Essentials of Analytical Geometry and Linear Algebra. Lecture 1.

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September 4, 2020

Outline

- Part 1. About the course
- Part 2. Applications of Analytical Geometry and Linear Algebra
- Part 3. Introduction. Vector spaces. Linear independence. Basis

Main questions for today's lecture

- What is this course about?

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- How to get high grade in this course?
- How to use this course in your projects?

Main questions for today's lecture

What is this course about?

Topics of the course

1. Vector spaces, matrices and transformations in 2D and 3D
2. Lines and planes
3. Conics or quadric curves
4. Quadratic surfaces
5. Polar and spherical coordinates

Goals of this course

What you will learn in this course?

1. to use vectors and matrices to solve applied problems
2. to change basis in a vector space
3. to calculate determinants
4. to recognise different transformations, such as rotation, reflection, shear, etc.
5. to work with lines and planes in 2D and 3D
6. to operate with quadric curves, such as ellipse, hyperbola and parabola
7. many more + examples in Python :)

Main questions for today's lecture

How to get a high grade in this course?

Grading in the course

1. Labs 5%
2. Test 1 15%
3. Midterm 30%
4. Test 2 15%
5. Final Exam 35%

In total, 100 %

How to get the highest grade?

1. Attend classes (either online or offline)
    1. Labs
    2. Tutorials
    3. Lectures
2. Solve assignments (also at home) on your own and in groups
3. Read books (check the list in moodle)
4. Come to office hours (either online or offline)

Repeat :)

What is the exact process you can follow?

1. **Friday**
   1. attend lecture
   2. attend tutorial
   3. review materials after classes

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    2. ask your questions
    3. participate in labs

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    2. ask your questions
    3. participate in labs
4. **Tuesday - Thursday**
    1. come to office hours
    2. apply your knowledge by some programming (yay!)

Team of the course and Materials

- Vladimir Ivanov (PhD), Principal Instructor, Lectures
- Mohammedreza Bahrami (PhD), Tutorials
- Anastasia Puzankova, Labs
- Oleg Bulichev, Labs

Resources: Books, Assignments, Useful links, etc.

Please, check Moodle!

Break, 5 min.

Main questions for today's lecture

How to use this course in your projects?

or more general,

What are the applications of Linear Algebra and Analytical Geometry?

Main questions for today's lecture

Applications of Linear Algebra and Analytical Geometry

Applications of AGLA in Computer Science and Engineering

Areas:

1. Computer Graphics and Computer Games
2. Machine Learning, Data Analysis
3. Natural Language Processing
4. Robotics
5. Computer Vision
6. and many, many other areas...

Applications

Computer Graphics and Computer Games

- 2D/3D graphics
- Projective geometry, Homogeneous coordinates
- Collision detection in games. Calculation of trajectories

Machine Learning, Data Analysis

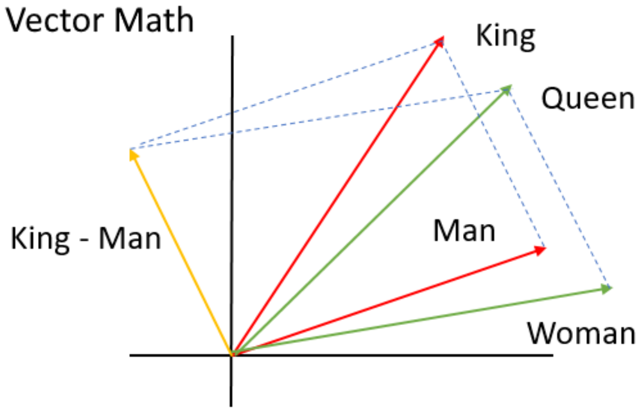
- Linear Regression
- Eigendecomposition
- Singular Value Decomposition
- Covariance matrix
- Linear Layers, Attention Mechanisms in Neural Networks

Demo 1

Natural Language Processing

- Document-term matrix
- Latent semantic analysis
- Words vectors (in a semantic vector space): word2vec
- Semantic similarity as a cosine measure between words

Demo 2



https://cfss.uchicago.edu/slides/text-analysis-fundamentals-and-sentiment-analysis/#8

Applications

Robotics

- Orientation in 3D space
- Representation of movement
- Representation of forces, velocities, moments...

