Tutorial 11 : Quadratic Curves (cntd.)

Dr. Mohammad Reza Bahrami

Innopolis University
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Last weeks' topics

- Quadratic Curves
  - Parabolas
  - Circles
  - Ellipses

Content

Conic sections are the curves obtained by intersecting a plane and a right circular cone.

- ❖ A plane perpendicular to the cone's axis cuts out a circle;
- ❖ A plane parallel to a side of the cone produces a parabola;
- ❖ A plane at an arbitrary angle to the axis of the cone forms an ellipse;
- ❖ A plane parallel to the axis cuts out a hyperbola.

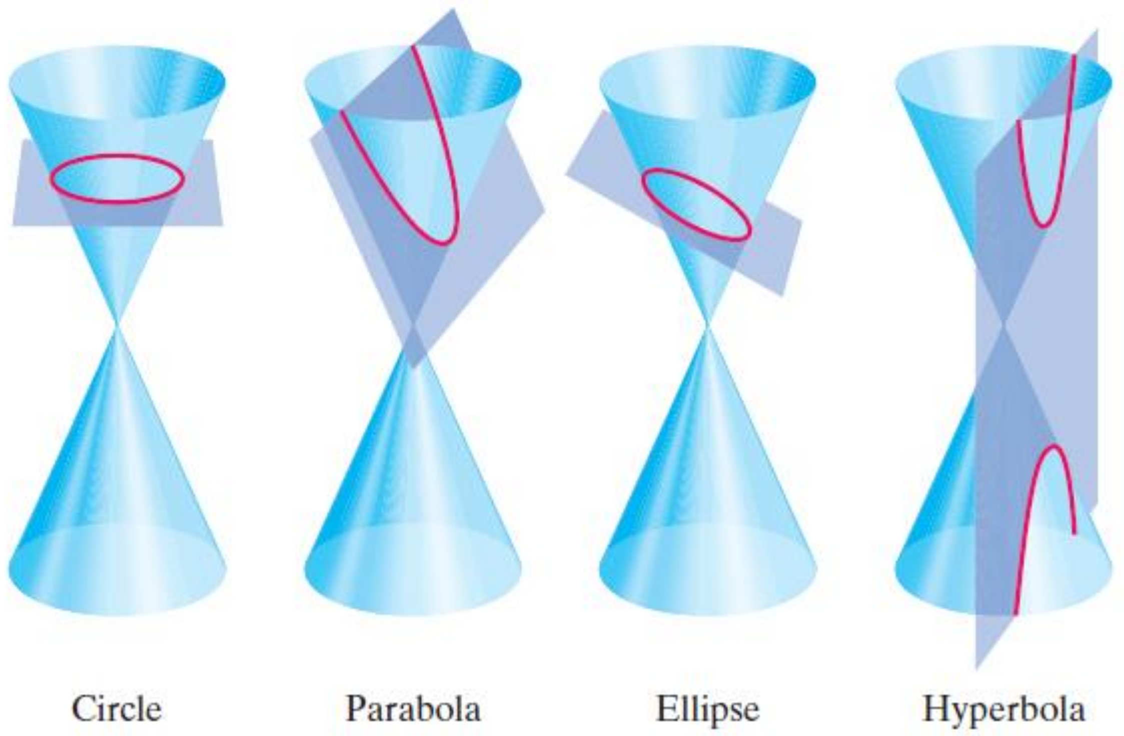
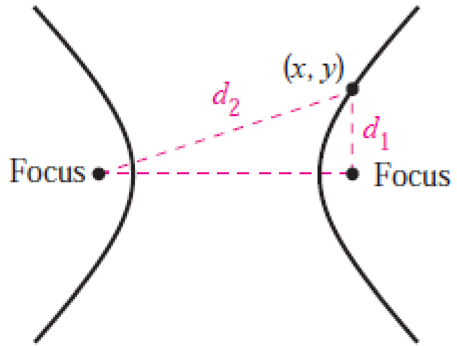
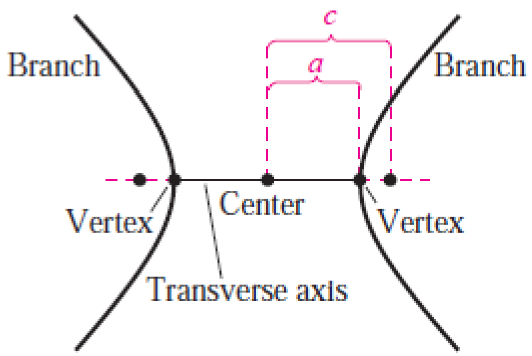


*Figure from internet.

