# Essentials of Analytical Geometry and Linear Algebra. Lecture 3.

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# End of Lecture #2

## Review. Lecture 2

- Part 1. The Dot Product and its properties
  - Norm of a vector
  - Cauchy-Schwarz inequality
  - Triangle Inequality
- Part 2. Vector Cross Product
- Part 3. Matrices (2x2, 3x3).

# Quiz in class

Go to `http://b.socrative.com`

Type Room: **LINAL**

Answer questions.

# Lecture 3. Outline

- Part 1 (recap). Matrices. Transpose, Addition, Scalar multiplication
- Part 2. Matrix multiplication
- Part 3. Determinants. Scalar Triple Product
- Part 4. Change of basis and coordinates

# Hey, Professor, there is a truck behind you!

## Part 1. Matrices

# Definition

Matrix $A$ is a rectangular table of numbers with $m$ rows and $n$ columns.

Example of a $3 \times 3$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Example of a $2 \times 3$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

# Different kinds of matrices

$A$ is a $m \times n$ matrix

1. Square ($m = n$)
2. Rectangular matrix ($m \neq n$)
3. Symmetric matrix ($A^\top = A$)
4. (Upper) Triangular matrix ($\forall i, j$, such that $i > j$ : $a_{i,j} = 0$)
5. Diagonal matrix ($\forall i, j$, such that $i \neq j$ : $a_{i,j} = 0$)
6. Identity matrix ($IA = AI = A$)
7. Zero matrix ($\mathbf{0} + A = A$)

# Examples

(1) Square matrix : (#rows = #columns)

$$\begin{bmatrix} 3 & -1 & -3 \\ 2 & 4 & 0 \\ -1 & 5 & 6 \end{bmatrix}$$

Main diagonal

(9) Column matrix ($n$ – vector)

$$\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \text{ is a 3-vector}$$

(8) Row matrix ($n$ – vector):

$$\begin{bmatrix} 1 & 7 & -3 \end{bmatrix} \text{ is a 3-vector}$$

(2) Upper triangular matrix :

$$\begin{bmatrix} 3 & -1 & -3 \\ 0 & 4 & 7 \\ 0 & 0 & 6 \end{bmatrix}$$

(3) Lower triangular matrix :

$$\begin{bmatrix} 3 & 0 & 0 \\ -2 & 4 & 0 \\ 2 & 7 & -6 \end{bmatrix}$$

(5) Diagonal matrix :

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

(6) Identity matrix :

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(7) Zero matrix :

$$O_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; O_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Source: `https://medium.com/@nithishraghav/linear-algebra-for-aspiring-data-scientists-part-i-37a9b63c031f`

# Operations. Transpose a matrix

## Transpose of matrix

If $A$ is an $m \times n$ matrix, the *transpose* $A^T$ is an $n \times m$ matrix defined by $(A^T)_{ij} = A_{ji}$.

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}^\top = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\forall A, (A^\top)^\top = A$$

## Operations. Addition, multiplication by a scalar

Element-wise addition:

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} = \begin{bmatrix} 1+a & 4+d \\ 2+b & 5+e \\ 3+c & 6+f \end{bmatrix}$$

**Properties. A, B, C are matrices of the same size (!)**

1. $A + B = B + A$ (commutative)
2. $A + (B + C) = (A + B) + C$ (associative)
3. $B = \lambda A, \lambda \in \mathbb{R}$ (multiplication by a scalar $\lambda$, element-wise)

$$B = \lambda A, \quad \forall 1 \leq i \leq m; 1 \leq j \leq n : b_{ij} = \lambda a_{ij}$$

# Trace of a matrix

**Definition of trace of a square matrix $A$**

$$Tr(A) = \sum_{i=1}^{m} a_{ii}$$

$$Tr(A + B) = Tr(A) + Tr(B)$$

$$\forall \lambda \in \mathbb{R}, \quad Tr(\lambda A) = \lambda Tr(A)$$

Linearity of the trace operator means

$$Tr(\alpha A + \beta B) = \alpha Tr(A) + \beta Tr(B)$$

Part 2. Matrix multiplication

# Definition

**Definition**

Let
$A$ be $m \times n$ matrix;
$B$ be $n \times p$ matrix
Then exists $C = AB$,
$C$ must be $m \times p$ matrix

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj},$$

for $i = 1, ..., m$ and $j = 1, ..., p$

# Most important!

Before you multiply two matrices $A$ and $B$. $A$ is $m \times n$ matrix; $B$ is $k \times p$ matrix

- Commit into your memory: **matrix multiplication is not commutative.** So, in general:

$$AB \neq BA$$

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- **Check sizes** of the two matrices:
  - if you multiply $AB$ $(m \times n)( k \times p)$, then check that $n = k$

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- **Check sizes** of the two matrices:
  - if you multiply $AB$ $(m \times n)(\, k \times p)$, then check that $n = k$
- **Calculate the size** of the result:
  - if you multiply $AB$ $(m \times n)(\, k \times p)$, then the result is a $m \times p$ matrix.

