Tutorial 3: Matrices

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Course of Essentials of Analytical Geometry and Linear Algebra I

September 17, 2020

Review (Last week topics)

Vectors

- Linear Dependence and Independence
- Dot Product
- Vector Cross Product

Content

Matrices

- Matrix operations: Transpose, Addition, Scalar multiplication
- Matrix multiplication
- Change of basis

Problem 1

If a matrix A is 5×3 and the product AB is 5×7, what is the size of B?

Solution

Since A has 3 columns, B must match with 3 rows. Otherwise, AB is undefined.

Since AB has 7 columns, so does B. Thus, B is 3×7.

Problem 2

Compute matrix sum or product if it is defined. If an expression in undefined, explain why.

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}, B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, D = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}, E = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

1. $-2A,$
2. $B - 2A,$
3. $AC,$
4. CD

Solution

1. $-2A = (-2)\begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 2 \\ -8 & 10 & -4 \end{bmatrix}.$

2. $B - 2A = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} + \begin{bmatrix} -4 & 0 & 2 \\ -8 & 10 & -4 \end{bmatrix} = \begin{bmatrix} 3 & -5 & 3 \\ -7 & 6 & -7 \end{bmatrix}$

3. The product AC is not defined because the number of columns of A does not match the number of rows of C.

4. $CD = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 1\cdot 3 + 2(-1) & 1\cdot 5 + 2\cdot 4 \\ -2\cdot 3 + 1(-1) & -2\cdot 5 + 1\cdot 4 \end{bmatrix} = \begin{bmatrix} 1 & 13 \\ -7 & -6 \end{bmatrix}$

Problem 3

Let $\mathbf{u} = \begin{bmatrix} -3 \\ 2 \\ -5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Compute $\mathbf{u}^T\mathbf{v}, \mathbf{v}^T\mathbf{u}, \mathbf{u}\mathbf{v}^T$ and $\mathbf{v}\mathbf{u}^T$.

Solution

The product $\mathbf{u}^T\mathbf{v}$ is a 1×1 matrix, which usually is identified with a real number and is written without the matrix brackets.

$$\mathbf{u}^T\mathbf{v} = \begin{bmatrix} -3 & 2 & -5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = -3a + 2b - 5c \,,$$

$$\mathbf{u}\mathbf{v}^T = \begin{bmatrix} -3 \\ 2 \\ -5 \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} -3a & -3b & -3c \\ 2a & 2b & 2c \\ -5a & -5b & -5c \end{bmatrix}$$

$$\mathbf{v}^T\mathbf{u} = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ -5 \end{bmatrix} = -3a + 2b - 5c$$

$$\mathbf{v}\mathbf{u}^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} -3 & 2 & -5 \end{bmatrix} = \begin{bmatrix} -3a & 2a & -5a \\ -3b & 2b & -5b \\ -3c & 2c & -5c \end{bmatrix}$$

Problem 4

a) If $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbb{R}^n$, how are $\mathbf{u}^T\mathbf{v}$ and $\mathbf{v}^T\mathbf{u}$ related?

b) How are $\mathbf{u}\mathbf{v}^T$ and $\mathbf{v}\mathbf{u}^T$ related?

Solution

a) Since the inner product $\mathbf{u}^T\mathbf{v}$ is a real number, it equals its transpose. That is, $\mathbf{u}^T\mathbf{v} = (\mathbf{u}^T\mathbf{v})^{\boldsymbol{T}} = \mathbf{v}^{\boldsymbol{T}}(\mathbf{u}^T)^{\boldsymbol{T}} = \mathbf{v}^{\boldsymbol{T}}\mathbf{u}$, by Theorem $(AB)^T = B^T A^T$ regarding the transpose of a product of matrices and by Theorem $(A^T)^T = A$.

b) The outer product $\mathbf{u}\mathbf{v}^T$ is an $n \times n$ matrix. By Theorems (mentioned above), $\mathbf{u}\mathbf{v}^T = (\mathbf{u}\mathbf{v}^T)^{\boldsymbol{T}} = \mathbf{v}\mathbf{u}^T$

Problem 5

Suppose the last column of AB is entirely zeros but B itself has no column of zeros. What can be said about the columns of A?

Solution

Let $\mathbf{b}_p$ be the last column of B. By hypothesis, the last column of AB is zero. Thus, $A\mathbf{b}_p = \mathbf{0}$.

However, $\mathbf{b}_p$ is not the zero vector, because B has no column of zeros. Thus, the equation $A\mathbf{b}_p = \mathbf{0}$ is a linear dependence relation among the columns of A, and so the columns of A are linearly dependent.

Problem 6

True or false: If $AB = 0$ then either $A = 0$ or $B = 0$.

Solution

Fales.

