Essentials of Analytical Geometry and Linear Algebra. Lecture 5.

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End of Lecture #4

- Part 1. Change of basis and coordinates
- Part 2. Matrix rank
- Part 3. Matrix inverse

Quiz in class

Go to `http://b.socrative.com`

Type Room: **LINAL**

Answer questions.

Lecture 5. Outline

- Part 1. Matrix inverse recap. General method
- Part 2. Applications
- Part 3. Summary of the block / What is in the next block?

Part 1. Matrix inverse recap. General method

Step by step

Find inverse for a square matrix A

Step 1. Find $det(A)$
Step 2. Build a matrix with minors: $M = m_{ij}$
Step 3. Build a matrix with ± 1: $H = h_{ij}, \quad h_{ij} = (-1)^{i+j}$
Step 4. Build a cofactor matrix: $C = H \odot M, \quad c_{ij} = h_{ij} m_{ij}$
Step 5. Transpose and scale $\frac{1}{det(A)} C^\top$

Example

Here we inverse a 3×3 matrix

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

Example

Step 1. Find $det(A)$

$$\begin{vmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{vmatrix} = -1$$

Example

Step 2. Build a matrix of first minors: M_{ij}

(First) Minor M_{ij} of matrix A is **the determinant** of the submatrix formed by deleting the i-th row and j-th column.

$$A=\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

$$M_{11}=\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -2$$

$$M_{23}=\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = -1$$

Result

Step 2. Build a matrix of first minors: M_{ij} M= $\begin{bmatrix} -2 & -1 & -1 \\ -4 & -2 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

Example

Step 3. Build a matrix with ± 1: $H = h_{ij}, \quad h_{ij} = (-1)^{i+j}$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

Example

Step 3. Build a matrix with ± 1: $H = h_{ij}, \quad h_{ij} = (-1)^{i+j}$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

Step 4. Build a cofactor matrix: $C = H \odot M, \quad c_{ij} = h_{ij} m_{ij}$

$$C = H \odot M =$$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \odot \begin{bmatrix} -2 & -1 & -1 \\ -4 & -2 & -1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & -1 \\ 4 & -2 & 1 \\ 1 & -1 & 1 \end{bmatrix} = C$$

Example

Step 5. Transpose and scale $\frac{1}{det(A)}C^\top$

$|A| = -1;$

Example

Step 5. Transpose and scale $\frac{1}{det(A)}C^{\top}$

$$|A| = -1;\ C = \begin{bmatrix} -2 & 1 & -1 \\ 4 & -2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

Example

Step 5. Transpose and scale $\frac{1}{det(A)} C^\top$

$$|A| = -1; C = \begin{bmatrix} -2 & 1 & -1 \\ 4 & -2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$C^\top = \begin{bmatrix} -2 & 4 & 1 \\ 1 & -2 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

Example

Step 1. Find $det(A)$
Step 2. Build a matrix with minors: $M = m_{ij}$
Step 3. Build a matrix with ± 1: $H = h_{ij}, \quad h_{ij} = (-1)^{i+j}$
Step 4. Build a cofactor matrix: $C = H \odot M, \quad c_{ij} = h_{ij} m_{ij}$
Step 5. Transpose and scale $\frac{1}{det(A)} C^\top$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 2 & -4 & -1 \\ -1 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

Check: $A^{-1}A = I$

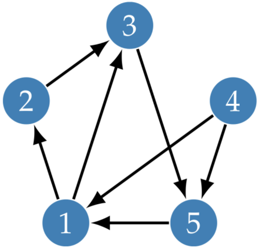
Homework

Implement the method to find inverse for a square matrix A.

Part 2. Applications

Graphs and Matrices

Given a graph you can define its **adjacency** matrix, A



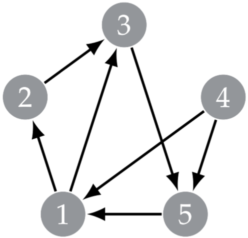
$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix represents paths of length 1 (e.g. one ‘hop’ between 4 and 1)

Graphs and Matrices: Powers of A

Given an adjacency matrix, A you can find its power ($A^2 = AA$)

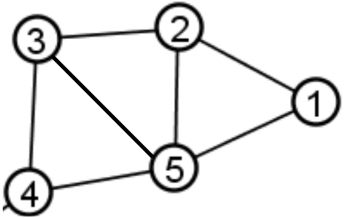
$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$



Matrix represents paths of length 2 (e.g. two ‘hops’ to reach 5 from 1)

Graphs and Matrices: Example

Given a graph G



Build its adjacency matrix, A
Find A^3.
Find the trace of A^3, $Tr(A^3)$

How can you interpret it?

