Mathematical Analysis. Assignment 3.
Derivatives

1. Find the derivatives of the following functions:

   (a) $f(x) = \arctan^3 x$;

   (b) $f(x) = \frac{\arcsin x}{e^{3x}}$;

   (c) $f(x) = \ln \sin x$;

   (d) $f(x) = \ln\left(x^2 + \sqrt{x^4+1}\right)$;

   (e) $f(x) = \cos(3\arccos x)$;

   (f) $f(x) = \frac{1}{2\sqrt{6}} \ln\left(\frac{\sqrt{2}+x\sqrt{3}}{\sqrt{2}-x\sqrt{3}}\right)^2$;

   (g) $f(x) = \arctan\left(e^{x/2}\right) - \ln\sqrt{\frac{e^x}{e^x+1}}$;

   (h) $f(x) = \log_x 7$;

   (i) $f(x) = (\cosh x)^{\tanh x}$.

2. Find the derivative of a function at a given point:

   (a) $f(x) = \prod\limits_{k=0}^{2019}(x-k)$ at $x_0 = 0$;

   (b) $f(x) = (1+x)\sqrt{2+x^2}\sqrt[3]{3+x^3}$ at $x_0 = 0$.

3. Find all values of α such that function $f(x) = \begin{cases} |x|^\alpha \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases}$

   (a) is continuous at $x_0 = 0$;

   (b) has a derivative at point $x_0 = 0$;

   (c) has a continuous derivative at $x_0 = 0$.

4. Find such values of α and β such that function $f(x) = \begin{cases} (x+\alpha)e^{-\beta x}, & \text{if } x < 0, \\ \alpha x^2 + \beta x + 1, & \text{if } x \geqslant 0 \end{cases}$ is differentiable for all $x \in \mathbb{R}$.

5. Prove or refute the following statements:

   (a) for a differentiable function to have an even derivative it is sufficient that the function is odd;

   (b) for a differentiable function to have an even derivative it is necessary that the function is odd.

6. Find the derivative of the inverse function at a given point:

   (a) $y = 2x - \cos\frac{x}{2}$, $y_0 = -\frac{1}{2}$;

   (b) $y = x + \frac{x^5}{5}$, $y_0 = \frac{6}{5}$.

7. Find the derivative y'_x of a function $y(x)$ given parametrically by $x = t^2 + 6t + 5$, $y = \frac{t^3-54}{t}$, $t > 0$.

8. Find the derivative y'_x of a function $x(y)$ given parametrically by $x = \cot 2t$, $y = \frac{2\cos 2t - 1}{2\cos t}$, $0 < t < \frac{\pi}{2}$.

9. Find the differential of a function $y = y(x)$ given implicitly by an equation

   (a) $x^4 + y^4 - 8x^2 - 10y^2 + 16 = 0$ at point $(1;3)$;

   (b) $xe^{\frac{x}{y^2}-1} - 2y = 0$ at point $(4;2)$.

10. Express the differentials of the following functions through u, v, du and dv:

    (a) $y = u^3 v$;

    (b) $y = \frac{u^2}{v - u^3}$;

    (c) $y = e^{uv}$.

11. Replacing the increment of a function $\sqrt[3]{x}$ by its differential find the approximate value of $\sqrt[3]{65}$.

12. Find the second differential of $f(x) = \arctan \frac{2+x^2}{2-x^2}$ at $x = 0$.

13. Find the second differential $d^2 y$ considering du, $d^2 u$, dv, $d^2 v$ to be known:

    (a) $y = u(2 + v)$;

    (b) $y = u^v$.

14. Find the second derivative $\frac{d^2 y}{dx^2}$ of a function given parametrically by

    (a) $x = \frac{t^2}{1+t^3}$, $y = \frac{t^3}{1+t^3}$;

    (b) $x = t \cosh t - \sinh t$, $y = t \sinh t - \cosh t$.

15. Function $y(x)$ is given implicitly by an equation $x^3 y + \arcsin(y - x) = 1$. Find $d^2 y$ at point $(1; 1)$.

16. Find $y^{(n)}(x)$ for the following functions:

    (a) $y = \ln(ax + b)$;

    (b) $y = \sin \alpha x \sin \beta x$;

    (c) $y = \cosh \alpha x \cosh \beta x$;

    (d) $y = \frac{1}{\sqrt{1-2x}}$;

    (e) $y = \frac{x}{x^2 - 4x - 12}$;

    (f) $y = \frac{3 - 2x^2}{2x^2 + 3x - 2}$;

    (g) $y = (2x - 1) \cdot 2^{4x} \cdot 3^{2x}$;

    (h) $y = x \ln \frac{3+x}{3-x}$;

    (i) $y = (x^2 + x) \cos^2 2x$.