A **hyperbola** is the set of points in a plane such that the absolute value of the difference of the distance of each point from two fixed points is constant. Each fixed point is called a *focus*, and the point midway between the foci is called the *center*. The line containing the foci is the **transverse axis**. The graph is made up of two parts called **branches**. Each branch intersects the transverse axis at a point called the *vertex*.

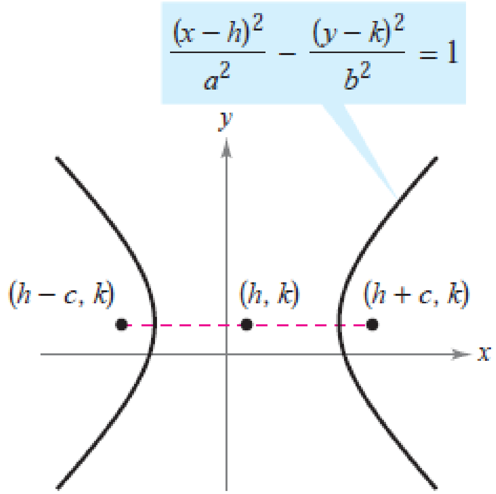


$d_2 - d_1$ is a positive constant.

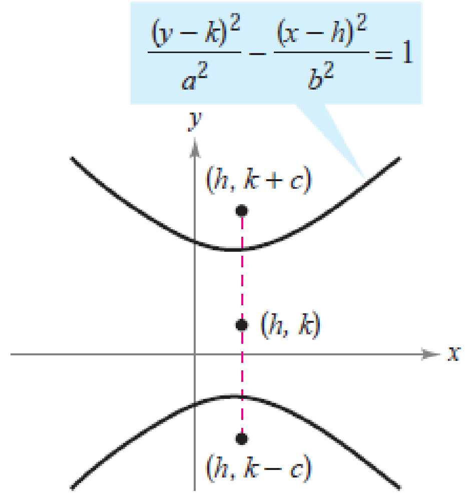


The standard form of the equation
of a hyperbola with center at (h, k) can be seen in figure.

The vertices are a units from the center, and the foci are c units from the center. Moreover, $c^2 = a^2 + b^2$.



Transverse axis is horizontal.



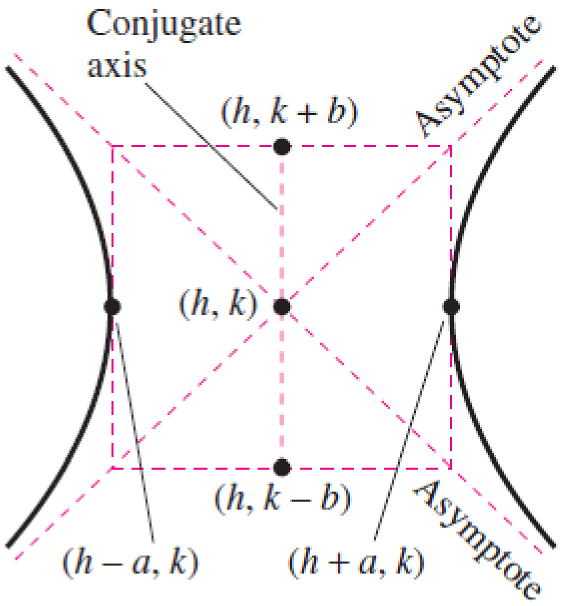
Transverse axis is vertical.

Asymptotes of a Hyperbola

Each hyperbola has two asymptotes that intersect at the center of the hyperbola.

The asymptotes pass through the corners of a rectangle of dimensions $2a$ by $2b$, with its center at (h, k), as shown in figure.