Systems of equations

Systems of equations

Given a system:

$$2x + 3y = 5$$

$$-x + 7y = 7$$

Systems of equations

Given a system:

$$2x + 3y = 5$$

$$-x + 7y = 7$$

We transform it into a matrix

$$\left[\begin{array}{cc|c} 2 & 3 & 5 \\ -1 & 7 & 7 \end{array}\right]$$

Systems of equations. Gaussian Elimination

Goal of the Gaussian Elimination

Derive a upper triangle matrix in the left part.

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \end{array}\right]$$

Method. You can:

Swap rows
Multiply a row by a nonzero number
You can add a multiply of any row to any other row

Systems of equations. Gaussian Elimination

Goal of the Gaussian Elimination

Derive a upper triangle matrix in the left part.

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Method. You can:

Swap rows
Multiply a row by a nonzero number
You can add a multiply of any row to any other row

$$R_j := R_j + \alpha R_k$$

Systems of equations. Some exceptions :)

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & p \end{array}\right]$$

where $p \neq 0$

What does it mean?

Systems of equations. Some exceptions :)

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & \mathbf{0} & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

What does it mean?

Demo

Here we check source code

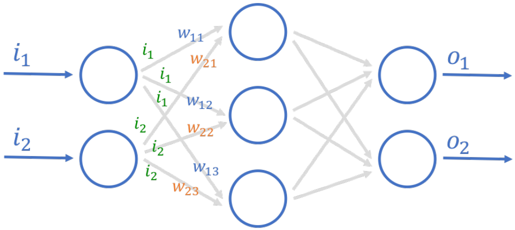
Linear models and Matrices

$$\begin{bmatrix} y_1 \\ \vdots \\ \vdots \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & X_{11} & \cdots & \cdots & \cdots & X_{1p} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & X_{n1} & \cdots & \cdots & \cdots & X_{np} \end{bmatrix} \begin{bmatrix} b_0 \\ \vdots \\ \vdots \\ b_p \end{bmatrix} + \begin{bmatrix} e_1 \\ \vdots \\ \vdots \\ \vdots \\ e_n \end{bmatrix}$$

$$Y = X\beta + \varepsilon$$

Source: `https://www.brainvoyager.com/bv/doc/UsersGuide/StatisticalAnalysis/TheGeneralLinearModel.html`

Neural Networks and Matrices



$$\begin{bmatrix} w_{11} & w_{21} \\ w_{12} & w_{22} \\ w_{13} & w_{23} \end{bmatrix} \cdot \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} (w_{11} \times i_1) + (w_{21} \times i_2) \\ (w_{12} \times i_1) + (w_{22} \times i_2) \\ (w_{13} \times i_1) + (w_{23} \times i_2) \end{bmatrix}$$

+ Non-linear transformation of result !

Source: `https://sausheong.github.io/posts/`

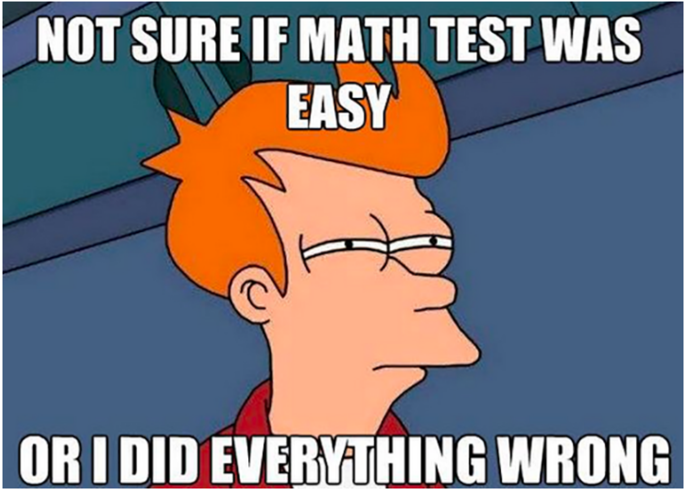
Part 3. Summary / What is in the next block?

End of Lecture #5

Next week:

- Lines in space
- Equations of line
- Finding distances

Good luck!



Useful links

- https://www.geogebra.org
- https://youtu.be/fNk_zzaMoSs
- http://immersivemath.com/ila