Discrete Mathematics
Tutorial 1

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Introduction

The well-known example

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

What are the proofs of this?

Introduction

Basic proof techniques

- Directly
- By construction
- By contradiction
- By induction

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Basic proof techniques

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- By construction (bad here)
- By contradiction
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Direct proof

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$$2 \times (1 + 2 + \cdots + (n-1) + n) =$$

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$$2 \times (1 + 2 + \cdots + (n-1) + n) =$$

$$= 1 + 2 + \cdots + (n-1) + n +$$
$$+ n + (n-1) + \cdots + 2 + 1 =$$

Direct proof

Direct proof

$$2 \times (1 + 2 + \cdots + (n-1) + n) =$$

$$= 1 + 2 + \cdots + (n-1) + n +$$
$$+ n + (n-1) + \cdots + 2 + 1 =$$

$$\underbrace{(n+1) + (n+1) + \cdots (n+1) + (n+1)}_{n \text{ times}} = n \times (n+1)$$

Direct proof

Direct proof

Hence,

$$2 \times (1 + 2 + \cdots + (n-1) + n) = n \times (n+1)$$

So,

$$1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$$

Direct proof

The method

Given propositions $P_1, \cdots P_n$, we prove P_0.

- P_1

$$\cdots$$

- P_n
- P_{n+1} (it follows from P_{i_1} and P_{j_1})
- P_{n+2} (it follows from P_{i_2} and P_{j_2})

$$\cdots$$

- P_0 (it follows from P_{i_k} and P_{j_k})

By construction

By construction. $n = 1$

$$1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$$

$$1 = \frac{1 \times (1+1)}{2}$$

It is true!

By construction

By construction. $n = 2$

$$1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$$

$$1 + 2 = \frac{2 \times (2+1)}{2}$$

It is also true!

By construction

By construction. $n = 3$

$$1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$$

$$1 + 2 + 3 = \frac{3 \times (3+1)}{2}$$

It is also true!

By construction

By construction (bad here)

$$1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$$

This statement is true for $n = 1, 2, 3$. Hence, it holds for any n.

By construction

By construction (bad here)

$$1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$$

This statement is true for $n = 1, 2, 3$. Hence, it holds for any n.

This is wrong! (in general)

By construction

By construction (bad here)

We need to prove (or to construct) $P(n)$ for any n.

Suppose that we prove $P(1), P(2), \ldots P(n_0)$ for some n_0.

This is wrong to say $P(n)$ for any n.

By construction

By construction (the bad example)

Assume that 1 is also a prime number.

$1, 3, 5, 7$ are prime.

Hence, all odd numbers are prime. This is wrong!

We know $9 = 3 \times 3$ is not prime!

By contradiction

We need to prove

$$1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}, \text{ for any } n \geq 1$$

Assume for a contradiction that this is false!

$$1 + 2 + \cdots + (n-1) + n \neq \frac{n(n+1)}{2}, \text{ for some } n \geq 1$$

Choose the minimal such $n = n_0$.

By contradiction

Choose the minimal such $n = n_0 > 1$ (the case $n = 1$ see above), i.e.,

$$1 + 2 + \cdots + (n_0 - 1) + n_0 \neq \frac{n_0(n_0 + 1)}{2}$$

But

$$1 + 2 + \cdots + (n_0 - 1) = \frac{(n_0 - 1)(n_0 - 1 + 1)}{2} = \frac{(n_0 - 1)n_0}{2}$$

By contradiction

Since

$$1 + 2 + \cdots + (n_0 - 1) = \frac{(n_0 - 1)(n_0 - 1 + 1)}{2} = \frac{(n_0 - 1)n_0}{2},$$

we have

$$1 + 2 + \cdots + (n_0 - 1) + n_0 = \frac{(n_0 - 1)n_0}{2} + n_0 = n_0\left(\frac{n_0 - 1}{2} + 1\right)$$

This contradicts with

$$1 + 2 + \cdots + (n_0 - 1) + n_0 \neq \frac{n_0(n_0 + 1)}{2}$$

By contradiction

Conclusion

From the contradiction it follows that the assumption

$$1 + 2 + \cdots + (n-1) + n \neq \frac{n(n+1)}{2}, \text{ for some } n \geq 1$$

is wrong!

Hence,

$$1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}, \text{ for any } n \geq 1$$

By contradiction

By contradiction

To prove P,

- assume not P,
- obtain X and not X for some X.
- This gives the assumption "not P" is wrong.

So, you have prove P.

By induction

We need to prove

$$P(n), \text{ for any } n \geq 1$$

Initial step $n = 1$

Prove $P(1)$

Inductive hypothesis

Suppose that

$$P(1), P(2), \ldots, P(k)$$

Inductive step

Prove $P(k+1)$

By induction

We need to prove

$$1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}, \text{ for any } n \geq 1$$

Initial step $n = 1$

$$1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$$

$$1 = \frac{1 \times (1+1)}{2}$$

It is true!

By induction

Inductive hypothesis $n = k$

Suppose that

$$1 + 2 + \cdots + (k-1) + k = \frac{k(k+1)}{2}$$

By induction

Inductive step

We need to prove

$$1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}, \text{ for } n = k+1$$

I.e.,

$$1 + 2 + \cdots + (k-1) + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

By induction

We need to prove

$$1 + 2 + \cdots + (k-1) + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

The inductive hypothesis is $1 + 2 + \cdots + (k-1) + k = \frac{k(k+1)}{2}$

$$1+2+\cdots+(k-1)+k+(k+1) = (1+2+\cdots+(k-1)+k)+(k+1) =$$

$$= \frac{k(k+1)}{2} + (k+1) = (k+1)\left(\frac{k}{2}+1\right) = \frac{(k+1)(k+2)}{2}$$

This completes the inductive step.

By induction

Conclusion

Therefore,

$$1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2} \text{ for any } n$$

By induction

We need to prove

$$P(n), \text{ for any } n \geq 1$$

Initial step $n = 1$

Prove $P(1)$

Inductive step

Prove

$$P(1), P(2), \ldots, P(k) \Rightarrow P(k+1)$$

By induction

How does it work?

Have $P(1)$ and $P(k) \Rightarrow P(k+1)$.

- $P(1)$
- $P(1) \Rightarrow P(2)$
- $P(2)$
- $P(2) \Rightarrow P(3)$
- $P(3)$

$$\dots$$

- $P(1000000)$
- $P(1000000) \Rightarrow P(1000001)$
- $P(1000001)$

$$\dots$$

Thank you for your attention!