Essentials of Analytical Geometry and Linear Algebra. Lecture 4.

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End of Lecture #3

Review. Lecture 3

- Part 1 (recap). Matrices. Transpose, Addition, Scalar multiplication
- Part 2. Matrix multiplication
- Part 3. Determinants. Scalar Triple Product

Quiz in class

Go to `http://b.socrative.com`

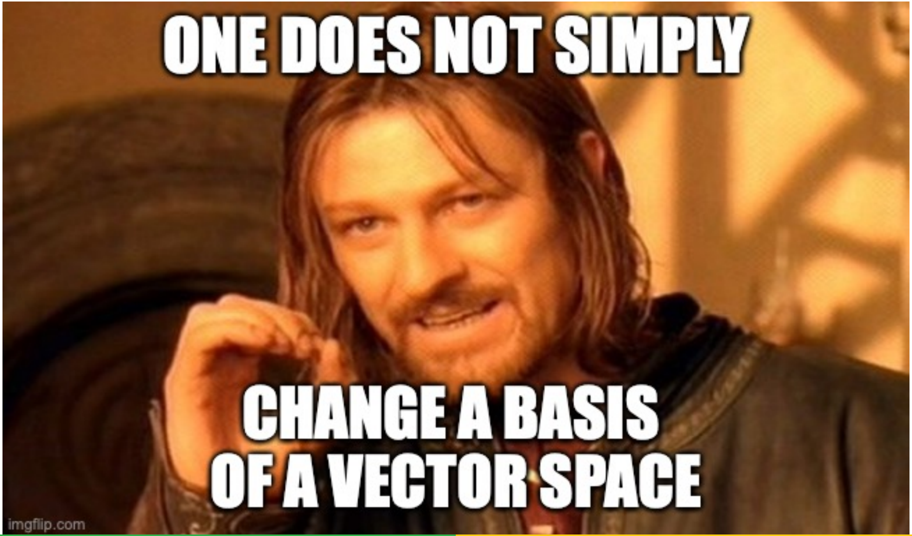
Type Room: **LINAL**

Answer questions.

Lecture 4. Outline

- Part 1. Change of basis and coordinates
- Part 2. Matrix rank
- Part 3. Matrix inverse

Change of basis and coordinates


ONE DOES NOT SIMPLY
CHANGE A BASIS
OF A VECTOR SPACE
imgflip.com

Here we are going to derive the formula.

Break, 5 min.

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Matrix rank

Consider the following matrices $(a \neq b \neq 0)$

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} a & 0 & a \\ 0 & b & b \end{bmatrix}$$

$$D = \begin{bmatrix} a & 0 & a & -2a \\ 0 & b & b & -2b \end{bmatrix}$$

$$E = \begin{bmatrix} a & 0 & a & -2a & 3a \\ 0 & b & b & -2b & 2b \end{bmatrix}$$

What can you say about columns-vectors inside each matrix?
Which matrices contain basis for $\mathbb{R}^2$?
Which matrices contain ‘redundant’ information about space spanned by column-vectors?

Matrix rank

column rank

The *column rank* of a matrix is the maximum number of linearly independent columns.

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The *row rank* of a matrix is the maximum number of linearly independent rows.

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Theorem

The column rank and row rank are equal for any $m \times n$ matrix.

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Theorem

The column rank and row rank are equal for any $m \times n$ matrix.

So, there is only one matrix rank. $rank(A) = rank(A^\top)$
Notation: r, $rank(A)$

Examples. Calculate rank of a matrix and its transpose

A = $\begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $rank(A) =?$, $rank(A^\top) =?$

B = $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \end{bmatrix}$, $rank(B) =?$, $rank(B^\top) =?$

C = $\begin{bmatrix} a & 0 & a \\ 0 & b & b \end{bmatrix}$, $rank(C) =?$, $rank(C^\top) =?$

D = $\begin{bmatrix} a & 0 & a & -2a \\ 0 & b & b & -2b \end{bmatrix}$, $rank(D) =?$, $rank(D^\top) =?$

More examples

Important properties of rank

Given $m \times n$ matrix A.

- **maximum** possible rank of A equals $\min(m, n)$:
- $rank(A) = rank(AA^\top) = rank(A^\top A) = rank(A^\top)$
- $rank(A + B) \leq rank(A) + rank(B)$
- $rank(AB) \leq \min(rank(A), rank(B))$

What about $rank(\lambda A)$?
$\lambda \in \mathbb{R}$

Break, 5 min.

