_beta_binomial(1,1,20,120)
```



(e) $Y = 234$ in $n = 468$ trials

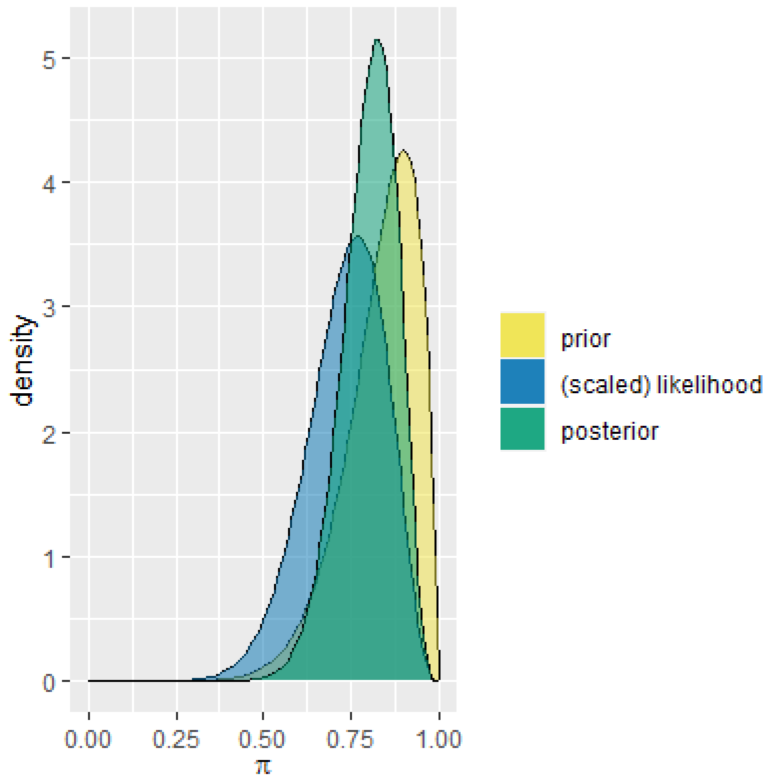
```
> summarize_beta_binomial(1,1,234,468) |> mutate_if(is.numeric, round, digits
      model alpha beta mean mode    var     sd
1     prior     1    1  0.5  NaN 0.0833 0.2887
2 posterior   235  235  0.5  0.5 0.0005 0.0230
> plot_beta_binomial(1,1,234,468)
```



3. (*Bayes Rules!* exercise 4.12) **(Different data, informative prior)** Repeat the previous exercise, but with a $\pi \sim \text{Beta}(10, 2)$ prior.

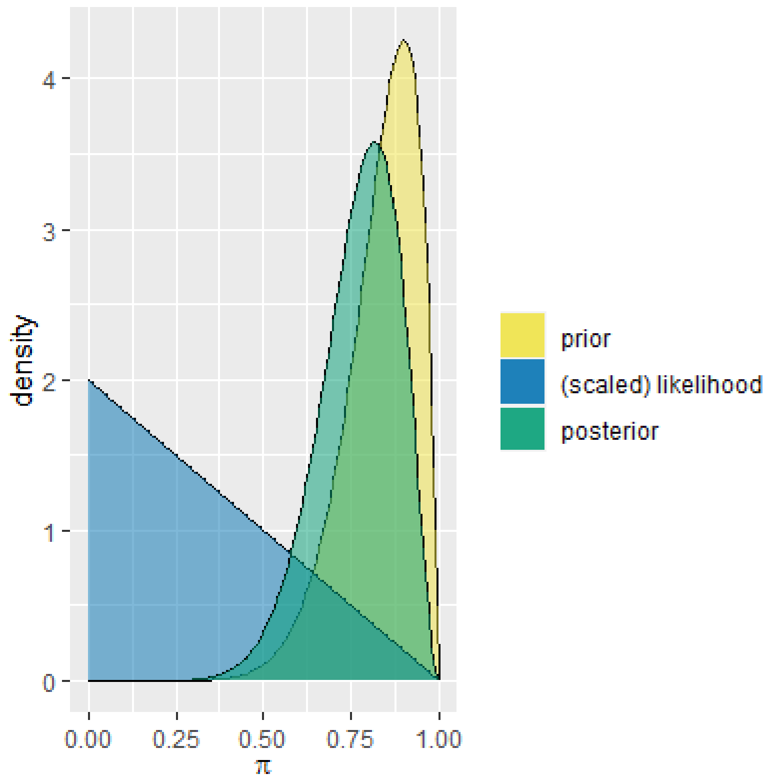
   (a) $Y = 10$ in $n = 13$ trials

```
> summarize_beta_binomial(10,2,10,13) |> mutate_if(is.numeric, round, digits
      model alpha beta   mean   mode    var     sd