Computer Vision and Digital Signal Processing

- Fast Fourier Transform
- Convolutions
  (filters applied to images are, in fact, matrices)
- Gram Matrix in Neural Style Transfer
- Haar Transform, Haar Cascades

Applications for faster computation

- Modern computer architectures allow parallel calculations
- Numpy is a Python library that leverages this
- So, a programmer does not need to resort to explicit loops of individual scalar operations

Array programming

For more information, check

- **MIMD**: Multiple instruction, Multiple data
- **SIMD**: Single instruction, Multiple data

https://en.wikipedia.org/wiki/Array_programming

Break, 5 min.

Good to know: Google’s original PageRank algorithm for ranking webpages by “importance” can be explained as the search for an **eigenvector** of a matrix.

Very important note!

- The only way to learn mathematics is **to solve** math problems.
- Watching and re-watching video lectures is important and helpful, but it’s not enough.
- If you really want to learn linear algebra, you need to solve problems **by hand**.
- Checking your work on a computer is a recommended second step.

Agenda: This week

Vectors and Matrices

- Points and Vectors
- Vector Addition. Scalar Vector Multiplication
- Properties of Vector Arithmetic
- Vector spaces, Subspaces
- Span, Linear Independence
- Vector Bases and Coordinates

Agenda: Week 2

Vectors and Matrices

- The Dot Product and its properties
- Vector Length. Vector Orthogonality
- Outer Product
- Vector Cross Product

Agenda: Week 3

Vectors and Matrices

- Matrices
- Operations with matrices: Transpose, Addition, Scalar multiplication
- Matrix multiplication
- Change of basis

Agenda: Week 4

Vectors and Matrices

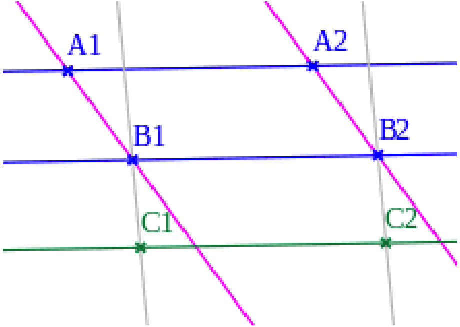
- Determinants
- Inverse matrix
- Rank
- Transformations in 2D and 3D

Notation

- We denote points by capital italic letters, e.g., $A, B, ..., Q, ...$
- We denote numbers by Greek letters, e.g., $\alpha, \beta, ..., \lambda, \theta, ...$ and sometimes by Latin letters, $a, b, ..., v, u, x, ...$
- We denote vectors by **bold** letters, e.g., $\mathbf{a}, \mathbf{b}, ..., \mathbf{v}, \mathbf{u}, \mathbf{x}, ...,$
- and also we denote vectors by a letter with an arrow, e.g. $\vec{a}, \vec{b}, \vec{u}$
- and sometimes we denote vectors by end-points, e.g. $\overline{AB}, \overline{BC}, \overline{OA}$
- $\mathbb{R}$ is the set of real numbers
- $\mathbb{C}$ is the set of complex numbers

Introduction

Aside: affine geometry



- affine geometry considers points and ‘parallel’ lines
- **no** notion of angles and distance, so you cannot measure them
- **no** notion of perpendicularity

Is affine geometry different from the Euclinean geometry? Why?

Points and Vectors (informally). Direction

Vector. Geometrical point of view. Vectors as ‘arrows’ in plane or in 3D space

Let A and B be two points.
A directed line segment from A to B is denoted by: $\overline{AB}$
This directed line segment constitutes a vector.

Points and Vectors (informally). Direction

Vector. Geometrical point of view. Vectors as ‘arrows’ in plane or in 3D space

Let A and B be two points.
A directed line segment from A to B is denoted by: $\overline{AB}$
This directed line segment constitutes a vector.

Thus, each vector can be associated with a notion of *direction*. In this case, we can think of a vector as an “arrow” in space.

Points and Vectors (informally). Magnitude

Length (or Magnitude) of a Vector

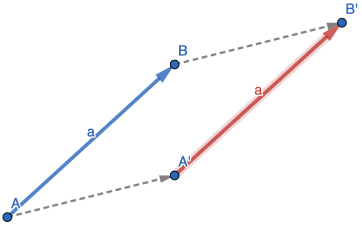
Also, often (**but not always!**) vector has a *length* (or a magnitude). The length of a vector is a number is denoted by $\|\mathbf{v}\|$.

