Discrete Mathematics

Lab Session 4

October 6, 2020

Agenda

- Test discussion
- Relations
- Functions
- Homework

Relations

Cartesian product

Let $A = \{a_1, a_2, \ldots a_k\}$ and $B = \{b_1, b_2, ..b_m\}$.
The Cartesian product $A \times B$ is defined by a set of pairs

$$\{(a_1 b_1), (a_1, b_2), \ldots (a_1, b_m), \ldots, (a_k, b_m)\}$$

Cartesian product defines a product set, or a set of all ordered arrangements of elements in sets in the Cartesian product.

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

Cartesian product of an empty set

Proposition

If A is a set, then $A \times \emptyset = \emptyset$ and $\emptyset \times A = \emptyset$.

Proof.

We argue by contradiction using the definition of Cartesian product: Suppose $A \times \emptyset \neq \emptyset$ and consider $(x, y) \in A \times \emptyset$. Then, by definition of Cartesian product, $y \in \emptyset$, a contradiction. Therefore, the set $A \times \emptyset$ must be empty. The proof that $\emptyset \times A = \emptyset$ is similar, and is left as an exercise. □

Binary relation

Definition

Let A and B be two sets. A binary relation from A to B is a subset of a Cartesian product $A \times B$.

- $R \subseteq A \times B$ means R is a set of ordered pairs of the form (a,b) where $a \in A$ and $b \in B$
- We use the notation aRb to denote $(a, b) \in R$ and $a\not R b$ to denote $(a, b) \notin R$. If aRb, we say a is related to b by R.

Exercise 1

Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$

1. Is $R = \{(a, 1), (b, 2), (c, 2)\}$ a relation from A to B?
2. Is $Q = \{(1, a), (2, b)\}$ a relation from A to B?
3. Is $P = \{(a, a), (b, c), (b, a)\}$ a relation from A to A?

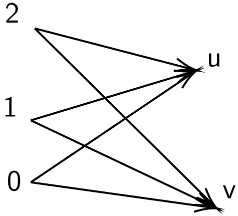
Representing binary relations graphically

We can graphically represent a binary relation R as follows:

- If $a\,R\,b$ then draw an arrow from a to b. $a \rightarrow b$

Example:

- Let $A = \{0, 1, 2\}, B = \{u, v\}$ and $R = \{(0, u), (0, v), (1, u), (1, v), (2, u), (2, v)\}$
- Note : $R \subseteq A \times B$
- **Graph**



Representing binary relations by tables

We can represent a binary relation R by a table showing (making) the ordered pairs of R.

Example:

- Let $A = \{0, 1, 2\}, B = \{u, v\}$ and $R = \{(0, u), (0, v), (1, v), (2, u)\}$
- **Table:**

R	u	v
0	x	x
1		x
2	x	

OR

R	u	v
0	1	1
1	0	1
2	1	0

Number of binary relations

Theorem

The number of binary relations on a set A, where $|A| = n$ is: 2^{n^2}

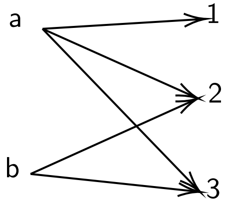
Proof.

If $|A| = n$ then the cardinality of the Cartesian product $|A \times A| = n^2$

- R is a binary relation on A if $R \subseteq A \times A$ (that is, R is a subset of $A \times A$)
- The number of subsets of a set with k elements : 2^k
- The number of subsets of $A \times A$ is : $2^{|AxA|} = 2^{n^2}$ $\square$

Relations and functions - question

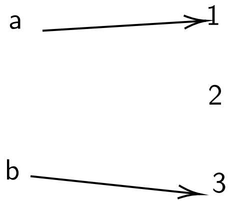
Relations represent one to many relationships between elements in A and B.



What is the difference between a relation and a function from A to B?

