

Essentials of Analytical Geometry and Linear Algebra. Lecture 2.

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End of Lecture #1

Review

- Points and Vectors
- Vector Addition. Scalar Vector Multiplication
- Properties of Vector Arithmetic
- Vector spaces, Subspaces
- Span, Linear Independence
- Vector Bases and Coordinates

Quiz in class

Go to <http://b.socrative.com>

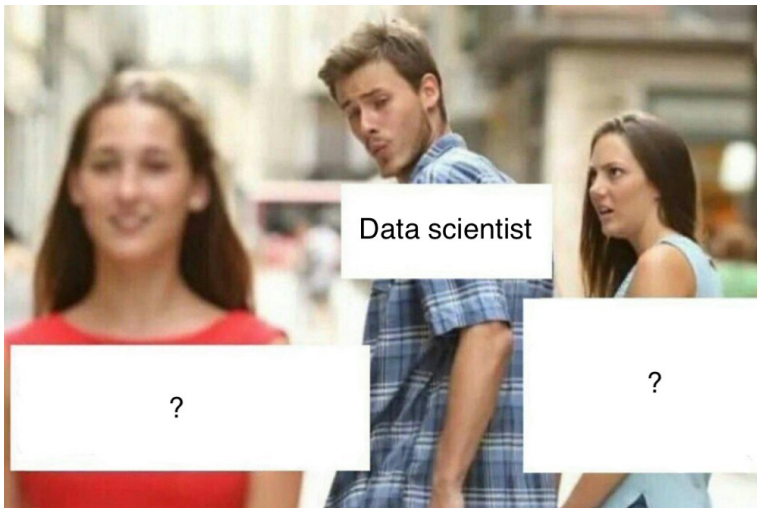
Type Room: **LINAL**

Answer 3 questions.
(you can use anonymous name)

Lecture 2. Outline

- **Quiz.** Dimension, Subspaces and subsets
- Part 1. The Dot Product and its properties
 - Norm of a vector
 - Cauchy-Schwarz inequality
 - Triangle Inequality
- Part 2. Vector Cross Product
- Part 3. Matrices (2x2, 3x3).

How would you label the blanks?

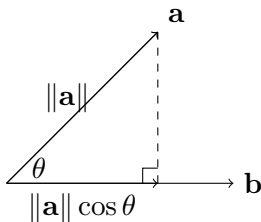


Part 1. Dot Product

Geometric view (in \mathbb{R}^2 and \mathbb{R}^3)

Scalar/dot product

$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$, where θ is the angle between \mathbf{a} and \mathbf{b} .



Examples

Scalar projection

Scalar projection of vector \mathbf{a} on vector \mathbf{b} is **a scalar**: $a_b = \|\mathbf{a}\| \cos \theta$

Find the scalar projections a_b and b_a .

$$\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

Orthogonal projection

Orthogonal projection of vector \mathbf{a} on vector \mathbf{b} is **a vector**: $a_b = \hat{\mathbf{b}} \|\mathbf{a}\| \cos \theta$

Dot Product. Algebraic view

Definition

Let V be a vector space over \mathbb{R} .

By a dot product on V we mean a real valued function $\mathbf{u} \cdot \mathbf{v}$ on $V \times V \rightarrow \mathbb{R}$ which satisfies the following axioms:

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Notation

$$\mathbf{u} \cdot \mathbf{v} = (\mathbf{u}, \mathbf{v}) = \mathbf{u}^T \mathbf{v} = \langle \mathbf{u}, \mathbf{v} \rangle$$

Dot Product. Calculation

Dot product in \mathbb{R}^n

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + \dots + u_nv_n = \sum_{i=1}^n u_iv_i$$

If \mathbf{u}, \mathbf{v} are column vectors, then

$$\mathbf{u}^\top \mathbf{v} = u_1v_1 + \dots + u_nv_n = \sum_{i=1}^n u_iv_i = \mathbf{u} \cdot \mathbf{v}$$

Examples

Question. Find the angle between \mathbf{a} and \mathbf{b}

$$\mathbf{a} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -5 \\ -1 \\ -1 \end{bmatrix}$$

Hint

$$\|\mathbf{u}\| \equiv \sqrt{\mathbf{u} \cdot \mathbf{u}}$$

Norm (or a length) of a vector

A norm on any vector space is defined as follows:

Definition

We say $\| \mathbf{u} \|$ is a norm on a vector space V if $\forall \mathbf{u}, \mathbf{v} \in V$ and $\alpha \in \mathbb{R}$,

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- d) $\| \mathbf{u} + \mathbf{v} \| \leq \| \mathbf{u} \| + \| \mathbf{v} \|$

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Check

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^n u_i v_i = \| \mathbf{u} \| \| \mathbf{v} \| \cos \theta$$

Problem

In the triangle ABC the median AD is divided into three equal segments: AE, EF and FD.

$$\overline{BA} \cdot \overline{CA} = 4$$

$$\overline{BF} \cdot \overline{CF} = -1$$

Find $\overline{BE} \cdot \overline{CE}$.

Cauchy-Schwarz inequality

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For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|.$$

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For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|.$$

Proof

Consider the expression $\|\mathbf{x} - \lambda \mathbf{y}\|^2$. We must have

$$\|\mathbf{x} - \lambda \mathbf{y}\|^2 \geq 0$$

$$(\mathbf{x} - \lambda \mathbf{y}) \cdot (\mathbf{x} - \lambda \mathbf{y}) \geq 0$$

$$\lambda^2 \|\mathbf{y}\|^2 - \lambda(2\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{x}\|^2 \geq 0.$$

Cauchy-Schwarz inequality

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For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

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Consider the expression $\|\mathbf{x} - \lambda \mathbf{y}\|^2$. We must have $\lambda^2 \|\mathbf{y}\|^2 - \lambda(2\mathbf{x} \cdot \mathbf{y}) + \|\mathbf{x}\|^2 \geq 0$.

Viewing this as a quadratic in λ , we see that the quadratic is non-negative. Thus, it cannot have 2 different real roots. The discriminant $\Delta = b^2 - 4ac \leq 0$. So

$$4(\mathbf{x} \cdot \mathbf{y})^2 \leq 4\|\mathbf{y}\|^2 \|\mathbf{x}\|^2$$

$$(\mathbf{x} \cdot \mathbf{y})^2 \leq \|\mathbf{x}\|^2 \|\mathbf{y}\|^2$$

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|.$$

Write some code

Here we open Google Colab...

... to check Cauchy-Schwarz inequality

```
https://colab.research.google.com/drive/  
1QKCs22fjRaLks5oSA2QjssqXYgBHMn1A?usp=sharing
```

Triangle inequality

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$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

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$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

Proof

$$\begin{aligned}\|\mathbf{x} + \mathbf{y}\|^2 &= (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) \\ &= \|\mathbf{x}\|^2 + 2\mathbf{x} \cdot \mathbf{y} + \|\mathbf{y}\|^2 \\ &\leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 \\ &= (\|\mathbf{x}\| + \|\mathbf{y}\|)^2.\end{aligned}$$

Orthogonality

Definition

Let V be vector space with a dot product.

Vectors $\mathbf{u}, \mathbf{v} \in V$ are said to be **orthogonal** if

$$\mathbf{u} \cdot \mathbf{v} = 0$$

Examples

Here we open the Geogebra :)

Homework

Show that the difference between a vector \mathbf{a} and its orthogonal projection on a vector \mathbf{b} is orthogonal to the vector \mathbf{b} .

If

$$\mathbf{p} = \mathbf{a} - \mathbf{a}_b$$

then

$$\mathbf{p} \cdot \mathbf{b} = 0$$

Break

5 min. break

Part 2. Vector Cross Product

Vector Cross product

Apart from the scalar product, we can also define the *vector product*. However, this is defined only for \mathbb{R}^3 space, but not spaces in general.

Vector cross product

Consider $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. Define the *vector cross product*

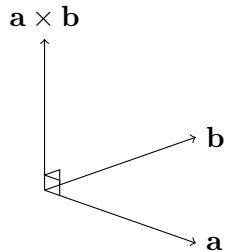
$$\mathbf{a} \times \mathbf{b} = \hat{\mathbf{n}} \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta,$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to both \mathbf{a} and \mathbf{b} .