Unit vector

A *unit vector*, $\mathbf{u}$ is a vector with unit length (so $\|\mathbf{u}\|$=1). We can derive a unit vector as $\mathbf{u} = \mathbf{v}/\|\mathbf{v}\|$.

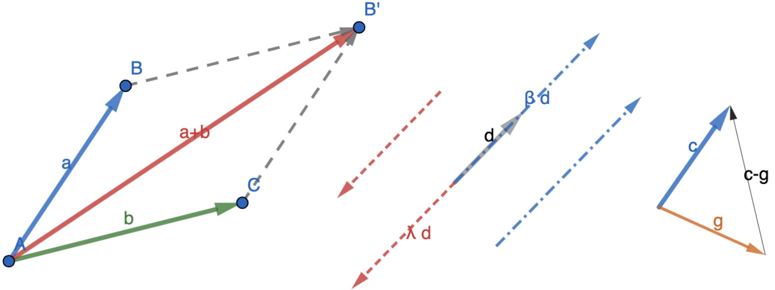
The length of a vector is closely related to the **dot product**, an operation which will be discussed in the next lecture. Therefore, $\mathbf{v}/\|\mathbf{v}\|$ is called a normalized vector.

If you move the line segment to another line segment with the same direction and length, they constitute **the same vector**.



Examples: Points and Vectors (informally)

Note that vector $\lambda\mathbf{d}$ is either parallel ($\lambda > 0$) to or anti-parallel ($\lambda < 0$) to $\mathbf{d}$.



$$\lambda, \beta \in \mathbb{R}$$

In this figure: $\lambda > 0$?

What if $\lambda = 0$?

Vector spaces

Vector space definition

Vector space

A *vector space* V over $\mathbb{R}$ (or $\mathbb{C}$) is a collection of vectors $\mathbf{v} \in V$, together with two operations:

- $\mathbf{a} + \mathbf{b}$, addition of two vectors and
- $\lambda \mathbf{a}$, multiplication of a vector with a scalar ($\lambda \in \mathbb{R}$)

A scalar is a number from $\mathbb{R}$ or $\mathbb{C}$, respectively.

Addition and multiplication SHOULD satisfy following axioms

Vector addition axioms

Vector addition $\mathbf{a} + \mathbf{b}$ is defined $\forall \mathbf{a}, \mathbf{b} \in V$

Vector addition has to satisfy the following axioms:

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (commutativity)
2. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ (associativity)
3. There is a vector $\mathbf{0}$ (zero vector) such that $\mathbf{a} + \mathbf{0} = \mathbf{a}$. (identity)
4. For each vector $\mathbf{a}$, there exists a vector $(-\mathbf{a})$ such that $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$ (inverse)

Scalar multiplication axioms

$\lambda\mathbf{a}$ is defined $\forall\lambda \in \mathbb{R}, \forall\mathbf{a} \in V$

Scalar multiplication has to satisfy the following axioms:

1. $\lambda(\mathbf{a}+\mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$.
2. $(\lambda+\mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$.
3. $\lambda(\mu\mathbf{a}) = (\lambda\mu)\mathbf{a}$.
4. $1\mathbf{a} = \mathbf{a}$.

The scalar is called a *scalar*, because it **scales** a vector :)

Vectors as lists of numbers

Column vectors. Examples

$\begin{bmatrix}1\\2\end{bmatrix}$, $\begin{bmatrix}1\\2\\3\end{bmatrix}$, $\begin{bmatrix}a\\b\\c\end{bmatrix}$ — we will use **this notation**! We represent vectors as **columns**!

Vectors as lists of numbers

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Row vectors. Examples

$\begin{bmatrix}3 & 4\end{bmatrix}$, $\begin{bmatrix}3 & 4 & 5\end{bmatrix}$, $\begin{bmatrix}x & y & z\end{bmatrix}$ — **not** this notation! Even though vectors can be represented as rows.

