Complete the following tasks. Show your work where applicable. Some answers have been provided. Assemble your work into one PDF document and upload the PDF back into our CatCourses page.

1. (*Bayes Rules!* exercise 3.18)

   (a) Patrick has a Beta(3,3) prior for π, the probability that someone in their town attended a protest in June 2020. In their survey of 40 residents, 30 attended a protest. Summarize Patrick's analysis using `summarize_beta_binomial()` and `plot_beta_binomial()`.

   (b) Harold has the same prior as Patrick, but lives in a different town. In their survey, 15 out of 20 people attended a protest. Summarize Harold's analysis using `summarize_beta_binomial()` and `plot_beta_binomial()`

   (c) How do Patrick and Harold's posterior models compare? Briefly explain what causes these similarities and differences.

2. (*Bayes Rules!* exercise 4.11) **(Different data, uninformative prior)** In each situation below we have the same prior on the probability of a success, $\pi \sim \text{Beta}(1, 1)$, but different data. Identify the corresponding posterior model and utilize `plot_beta_binomial()` to sketch the prior pdf, likelihood function, and posterior pdf.

   (a) $Y = 10$ in $n = 13$ trials

   (b) $Y = 0$ in $n = 1$ trials

   (c) $Y = 100$ in $n = 130$ trials

   (d) $Y = 20$ in $n = 120$ trials

   (e) $Y = 234$ in $n = 468$ trials

3. (*Bayes Rules!* exercise 4.12) **(Different data, informative prior)** Repeat the previous exercise, but with a $\pi \sim \text{Beta}(10, 2)$ prior.

   (a) $Y = 10$ in $n = 13$ trials

   (b) $Y = 0$ in $n = 1$ trials

   (c) $Y = 100$ in $n = 130$ trials

   (d) $Y = 20$ in $n = 120$ trials

   (e) $Y = 234$ in $n = 468$ trials

4. (*Bayes Rules!* exercise 4.13) **(Bayesian Bummer)** Bayesian methods are great! But, like anything, we can screw it up. Suppose a politician specifies their prior understanding about their approval rating, π, by:

$$\pi \sim \text{Unif}(0.5, 1) \quad \text{with} \quad f(\pi) = \begin{cases} 0, & 0 < \pi < 0.5 \\ 2, & 0.5 < \pi < 1 \end{cases}$$

(a) Sketch the prior pdf [either by hand or using computer aid]

(b) Describe the politician's prior understanding of π

(c) The politician's aides show them a poll in which 0 of 100 people approve of their job performance. Construct a formula for and sketch the politician's posterior pdf of π.

(d) Describe the politician's posterior understanding of π. Use this to explain the mistake the politician made in specifying their prior.

5. (*Bayes Rules!* exercise 4.14)

   (a) In the Beta-Binomial setting, show that we can write the posterior mode of π as the weighted average of the prior mode and observed sample success rate:

$$\text{Mode}(\pi|Y = y) = \frac{\alpha + \beta - 2}{\alpha + \beta + n - 2} \cdot \text{Mode}(\pi) + \frac{n}{\alpha + \beta + n - 2} \cdot \frac{y}{n}$$

   (b) To what value does the posterior mode converge as our sample size n increases?

6. (*Bayes Rules!* exercise 4.15) **(One at a time)** Let π be the probability of success for some event of interest. You place a Beta(2, 3) prior on π, and are really impatient. Sequentially update your posterior for π with each new observation below.

   (a) First observation: Success

   (b) Second observation: Success

   (c) Third observation: Failure

   (d) Fourth observation: Success

7. (*Bayes Rules!* exercise 4.16) **(Five at a time)** Let π be the probability of success for some event of interest. You place a Beta(2, 3) prior on π, and are impatient, but you have been working on that aspect of your personality. So you sequentially update your posterior model of π after every five new observations. For each set of five new observations, report the updated posterior model for π.

   (a) First set of observations: 3 successes

   (b) Second set of observations: 1 success

   (c) Third set of observations: 1 success

   (d) Fourth set of observations: 2 successes

8. (*Bayes Rules!* exercise 4.17) **(Different data, different posteriors)** A shoe company develops a new internet ad for their latest sneaker. Three employees share the same Beta(4, 3) prior model for π, the probability that a user will click on the ad when shown. However, the employees run three different studies, thus each has access to different data. The first employee tests the ad on 1 person – they do not click on the ad. The second tests 10 people, 3 of whom click on the ad. The third tests 100 people, 20 of whom click on the ad.

(a) Sketch the prior pdf using `plot_beta()`. Describe the employees' prior understanding of the chance that a user will click on the ad.

(b) Plot the prior pdf, likelihood function, and posterior pdf for each employee.

(c) Specify the unique posterior model of π for each of the three employees.

(d) Summarize and compare the employees' posterior models of π.

9. (*Bayes Rules!* exercise 4.18) **(A sequential employee)** The shoe company described in [the previous exercise] brings in a fourth employee. They start with the same Beta(4, 3) prior for π as the first three employees but, not wanting to re-create work, don't collect their own data. Instead, in their first day on the job, the new employee convinces the first employee to share their data. On the second day they get access to the second employee's data and on the third day they get access to the third employee's data.

   (a) Suppose the new employee updates their posterior model of π at the end of each day. What's their posterior at the end of day one? At the end of day two? At the end of day three?

   (b) Sketch the new employee's prior and three (sequential) posteriors. In words, describe how their understanding of π evolved over their first three days on the job.

   (c) Suppose instead that the new employee didn't update their posterior until the end of their third day on the job, after they'd gotten data from all three of the other employees. Specify their posterior model of π and compare this to the day three posterior from part (a).

Here are some incomplete answers.

1. (a) Beta(33,13)
   (b) Beta(18,8)
   (c)
2. (a) Beta(11,4)
   (b) Beta(1,2)
   (c) Beta(101,31)
   (d) Beta(21, 101)
   (e) Beta(235, 235)
3. (a) Beta(20,5)
   (b) Beta(10,3)
   (c) Beta(110,32)
   (d) Beta(30,102)
   (e) Beta(244,236)
4.
5.
6. (a)
   (b)
   (c) Beta(4,4)
   (d) Beta(5,4)
7. (a)
   (b)
   (c) Beta(7,13)
   (d) Beta(9,16)
8.
9. (a)
   - after day 1:
   - after day 2: Beta(7,11)
   - after day 3: Beta(27, 91)

   (b)
   (c)