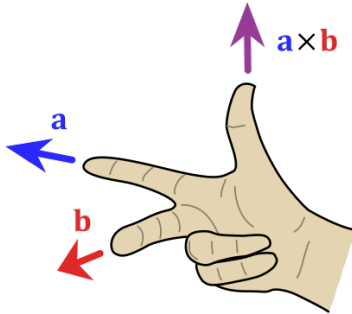
Since there are two (opposite) unit vectors that are perpendicular to both of them, we pick $\hat{\mathbf{n}}$ to be the one that is perpendicular to \mathbf{a}, \mathbf{b} in a *right-handed* sense.

Vector cross product defined only for 3-dimensional vectors!!!

Geometric view



Right hand rule



The vector product satisfies the following properties:

- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}.$
- $\mathbf{a} \times \mathbf{a} = \mathbf{0}.$
- $\mathbf{a} \times \mathbf{b} = \mathbf{0} \Rightarrow \mathbf{a} = \lambda \mathbf{b}$ for some $\lambda \in \mathbb{R}$ (or $\mathbf{b} = \mathbf{0}$).
- $\mathbf{a} \times (\lambda \mathbf{b}) = \lambda(\mathbf{a} \times \mathbf{b}).$
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}.$

There is a way of calculating vector products:

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}) \times (b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}) \\ &= (a_2b_3 - a_3b_2)\hat{\mathbf{i}} - (a_1b_3 - a_3b_1)\hat{\mathbf{j}} + (a_1b_2 - a_2b_1)\hat{\mathbf{k}} \\ &= (a_2b_3 - a_3b_2)\hat{\mathbf{i}} + (a_3b_1 - a_1b_3)\hat{\mathbf{j}} + (a_1b_2 - a_2b_1)\hat{\mathbf{k}}\end{aligned}$$

There are more convenient ways to calculate vector cross products.

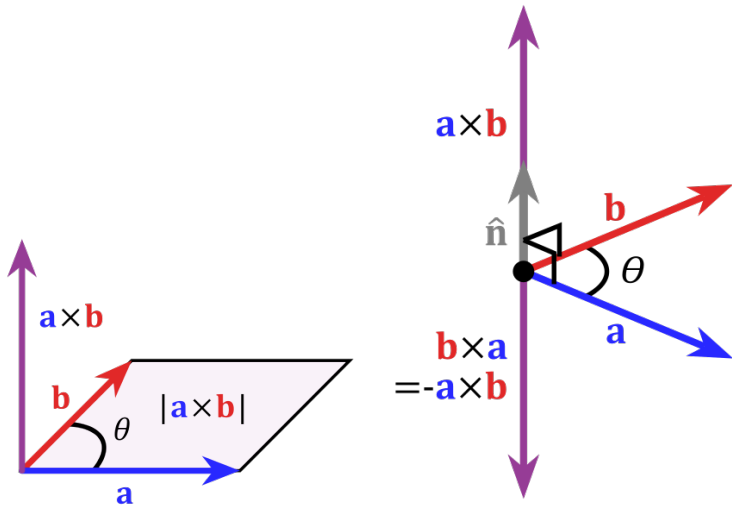
Examples

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2c - 3b \\ 3a - 1c \\ 1b - 2a \end{bmatrix}$$

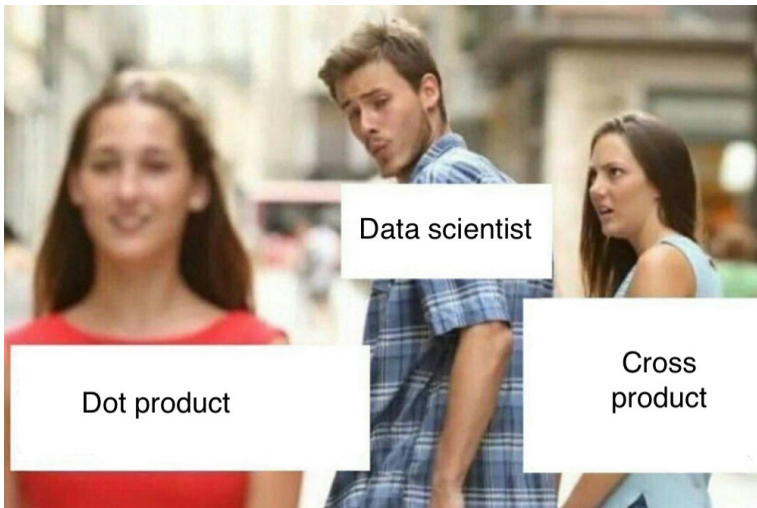
Simplify expression

$$(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b})$$

Interpretation



Why would you assign such labels?