Vectors as lists of numbers

Column vectors. Examples

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$$\begin{bmatrix}3 & 4\end{bmatrix} \neq \begin{bmatrix}3\\4\end{bmatrix}$$

Transposition

Transposition

$$\begin{bmatrix} 3 & 4 \end{bmatrix}^\top = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix}^\top = \begin{bmatrix} 3 & 4 \end{bmatrix} \tag{2}$$

This operation transforms a row vector a to column vector and back

For any vector

$$(\mathbf{v}^\top)^\top = \mathbf{v}$$

Examples

$\mathbb{R}^n$ is a vector space with component-wise addition and scalar multiplication. Note that the vector space $\mathbb{R}$ is a line, but not all lines are vector spaces. For example, $x + y = 1$ is not a vector space since **it does not contain $\mathbf{0}$**. So this is not a vector space.

Try to answer why zero vector $\mathbf{0}$ should be in any vector space?

Another example

Vector space V consisting of all functions $f(x)$ that are continuous on $\mathbb{R}$

$$V = \{f(x), \texttt{such that} f(x) \texttt{is continuous on } \mathbb{R}\}$$

Linear combination

Vector $\mathbf{w} \in V$ is a linear combination of vectors $\mathbf{v_1}, \ldots, \mathbf{v_n} \in V$
if $\exists c_k \in \mathbb{R}; (k = 1..n)$
such that

$$\mathbf{w} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n}$$

Examples

Subspace

Definition

W is a subspace of V if

a)	$W \subset V$	(subset)
b)	$\mathbf{u}, \mathbf{v} \in W \Rightarrow \mathbf{u} + \mathbf{v} \in W$	(closure under addition)
c)	$\mathbf{u} \in W, \lambda \in \mathbb{R} \Rightarrow \lambda \mathbf{u} \in W$	(closure under scalar multiplication)

Examples

Span

Span

Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\} \subset V$.

$$span(S) \equiv \left\{ \mathbf{w} \in V : \mathbf{w} = \sum_{k=1}^{n} c_k \mathbf{v_k}, \quad \forall c_k \in \mathbb{R} \right\}$$

In words, $W = span(S)$ is the set of all (possible) linear combinations of the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$.
Note that W is a subspace of V.

Examples

Linear independence in $\mathbb{R}^2$ and in $\mathbb{R}^3$

Linearly independent vectors in $\mathbb{R}^2$

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are *linearly independent* if for $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1\mathbf{a} + \alpha_2\mathbf{b} = \mathbf{0}$ if and only if $\alpha_1 = \alpha_2 = 0$.

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Linearly independent vectors in $\mathbb{R}^3$

Vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are *linearly independent*
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Try to give a definition for Linearly independent vectors in $\mathbb{R}^n$

Basis in $\mathbb{R}^2$

Basis in $\mathbb{R}^2$

A **set** of vectors is a *basis* of $\mathbb{R}^2$ if it spans $\mathbb{R}^2$ and this set is **linearly independent.**

Standard basis in $\mathbb{R}^2$

$\{\hat{\mathbf{i}}, \hat{\mathbf{j}}\} = \{(1,0),(0,1)\}$ is a basis of $\mathbb{R}^2$. They are the standard basis in $\mathbb{R}^2$.

Standard basis in $\mathbb{R}^3$

$\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\} = \{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis of $\mathbb{R}^2$. They are the standard (canonical) basis in $\mathbb{R}^3$.

Examples

Representation of a Vector in Vector Space

Theorem

Let V be a vector space over $\mathbb{R}^m$ and let $\{\mathbf{e_1}, ..., \mathbf{e_m}\}$ be a basis.
Then each vector $\mathbf{u}$ can be identified with its coordinates $\{u_1, ..., u_m\}$ in the basis.

$$\mathbf{u} = \sum_{k=1}^{m} u_k \mathbf{e_k}$$

Homework Assignment

Let P_3, a set of all polynomials of degree 3 or less. It is a vector space over $\mathbb{R}$

Show that P_3 is a vector space over $\mathbb{R}$.

What could be a basis of P_3?
Give examples of two bases in P_3.
Express the polynomial $x^3 - 2x^2 + 3$ in the basis.

End of Lecture #1

Useful links

- https://www.geogebra.org
- https://youtu.be/fNk_zzaMoSs
- http://immersivemath.com/ila