Tutorial 8: Lines and Planes in Space

(more examples)

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Course of Essentials of Analytical Geometry and Linear Algebra I

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Last weeks' topics

- ❑ Planes in Space
  - ➢ General Equation of a Plane
  - ➢ Equation of a Plane Passing through Three Points
  - ➢ Other Forms of Equations of a Plane
  - ➢ Angle Between Two Planes
  - ➢ Distance From a Point to a Plane
  - ➢ Relative Position of Planes
  - ➢ Relative Position of a Plane and a Line

Content

❑ More Examples on:

  - ➢ Lines in Space
  - ➢ Planes in Space

Example 1

$\boxed{\ell = 4P}$

- Find the equation of a line passing through the point $(4, -7)$ parallel to the line $4x + 6y = 9$.

$y = \boxed{m} x + b$

$y = \boxed{k} x + b$

$\overset{\text{slope}}{m}$

Solution:

Step 1 — Find the slope of the line $4x + 6y = 9 \longrightarrow y = -\frac{2}{3}x + \frac{3}{2}$

Step 2 — Use ~~the~~ slope to find the y-intercept. $\left(m = k = -\frac{2}{3}\right)$

$$y = -\frac{2}{3}x + b \quad \xrightarrow{(4,-7)} \quad -7 = \left(-\frac{2}{3}\right)(4) + b \longrightarrow b = -\frac{13}{3}$$

Step 3 — Write the answer

$$\boxed{y = -\frac{2}{3} - \frac{13}{3}}$$

Example 2

- Find the equation of a line passing through the point $(-3, 8)$ perpendicular to the line $2x - 7y = -11$.

Solution:

$$\boxed{m_1 m_2 = -1}$$

$L_1 \perp L_2$

Step 1 — Find the slope of the line $2x - 7y = -11$

$$\rightarrow y = \underset{m_1}{\boxed{\frac{2}{7}}} x + \frac{11}{7}$$

Step 2 — Use the slope to find the y-intercept. $\left(m_2 = -\frac{7}{2}\right)$

$$y = mx + b \implies y = -\frac{7}{2}x + b \xrightarrow{(-3,8)} 8 = -\frac{7}{2}(-3) + b$$

$$\implies b = -\frac{5}{2}$$

Step 3 _ Write the answer

$$y = -\frac{7}{2}x - \frac{5}{2}$$

Example 3

L_1: $A_1x+B_1y+C_1=0$ $\quad$ $A_2x+B_2y+C_2=0$

- Find the distance between parallel lines given by the equations $Ax + By + C1 = 0$ and $Ax + By + C2 = 0$.

Solution:

$$\boxed{A_1=A_2=A \;/\; B_1=B_2=B} \qquad \frac{A_2}{A_1}=\frac{B_2}{B_1}\neq\frac{C_2}{C_1}$$

$P(x_0, y_0)$ on L_1

we use the formula of distance between a point and a line.

$$d=\left|\frac{A_2x_0+B_2y_0+C_2}{\sqrt{A_2^2+B_2^2}}\right| \longrightarrow d=\left|\frac{\boxed{A_1x_0+B_1y_0}+C_2}{\sqrt{A_2^2+B_2^2}}\right|$$

$$L_1: A_1x_0+B_1y_0+C_1=0 \Longrightarrow \boxed{\boxed{A_1x_0+B_1y_0}=-C_1}$$

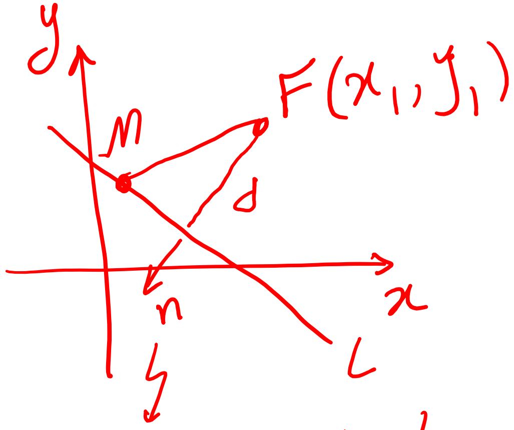
$$\Longrightarrow d=\left|\frac{-C_1+C_2}{\sqrt{A_2^2+B_2^2}}\right|$$