# Illustration

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix}$$

# Python Code

# How to calculate the result? Example $2 \times 2$ matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} =$$

## How to calculate the result? Example $2 \times 2$ matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$AB = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ 3 & 4 \end{bmatrix} \begin{bmatrix} \mathbf{a} & b \\ \mathbf{c} & d \end{bmatrix} = \begin{bmatrix} 1a + 2c & * \\ * & * \end{bmatrix}$$

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# Your turn!

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$BA = ?$$

# Your turn!

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$BA$ = ?

**True or False?**

$AB = BA$?

# Your turn!

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$BA = ?$$

**True or False?**

$AB = BA$?

(AB)C = A(BC) = ABC?

# Exercise

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 6 \end{bmatrix}, B = \begin{bmatrix} x & u & a \\ y & v & b \\ z & w & c \end{bmatrix}$$

$$AB = ?$$

# Three other ways to think about matrix multiplication

- row-oriented view
- column-oriented view
- layer-oriented view

# Three other ways to think about matrix multiplication

- row-oriented view

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1\begin{bmatrix} a & b \end{bmatrix} + 2\begin{bmatrix} c & d \end{bmatrix} \\ 3\begin{bmatrix} a & b \end{bmatrix} + 4\begin{bmatrix} c & d \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1a & 1b \end{bmatrix} + \begin{bmatrix} 2c & 2d \end{bmatrix} \\ \begin{bmatrix} 3a & 3b \end{bmatrix} + \begin{bmatrix} 4c & 4d \end{bmatrix} \end{bmatrix}$$

Here result is still a $2 \times 2$ matrix.
It has two rows, but each row is a $1 \times 2$ vector (!)

# Three other ways to think about matrix multiplication

- column-oriented view

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c \begin{bmatrix} 2 \\ 4 \end{bmatrix}, & b \begin{bmatrix} 1 \\ 3 \end{bmatrix} + d \begin{bmatrix} 2 \\ 4 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1a \\ 3a \end{bmatrix} + \begin{bmatrix} 2c \\ 4c \end{bmatrix}, & \begin{bmatrix} 1b \\ 3b \end{bmatrix} + \begin{bmatrix} 2d \\ 4d \end{bmatrix} \end{bmatrix}$$

Here result is still a $2 \times 2$ matrix.
It has two columns, but each column is a $2 \times 1$ vector (!)

# Three other ways to think about matrix multiplication

- layer-oriented view

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1a & 1b \\ 3a & 3b \end{bmatrix} + \begin{bmatrix} 2c & 2d \\ 4c & 4d \end{bmatrix} = \begin{bmatrix} 1a + 2c & 1b + 2d \\ 3a + 4c & 3b + 4d \end{bmatrix}$$

Here result is still a $2 \times 2$ matrix. It is represented as a sum of ‘simpler’ matrices.

# Assignment

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$AB =$

# Order of operations

$$(ABCD)^\top = D^\top C^\top B^\top A^\top$$

# Very special and important case

Matrix - vector multiplication

$$A\mathbf{x}$$

$$\mathbf{x}^\top A$$

# Matrix - vector multiplication

Result is always a vector!

$$A\mathbf{x}$$

$$(m \times n)(n \times 1) \rightarrow (m \times 1) \quad \texttt{is a column-vector}$$

$$\mathbf{x}^\top A$$

$$(1 \times m)(m \times n) \rightarrow (1 \times n) \quad \texttt{is a row-vector}$$

So, we can see that matrix multiplication transforms vectors. Matrix A is a linear map.

# Matrix as a linear transformation

Again, it is important!
Matrix $A$ is a linear map.

Vector $\mathbf{x}$ was a $(n \times 1)$ `column-vector`

$$A\mathbf{x}$$

$$(m \times n)(n \times 1) \to (m \times 1) \quad \texttt{column-vector}$$

Result is $(m \times 1)$ `column-vector`

$A$ maps vectors in $\mathbb{R}^n$ to vectors in $\mathbb{R}^m$

# Examples of transformations. Rotation

## Rotation matrix

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$A$ rotates any vector $\mathbf{x} = [x_1, x_2]^\top$ by an angle $\theta$ counter-clockwise!

$$A\mathbf{x} = \begin{bmatrix} x_1 \cos(\theta) - x_2 \sin(\theta) \\ x_1 \sin(\theta) + x_2 \cos(\theta) \end{bmatrix}$$

Demo in Geogebra

# Coding

Here we run some code in Colab.

https://colab.research.google.com/drive/1Kfv4253b5duaP-KjQTk4Ail9rjR45pCX#scrollTo=RNdMstvGEUf0

# A very interesting case

What if multiplication $A\mathbf{w}$ work as follows?

$$A\mathbf{w} = \lambda\mathbf{w}, \quad \lambda \in \mathbb{R}$$

## Example

$$\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

## Interesting indeed!

$\lambda$ is called eigenvalue
$\mathbf{w}$ is called eigenvector

Break, 5 min.

