Mathematical Analysis. Assignment 4.
Mean Value Theorems & l'Hôpital's rule

1. Prove that the derivative of a function $f(x) = x(x-1)(x-2)(x-3)(x-4)$ has four distinct roots that belong to intervals $(0;1)$, $(1;2)$, $(2;3)$, $(3;4)$, the multiplicity of each root being equal to 1.

2. Using mean value theorems prove the inequalities[1]

   (a) $\frac{x}{x+1} < \ln(1+x) < x$ for $x > 0$;

   (b) $e^x \geqslant 1 + x$, $x \in \mathbb{R}$.

3. Rolle's theorem states that if function $f(x)$

   (a) is continuous on $[a;b]$,

   (b) is differentiable on $(a;b)$,

   (c) has equal values at the endpoints of the interval, that is $f(a) = f(b)$ then there exists $c \in (a;b)$ such that $f'(c) = 0$.

   Show that all conditions of the theorem are substantial, i.e. that the theorem does not hold if you omit at least one of them[2].

4. Find the following limits (use l'Hôpital's rule[3]):

   (a) $\lim\limits_{x\to 1} \frac{x^{100}-100x+99}{x^{50}-50x+49}$;

   (b) $\lim\limits_{x\to 0} \frac{e^{\sin x}-e^x}{\sin x - x}$;

   (c) $\lim\limits_{x\to 0^+} \frac{3+\ln x}{2-3\ln(\sin x)}$;

   (d) $\lim\limits_{x\to 0} \sin x \ln(\cot x)$;

   (e) $\lim\limits_{x\to +\infty} (\pi - 2\arctan\sqrt{x})\sqrt{x}$;

   (f) $\lim\limits_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x-1}\right)$;

   (g) $\lim\limits_{x\to 1} x^{\frac{1}{x-1}}$;

   (h) $\lim\limits_{x\to 0} \left(\frac{(1+x)^{\frac{1}{x}}}{e}\right)^{\frac{1}{x}}$;

   (i) $\lim\limits_{x\to \frac{\pi}{2}^-} (\tan x)^{\cos x}$;

   (j) $\lim\limits_{x\to 0} \frac{\ln\frac{1+x}{1-x} - 2x}{x - \sin x}$;

   (k) $\lim\limits_{x\to 0} \frac{(a+x)^x - a^x}{x^2}$, $a > 0$.

   (l) $\lim\limits_{x\to 0} \frac{2\tan 3x - 6\tan x}{3\arctan x - \arctan 3x}$;

   (m) $\lim\limits_{x\to 0^+} x^\alpha \ln^\beta\left(\frac{1}{x}\right)$;

   (n) $\lim\limits_{x\to +\infty} x^\alpha a^x$, $a > 0$, $a \neq 1$;

   (o) $\lim\limits_{x\to +\infty} \left(x^{\frac{7}{8}} - x^{\frac{6}{7}} \ln^2 x\right)$;

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[1](a) Apply Lagrange mean value theorem to function $f(t) = \ln(1+t)$, $t \in [0;x]$. (b) If $x \in (-1;0)$ consider $g(t) = e^t - t$, $t \in [x;0]$; if $x > 0$ consider $g(t) = e^t - t$, $t \in [0;x]$. Otherwise this inequality is obvious.

[2]It implies that you have to provide 3 counterexamples.

[3]The dreams come true...

(p) $\lim\limits_{x\to 1}\left(\frac{\alpha}{1-x^{\alpha}}-\frac{\beta}{1-x^{\beta}}\right)$;

(q) $\lim\limits_{x\to +\infty}(3x^2+3^x)^{\frac{1}{x}}$.

Answer. (a) $\frac{198}{49}$; (b) 1; (c) $-\frac{1}{3}$; (d) 0; (e) $\frac{1}{2}$; (g) e; (h) $e^{-\frac{1}{2}}$; (i) 1; (j) 4; (k) $\frac{1}{a}$; (l) 2; (m) 0; (n) 0 if $0<a<1$; $+\infty$ if $a>1$; (o) $+\infty$; (p) $\frac{\alpha-\beta}{2}$; (q) 3.

5. Show that l'Hôpital's rule is not applicable for the limits below and calculate them using some other methods:

   (a) $\lim\limits_{x\to\infty}\frac{x+\cos x}{x-\cos x}$;

   (b) $\lim\limits_{x\to 0}\frac{x^3\sin\frac{1}{x}}{\sin^2 x}$.

   Answer. (a) 1; (b) 0.

6. Let us suppose that $f(x)$ has at least three derivatives in the neighborhood of point a. Calculate the limits

   (a) $\lim\limits_{h\to 0}\frac{f(a+h)+f(a-h)-2f(a)}{h^2}$;

   (b) $\lim\limits_{h\to 0}\frac{f(a+3h)-3f(a+2h)+3f(a+h)-f(a)}{h^3}$.

   Answer. (a) $f''(a)$; (b) $f'''(a)$.

7. Let us consider $f(x)=\begin{cases} e^{-\frac{1}{x^2}}, & x\neq 0,\\ 0, & x=0.\end{cases}$ Prove that this function is infinitely differentiable for all $x\in\mathbb{R}$. Find $f^{(k)}(0)$ as well.

   Answer. $f^{(k)}(0)=0$, $k\in\mathbb{N}$.

8. Find the following limits:

   (a) $\lim\limits_{x\to 1^-}\ln x\cdot\ln(1-x)$;

   (b) $\lim\limits_{x\to 0^+}\frac{\ln x\cdot\ln(1+x)}{\sqrt{x}}$;

   (c) $\lim\limits_{x\to 0}\frac{\cos x-\cos 3x+x^3\cos\frac{\pi}{x}}{x^2}$.

   Answer. (a) 0; (b) 0; (c) 4.