Discrete Math

Lab 7 – November, 3

Agenda

- Homework review
- Discrete probability
- Lab 6 solutions
- Homework

Homework review

Homework

1. How many full houses are there in poker? The deck has 52 cards of 4 denominations. A full house has 5 cards, 3 of one kind and 2 of another. E.g.: 3 5's and 2 Kings.
2. How many ways are there of choosing k numbers from {1, . . . , n} if 1 and 2 can't both be chosen? (Suppose n, k ≥ 2, n ≥ k)
3. How many positive integers not exceeding 100 are divisible either by 4 or by 6?
4. A multiple-choice test contains 10 questions. There are four possible answers for each question. In how many ways can a student answer the questions on the test if:

    a) the student answers every question?

    b) the student can leave answers blank?

Homework continued

5. How many integer solutions does the equation $x_1 + x_2 + x_3 + x_4 = 15$ have, if we require that $x_1 \geq 2$, $x_2 \geq 3$, $x_3 \geq 10$ and $x_4 \geq -3$?
6. We go to a pizza party, and there are 5 types of pizza. We have starved for days, so we can eat 13 slices, but we want to sample each type at least once. In how many ways can we do this? Order does not matter
7. There are 8 types(the same type can be used several times) of cookies available in a store. Count the number of ways
   (a) to pick 6 of them and arrange them in a line.
   (b) to pick 6 of them and place them into lines named A and B, with 3 in each.
   (c) to pick 6 of them and place them into two equal-sized unlabeled lines.

Elements of probability theory

Standard probability definition

An **experiment** is a procedure that yields one of a given set of possible outcomes.
The **sample space** of the experiment is the set of possible outcomes.
An **event** is a subset of the sample space

If S is a finite nonempty sample space of equally likely outcomes, and E is an event, that is, a subset of S, then the **probability** of E is p(E) = $\frac{|E|}{|S|}$.

$0 \leq p(E) \leq 1$

Probability of an impossible event E: $p(E) = 0$

Probability of a sure event E: $p(E) = 1$

Example

An urn contains four blue balls and five red balls. What is the probability that a ball chosen at random from the urn is blue? What is an experiment, a possible outcome, a sample space, an event in this task?

Solution:
an experiment – choosing a ball from the urn,
sample space S = $\{b_1, b_2, b_3, b_4, r_1, r_2, r_3, r_4, r_5\}$
Possible outcome – element of the set S
an event – "the ball is blue", E = $\{x \in S \mid x = b_i\} = \{b_1, b_2, b_3, b_4\}$
p(E) = |E|/|S|=4/9

Assigning Probabilities

Let S be the sample space of an experiment with a finite or countable number of outcomes. We assign a probability p(s) to each outcome s. We require that two conditions be met:

- The probability of each outcome is a nonnegative real number ≤ 1.
- The sum of the probabilities of all possible outcomes should be 1

The function p from the set of all outcomes of the sample space S is called a probability distribution.

The **probability of the event** E is the sum of the probabilities of the outcomes in E. That is, $p(E) = \sum_{s \in E} p(s)$

Probabilities of Complements and Unions of Events

Let E be an event in a sample space S. The probability of the event $\bar{E} = S - E$, the complementary event of E, is given by $\mathrm{p}(\bar{E}) = 1 - p(E)$

Let A and B be events in the sample space S. Then
$p(A \cup B) = p(A) + p(B) - p(A \cap B)$

Independent events

The events A and B are **independent** if and only if $p(A \cap B) = p(A)p(B)$

The events $E_1, E_2, \ldots, E_n$ are **pairwise independent** if and only if $p(E_i \cap E_j) = p(E_i)p(E_j)$ for all pairs of integers i and j with $1 \leq i < j \leq n$. These events are **mutually independent** if $p(E_{i1} \cap E_{i2} \cap \cdots \cap E_{im}) = p(E_{i1})p(E_{i2}) \cdots p(E_{im})$ whenever ij, $j = 1, 2, \ldots, m$, are integers with $1 \leq i1 < i2 < \cdots < im \leq n$ and $m \geq 2$.

Every set of n mutually independent events is also pairwise independent. However, n pairwise independent events are not necessarily mutually independent.

Mutually exclusive events

The events A and B are mutually exclusive if they cannot both occur.
Probability of two mutually exclusive events A and B: $p(A \cup B) = p(A) + p(B)$

Events $A_1, A_2, ..., A_n$ are said to be mutually exclusive if the occurrence of any one of them implies the non-occurrence of the remaining $n - 1$ events.

