Essentials of Analytical Geometry and Linear Algebra. Lecture 7.

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End of Lecture #6

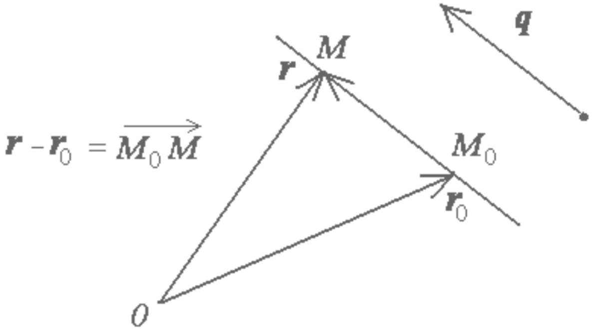
- Part 1. Straight line in plane
- Part 2. Equations of a line
- Part 3. Pair of lines

Lecture 7. Outline

- Part 1. Straight line in 3D space.
- Part 2. Plane in 3D space. Equations

Part 1. Straight line in 3D space.
(+ recap about equations of lines)

Parametric Vector Equation



Equation:

$$\mathbf{r} - \mathbf{r_0} = t\mathbf{q}$$

where t is a parameter.

Parametric Equation in 3D

In rectangular Cartesian coordinate system

$$\text{Equation of a line: } \begin{cases} \mathsf{x} = \mathsf{x}_0 + q_x t \\ \mathsf{y} = \mathsf{y}_0 + q_y t \\ \mathsf{z} = \mathsf{z}_0 + q_z t \end{cases}$$

$$\mathbf{r} - \mathbf{r_0} = [x - x_0, y - y_0, z - z_0]^\top$$

$$\mathbf{q} = [q_x, q_t, q_z]^\top$$

Canonical equation of a line

Eliminating t from the system:

$$\begin{cases} \mathsf{x} = \mathsf{x}_0 + q_x t \\ \mathsf{y} = \mathsf{y}_0 + q_y t \\ \mathsf{z} = \mathsf{z}_0 + q_z t \end{cases}$$

we get the **Canonical equation**

$$\frac{x - x_0}{q_x} = \frac{y - y_0}{q_y} = \frac{z - z_0}{q_z}$$

Canonical equation of a line

Eliminating t from the system:

$$\begin{cases} x = x_0 + q_x t \\ y = y_0 + q_y t \\ z = z_0 + q_z t \end{cases}$$

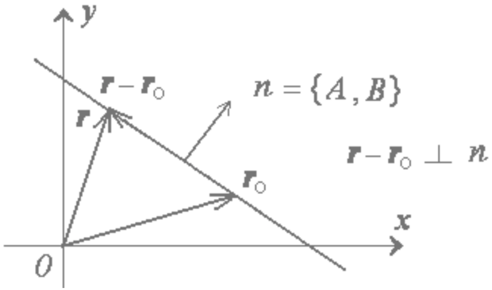
we get the **Canonical equation**

$$\frac{x - x_0}{q_x} = \frac{y - y_0}{q_y} = \frac{z - z_0}{q_z}$$

Give two points: M_0 and M_1:

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}$$

Recap on 2D case



$$Ax + By + C = 0$$

for point M_0 on a line:

$$Ax_0 + By_0 + C = 0$$

$$(\mathbf{r} - \mathbf{r_0}) = [x - x_0, y - y_0]^\top; \mathbf{n} = [A, B]^\top$$

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$$

Angle Between Two Lines

1. The angle between two lines is the angle between direction vectors of the lines.

$$\cos\theta = \frac{\mathbf{p} \cdot \mathbf{q}}{\|\mathbf{p}\|\|\mathbf{q}\|}$$

Where $\mathbf{p}$ and $\mathbf{q}$ are direction vectors of lines.