$$\Longrightarrow d=\left|\frac{C_2-C_1}{\sqrt{A^2+B^2}}\right|$$

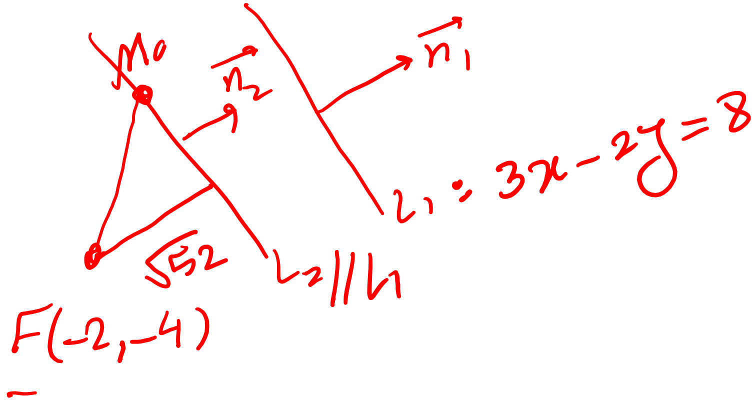
Example 4

- ➢ Find the equations of the lines parallel to $3x - 2y = 8$ and having distance $\sqrt{52}$ from point $F(-2,-4)$.

Solution:



normal vector to L



$Ax+By+c=0$

$3x-2y=c$

$x=0 \rightarrow -2y=c \rightarrow y=-\frac{c}{2}$

$(0,-\frac{c}{2})$

$M(x_0,y_0)$ — any point on the line L

$$d=\left|Proj_{\vec{n}}\vec{FM}\right| = \left|\frac{\vec{FM}\cdot\vec{n}}{\|n\|}\right| = \left|\frac{A(x_1-x_0)+B(y_1-y_0)}{\sqrt{A^2+B^2}}\right| \Rightarrow \sqrt{52}=\left|\frac{3(-2-0)-2(-4-\frac{c}{2})}{\underset{\sqrt{13}}{\cancel{\sqrt{(3)^2+(-2)^2}}}}\right|$$

$$\Rightarrow \sqrt{13}\sqrt{13}\cdot 2 = \left|-6+8-c\right| \Rightarrow 26=\left|2-c\right| \Rightarrow 2-c=26 \text{ or } 2-c=-26$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$c=-24 \qquad\qquad c=28$$

Ans: $3x-2y=-24$

or

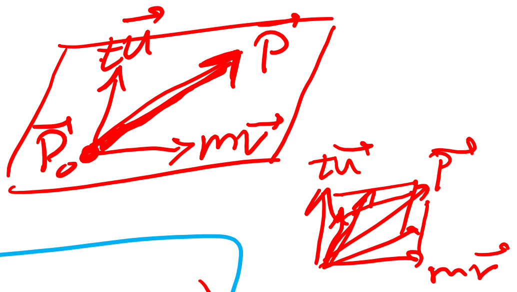
$3x-2y=28$

Example 5

- Find the parametric equation of the plane given by equation $\overset{\vec{P}}{\boxed{x - 2y + 3z = 1.}}$

normal vector to the plane

$$\vec{P} = \vec{P_0} + t\vec{u} + m\vec{v}$$



Solution:

$$\text{Setting} \begin{cases} y=0 \\ z=0 \end{cases} \Longrightarrow x=1 \Longrightarrow P_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Vector direction of the line

$$\vec{a}(\alpha,\beta,\gamma) \parallel \vec{P} \quad \text{plane} \quad \vec{a}\cdot\vec{n}=0 \quad \boxed{\alpha - 2\beta + 3\gamma = 0}$$

$$\text{Setting} \quad \beta_1 = 1 \quad \gamma_1 = 0 \Longrightarrow \alpha_1 = 2$$

$$\beta_2 = 0 \quad \gamma_1 = 1 \Longrightarrow \alpha_2 = -3$$

$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + m\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}}$$

$$x = 2y - 3z + 1$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2y-3z \\ y \\ z \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \underset{t}{\cancel{y}}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \underset{m}{\cancel{z}}\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}}$$

Example 6

- Find the orthocenter of a triangle with the vertices R$(3, 9)$, $M(1, 3)$, and $E(10, 2)$.