Part 3. Matrices

Definition

Matrix A is a rectangular table of numbers with m rows and n columns.

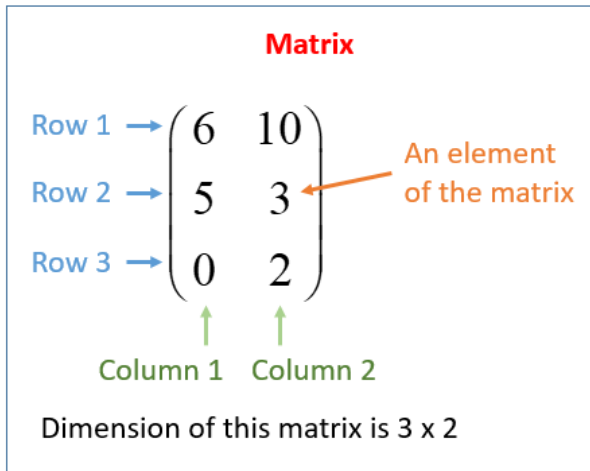
Example of a 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Example of a 2×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Sizes



Quick check. What are the sizes?

$$A = \begin{bmatrix} 3 & 4 & 9 \\ 12 & 11 & 35 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & -6 & 13 \\ 32 & -7 & -23 \\ -9 & 9 & 15 \\ 8 & 25 & 7 \end{bmatrix}$$

Different kinds of matrices

- Square and rectangular matrix
- Symmetric matrix
- Triangle matrix
- Diagonal matrix
- Identity matrix
- Zero matrix

Examples

(1) Square matrix : (#rows = #columns)

$$\begin{bmatrix} 3 & -1 & -3 \\ 2 & 4 & 0 \\ -1 & 5 & 6 \end{bmatrix}$$

Main diagonal

(9) Column matrix (n -vector)

(8) Row matrix (n -vector):
[1 7 -3] is a 3-vector

$$\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \text{ is a 3-vector}$$

(2) Upper triangular matrix :

$$\begin{bmatrix} 3 & -1 & -3 \\ 0 & 4 & 7 \\ 0 & 0 & 6 \end{bmatrix}$$

(3) Lower triangular matrix :

$$\begin{bmatrix} 3 & 0 & 0 \\ -2 & 4 & 0 \\ 2 & 7 & -6 \end{bmatrix}$$

(5) Diagonal matrix :

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

(6) Identity matrix :

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(7) Zero matrix :

$$O_{2 \times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, O_{2 \times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Source: <https://medium.com/@nithishraghav/linear-algebra-for-aspiring-data-scientists-part-i-37a9b63c031f>

Operations. Transpose a matrix

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\forall A, A^{TT} = A$$

What does it mean if $A^T = A$

Operations. Addition, multiplication by a scalar

Element-wise addition:

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} = \begin{bmatrix} 1+a & 4+d \\ 2+b & 5+e \\ 3+c & 6+f \end{bmatrix}$$

Properties. A, B, C are matrices

- $A + B = B + A$ (commutative)
- $A + (B + C) = (A + B) + C$ (associative)
- $B = \lambda A, \lambda \in \mathbb{R}$ (element-wise multiplication)

$$B = \lambda A, \quad \forall 1 \leq i \leq m; 1 \leq j \leq n : b_{ij} = \lambda a_{ij}$$

Addition defined only for matrices of the same size!

Examples

Trace of a matrix

A is $(m \times m)$ matrix

$$Tr(A) = \sum_{i=1}^m a_{ii}$$

Trace can be applied to square matrices only!

Examples

Homework. Linearity of the trace

Write a program to demonstrate Linearity of the trace

$$\text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B)?$$

$$\lambda \in \mathbb{R}, \quad \text{Tr}(\lambda A) = \lambda \text{Tr}(A)?$$

End of Lecture #2

Useful links

- <https://www.geogebra.org>
- https://youtu.be/fNk_zzaMoSs
- <http://immersivemath.com/ila>