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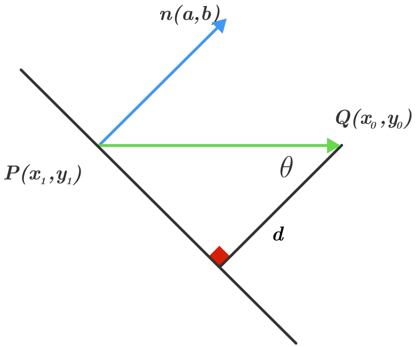
$$\cos\theta = \frac{\mathbf{p}\cdot\mathbf{q}}{\|\mathbf{p}\|\|\mathbf{q}\|}$$

Where $\mathbf{p}$ and $\mathbf{q}$ are direction vectors of lines.
2. The angle between two lines is the angle between normal vectors of the lines.

$$\cos\theta = \frac{\mathbf{n_1}\cdot\mathbf{n_2}}{\|\mathbf{n_1}\|\|\mathbf{n_2}\|}$$

Where $\mathbf{n_1}$ and $\mathbf{n_2}$ are normal vectors of lines.

Distance From a Point to a Line



$$d = ?$$

$$d = \frac{|\mathbf{n} \cdot \overline{PQ}|}{\|\mathbf{n}\|} = \ldots$$

Break, 5 min.

Part 2. Planes

General Equation of a Plane

In a rectangular Cartesian coordinate system

$$Ax + By + Cz + D = 0$$

x, y, z are arbitrary coordinates of a point on a plane.

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In a rectangular Cartesian coordinate system

$$Ax + By + Cz + D = 0$$

x, y, z are arbitrary coordinates of a point on a plane.
What if some of coefficients are zero?
C = 0, then is not it a line (???):

$$Ax + By + D = 0$$

General Equation of a Plane

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General Equation of a Plane

In a rectangular Cartesian coordinate system

$$Ax + By + Cz + D = 0$$

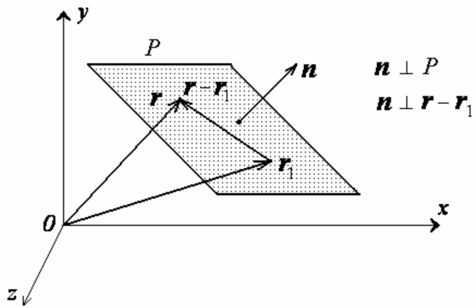
Given $M_1(x_1, y_1, z_1)$ is a point in plane:

$$Ax_1 + By_1 + Cz_1 + D = 0$$

We get the general equation of the plane

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$

Vector form



$$\mathbf{r} - \mathbf{r_1} = [x - x_1, y - y_1, z - z_1]^\top$$

Hence, the general equation of the plane $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$
Can be presented in the vector form:

$$(\mathbf{r} - \mathbf{r_1}) \cdot \mathbf{n} = 0$$

Example

A plane is given by the equation: $x - 2y + 3z - 6 = 0$.
Find: a unit normal vector **u** to the plane and find any two points in the plane.

Equation of a Plane Passing Through Three Points

Given three points: M_1, M_2, M_3,

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Consider vectors

$$\mathbf{r} - \mathbf{r_1} = [x - x_1, y - y_1, z - z_1]^\top$$

$$\mathbf{r_2} - \mathbf{r_1} = [x_2 - x_1, y_2 - y_1, z_2 - z_1]^\top$$

$$\mathbf{r_3} - \mathbf{r_1} = [x_3 - x_1, y_3 - y_1, z_3 - z_1]^\top$$

Equation of a Plane Passing Through Three Points

Given three points: M_1, M_2, M_3,
Consider vectors

$$\mathbf{r} - \mathbf{r_1} = [x - x_1, y - y_1, z - z_1]^\top$$

$$\mathbf{r_2} - \mathbf{r_1} = [x_2 - x_1, y_2 - y_1, z_2 - z_1]^\top$$

$$\mathbf{r_3} - \mathbf{r_1} = [x_3 - x_1, y_3 - y_1, z_3 - z_1]^\top$$

Their scalar triple product is zero.
(**Why?**)

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Equation of a Plane Passing Through Three Points

Triple scalar product:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

Example

Let $M_1(2, 5, -1)$, $M_2(2, -3, 3)$ and $M_3(4, 5, 0)$ be points in a plane.
Find an equation of that plane.

