Discrete Mathematics and Logic
Lecture 9

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Graph Theory. The Basics

Definition

A pair $G = (V, E)$ is called an (undirected) **graph**, if

$$E \subseteq E(V) = \{\{u, v\} \mid u, v \in V \,\&\, u \neq v\}$$

The elements of V is called **vertices** of G, and those of E the **edges** of G.

Remark

Here, G is a simple graph.

Vertices are also called nodes or points; edges are called lines or links.

An edge $\{x, y\}$ is usually written as xy.

Graph Theory. The Basics

Definitions

1) Two vertices x, y of G are **adjacent** of **neighbours**, if xy is an edge of G.

2) A vertex v is **incident** with an edge e, if $v \in e$.

3) Two vertices incident with an edge are its **endvertices** or **ends**.

4) Two edges v, w of G are **adjacent** of **neighbours**, if one of their ends is the same.

Graph Theory. The Basics

Definitions

1) The set of neighbours of a vertex v in $G = (V_G, E_G)$ is denoted by $N_G(v)$ or shortly by $N(v)$.

$$N_G(v) = \{u \in V_G \mid vu \in E_G\}$$

2) The degree (or valency) $d_G(v) = d(v)$ of a vertex v is the number of its neighbours:

$$d_G(v) = |N_G(v)|$$

Graph Theory. The Basics

If several people shake hands, then the number of hands shaken is even.

Lemma (Handshaking lemma)

For each graph $G = (V_G, E_G)$,

$$\sum_{v \in V_G} d_G(v) = 2 \cdot |E_G|.$$

Graph Theory. The Basics

If several people shake hands, then the number of hands shaken is even.

Lemma (Handshaking lemma)

For each graph $G = (V_G, E_G)$,

$$\sum_{v \in V_G} d_G(v) = 2 \cdot |E_G|.$$

Proof

Every edge $e \in E_G$ has two ends.

Graph Theory. The Basics

Definition

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. We call G and G' **isomorphic**, and write $G \cong G'$, if there is a bijection $\varphi : V \to V'$ such that for all $x, y \in V$

$$xy \in E \Leftrightarrow \varphi(x)\varphi(y) \in E'.$$

Such a map φ is called **isomorphism**. If $G = G'$ then it is called an **automorphism**.

Note that the degrees of G do not determine G. Indeed, there are graphs $G = (V, E_G)$ and $H = (V, E_H)$ on the same set of vertices that are not isomorphic, but for which $d_G(v) = d_H(v)$ for all $v \in V$.

Graph Theory. The Basics

Definition

The graph G is the **complete graph**, if every two vertices are adjacent.

The order of a graph $G = (V, E)$ is the number $|V|$.

Lemma

All complete graphs of order n are isomorphic with each other, and they will be denoted by K_n.

Graph Theory. The Basics

Definition

Let $G = (V, E)$ be a graph, a vertex $v \in V$ be one of ends of an edge $e_1 \in E$, a vertex $w \in V$ be one of ends of an edge $e_k \in E$. The sequence $W = \{e_1, e_2, \dots, e_k\}$ is called **walk** of length k from v to w, if the edges e_i and e_{i+1} are neighbours for all $i \in \{1, \dots, k-1\}$.

Definition

Let $W = \{e_1, e_2, \dots, e_k\}$ be a walk $(e_i = u_i u_{i+1})$. We say that

W is **closed**, if $u_1 = u_{k+1}$.

W is **path**, if $u_i \neq u_j$ for all $i \neq i$.

W is **cycle**, if it is closed, and $u_i \neq u_j$ for all $i \neq i$ except $u_1 = u_{k+1}$.

Connectivity

Definition

A non-empty graph G is called **connected** if, for any its vertices v, w, G contains a path from v to w.

Otherwise, G is called disconnected.

Definition

Let $G = (V, E)$ be a graph. A maximal connected subgraph of G is called a **component** of G.

$G' = (V', E')$ is subgraph of G, if $V' \subseteq V$ and $E' \subseteq E$.

Connectivity

Definition

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. We set

$$G \cup G' \leftrightharpoons (V \cup V', E \cup E')$$

$$G \cap G' \leftrightharpoons (V \cap V', E \cap E')$$

Proposition

Any graph is a disjoint union of all its connected components.

Proof

It is obvious. :)

Connectivity

Definition

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. We set

$$G - G' \leftrightharpoons (V, E \setminus E')$$

$$\overline{G} = K_{|V|} - G$$

Let $G = (V, E)$ be a graph, and $E' \subseteq E$ and $e \in E$.

$$G - E' \text{ denotes } G - (V, E')$$

$$G - e \text{ denotes } G - (V, \{e\})$$

.

Connectivity

Definition

Let $G = (V, E)$ be a graph and $U \subseteq V$. Suppose that E' contains all the edges $xy \in E$ with $x, y \in U$. Then we write $G[U] = (U, E')$ and call it as **induced subgraph** of G.

If $G' = (V', E')$ is a subgraph of G, then $G[G'] = G[V']$.

If U is subset of the vertex set V of a graph G, we write $G - U$ for $G[V \setminus U]$.

Connectivity

Definition

Let G be a connected graph. Its edge e is called **bridge**, if $G - e$ is disconnected.

Definition

A graph G is called k-**connected** (for $k \in \mathbb{N}$) if $k < |G|$ and $G - X$ is connected for every set $X \subseteq V_G$ with $|X| < k$.

0-connected graphs = (non-empty) graphs

1-connected graphs = connected graphs

exactly 2-connected graphs = connected graphs with bridges

Trees

Definition

A graph is called a **forest** if it does not contain any cycles.

A connected forest is called a **tree**.

Theorem (home-work)

The following are equivalent for a graph T:

1) T is tree,
2) any two vertices of T are linked by a unique path in T,
3) T is minimally connected, i.e., any its edge is a bridge,
4) T is maximally acyclic, i.e., T contains no cycle but $T \cup (\{x, y\}, \{\{x, y\}\})$ does, for any two non-adjacent vertices $x, y \in V_T$.

Trees

Definition

Let $G = (V, E)$ be a connected graph. $G' = (V', E') \subseteq G$ is called **spanning tree**, if G' is tree and $V' = V$.

Theorem

Any connected graph has a spanning tree.

Trees

Proposition (home-work)

Any tree has a vertex with degree 1.

Theorem

A connected graph G with n vertices is a tree iff it has $n-1$ edges.

Proof (by induction)

Base. Let $n = 1$. G have no edges.

Hypothesis. Suppose that the theorem holds for any $k < n$.

Inductive step. ($\Rightarrow$). Let G be a tree with n vertices, and v have the degree 1. Then $G - v$ is a tree with $n-1$ vertices and hence (by induction hypothesis) $G - v$ has $n-2$ edges. Therefore, G has $n-1$ edges.

Trees

Proposition

Any tree has a vertex with degree 1.

Theorem

A connected graph G with n vertices is a tree iff it has $n-1$ edges.

Proof (by induction)

Inductive step. ($\Leftarrow$). Let G' be a connected graph with $n-1$ edges. Suppose that G' is a spanning tree of G. Since G' has n vertices and $n-1$ edges, by the first implication it follows that $G' = G$.

Thank you for your attention!