Discrete Math

Lab 3 - September, 22

Agenda

- Homework review
- Naïve Set Theory
- Logic

Naïve Set Theory

Definitions

- A **set** is an unordered collection of objects, called **elements** or members of the set. A set is said to contain its elements.

  $a \in A$ denotes that a **is an element** of the set A.

  $a \notin A$ denotes that a **is not an element** of the set A.

  $\emptyset$ denotes an **empty set** (a set that has no elements)

- The cardinality $|A|$ of a finite set A is a number of distinct elements in A.

- The infinite sets A and B have the same cardinality if and only if there is a one-to-one correspondence from A to B.

Set builder notation

- **Set builder notation** – a way to describe a set by stating the properties the elements must have to be members

  Example: $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

  $O = \{x \in Z+ \mid x \text{ is odd and } x < 10\}$

  Domain (the set of entities over which x may range)

- The set $\{ x \in U \mid P(x) \}$ is the set of all x from U such that P(x) is true.

  Predicate P(x) is a statement about some object x that is either true or false.

Discussion

- Why the notion of a set is important in Computer Science?

Definitions

- Two sets are equal if and only if they have **the same elements**.
  $A = B$ if and only if $\forall x\ (x \in A \leftrightarrow x \in B)$
  Example: $\{1, 2\} = \{2, 1\}$

- Two sets are equivalent if they have **the same number of elements**.
  Example: {1,3,5} and {January, March, May} are equivalent

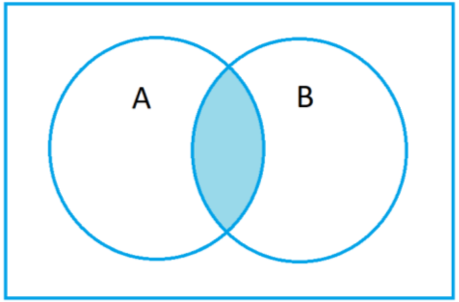
- The set A is a **subset** of B if and only if every element of A is also an element of B. $A \subseteq B$ if and only if $\forall x\ (x \in A \rightarrow x \in B)$

Exercises I

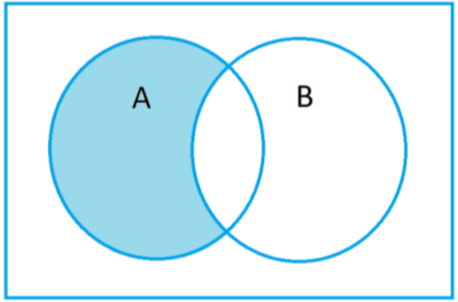
1. Are sets A and B equal? $A = \{1, 2, 3\}$, $B = \{3, 2, 2, 1\}$
2. Are sets A and B equal? $A = \{\emptyset\}$, $B = \emptyset$
3. Let $S = \{ \{1, 2\}, \{2, 3\}, 4\}$. Is 1 an element of S? List all elements of S.
4. Are sets Z and Q (of integer and rational numbers) equivalent?
5. True or false? $\{a, b\} \subseteq \{b,a,c\}$
6. True or false? $\{a\} \subseteq \{\{a\}\}$
7. True or false? $\{a\} \in \{\{a\}\}$
8. True or false? $\emptyset \subseteq \{b,a,c\}$
9. True or false? $\emptyset \in \{a,b,c\}$
10. True or false? $\{a, b\} \subset \{a,a,b\}$

Set Theory: operations on Venn diagrams

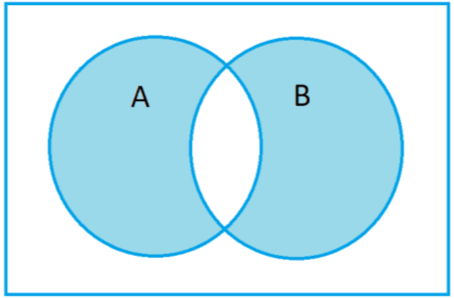
Intersection $A \cap B$



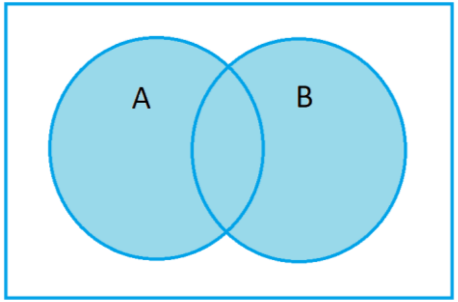
Difference $A \setminus B$



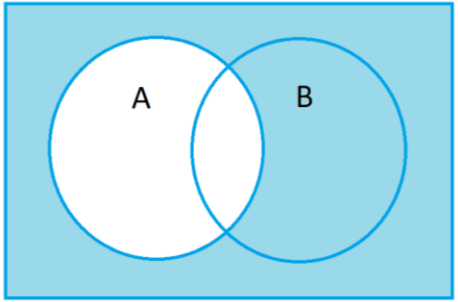
Symmetric difference $A \oplus B$



Union $A \cup B$



Complement $\bar{A}$



Fundamental Set Properties

Idempotence

$A \cup A = A$

$A \cap A = A$

Associativity

$(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

Commutativity

$A \cup B = B \cup A$

$A \cap B = B \cap A$

Distributivity ($\cap$ over $\cup$)

$[A \cap (B \cup C)] = [(A \cap B) \cup (A \cap C)]$

$[(A \cup B) \cap C] = [(A \cap C) \cup (B \cap C)]$

Complement

$A \cup \overline{A} = U$

$A \cap \overline{A} = \varnothing$

Involution

$\overline{(\overline{A})} = A$

Domination

$A \cup U = U$

$A \cap \varnothing = \varnothing$

Identity

$A \cup \varnothing = A$

$A \cap U = A$

De Morgan's Laws

$\overline{A \cup B} = \overline{A} \cap \overline{B}$

$\overline{A \cap B} = \overline{A} \cup \overline{B}$

Distributivity ($\cup$ over $\cap$)