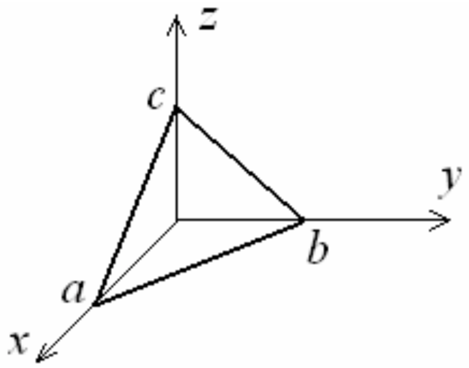
Other forms of equation

Given two vectors: $\mathbf{p}, \mathbf{q}$ that are parallel to a plane, and a point $M_1(x_1, y_1, z_1)$ on the plane

Consider arbitrary vector $\mathbf{r} = [x, y, z]$. Then three vectors $\mathbf{r} - \mathbf{r_1}, p, q$ are coplanar.

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ p_x & p_y & p_z \\ q_x & q_y & q_z \end{vmatrix} = 0$$

Equation of a plane in the intercept form



Equation of a plane in the intercept form

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Angle Between Two Planes

Definition

The angle between two planes equals the angle between their normal vectors.

$$\cos\theta = \frac{\mathbf{n_1} \cdot \mathbf{n_2}}{\|\mathbf{n_1}\|\|\mathbf{n_2}\|}$$

Find $\cos\theta$ if two planes are given by equations in the general form.

Example

Find the angle between two planes

Given that

- Three points of plane 1 are $M_1(-2, 2, 2)$, $M_2(0, 5, 3)$ and $M_3(-2, 3, 4)$
- Equation of plane 2: $3x - 4y + z + 5 = 0$

$\bar{n}_1 = (3, -4, 1)$

$$\begin{vmatrix} x+2 & y-2 & z-2 \\ 2 & 3 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 0$$

$$(x+2)5 - (y-2)4 + (z-2)2 = 0$$

$$5x + 10 - 4y + 8 + 2z - 4 = 0$$

$$5x - 4y + 2z + 14 = 0 \Rightarrow \bar{n}_2 = [5, -4, 2]$$

$$\cos\theta = \frac{\bar{n}_1 \cdot \bar{n}_2}{\|n_1\|\|n_2\|} = \frac{15 + 16 + 2}{\sqrt{26} \cdot \sqrt{45}} =$$

$$= \frac{33}{3\sqrt{130}} = \frac{11}{\sqrt{130}}.$$

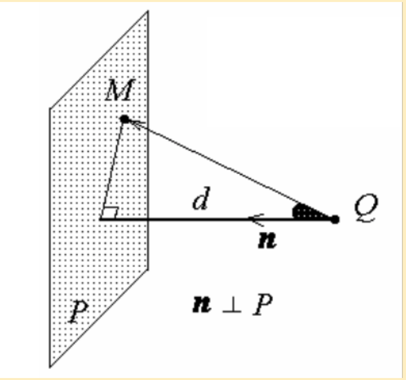
Alternative method for $\bar{n}_2$:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 1 \\ 0 & 1 & 2 \end{vmatrix} = (6-1)\hat{i} - (4-0)\hat{j} + (2-0)\hat{k}$$

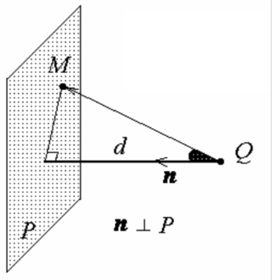
$$\Rightarrow \bar{n}_2 = [5, -4, 2].$$

Distance From a Point To a Plane

- Equation of plane: $Ax + By + Cz + D = 0$
- Point $Q(x_1, y_1, z_1)$ is **not** in plane.
- Point $M(x, y, z)$ is an arbitrary point in plane.



Distance From a Point To a Plane



$$d = \frac{|\mathbf{n} \cdot \overline{QM}|}{\|\mathbf{n}\|} = \dots$$

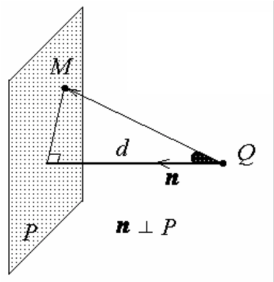
$\bar{n} = [A, B, C]$
(normal)