Conditional probability

Let A and B be events with $p(B) > 0$. The conditional probability of A given B, denoted by $p(A \mid B)$, is defined as:

$$p(A \mid B) = p(A \cap B) \,/\, p(B)$$

Bernoulli Trials

Suppose that an experiment can have only two possible outcomes. Each performance of an experiment with two possible outcomes is called a **Bernoulli trial**. In general, a possible outcome of a Bernoulli trial is called a **success** or a **failure**. If p is the probability of a success and q is the probability of a failure, it follows that $p + q = 1$

Bernoulli trials are **mutually independent** if the conditional probability of success on any given trial is p, given any information about the outcomes of the other trials

The **probability of exactly k successes** in n independent Bernoulli trials, with probability of success p and probability of failure $q=1-p$, is $\mathbf{C(n, k)p^k q^{n-k}}$

Exercises

1. What is the probability that a die comes up an odd number when it is rolled? What is an experiment, a possible outcome, a sample space, an event in this task? What would be a sample space and the number of possible outcomes if we roll the die twice?

2. Determine the probability of following results when rolling 2 dice:
    a) sum equals to 8
    b) sum is divisible by 5
    c) Sum is even

3. A gambler playing with 3 dice wants to know weather to bet on sum 11 or 12. Which of the sums will occur more probably?

Exercises II

4. There are 16 cola bottles on the table. 10 of them are filled by Coca Cola and 6 of them are filled by Pepsi. Determine the probability of 4 randomly selected bottles to include 2 Coca Cola and 2 Pepsi bottles.

5. In a game of chance you draw 6 numbers out of 49. Determine the probability of reaching:
a) matching 3 of 6 winning numbers
c) matching 6 of 6 winning numbers

6. 32 playing cards include 4 aces and 12 figures. Determine the probability of a randomly selected card to be an ace or a figure.

Exercises III

7. In a town there are 4 crossroads with traffic lights. Each traffic light opens or closes the traffic with the same probability of 0,5. Determine the probability of the crossing all the crossroads without stopping.

8. Determine the probability of 3 of 5 born children being sons if the probability of a children to be a boy equals p(A) = 0,51

9. 82 170 of 100 000 children live 40 years and 37 930 of 100 000 children live 70 years. Determine the probability of a 40 years old person to live 70 years.

10. A coin is biased so that the probability of heads is 2/3. What is the probability that exactly four heads come up when the coin is flipped seven times, assuming that the flips are independent?

Lab 6 exercises.
Principle of inclusion-exclusion

Exercises - II

7. You are going to an amusement park. There are four attractions, (haunted house, roller coaster, a carousel, water ride). You buy 25 tokens. Each attraction cost 3 tokens each ride, except the roller coaster that costs 5. Obviously, you want to ride each ride at least once, but the order of the rides does not matter. In how many ways can you spend your tokens? You may have some remaining tokens in the end of the day.
8. How many words can you create of length 6, from the letters a, b, c and d if
    - you must include each letter at least once, **and**
    - “a” must appear exactly once
9. How many bit strings of length 10 over the alphabet {a, b, c} have either exactly three a’s or exactly four b’s

Exercises - III

10. There are five people of different height. In how many ways can they stand in a line, so that there is no 3 consecutive people with increasing height?
11. We have a smorgasbord, with 50 dishes, — 5 countries are represented, and there are 10 dishes from each. We want to make a plate with 8 dishes (no duplicates), but make sure that no country is missing. In how many ways we can do it?

Exercise 7

After we rode each attraction once, we have 25-3*3-5 = 11 tickets left. We can ride 0, 1, 2, or 3 attractions (4 cheapest attractions require 12 tickets).

- 0 – there is 1 way to do it
- 1 – C(4, 1) = 4 ways
- 2 – C(4+2-1, 2) = C(5,2) = 5!/(3!*2!) = 10 ways
- 3 – we may ride roller coaster only 0 or more times as if we ride it 2 times, we have 1 ticket left which is not enough for the third ride
  - Ride roller coaster once: (3+2-1, 2) = 4!/(2!*2!) = 6 ways
  - Ride roller coaster 0 times: (3+3-1, 3) = 5!/(3!*2!) = 10 ways.

So the total number of ways is 1+ 4+10+6+10 = 31

Exercise 8

We need to decide which two additional letters to add to abcd.

- We add two different letters. There are C(3,2)=3 ways to choose a letter to add (as we cannot add a), so we choose 3 letters out of 5. To calculate the number of words we can make from a word with 2 pairs of same letters and two other different letters, we use a multinomial coefficient, 6!/(2!*2!*1!*1!). So the total number is 3*(3*4*5*3) = 540
- We add the same letter twice. There are C(3,1)=3 ways to choose a letter to add, and each of these options gives 6!/(3!*1!*1!*1!) words. In total there are 3*(4*5*6) = 360 ways
- Summing up these two numbers we get the total result: 540+360 = 900.