1     prior    10    2 0.8333 0.9000 0.0107 0.1034
2 posterior    20    5 0.8000 0.8261 0.0062 0.0784
> plot_beta_binomial(10,2,10,13)
```



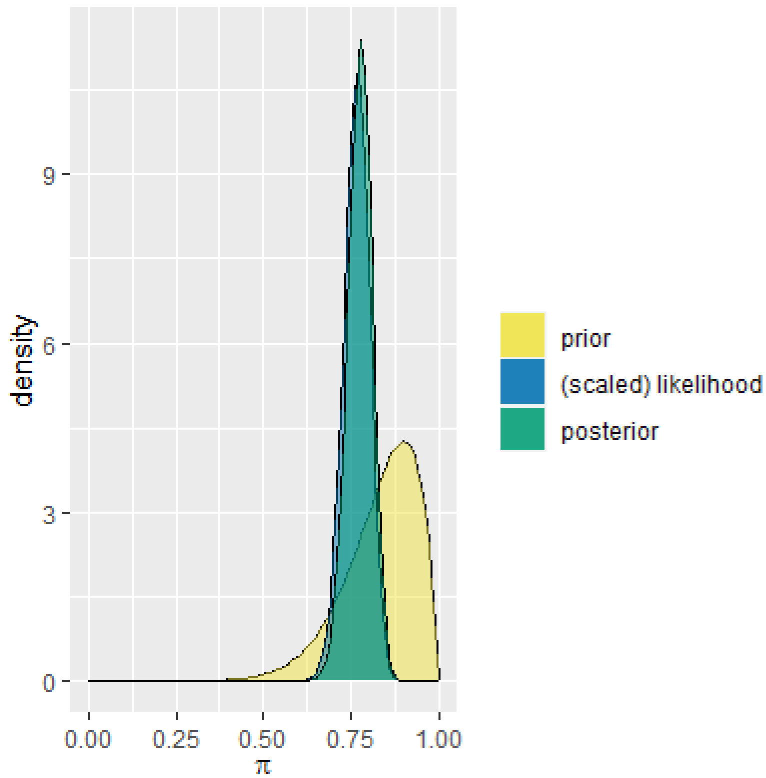
(b) $Y = 0$ in $n = 1$ trials

```
> summarize_beta_binomial(10,2,0,1) |> mutate_if(is.numeric, round, digits =
      model alpha beta   mean   mode    var     sd
1     prior    10    2 0.8333 0.9000 0.0107 0.1034
2 posterior    10    3 0.7692 0.8182 0.0127 0.1126
> plot_beta_binomial(10,2,0,1)
```



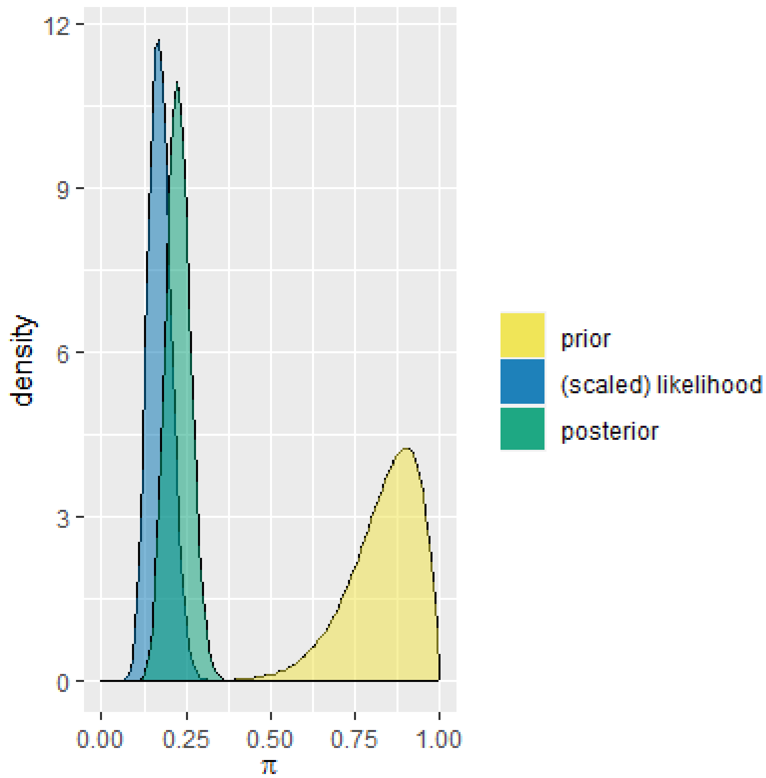
(c) $Y = 100$ in $n = 130$ trials

```
> summarize_beta_binomial(10,2,100,130) |> mutate_if(is.numeric, round, digit
      model alpha beta   mean   mode    var     sd