For example, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Problem 7

Let $\mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}, \mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, and consider the bases for $\mathbb{R}^2$ given by $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

Solution

The matrix $P_{C\leftarrow B}$ involves the C-coordinate vectors of $\mathbf{b}_1$ and $\mathbf{b}_2$. Let $[\mathbf{b}_1]_C = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $[\mathbf{b}_2]_C = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Then, by definition,

$$[\mathbf{c}_1 \quad \mathbf{c}_2 \mid \mathbf{b}_1 \quad \mathbf{b}_2] = \left[\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 6 & 4 \\ 0 & 0 & -5 & -3 \end{array}\right]$$

To solve both systems simultaneously, augment the coefficient matrix with $\mathbf{b}_1$ and $\mathbf{b}_2$, and row reduce:

Thus $[\mathbf{b}_1]_C = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ and $[\mathbf{b}_2]_C = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$

The desired change-of-coordinates matrix is therefore

$$P_{C\leftarrow B} = [[\mathbf{b}_1]_C \quad [\mathbf{b}_2]_C] = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$$

Problem 8

Let $D = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ and $F = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ be bases a vector space V, and suppose $\mathbf{f}_1 = 2\mathbf{d}_1 - \mathbf{d}_2 + \mathbf{d}_3, \mathbf{f}_2 = 3\mathbf{d}_2 + \mathbf{d}_3$ and $\mathbf{f}_3 = -3\mathbf{d}_1 + 2\mathbf{d}_3$.

I. Find the change of coordinates matrix from F to D.

II. Find $[\mathbf{x}]_D$ for $x = \mathbf{f}_1 - 2\mathbf{f}_2 + 2\mathbf{f}_3$.

Solution

a. Since $\mathbf{f}_1 = 2\mathbf{d}_1 - \mathbf{d}_2 + \mathbf{d}_3, \mathbf{f}_2 = 3\mathbf{d}_2 + \mathbf{d}_3$ and $\mathbf{f}_3 = -3\mathbf{d}_1 + 2\mathbf{d}_3$

$$[\mathbf{f}_1]_D = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, [\mathbf{f}_2]_D = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, [\mathbf{f}_3]_D = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} \text{ and } P_{D\leftarrow F} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

b. Since $x = \mathbf{f}_1 - 2\mathbf{f}_2 + 2\mathbf{f}_3$

$$[\mathbf{x}]_D = P_{D\leftarrow F}[\mathbf{x}]_F = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \\ 3 \end{bmatrix}$$

Problem 9

Find the vector $\mathbf{x}$ determined by the given coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ and given basis $\mathcal{B}$.

1. $\mathcal{B} = \left\{\begin{bmatrix}3\\-5\end{bmatrix}, \begin{bmatrix}-4\\6\end{bmatrix}\right\}, [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}5\\3\end{bmatrix}$

2. $\mathcal{B} = \left\{\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}-4\\1\end{bmatrix}\right\}, [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}-2\\5\end{bmatrix}$

3. $\mathcal{B} = \left\{\begin{bmatrix}1\\-2\\3\end{bmatrix}, \begin{bmatrix}5\\0\\-2\end{bmatrix}, \begin{bmatrix}4\\-3\\0\end{bmatrix}\right\}, [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}1\\0\\-2\end{bmatrix}$

4. $\mathcal{B} = \left\{\begin{bmatrix}-2\\2\\0\end{bmatrix}, \begin{bmatrix}3\\0\\2\end{bmatrix}, \begin{bmatrix}4\\-1\\3\end{bmatrix}\right\}, [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}-3\\2\\-1\end{bmatrix}$

Solution

1. $\mathbf{x} = 5\begin{bmatrix}3\\-5\end{bmatrix} + 3\begin{bmatrix}-4\\6\end{bmatrix} = \begin{bmatrix}3\\-7\end{bmatrix}$

2. $\mathbf{x} = -2\begin{bmatrix}3\\2\end{bmatrix} + 5\begin{bmatrix}-4\\1\end{bmatrix} = \begin{bmatrix}-26\\1\end{bmatrix}$

3. $\mathbf{x} = 1\begin{bmatrix}1\\-2\\3\end{bmatrix} + 0\begin{bmatrix}5\\0\\-2\end{bmatrix} - 2\begin{bmatrix}4\\-3\\0\end{bmatrix} = \begin{bmatrix}-7\\4\\3\end{bmatrix}$

4. $\mathbf{x} = -3\begin{bmatrix}-2\\2\\0\end{bmatrix} + 2\begin{bmatrix}3\\0\\2\end{bmatrix} - 1\begin{bmatrix}4\\-1\\3\end{bmatrix} = \begin{bmatrix}8\\-5\\1\end{bmatrix}$

Problem 10

Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of $\mathbf{x}$ relative to the given basis $\mathcal{B} = \{\mathbf{b}_1 \cdots \mathbf{b}_n\}$.

1. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Solution

The matrix $[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{x}]$ row reduces to $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$, so $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

2. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$

Solution

The matrix $[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{x}]$ row reduces to $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, so $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Problem 11 (1)

Find the coordinate vector to test the linear independence of the sets of polynomials.

$a)\;\; 1+2t^3, 2+t-3t^2, -t+2t^2-t^3$

Solution

The coordinate mapping produces the coordinate vectors (1, 0, 0, 2), (2, 1, -3, 0), and (0, -1, 2, -1) respectively. We test for linear independence of these vectors by writing them as columns of a matrix and row reducing:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -3 & 2 \\ 2 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}$$

Since the matrix has a pivot in each column, its columns (and thus the given polynomials) are linearly independent.

Problem 11 (2)

$b)\ (1-t)^2, t-2t^2+t^3, (1-t)^3$

Solution

The coordinate mapping produces the coordinate vectors (1, -2, 1, 0), (0, 1, -2, 1), and (1, -3, 3, -1) respectively. We test for linear independence of these vectors by writing them as columns of a matrix and row reducing:

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -3 \\ 1 & -2 & 3 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the matrix does not have a pivot in each column, its columns (and thus the given polynomials are linearly dependent.