Relations and functions - answer

What is the difference between a relation and a function from A to B?
A function defined on sets $A, B \quad A \to B$ assigns to each element in the domain set A exactly one element from B. So it is a special relation.



Exercise 2

State whether each of the following relations represent a function or not.

(a)

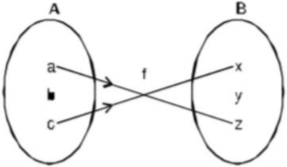


Fig.15.14

(b)

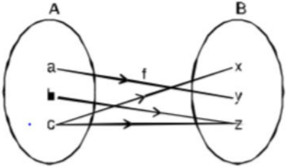


Fig.15.15

(c)

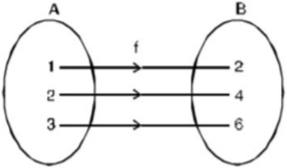


Fig. 15.16

(d)

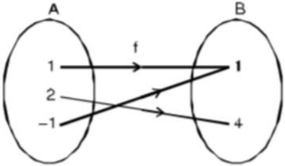


Fig.15.17

Inverse of a relation

If R is a relation from A to B, then a relation R^{-1} from B to A can be defined by interchanging the elements of all the ordered pairs of R.

Definition

Let R be a relation from A to B. define the inverse relation R^{-1} from B to A as follows:

$$R^{-1} = \{(y, x) \in (B \times A) \mid (x, y) \in R\}$$

Definition

This definition can be written operationally as follows: For all $x \in A$ and $y \in B, (y, x) \in R^{-1} \Leftrightarrow (x, y) \in R$

Exercise 3

Let $A = \{2, 3, 4\}$ and $B = \{2, 6, 8\}$ and let R be the "divides" relation from A to B : for all $(x, y) \in (A \times B)$

$$xRy \Leftrightarrow x \mid y$$

x divides y

1. State explicitly which ordered pairs are in R and R^{-1}, and draw arrow diagrams for R and R^{-1}
2. Describe R^{-1} in words.

Relation on the set - 1

Definition

A **relation on the set** A is a relation from A to itself.

Example

- Let $A = \{1, 2, 3, 4\}$ and $R_{div} = \{(a, b) \mid \text{a divides b}\}$
- What does R_{div} consist of?
- $R_{\text{div}} = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$
-

R	1	2	3	4
1	x	x	x	x
2		x		x
3			x	
4				x

Relation on the set - 2

Example

- Let $A = \{1, 2, 3, 4\}$
- Define $aR_{\neq}b$ if and only if a $\neq b$

$R_{\neq} = \{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),$

(4,1),(4,2),(4,3)$\}$

R	1	2	3	4
1		x	x	x
2	x		x	x
3	x	x		x
4	x	x	x	

Reflexive relations

Definition

A relation R on a set A is called **reflexive** if $(a, a) \in R$ for every element a $\in A$.

A relation R is reflexive if and only if MR has 1 in every position on its main diagonal.

Exercise 4

- Assume relation $R_{div} = \{(a, b) | a \text{ divides } b\}$ on $A = \{1, 2, 3, 4\}$
- Is R_{div} reflexive?

Solution

$R_{div} = \{(ab), \text{ if a} \mid b\}$ on $A = \{1, 2, 3, 4\}$
$R_{div} = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

$$
MR_{\text{div}} \quad \begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}
$$

R_{div} is reflexive as $(1,1), (2,2), (3,3)$, and $(4,4) \in R_{div}$

Exercise 5

- Relation R_{fun} on $A = \{1, 2, 3, 4\}$ is defined as

$$R_{fun} = \{(1, 2), (2, 2), (3, 3)\}$$

- Is R_{fun} reflexive?

Irreflexive relations

Definition

A relation R on a set A is called **irreflexive** if $(a, a) \notin R$ for every a $\in A$

A relation R is irreflexive if and only if MR has 0 in every position on its main diagonal.