$[A \cup (B \cap C)] = [(A \cup B) \cap (A \cup C)]$

$[(A \cap B) \cup C] = [(A \cup C) \cap (B \cup C)]$

Complement (continued)

$\overline{\varnothing} = U$

$\overline{U} = \varnothing$

Exercises II

1. Let A={1,2,3,4,5}, B={0,3,6}. Find A ∩ B, A ∪ B, A \ B, B \ A, A⨁B

2. Use Venn diagrams to prove or disprove the following equation:
   [(A ∩ B) \ C] ∪ [(B ∩ C) \ A] = [(A ∩ B) ∪ (B ∩ C) \ (A ∩ B ∩ C)]

3. Simplify: $\bar{A} \cup \left(\overline{A \cup \bar{B} \cup \bar{C}}\right) \cup \left(B \cap \overline{(A \cup C)}\right)$

4. There are 35 students. Each is using at least one way of transportation: subway, bus or tram. Only 6 students are using all 3 ways. Both subway and bus are used by 15 students, subway+tram - 13 students, tram+bus - 9 students. How many students are using the only way of transportation?

Propositional Logic

Propositions

- A **proposition** is a declarative sentence that is either true or false, but not both
- Examples of the sentences that are NOT propositions:
    1. What time is it?
    2. Read this carefully.
    3. $x + 1 = 2$.
    4. $x + y = z$.

- A **tautology** - a compound proposition that is always **true**
- A **contradiction** - a compound proposition that is always **false**

Well-formed propositional formulas

- **Propositional formulas** are constructed from atomic propositions by using logical operations (see formal definition in the lecture 2).

- The truth of a propositional formula $S(x_1, x_2, \dots, x_n)$ is a function of the truth values of the atomic propositions $x_1, x_2, \dots, x_n$ it contains.

- A **truth table** shows whether a propositional formula is true or false for each possible truth assignment.

- When two compound propositions always have the same truth value we call them **equivalent**

Operations & truth tables

X	Y	X&Y conjunction	XvY disjunction	X→Y Implication	X↔Y equivalence
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

X	¬X negation
1	0
0	1

Implications

Let’s look at the formula X→Y

- X is the **antecedent** or hypothesis,
- Y is the **consequent** or conclusion
- Y→X is the **converse** of X→Y
- ¬Y→ ¬X is the **contrapositive** of X→Y
- ¬X→ ¬Y is the **inverse** of X→Y

X	Y	¬X	¬Y	X→Y	Y→X	¬Y→ ¬X	¬X→ ¬Y
0	0			1			
0	1			1			
1	0			0			
1	1			1			

Exercise:
Show that a conditional statement and its contrapositive are equivalent

Quantifiers

- A **predicate** (propositional function) - a statement that may be true or false depending on the values of its variables

- **Quantification** expresses the extent to which a predicate is true over a range of elements

Statement		**Negation**	
$\forall x\, P(x)$	P(x) is true for every x	$\exists x\, \neg P(x)$	There is an x for which P(x) is false
$\exists x$ P(x)	There is an x for which P (x) is true.	$\forall x\, \neg P(x)$	P(x) is false for every x

Logical equivalences

Equivalence	Name
p & 1 ≡ p p ∨ 0 ≡ p	Identity laws
p ∨ 1 ≡ 1 p & 0 ≡ 0	Domination laws
p ∨ p ≡ p p & p ≡ p	Idempotent laws
¬(¬p) ≡ p	Double negation law
p ∨ q ≡ q ∨ p p & q ≡ q & p	Commutative laws

Equivalence	Name
(p ∨ q) ∨ r ≡ p ∨ (q ∨ r) (p & q) & r ≡ p & (q & r)	Associative laws
p ∨ (q & r) ≡ (p ∨ q) & (p ∨ r) p & (q ∨ r) ≡ (p & q) ∨ (p & r)	Distributive laws
¬(p & q) ≡ ¬p ∨ ¬q ¬(p ∨ q) ≡ ¬p & ¬q	De Morgan’s laws
p ∨ (p & q) ≡ p p & (p ∨ q) ≡ p	Absorption laws
p ∨ ¬p ≡ 1 p & ¬p ≡ 0	Negation laws

Exercises III

1. Find the truth table of the compound propositions:
   1. $(p \,\&\, q) \to (p \vee \neg r)$.
   2. $(p \to q) \,\&\, (\neg p \to r)$
2. Give the contrapositive of the statement:
   If $|x| = x$, then $x \geq 0$.
3. Show that $p \leftrightarrow q$ and $(p \,\&\, q) \vee (\neg p \,\&\, \neg q)$ are logically equivalent, using truth tables.
4. Show that $\neg(p \vee (\neg p \,\&\, q))$ and $\neg p \,\&\, \neg q$ are logically equivalent by developing a series of logical equivalences.
5. Show that the proposition is always true: $(\neg q \,\&\, (p \to q)) \to \neg p$

Homework

Submit on Moodle by 10pm September 25 **the pdf file named gg-surname.pdf**

1. Show that $(p \to q) \to r$ and $p \to (q \to r)$ are not logically equivalent.

2. Show that each of these conditional statements is a tautology by using truth tables.

a) $(p \,\&\, q) \to p$

b) $p \to (p \lor q)$

3. Use a truth table to verify the first De Morgan law: $\neg(p \,\&\, q) \equiv \neg p \lor \neg q$.

4. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for all sets A, B, and C.

5. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{0, 3, 6\}$. Find

a) $A \cup B$ b) $A \cap B$ c) $A \setminus B$ d) $B \setminus A$.