Discrete Mathematics and Logic
Lecture 8

Andrey Frolov

Innopolis University

Combinatorics

	without repetitions	with repetitions
order matters	$P(n,k)$	n^k
order doesn't matter	$\binom{n}{k}$	$\binom{n+k-1}{k}$

Stirling numbers

$$2^A = P(A) = \{B \mid B \subseteq A\}$$

$$|P(A)| = 2^{|A|}$$

Stirling numbers

Example

Let $A = \{1, 2, 3\}$. Then

$$P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Stirling numbers

Example

Let $A = \{1, 2, 3\}$. Then

$P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

$k = 2$

$\{\{1\}, \{2, 3\}\}$,

$\{\{2\}, \{1, 3\}\}$,

$\{\{3\}, \{1, 2\}\}$

Stirling numbers

Definition

Let X be a set with $|A| = n$ and $k \geq 0$. Set $X_1, \ldots, X_k$ is called a k-splitting, if

$$X = \bigcup_{i=1}^{k} X_i \ \ \& \ \ X_i \bigcap X_j = \emptyset \ (i \neq j)$$

Stirling numbers

Definition

The number of k-splittings is called the Stirling number and denoted as S_n^k.

Example

$$S_3^2 = 3$$

$\{\{1\},\{2,3\}\},\{\{2\},\{1,3\}\},\{\{3\},\{1,2\}\}$

Stirling numbers

Theorem

1) $S_0^0 = 1$, $S_0^1 = 0$,
2) $S_n^k = S_{n-1}^{k-1} + kS_{n-1}^k$

Proof

1) $S_n^k = 0$ for $k > n$. Hence, $S_0^1 = 0$.

Stirling numbers

Proof

2) Fix $a \in X$. All k-splitting of X:

$$\{\{a\}, \underbrace{X_1, \ldots, X_{k-1}}_{(k-1)-\text{splitting}}\}$$

$$\{Y_1, \ldots, Y_k\}, \text{ where } |Y_i| \geq 2$$

The number of all splittings of the first type is S_{n-1}^{k-1} (by definition)

Stirling numbers

Proof

To find the number of all splittings such that

$$\{Y_1, \dots, Y_k\}, \text{ where } |Y_i| \geq 2,$$

let $Y = X \setminus \{a\}$, i.e., $|Y| = n - 1$.

The number of all k-splittings for Y is S^k_{n-1}.

We can add the element a to each splittings. So, the number of all such splittings is kS^k_{n-1}.

Hence, $S^k_n = S^{k-1}_{n-1} + kS^k_{n-1}$.

Stirling numbers

Corollary

$S_n^2 = 2^{n-1} - 1$ for $n \geq 2$

Proof

$$S_n^2 = S_{n-1}^1 + 2S_{n-1}^2$$

1. **Base.** Let $n = 2$. $S_2^2 = S_1^1 + 2S_1^2 = 1 + 2 \cdot 0 = 1 = 2^{2-1} - 1$.

2. **Hypothesis.** Suppose that the corollary holds for any $k < n_0$.

3. **Inductive step.**
$S_n^2 = S_{n-1}^1 + 2S_{n-1}^2 = 1 + 2(2^{n-2} - 1) = 2^{n-1} - 1$.

Linear recursive sequences

Definition

A sequence $f_0, f_1, f_2, \ldots$ is called linear recursive sequence, if

$$f_{n+k} = a_1 \cdot f_{n+k-1} + \cdots + a_k \cdot f_n = \sum_{i=1}^{k} a_i \cdot f_{n+k-i},$$

where all a_i are fixed coefficients.

Definition

A sequence $x_0, x_1, x_2, \ldots$ is a solution of a linear recursive sequence $f_{n+k} = \sum_{i=1}^{k} a_i \cdot f_{n+k-i}$, if the recursion holds for $f_n = x_n$.

Linear recursive sequences

Lemma

If sequences $x_0, x_1, x_2, \ldots$ and $y_0, y_1, y_2, \ldots$ are solutions of a linear recursive sequence, then

$\alpha x_0 + \beta y_0, \alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \ldots$ is also a solution of the linear recursive sequence.