1     prior    10    2 0.8333 0.9000 0.0107 0.1034
2 posterior   110   32 0.7746 0.7786 0.0012 0.0349
> plot_beta_binomial(10,2,100,130)
```



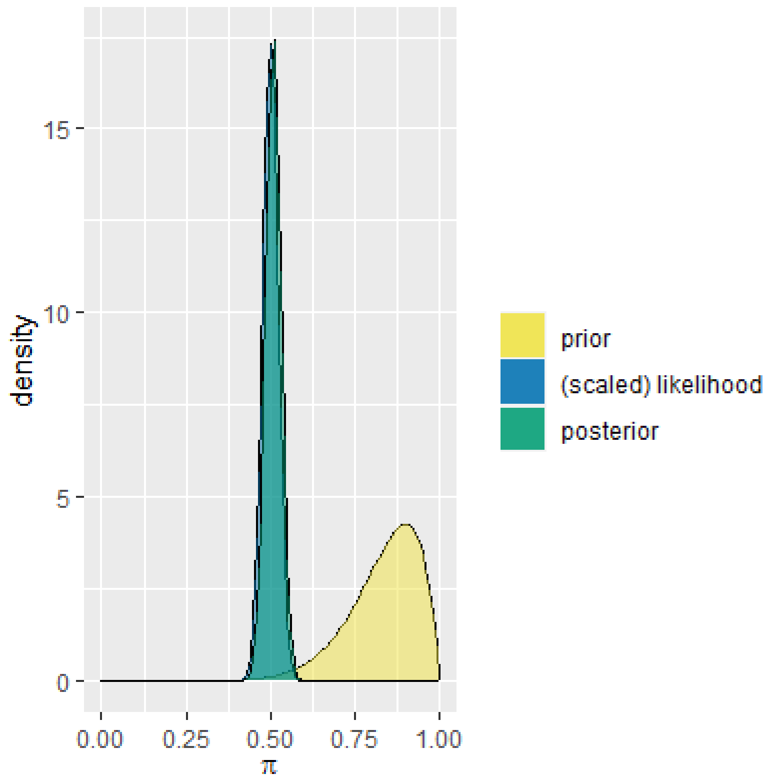
(d) $Y = 20$ in $n = 120$ trials

```
> summarize_beta_binomial(10,2,20,120) |> mutate_if(is.numeric, round, digits
      model alpha beta   mean   mode    var     sd
1     prior    10    2 0.8333 0.9000 0.0107 0.1034
2 posterior    30  102 0.2273 0.2231 0.0013 0.0363
> plot_beta_binomial(10,2,20,120)
```



(e) $Y = 234$ in $n = 468$ trials

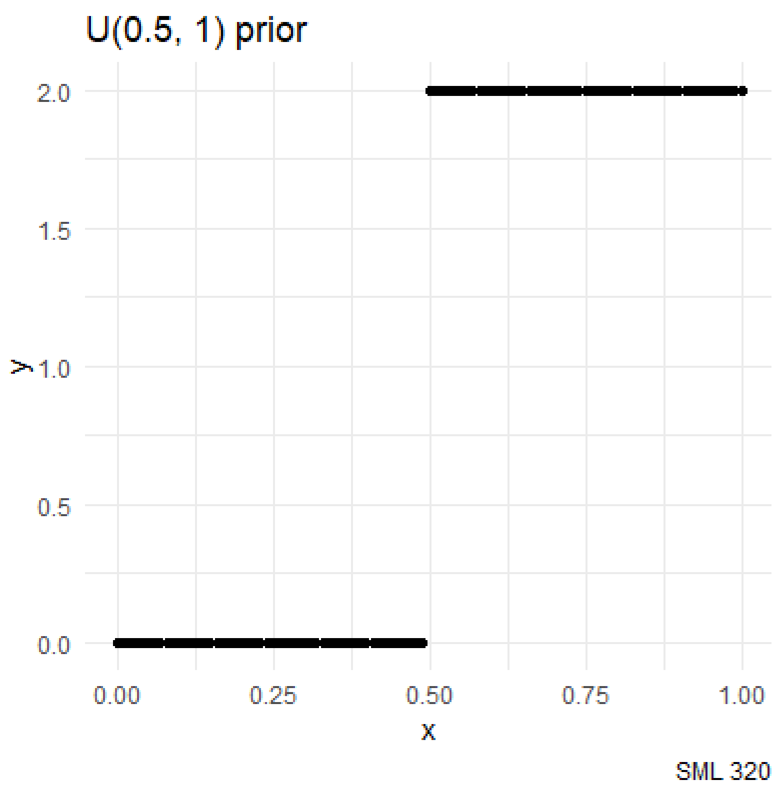
```
> summarize_beta_binomial(10,2,234,468) |> mutate_if(is.numeric, round, digit
      model alpha beta   mean   mode    var     sd
1     prior    10    2 0.8333 0.9000 0.0107 0.1034