Exercise 6

- Assume relation $R_{\neq}$, on $A = \{1, 2, 3, 4\}$, such that $aR_{\neq}b$ if and only if a $\neq b$
- Is $R_{\neq}$ irreflexive?
- $R_{\neq} = \dots$

Solution

- $R_{\neq}$ on $A = \{1, 2, 3, 4\}$, such that $aR_{\neq}b$ if and only if $a \neq b$
- $R_{\neq} = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2),$ (3,4), (4,1),(4,2),(4,3)$\}$

$$MR_{\neq} \quad \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

$R_{\neq}$ is irreflexive.

Exercise 7

Let $A = \{3, 4, 5\}$ and $B = \{4, 5, 6\}$ and let S be the "divides" relation. That is:
$\forall (x, y) \in A \times B, xSy \Leftrightarrow x \mid y$
State explicitly which ordered pairs are in S^{-1}

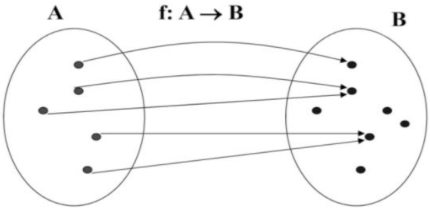
Properties of relations

- A relation R is **symmetric** if $\forall x, y, xRy \Leftrightarrow yRx$
- A relation R is **reflexive** if $\forall x, xRx$
- A relation R is **transitive** if $\forall x, y, z, (xRy \wedge yRz) \rightarrow xRz$
- An **equivalence relation** is a relation which is symmetric, reflexive and transitive

Functions

Function definition

Let A and B be non-empty sets. A **function** from A to B is an assignment of exactly one element of B to each element of A.
We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A.



If f is a function from A to B, we write $f : A \to B$

Function Definitions

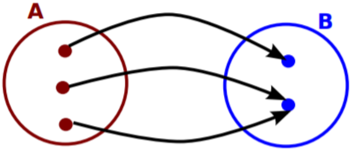
Definition

A function f from a set A to a set B assigns each element of A to exactly one element of B

- If f **maps** element $a \in A$ to element $b \in B$, we write $f(a) = b$
- A is called **domain** of f, and B is called **co-domain (range)** of f
- If $f(a) = b, b$ is called **image** of a; a is a **preimage** of b
- **Image of** f is the set of all images of elements in A

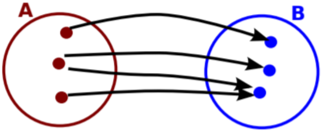
Exercise 2.1

Is it a function?



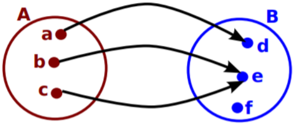
Exercise 2.2

Is it a function?



Exercise 2.3

Is it a function?



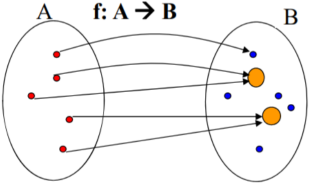
Injective functions

Definition

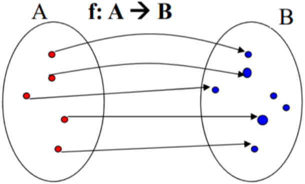
A function f is said to be **one-to-one, or injective**, if and only if $f(x) = f(y)$ implies $x = y$ for all x, y in the domain of f. A function is said to be an injection if it is one-to-one.

Alternative: A function is one-to-one if and only if $f(x) \neq f(y)$ whenever $x \neq y$. This is the contrapositive of the definition.

.



Not injective function



Injective function

Exercise 2.5

Determine whether the following functions are one-to-one (injective).

1. $f : R \to R$, defined by $y = f(x)$, where $x - 3y = 7$
2. $s : R \to R$, defined by $s(t) = 16t^2$
3. $M : P(N) \to N$, defined by:
   $M(S)$ is the minimum value of S, for $S \subseteq N$.