Matrix inverse

Simple picture

Matrix B is called inverse of a square matrix A if

$$AB = BA = I$$

Notation

$$B = A^{-1}$$

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$$AA^{-1} = A^{-1}A = I$$

Example

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 2 & -1 \\ -2 & 4 \end{bmatrix}$$

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$$AA^{-1} = \frac{1}{6}\begin{bmatrix} 4 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -2 & 4 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 4*2+1*(-2) & 4*(-1)+1*(4) \\ 2*2+2*(-2) & 2*(-1)+2*4 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = I$$

What if matrix A is **nonsquare**?

Left and Right inverse

Left inverse

Consider an $m \times n$ matrix A and $n \times m$ matrix B.
If $BA = I$, then we say B is the **left inverse** of A.

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If $AC = I$, then we say C is the **right inverse** of A.

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Right inverse

Consider an $m \times n$ matrix A and $n \times m$ matrix C.
If $AC = I$, then we say C is the **right inverse** of A.

Let A be a square matrix. Show that its left and right inverses are the same.
Hint: use associative property of matrix multiplication.

If A has an inverse, then A is ***invertible***.

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Are all matrices invertible?

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Are all matrices invertible?

Provide a simple counter-example of noninvertible 3×3 matrix.

Important property

If A and B are invertible and AB is invertible, then

$$(AB)^{-1} = B^{-1}A^{-1}$$

Prove it, using pen and paper.

Hint: multiply $(B^{-1}A^{-1})$ by $(AB)^{-1}$.

How to find an inverse of 2×2 matrix A?

$\mathsf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$

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Step 0: Find determinant: $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ = **ad-bc.** If $det(A) = 0$, then A^{-1} **does not exist**.

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Step 1: Swap **main** diagonal elements:

$$\begin{bmatrix} \mathbf{d} & b \\ c & \mathbf{a} \end{bmatrix},$$

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Step 1: Swap **main** diagonal elements:

$$\begin{bmatrix} \mathbf{d} & b \\ c & \mathbf{a} \end{bmatrix},$$

Step 2: Multiply off-diagonal elements by -1:

$$\begin{bmatrix} d & \mathbf{-b} \\ \mathbf{-c} & a \end{bmatrix}$$

Step 3: Divide by $det(A)$. So, $A^{-1} = \frac{1}{det(A)} \begin{bmatrix} d & \mathbf{-b} \\ \mathbf{-c} & a \end{bmatrix}$

Exercise

A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$; A^{-1} = $\frac{1}{det(A)}\begin{bmatrix} d & \textbf{-b} \\ \textbf{-c} & a \end{bmatrix}$

Check with pen and paper

$$A^{-1}A = ...$$

Example

Find the inverse and confirm that $AA^{-1} = A^{-1}A = I$

A= $\begin{bmatrix} 3 & 1 \\ 0 & b \end{bmatrix}$

Important case: Orthogonal matrix

$$A^{-1} = A^\top$$

Example

Rotation matrix is an example of an orthogonal matrix.

Rotation matrix

$$A = \begin{bmatrix} \cos(\theta) & \text{-}\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix};\ A^\top = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \text{-}\sin(\theta) & \cos(\theta) \end{bmatrix}$$

Find A^{-1}

General algorithm

We may skip this or postpone till the next lecture...

When a matrix is invertible?

For an $n \times n$ matrix A the following statements are equivalent:

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- The rank of the matrix A is n

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- The rank of the matrix A is n
- $A^\top$ is invertible
- The rows of matrix A form a basis for $\mathbb{R}^n$

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- A is invertible
- The determinant of matrix A is **nonzero** $det(A) \neq 0$
- The columns of matrix A form a basis for $\mathbb{R}^n$
- The rank of the matrix A is n
- $A^\top$ is invertible
- The rows of matrix A form a basis for $\mathbb{R}^n$
- $A\mathbf{x} = \mathbf{b}$ has exactly one solution
- $A\mathbf{x} = \mathbf{0}$ has only a *trivial* solution ($\mathbf{x} = \mathbf{0}, zerovector$)

Applications

Here we open Google Colab...

End of Lecture #4

Useful links

- https://www.geogebra.org
- https://youtu.be/fNk_zzaMoSs
- http://immersivemath.com/ila