Solution:

① Find the eqs. of lines forming sides MR & RE. $y=mx+b$

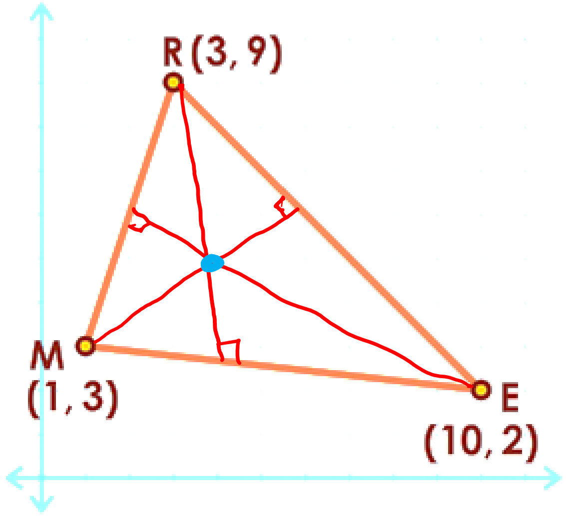
$$m = \frac{y_2-y_1}{x_2-x_1}$$

For MR : $m = \frac{9-3}{3-1} \Rightarrow m=3$

$y=3x+b$ $\xrightarrow{M(1,3)}$ $3=3(1)+b \Rightarrow b=0$

$\xrightarrow{R(3,9)}$ $9=3(3)+b \Rightarrow b=0$

$\Rightarrow$ eq of MR $\boxed{y=3x}$



For RE : $m = \frac{2-9}{10-3} = \frac{-7}{7} \Rightarrow m=-1$

$y=-1x+b$ $\xrightarrow{R(3,9)}$ $9=-1(3)+b \Rightarrow b=12$

eq. of line RE $\Rightarrow$ $\boxed{y=-x+12}$

② For MR : $m=3$ $\perp \Rightarrow m=-1/3$

For RE : $m=-1$ $\perp \Rightarrow m=1$

$\perp$: $\boxed{m_1 m_2 = -1}$

Example 6 (cntd.)

- Find the orthocenter of a triangle with the vertices R$(3, 9)$, $M(1, 3)$, and $E(10, 2)$.

Solution:

③ For side MR, its altitude is AE, with vertex E at $(10,2)$ and $m = \frac{-1}{3}$

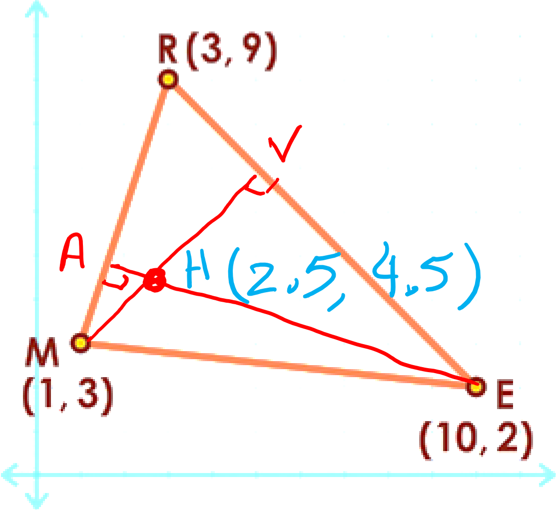
$y = mx + b \Rightarrow 2 = (-\frac{1}{3})10 + b \Rightarrow b = \frac{16}{3}$

The eq for altitude AE is $y = \frac{-1}{3}x + \frac{16}{3}$ *

For side RE, its altitude is VM, with vertex M at $(1,3)$ & $m = 1$

$y = mx + b \Rightarrow (3) = 1(1) + b \Rightarrow b = 2$

The eq for altitude VM is $y = x + 2$ **



④ $\begin{cases} y = -\frac{1}{3}x + \frac{16}{3} \\ y = x + 2 \end{cases}$ $x + 2 = -\frac{1}{3}x + \frac{16}{3} \rightarrow 3x + 6 = -x + 16 \rightarrow 4x = 10 \Rightarrow \boxed{x = 2.5}$

$\rightarrow y = 4.5$

Example 7

- Point $H(1,\ 2)$ is the orthocenter of a triangle, and $(4,-3)$ and $(-2,5)$ are the coordinates of vertices. Find the coordinates of the third vertices.