Exercise 9

We will apply the inclusion-exclusion principle.

- First let us calculate the number of these strings with exactly three a's. To specify such a string we need to choose the positions for the a's, which can be done in C(10, 3) ways. Then we need to choose either b or c to fill each of the other 7 positions in the string, which can be done in 2^7 ways. Therefore there are C(10, 3) · 2^7 = 15360 strings.
- Similarly, there are C(10, 4) · 2^6 = 13440 strings with exactly four b’s.
- To specify a string with exactly three a's and exactly four b's, we need to choose the positions for the a's, which can be done in C(10, 3) ways, and then choose the positions for the b's, which can be done in C(7,4) ways (only seven slots remained). Therefore there are C(10, 3) · C(7, 4) = 4200 such strings.
- Finally, by the inclusion-exclusion principle the number of strings having either exactly three a's or exactly four b's is 15360 + 13440 - 4200 = 24,600.

Exercise 10

To calculate the number of permutations with no 3 consecutive people with increasing height, we subtract from total number of permutations the number of permutations with 3 or more people with increasing height.

Let us number the spots in the line, 12345. Let A be the event that the people at spots 123 are in height order, B be the event that 234 are in order and C the event that 345 are in order.

Total number of orderings is 5!

Let us find $|A \cup B \cup C|$ via inclusion-exclusion.

$|A| = |B| = |C| = C(5,3)*2! = 20$ (choose 3 of 5 people, there is only 1 way how they can be arranged with increasing height, then multiply by arrangements of the remaining 2)

$|A \cap B| = |B \cap C| = C(5,4)*1! = 5$ (choose 4 of 5 people).

$|A \cap C| = 1$ and $|A \cap B \cap C| = 1$, since this means that the people in all five spots must appear in increasing order.

Result: $5! – (|A|+|B|+|C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|) = 120-(20*3-5*2-1+1) = 120–50 = 70$

Exercise 11

Let $A_1, \ldots, A_5$ be the possible plates where country i not represented. Then

$|A_1 \cup \cdots \cup A_n|$ is the number of plates where at least one country is not represented.

$|A_i| = C(40,8)$ can choose from all plates except of those of country i

$|A_i \cap A_j| = C(30,8)$ can choose from all plates except of those of country i and j.

There are C(5,2) = 5!/(3!*2!) = 10 such sets

$|A_i \cap A_j \cap A_k| = C(20,8)$ same logic. There are C(5,3)=10 such sets

$|A_i \cap A_j \cap A_k \cap A_l| = C(10,8)$ same logic. There are C(5,4)=5 such sets

$|A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5| = 0$ there are no plates that are of neither of countries 1…5

So the number of plates of 8 dishes with no country missing is the total number of ways minus $|A_1 \cup \cdots \cup A_5|$, which is

C(50,8) – 5*C(40,8)+10*C(30,8)-10*C(20,8)+5*C(10,8)-0

Homework 9

1. Read textbook Kenneth H. Rosen "Discrete Mathematics and Its Applications, 7th Edition“ – paragraphs 7.1,7.2

2. Solve problems 1-6 and submit on Moodle by Friday November 6 10pm

Homework problems - I

1. There are n people and n hats. People pick up hats at random. What is the probability that nobody picks up his hat?

2. Use mathematical induction to prove that if E_1, E_2, . . . , E_n is a sequence of n pairwise disjoint events in a sample space S, where n is a positive integer, then

$$p(\cup_{i=1}^{n} E_i) = \sum_{i=1}^{n} p(E_i)$$

3. There are 2 types of people in a big factory: prone to COVID (30% of workers), and non-prone to COVID (70% of workers). For a COVID-prone person, p(COVID) = 0.4. For a non-prone to COVID person, p(COVID) = 0.2. Compute :

a) the probability a random factory worker will have a COVID
b) given a person who has a COVID, what is p(person is COVID-prone).

Homework problems - II

4. A factory of iPhone production has three machines. Machine A produces 20% of the smartphones. Machine B produces 30% of the smartphones. Machine C produces 50% of the smartphones. For machine A, 6% of the products are defective. For machine B, 7% of the products are defective. For machine C, 8% of the products are defective. Select a product randomly and let E be the event that the product is defective. What is p(E)?

5. What is the probability of these events when we randomly select a permutation of {1,2,3,4}?

a) 1 precedes 4
b) 4 precedes 1
c) 4 precedes 1 and 4 precedes 2
d) 4 precedes 1, 4 precedes 2, and 4 precedes 3.
e) 4 precedes 3 and 2 precedes 1.

Homework problems - III

6. What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up heads?