Linear recursive sequences

Definition

Let $f_{n+k} = \sum_{i=1}^{k} a_i \cdot f_{n+k-i}$ be a linear recursive sequence. Then the character of it is

$$\chi(\lambda) = \lambda^k - a_1\lambda^{k-1} - a_2\lambda^{k-2} - \cdots - a_{k-1}\lambda^{k-1} - a_k.$$

Linear recursive sequences

Definition

Let $f_{n+k} = \sum_{i=1}^{k} a_i \cdot f_{n+k-i}$ be a linear recursive sequence. Then the character of it is

$$\chi(\lambda) = \lambda^k - a_1\lambda^{k-1} - a_2\lambda^{k-2} - \cdots - a_{k-1}\lambda^{k-1} - a_k.$$

Lemma

Let $f_{n+k} = \sum_{i=1}^{k} a_i \cdot f_{n+k-i}$ be a linear recursive sequence, $\chi(\lambda)$ be its character, and p be a root of $\chi(\lambda)$. Then the sequence

$$1, p, p^2, \ldots, p^n, \ldots$$

are a solution.

Linear recursive sequences

Theorem

Let $f_{n+2} = a_1 \cdot f_{n+1} + a_2 \cdot f_n$ and $a_1^2 + a_2^2 \neq 0$, and p_1, p_2 are roots of the character $\chi(\lambda)$. Then

1) If $p_1 \neq p_2$ then any solution is $x_n = \alpha \cdot p_1^n + \beta \cdot p_2^n$;

2) If $p_1 = p_2$ then any solution is $x_n = (\alpha \cdot n + \beta) \cdot p_1^n$.

Linear recursive sequences

Proof

1) From the lemma above it follows that $x_n = \alpha \cdot p_1^n + \beta \cdot p_2^n$ is a solution.

The back way. Let $p_1 \neq p_2$, and $\{x_0, x_1, x_2, \dots\}$ be any solution. The solution is defined by x_0, x_1, where

$$\begin{cases} \alpha \cdot p_1^0 + \beta \cdot p_2^0 & = x_0 \\ \alpha \cdot p_1^1 + \beta \cdot p_2^1 & = x_1 \end{cases}$$

$$\alpha = \frac{x_0 p_2 - x_1}{p_2 - p_1}, \beta = \frac{x_1 - p_1 x_0}{p_2 - p_1}$$

Linear recursive sequences

Proof

2) Since $p_1 = p_2 = p$, $\chi(\lambda) = \lambda^2 - a_1\lambda - a_2 = (\lambda - p)^2$. Hence,

$$a_1 = 2p, a_2 = -p^2.$$

Show that np^n is a solution:

$$a_1 f_{n+1} + a_2 f_n = a_1(n+1)p^{n+1} + a_2 np^n =$$

$$= 2(n+1)p^{n+2} - np^n + 2 = (n+2)p^{n+2} = f_{n+2}$$

Linear recursive sequences

Proof

2) The back way. Let $\{x_0, x_1, x_2, \ldots\}$ be any solution. The solution is defined by x_0, x_1, where

$$\begin{cases} (\alpha \cdot 0 + \beta) \cdot p^0 & = x_0 \\ (\alpha \cdot 1 + \beta) \cdot p^1 & = x_1 \end{cases}$$

$$\alpha = \frac{x_1 - px_0}{p}, \beta = x_0$$

Fibonacci numbers

$$f_0 = 1, f_1 = 1, f_{n+2} = f_{n+1} + f_n$$

$$\chi(\lambda) = \lambda^2 - \lambda - 1$$

$$p_1 = \frac{1+\sqrt{5}}{2}, p_2 = \frac{1-\sqrt{5}}{2}$$

$$f_n = \alpha \left(\frac{1+\sqrt{5}}{2} \right)^n + \beta \left(\frac{1-\sqrt{5}}{2} \right)^n$$

Fibonacci numbers

$$f_0 = 1, f_1 = 1, f_{n+2} = f_{n+1} + f_n$$

$$f_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} + \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right)$$

Thank you for your attention!