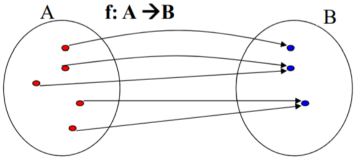
Surjective functions

Definition

A function from A to B is called **onto, or surjective,** if and only if for every $b \in B$ there is an element a $\in A$ such that $f(a) = b$

Alternative: all co-domain elements are covered

.



Exercise 2.6

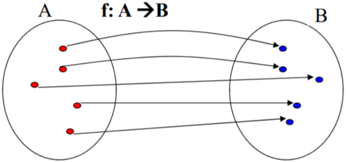
Determine whether the following functions are onto (surjective).

1. $f : R \to R$ defined by $y = f(x)$, where $x - 3y = 7$
2. $s : R \to R$ defined by $s(t) = 16t^2$
3. $M : P(N) \to N$ defined by $M(S) = \min(S)$, the minimum value of S, for $S \subseteq N$.

Bijective functions

Definition

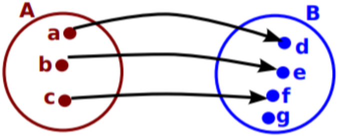
A function f is called a **bijection** if it is both one-to-one (injection) and onto (surjection).



.

Exercise 2.7

- a) Is this function onto?



- b) Consider the function $f(x) = x^2$ from the set of integers to the set of integers. Is f surjective?

Exercise 2.8

- Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$
  Define f as
  1. $1 \to c$
  2. $2 \to a$
  3. $3 \to b$
- Is f a bijection?

Exercise 2.9

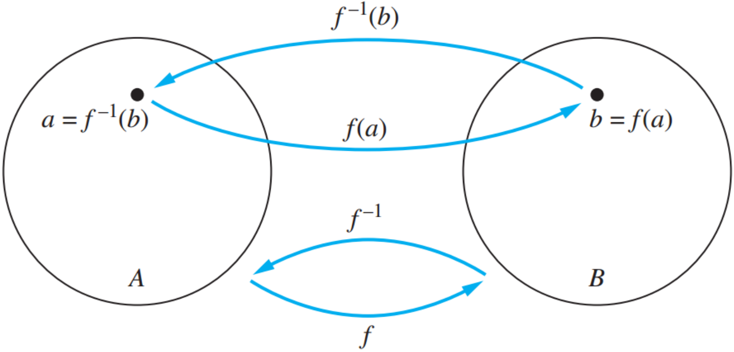
Define $g : W \to W$ (whole numbers), where $g(n) = \left\lfloor \frac{n}{2} \right\rfloor$ (floor function)

- $0 \to \lfloor 0/2 \rfloor = \lfloor 0 \rfloor = 0$
- $1 \to \lfloor 1/2 \rfloor = \lfloor 1/2 \rfloor = 0$
- $2 \to \lfloor 2/2 \rfloor = \lfloor 1 \rfloor = 1$
- $3 \to \lfloor 3/2 \rfloor = \lfloor 3/2 \rfloor = 1$
- ...

Is g a bijection?

Inverse Function

Let f be a one-to-one correspondence from the set A to the set B. The **inverse function** of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$



Example

Let $g : R \to R$, where $g(x) = 2x - 1$ What is the inverse function g^{-1}? Approach to determine the inverse:

$$\begin{aligned} y = 2x - 1 = y + 1 = 2x \\ => (y + 1)/2 = x \end{aligned}$$

Define $g^{-1}(y) = x = (y + 1)/2$ Test the correctness of inverse:

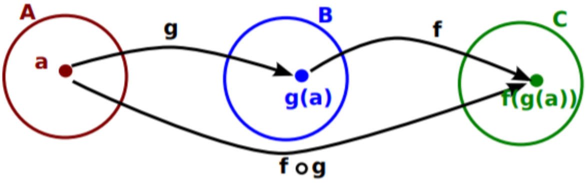
$$g(3) = \dots$$

Composition of the functions

Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The **composition of the functions f and g**, denoted for all $a \in A$ by $f \circ g$, is defined by

$$(f \circ g)(a) = f(g(a))$$

$$(f \circ g)(x) = f(g(x))$$



Composition - Example

$$
\begin{array}{ll}
g : A \to A, & f : A \to B \\
1 \to 3 & 1 \to b \\
2 \to 1 & 2 \to a \\
3 \to 2 & 3 \to d \\
f \circ g : A \to B : &
\end{array}
$$

Example

Let f and g be two functions from Z to Z, where

$$\begin{aligned}
f(x) = 2x \text{ and } g(x) &= x^2 \\
f \circ g : Z \to Z & \\
(f \circ g)(x) = & \quad f(g(x)) \\
&= f\left(x^2\right) \\
&= 2\left(x^2\right) \\
g \circ f : Z \to Z & \\
(g \circ f)(x) &= \quad ?
\end{aligned}$$

Answer

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f\left(x^2\right) \\
&= 2\left(x^2\right) \\
g \circ f : Z &\rightarrow 2\left(x^2\right) \\
(g \circ f)(x) &= g(f(x)) \\
&= g(2x) \\
&= (2x)^2 \\
&= 4x^2
\end{aligned}$$

NB: Note that the order of the function composition matters

Exercise 2.11

Given functions:
$f(x) = x^*e$
$g(x) = e^x$
$h(x) = x/(2 + \ln(3/e))$
Write $\left(h \circ g^{-1} \circ f\right)(x)$. Find domain, codomain, image. Calculate for $x = 3$

Extra Task

Points are in general position if there are no 3 collinear among them. Straight lines are in general position if non 3 intersect in a single point. Assuming that Euclidian plain is R^2, a point is a pair of numbers; a line is triple of numbers.

- Question 1: Guess what could it mean that plains are in general position?
- Question 2: Define (in set-theoretic terms) relation "points in general position on Euclidian plain".What is arity of this relation, what is the domain of the relation?
- Question 3: Define (in set-theoretic terms) relation "lines in general position on Euclidian plain".What is arity of this relation, what is the domain of the relation?

Hint for questions 2: Consider $\left\{X \in P\left(R^2\right) \mid \text{for all } (a,b),(c,d),(e,f) \in X : L_{(c,d)(e,f)}(a,b) \neq 0\right\}$ where is equation of a straight line that goes through points (c,d) and (e,f)

Homework - Part 1

Study other properties and complete this table:

Relation	Transitivity	Reflexivity	Symmetry
$x < y$			
$x \leq y$			
A divides B			
A fixes a car of B			

Homework - Part 2

Complete this table:

function	surjective	Injective	bijective	Pre-image	Image
$x^2 : R \to R$					
$log(x) : R \to R$					
$1/x : R \to R$					
$1/(x^2+1) : R \to R$					

Homework - Part 3

Why is f not a function from R to R if

1. $fx = 1/x$?
2. $f(x) = \sqrt{x}$?
3. $f(x) = \pm\sqrt{x^2+1}$?
   Determine whether the function: $f : Z \times Z \rightarrow Z$ is onto if
4. $f(m,n) = m + n$
5. $f(m,n) = m^2 + n^2$
6. $f(m,n) = m$
7. $f(m,n) = |n|$
8. $f(m,n) = m - n$

Homework - Part 4

Determine whether each of these function is a bijection from R to R

1. $f(x) = 2x + 1$
2. $f(x) = x^2 + 1$
3. $f(x) = x^3$
4. $f(x) = (x^2 + 1)/(x^2 + 2)$

Homework - Readings

Read textbook Kenneth H. Rosen "Discrete Mathematics and Its Applications, 7th Edition" – paragraphs 2.3, 9.1, 9.5