2 posterior   244  236 0.5083 0.5084 0.0005 0.0228
> plot_beta_binomial(10,2,234,468)
```



4. (*Bayes Rules!* exercise 4.13) **(Bayesian Bummer)** Bayesian methods are great! But, like anything, we can screw it up. Suppose a politician specifies their prior understanding about their approval rating, π, by:

$$\pi \sim \text{Unif}(0.5, 1) \quad \text{with} \quad f(\pi) = \begin{cases} 0, & 0 < \pi < 0.5 \\ 2, & 0.5 < \pi < 1 \end{cases}$$

(a) Sketch the prior pdf [either by hand or using computer aid]



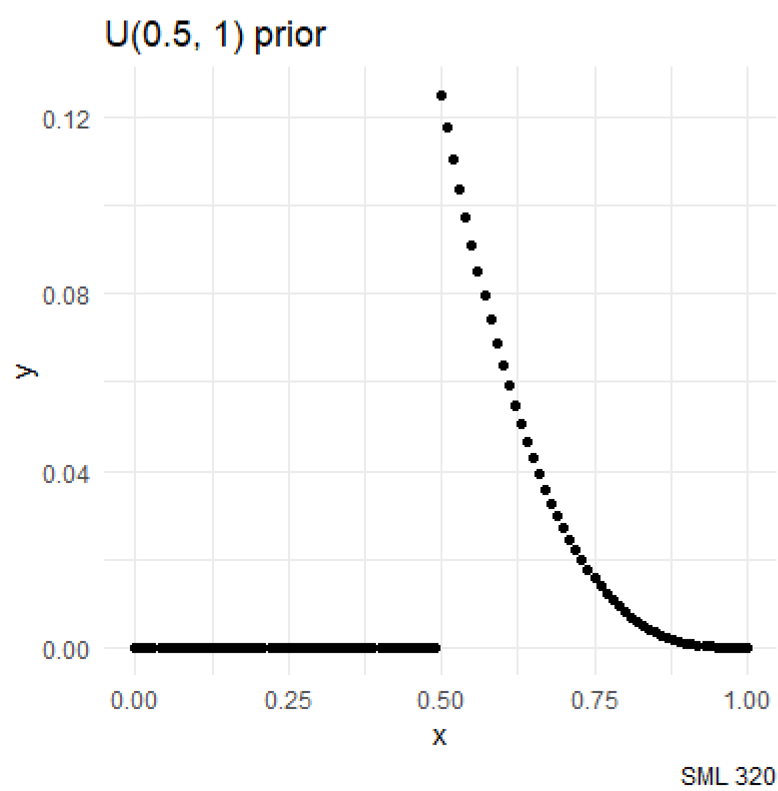
(b) Describe the politician's prior understanding of π

The politician though that a majority of their constituents approved of their job performance.