$Q(x_1, y_1, z_1)$

$M(x, y, z)$

$\overline{QM} = [x - x_1, y - y_1, z - z_1]$

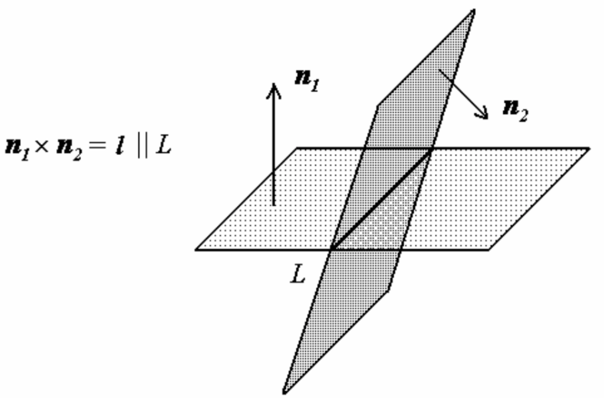
Distance From a Point To a Plane



$$d = \frac{|\mathbf{n} \cdot \overline{QM}|}{\|\mathbf{n}\|} = \dots$$

$$= \left| \frac{A(x - x_1) + B(y - y_1) + C(z - z_1)}{\sqrt{A^2 + B^2 + C^2}} \right|$$

Relative Position of Planes



Write a system of two equations (in general form).

- When the planes are parallel and coincide?
- When the planes are parallel and not coincide?
- When the planes are not parallel?

Break, 5 min.

Relative Position of a Plane and a Line

$$\begin{cases} Ax + By + Cz + D = 0 \\ A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

Which is where?

Relative Position of a Plane and a Line

$$\begin{cases} Ax + By + Cz + D = 0 \\ A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

Which is where?

There are three possible cases:

Relative Position of a Plane and a Line

$$
\begin{cases}
Ax + By + Cz + D = 0 \\
A_1x + B_1y + C_1z + D_1 = 0 \\
A_2x + B_2y + C_2z + D_2 = 0
\end{cases}
\Rightarrow
\begin{bmatrix} A & B & C \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{bmatrix}
\begin{bmatrix} x \\ y \\ z \end{bmatrix}
= -\begin{bmatrix} D \\ D_1 \\ D_2 \end{bmatrix}
$$

Which is where?

There are three possible cases:

- If the rank of the coefficient matrix equals 3, then $M_0(x_0, y_0, z_0)$ is the point of intersection of the plane and the line.

Relative Position of a Plane and a Line

$$\begin{cases} Ax + By + Cz + D = 0 \\ A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

can be considered as a plane!

Which is where?

There are three possible cases:

- If the rank of the coefficient matrix equals 3, then $M_0(x_0, y_0, z_0)$ is the point of intersection of the plane and the line.
- If system is consistent, and the rank of the coefficient matrix equals 2, then the line L lies in the plane P.

Relative Position of a Plane and a Line

$$\begin{cases} Ax + By + Cz + D = 0 \\ A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

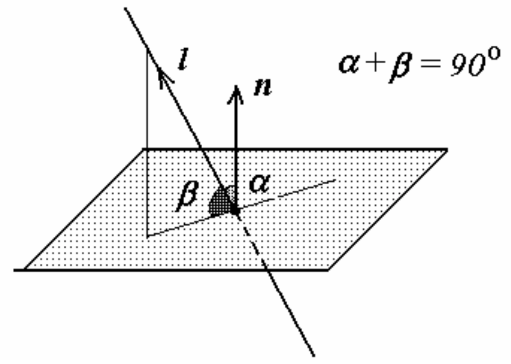
Which is where?

There are three possible cases:

- If the rank of the coefficient matrix equals 3, then $M_0(x_0, y_0, z_0)$ is the point of intersection of the plane and the line.
- If system is consistent, and the rank of the coefficient matrix equals 2, then the line L lies in the plane P.
- If system is inconsistent then the line L is parallel to the plane P.

The Angle Between a Plane and a Line

Angle Between a Plane and a Line ($\beta =?$)



- $\mathbf{n}$ is a normal vector of the plane
- $\mathbf{l}$ is a direction vector of the line
- β is the angle between the plane and the line

Useful links

- https://www.geogebra.org
- https://youtu.be/fNk_zzaMoSs
- http://immersivemath.com/ila