Equations of Asymptotes of a Hyperbola	
Asymptotes for horizontal transverse axis	Asymptotes for vertical transverse axis
$y = k \pm \frac{b}{a}(x - h)$	$y = k \pm \frac{a}{b}(x - h)$



The conjugate axis of a hyperbola is the line segment of length $2b$ joining $(h, k + b)$ and $(h, k - b)$ if the transverse axis is horizontal, and the line segment of length 2b joining $(h + b, k)$ and $(h - b, k)$ if the transverse axis is vertical.

Example 1

- Find the standard form of the equation of the hyperbola with foci (-1, 2) and (5, 2) and vertices (0, 2) and (4, 2).

Solution:

Example 2

- Sketch the hyperbola whose equation is $4x^2 - y^2 = 16$.

Solution:

Example 3

- Sketch the hyperbola given by $4x^2 - 3y^2 + 8x + 16 = 0$ and find the equations of its asymptotes.

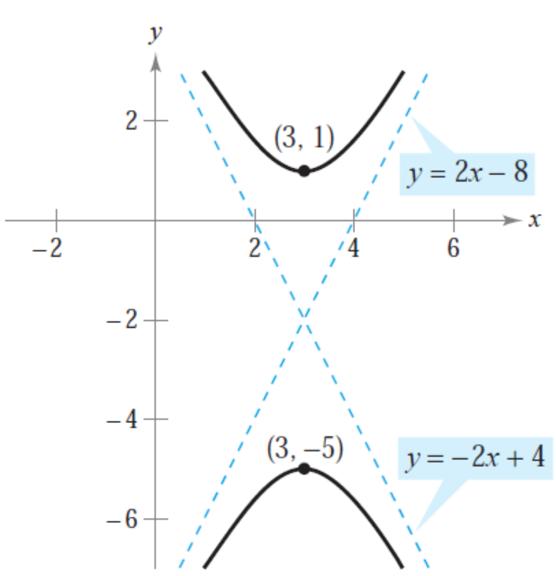
Solution:

Example 4

- Find the standard form of the equation of the hyperbola having vertices (3, -5) and (3, 1) and having asymptotes

$$y = 2x - 8 \text{ and } y = -2x + 4$$

as shown in figure.



Solution:

Example 5

- Two microphones, 1 mile apart, record an explosion. Microphone A receives the sound 2 seconds before microphone B. Where did the explosion occur? (Assume sound travels at 1100 feet per second.)

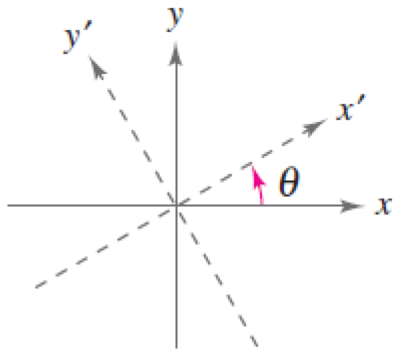
Solution:

It is known that the equation of a conic with axes parallel to the coordinate axes has a standard form that can be written in the general form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0. \quad \text{Horizontal or vertical axes}$$

In this section, you will study the equations of conics whose axes are rotated so that they are not parallel to either the *x*-axis or the *y*-axis. The general equation for such conics contains an xy-term.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad \text{Equation in } xy-\text{plane}$$



To eliminate this *xy*-term, you can use a procedure called **rotation of axes**. The objective is to rotate the *x*- and *y*-axes until they are parallel to the axes of the conic. The rotated axes are denoted as the x'-axis and the y'-axis, as shown in the figure.

Rotation of Axes to Eliminate an *xy*-Term

The general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

can be rewritten as

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$$

by rotating the coordinate axes through an angle θ, where $\cot 2\theta = \frac{A-C}{B}$ The coefficients of the new equation are obtained by making the substitutions

$$x = x' \cos\theta - y' \sin\theta \text{ and } y = x' \sin\theta + y' \cos\theta\,.$$

Rotation (2/2)

Rotation Invariants

The rotation of the coordinate axes through an angle θ that transforms the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ into the form $A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$
has the following rotation invariants.