(c) The politician's aides show them a poll in which 0 of 100 people approve of their job performance. Construct a formula for and sketch the politician's posterior pdf of π.

$$f(\pi|y=0) \propto f(\pi) \cdot L(\pi|y=0) \propto (1-\pi)^{100} \text{ for } \pi \in [0.5, 1]$$

and zero otherwise.



(d) Describe the politician's posterior understanding of π. Use this to explain the mistake the politician made in specifying their prior.

The politician's posterior view of the approval rating still assumes that a majority of people approve of the politician's work, but that view does not allow for the observed data.

5. (*Bayes Rules!* exercise 4.14)

(a) In the Beta-Binomial setting, show that we can write the posterior mode of π as the weighted average of the prior mode and observed sample success rate:

$$\text{Mode}(\pi|Y=y) = \frac{\alpha+\beta-2}{\alpha+\beta+n-2} \cdot \text{Mode}(\pi) + \frac{n}{\alpha+\beta+n-2} \cdot \frac{y}{n}$$

$$
\begin{aligned}
\text{Mode}(\pi|Y=y) &= \frac{(\alpha+y)-1}{(\alpha+y)+(\beta+n-y)-2} \\
&= \frac{\alpha+y-1}{\alpha+\beta+n-2} \\
&= \frac{\alpha-1}{\alpha+\beta+n-2} + \frac{y}{\alpha+\beta+n-2} \\
&= \frac{\alpha-1}{\alpha+\beta+n-2}\cdot\frac{\alpha+\beta-2}{\alpha+\beta-2} + \frac{y}{\alpha+\beta+n-2}\cdot\frac{n}{n} \\
&= \frac{\alpha+\beta-2}{\alpha+\beta+n-2}\cdot\frac{\alpha-1}{\alpha+\beta-2} + \frac{y}{\alpha+\beta+n-2}\cdot\frac{y}{n} \\
&= \frac{\alpha+\beta-2}{\alpha+\beta+n-2}\cdot\text{Mode}(\pi) + \frac{y}{\alpha+\beta+n-2}\cdot\frac{y}{n}
\end{aligned}
$$

(b) To what value does the posterior mode converge as our sample size n increases?

As mentioned in class, more weight is given to the new data observations $\frac{y}{n}$ as the sample size increases.

6. (*Bayes Rules!* exercise 4.15) **(One at a time)** Let π be the probability of success for some event of interest. You place a Beta(2, 3) prior on π, and are really impatient. Sequentially update your posterior for π with each new observation below.

   (a) First observation: Success

   Beta(3,3)

   (b) Second observation: Success

   Beta(4,3)

   (c) Third observation: Failure

   Beta(4,4)

   (d) Fourth observation: Success

   Beta(5,4)

7. (*Bayes Rules!* exercise 4.16) **(Five at a time)** Let π be the probability of success for some event of interest. You place a Beta(2, 3) prior on π, and are impatient, but you have been working on that aspect of your personality. So you sequentially update your posterior model of π after every five new observations. For each set of five new observations, report the updated posterior model for π.