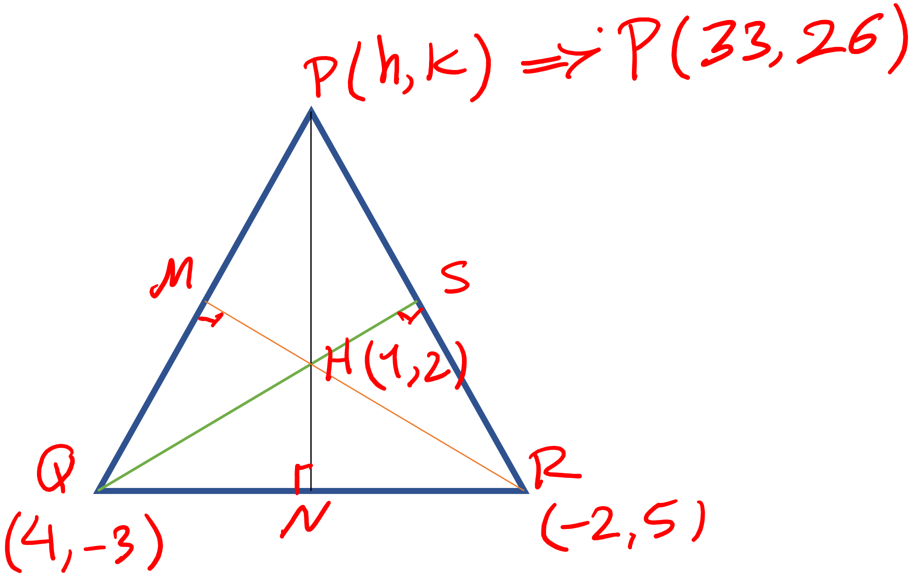
Solution:

for ⊥ lines : $m_1 m_2 = -1$

$m_{PQ}\, m_{MR} = -1$

$\left(\frac{k+3}{h-4}\right)\left(\frac{5-2}{-2-1}\right) = -1 \Rightarrow \left(\frac{k+3}{h-4}\right) = 1$

$\Rightarrow \boxed{k+7=h}$ ①



$m_{PN}\, m_{QR} = -1$

$\left(\frac{k-2}{h-1}\right)\left(\frac{5+3}{-2-4}\right) = -1 \Rightarrow \left(\frac{k-2}{h-1}\right) = \frac{3}{4} \Rightarrow \boxed{4k-3h=5}$ ②

$\xrightarrow{①} 4k-3(k+7)=15$

$\Rightarrow k=26 \xrightarrow{①} h=33$

Example 8

- ➢ Compose the equations of lines passing through point $A(3,2)$ and forming angles of $45°$ with the line $x - 2y = 3$.

Solution:

$$\tan\theta = \left|\frac{m_2 - m_1}{1 - m_1 m_2}\right|$$

$$m_1 : x - 2y = 3 \longrightarrow 2y = x - 3 \rightarrow y = \boxed{\frac{1}{2}}x - \frac{3}{2}$$

$$\underset{1}{\cancel{\tan 45}} = \left|\frac{m_2 - 1/2}{1 - \frac{m_2}{2}}\right| \Longrightarrow \left|\frac{2m_2 - 1}{\underset{2 - m_2}{2 \cancel{+} m_2}}\right| = 1$$

Hence, $\underbrace{\frac{2m_2 - 1}{2 \cancel{+} m_2} = 1}_{\Downarrow \; m_2 = \cancel{3} \; 1}$ or $\underbrace{\frac{2m_2 - 1}{2 \cancel{+} m_2} = -1}_{\Downarrow \; m_2 = \cancel{\frac{1}{3}} \; -1}$

$$y - y_1 = m_2(x - x_1)$$

$$y - 2 = m_2(x - 3) \longrightarrow \begin{cases} \cancel{m_2 = 3 ; \; 3x - y = 7} \; X \\ \cancel{m_2 = \frac{-1}{3} ; \; x + 3y = 9} \; X \end{cases}$$

$$\begin{cases} m_2 = 1 \Rightarrow x - y = 1 \; ✓ \\ m_2 = -1 \Rightarrow x + y = -1 \; ✓ \end{cases}$$

Next Week Topics

- ☐ Mid-Term Exam

Good Luck