1) $F = F'$

2) $\mathrm{A} + C = A' + C'$

3) $B^2 - 4AC = (B')^2 - 4A'C'$

Note that because $B' = 0$, the invariant $B^2 - 4AC$ reduces to

$$B^2 - 4AC = -4A'C'$$

Discriminant

Classification of Conics by the Discriminant

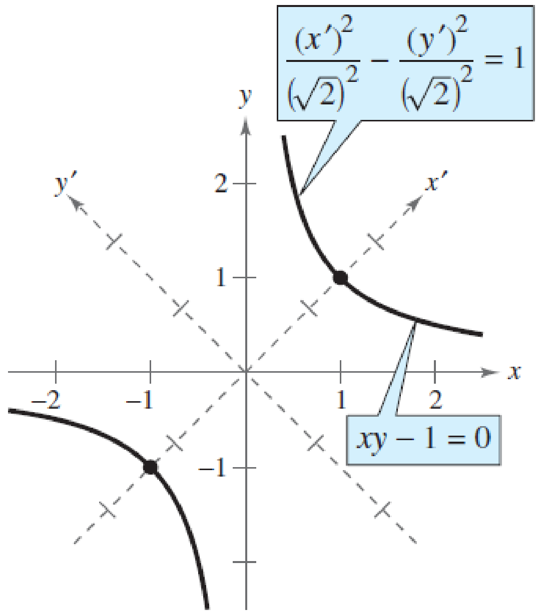
The graph of the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is, <u>except in degenerate cases</u>, determined by its discriminant as follows.

1. Ellipse or circle: $B^2 - 4AC < 0$
2. Parabola: $B^2 - 4AC = 0$
3. Hyperbola: $B^2 - 4AC > 0$

Example 6

➢ Rotate the axes to eliminate the xy-term in the equation $xy - 1 = 0$. Then write the equation in standard form and sketch its graph.

Solution:



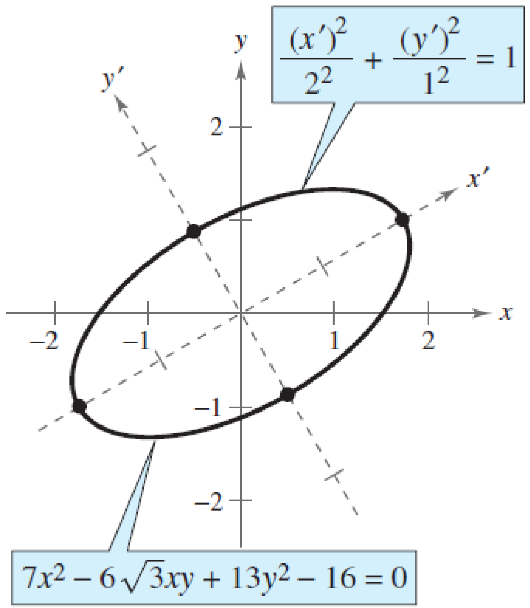
Example 7

- Rotate the axes to eliminate the *xy*-term in the equation

$$7x^2 - 6\sqrt{3}\,xy + 13y^2 - 16 = 0.$$

Then write the equation in standard form and sketch its graph.

Solution:



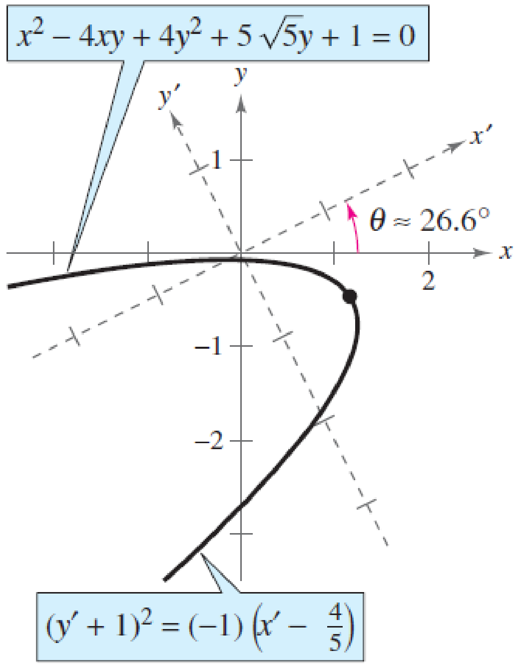
Example 8

- Rotate the axes to eliminate the xy-term in the equation

$$x^2 - 4xy + 4y^2 + 5\sqrt{5}y + 1 = 0.$$

Then write the equation in standard form and sketch its graph.

Solution:



Next Week Topics

- Quadratic Surfaces

Good Luck