(a) First set of observations: 3 successes

Beta(5,5)

(b) Second set of observations: 1 success

Beta(6,9)

(c) Third set of observations: 1 success

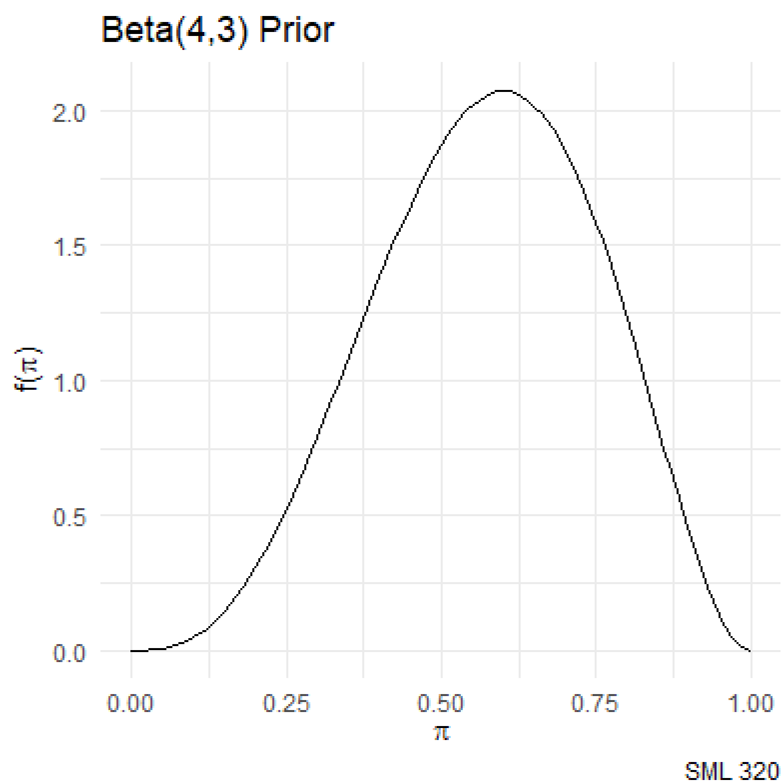
Beta(7,13)

(d) Fourth set of observations: 2 successes

Beta(9,16)

8. (*Bayes Rules!* exercise 4.17) **(Different data, different posteriors)** A shoe company develops a new internet ad for their latest sneaker. Three employees share the same Beta(4, 3) prior model for π, the probability that a user will click on the ad when shown. However, the employees run three different studies, thus each has access to different data. The first employee tests the ad on 1 person – they do not click on the ad. The second tests 10 people, 3 of whom click on the ad. The third tests 100 people, 20 of whom click on the ad.

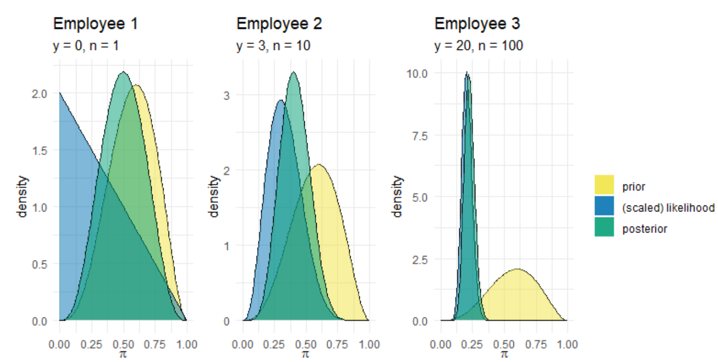
   (a) Sketch the prior pdf using `plot_beta()`. Describe the employees' prior understanding of the chance that a user will click on the ad.



The employees' prior understanding might include some belief that users will click on an advertisement slightly more than 50 percent of the time.

(b) Plot the prior pdf, likelihood function, and posterior pdf for each employee.

```
p1 <- bayesrules::plot_beta_binomial(4,3,0,1) +
  labs(title = "Employee 1", subtitle = "y = 0, n = 1") +
  theme_minimal() + theme(legend.position = "none")
p2 <- bayesrules::plot_beta_binomial(4,3,3,10) +
  labs(title = "Employee 2", subtitle = "y = 3, n = 10") +
  theme_minimal() + theme(legend.position = "none")
p3 <- bayesrules::plot_beta_binomial(4,3,20,100) +
  labs(title = "Employee 3", subtitle = "y = 20, n = 100") +
  theme_minimal()
p1 + p2 + p3
```



(c) Specify the unique posterior model of π for each of the three employees.

- Employee 1: Beta(4, 4)
- Employee 2: Beta(7, 10)
- Employee 3: Beta(24, 83)

(d) Summarize and compare the employees' posterior models of π.