# Part 3. Determinants

# Determinant. Concept and application

## Notation

$$det(A)$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \text{ is a } 2 \times 2 \text{ determinant,}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \text{ is a } 3 \times 3 \text{ determinant}$$

- Determinant is a **single** number $det(A) \in \mathbb{R}$
- Defined only for square matrices!
- $det(A) = 0$ if $A$ contains linearly dependent columns. Matrix in this case is called singular.

# Determinant. Concept and applications

## Applications

- Calculating Area/Volume of shape specified by coordinates in matrix
- Finding matrix inverse (later in this course).

## 2x2 Determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

What if we swap rows of the matrix?

$\begin{vmatrix} a & a\beta \\ b & b\beta \end{vmatrix}$ = ?

# Examples

$$\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = ?$$

$$\begin{vmatrix} -1 & 2 \\ 4 & -5 \end{vmatrix} = ?$$

# Examples

$$\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = ?$$

$$\begin{vmatrix} -1 & 2 \\ 4 & -5 \end{vmatrix} = ?$$

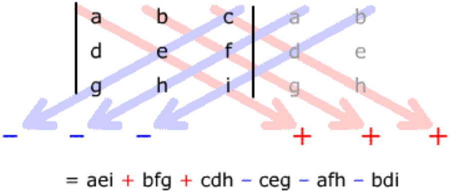
What $\lambda$ makes the following determinant zero?

$$\begin{vmatrix} 1 & 2 \\ 4 & \lambda \end{vmatrix} = ?$$

$$\begin{vmatrix} 5-\lambda & -1/3 \\ 3 & 5-\lambda \end{vmatrix} = ?$$

# 3x3 Determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - afh - bdi$$



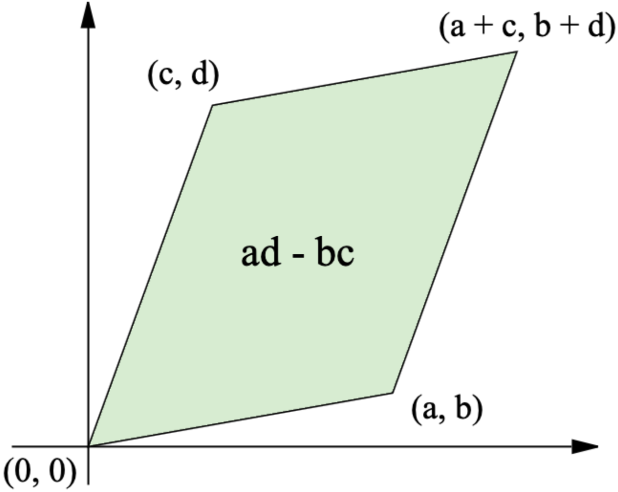

Source:
http://thejuniverse.org/PUBLIC/LinearAlgebra/LOLA/detDef/special.html

# Examples

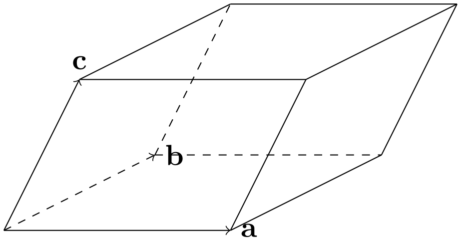
$$\begin{vmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \\ 4 & 3 & 0 \end{vmatrix} = ?$$

Yes, there exists one single general super formula for calculation of $det(A)$ for arbitrary square matrix $A$.
`https://en.wikipedia.org/wiki/Determinant`

# Meaning of the Determinant. Area of a parallelogram

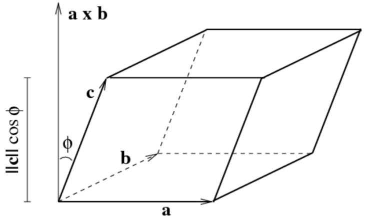


# Meaning of the Determinant. Volume of parallelepiped




$$V = \begin{vmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

# Scalar Triple Product




## Scalar Triple Product. Definition

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

Meaning: $V = \|a \times b\|(\|c\|| \cos(\phi)|)$ = Area of base * Height

# Check the following properties

- $det(A) = det(A^\top)$
- $det(AB) = det(A)det(B)$

Break, 5 min.

Part 4. Changing Basis and Coordinates

# Theory and derivation

We are going to derive a formula for changing basis
Check the following material in moodle **before** the lecture, please!

## Matrices. Changing of Basis and Coordinates

Lecture 3. Part 4

# Examples for changing basis

# Homework assignment

**Prove**

$Tr(BC) = Tr(CB)$

...

# End of Lecture #3

## Useful links

- https://www.geogebra.org
- https://youtu.be/fNk_zzaMoSs
- http://immersivemath.com/ila