```
> summarize_beta_binomial(4,3,0,1) |> mutate_if(is.numeric,round,digits=4)
      model alpha beta   mean mode    var     sd
1     prior     4    3 0.5714  0.6 0.0306 0.1750
2 posterior     4    4 0.5000  0.5 0.0278 0.1667
> summarize_beta_binomial(4,3,3,10) |> mutate_if(is.numeric,round,digits=4)
      model alpha beta   mean mode    var    sd
1     prior     4    3 0.5714  0.6 0.0306 0.175
2 posterior     7   10 0.4118  0.4 0.0135 0.116
> summarize_beta_binomial(4,3,20,100) |> mutate_if(is.numeric,round,digits=4)
      model alpha beta   mean  mode    var     sd
1     prior     4    3 0.5714 0.600 0.0306 0.1750
2 posterior    24   83 0.2243 0.219 0.0016 0.0401
```

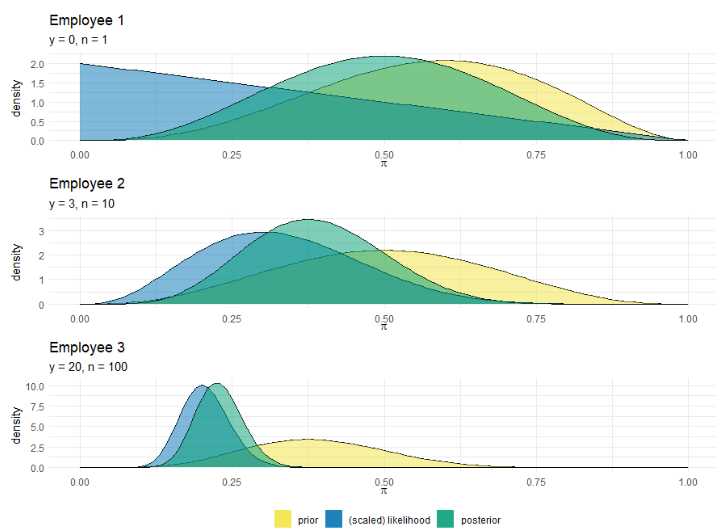
The mode for employee one barely moved. Since employees 2 and 3 surveyed more people, and both with low proportions in the observed data, the change to the posterior distributions are greater.

9. (*Bayes Rules!* exercise 4.18) **(A sequential employee)** The shoe company described in [the previous exercise] brings in a fourth employee. They start with the same Beta(4, 3) prior for π as the first three employees but, not wanting to re-create work, don't collect their own data. Instead, in their first day on the job, the new employee convinces the first employee to share their data. On the second day they get access to the second employee's data and on the third day they get access to the third employee's data.

(a) Suppose the new employee updates their posterior model of π at the end of each day. What's their posterior at the end of day one? At the end of day two? At the end of day three?

```
> summarize_beta_binomial(4,3,0,1) |> mutate_if(is.numeric,round,digits=4)
      model alpha beta   mean mode    var     sd
1     prior     4    3 0.5714  0.6 0.0306 0.1750
2 posterior     4    4 0.5000  0.5 0.0278 0.1667
> summarize_beta_binomial(4,4,3,10) |> mutate_if(is.numeric,round,digits=4)
      model alpha beta   mean  mode    var     sd
1     prior     4    4 0.5000 0.500 0.0278 0.1667
2 posterior     7   11 0.3889 0.375 0.0125 0.1118
> summarize_beta_binomial(7,11,20,100) |> mutate_if(is.numeric,round,digits=4
      model alpha beta   mean   mode    var     sd
1     prior     7   11 0.3889 0.3750 0.0125 0.1118
2 posterior    27   91 0.2288 0.2241 0.0015 0.0385
```

(b) Sketch the new employee's prior and three (sequential) posteriors. In words, describe how their understanding of π evolved over their first three days on the job.



The variance decreases in each subsequent posterior distribution, so the new employee may be more confident in the results after each iteration.

(c) Suppose instead that the new employee didn't update their posterior until the end of their third day on the job, after they'd gotten data from all three of the other employees. Specify their posterior model of π and compare this to the day three posterior from part (a).

Assuming that the employee experiments were independent, we have data invariance in the processes, so the final distribution from viewing all of the data all at once will once again be a Beta(27,91) distribution.

Here are some incomplete answers.

1. (a) Beta(33,13)
   (b) Beta(18,8)
   (c)
2. (a) Beta(11,4)
   (b) Beta(1,2)
   (c) Beta(101,31)
   (d) Beta(21, 101)
   (e) Beta(235, 235)
3. (a) Beta(20,5)
   (b) Beta(10,3)
   (c) Beta(110,32)
   (d) Beta(30,102)
   (e) Beta(244,236)
4.
5.
6. (a)
   (b)
   (c) Beta(4,4)
   (d) Beta(5,4)
7. (a)
   (b)
   (c) Beta(7,13)
   (d) Beta(9,16)
8.
9. (a)
   - after day 1:
   - after day 2: Beta(7,11)
   - after day 3: Beta(27, 91)

   (b)
   (c)