V1 B20-02 Dmitry Beresnev

I think you mean "even" instead of "equal"

① Direct Proof:

We have to consider two cases:

a) x and y are even. So there are some integers m amd n such as $x = 2m$, $y = 2n$. Hence,

$x + y = 2m + 2n = 2(m+n)$. As m amd n are integers, $(m+n)$ is an integer too. Obviously, $2(m+n) \vdots 2$. Therefore,

~~x+y is e~~ $2(m+n) = x+y$ is even. Q.E.D

b) x and y are odd. So there are some integers m and n such as $x = 2m+1$ and $y = 2n+1$. Hence,

$x + y = 2m + 1 + 2n + 1 = 2(m+n+1)$. As m and n are integers, $(m+n+1)$ is an integer too. Obviously, $2(m+n+1) \vdots 2$. Therefore, $2(m+n+1) = x+y$ is even Q.E.D.

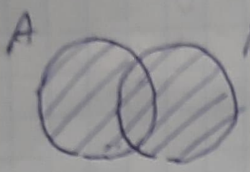
Proof by Contradiction:

Suppose $x+y$ is ~~odd~~ even, but x and y have different parity. Without loss of generality, let x be even and y be odd. So, there ~~is~~ are ~~an~~ integers m and n, such as $x = 2n$, $y = 2m+1$. Hence, $x+y = 2n + 2m + 1 = 2(m+n)+1$. As m and n are integers, $2(m+n)+1$ is an integer too. $2(m+n) \vdots 2$ but $2(m+n)+1 \not\vdots 2$. So, $2(m+n)+1 = x+y$ is odd. It is contradiction. Therefore our suggestion is false and x and y have to have the same parity. Q.E.D.

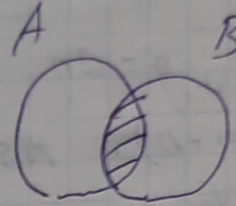
5. $A \cup B = \{1,2,3,4,5\}$

$(A \setminus B) \cup (B \setminus A) = \{1,2,5\}$ Find $A \cap B$ -?

Let's use the Euler circles notation.

$A \cup B$: 

$(A \setminus B) \cup (B \setminus A) =$ 

So we have: $A \cap B =$  $= (A \cup B) - ((A \setminus B) \cup (B \setminus A))$

Therefore, $A \cap B = \{1,2,3,4,5\} - \{1,2,5\} = \{3,4\}$

Answer: $A \cap B = \{3,4\}$

6. $P(\{\emptyset, b, \{a,b\}\})$. As $\{\emptyset, b, \{a,b\}\}$ have 2 elements (we don't count $\emptyset$, as it is subset for any set), $P(\{\emptyset, b, \{a,b\}\})$ have to have $2^2 = 4$ elements.

$P(\{\emptyset, b, \{a,b\}\}) = \{\{\emptyset\}, \{b\}, \{\{a,b\}\}, \{b, \{a,b\}\}\}$.

Answer: 4

7. $(A \setminus B = A) \to B \subset A$ is true or false?

Counter example

Let $A = \{1,2,3\}$ So, we have $A \setminus B = \{1,2,3\}$
$B = \{4,5\}$. $(A \setminus B = A) = 1$

~~$B \subset A = 0$~~

$B \subset A = 0$

$1 \to 0 = 0$ Therefore, proposition is false

③ $6^n - 1 \vdots 5 \quad \forall n > 0$.

Proof by Induction

Induction case: $n = 1 \Rightarrow 6^1 - 1 = 5 \vdots 5$. It is true.

Induction step: Let's suppose $6^n - 1 \vdots 5$ is true for some n.

~~Let's then~~ Let's prove that then $6^{n+1} - 1 \vdots 5$ is also true.

$$6^{n+1} - 1 = 6 \cdot 6^n - 1 = 6 \cdot 6^n - 6 + 5 = 6(6^n - 1) + 5.$$

$6(6^n - 1) + 5$ is an integer. $6^n - 1 \vdots 5 \Rightarrow 6(6^n - 1) \vdots 5$ (a)

$5 \vdots 5$ (b)

From (a) and (b) follows that $6(6^n - 1) + 5 \vdots 5$. Q.E.D.

④ $f_0 = 0,\ f_1 = 1 \quad f_n = f_{n-1} + f_{n-2}$ for $n \in \mathbb{N},\ n \geq 2$

Prove: $f_1^2 + f_2^2 + \ldots + f_n^2 = f_n \cdot f_{n+1}$

Proof by induction:

Induction case: $f_1 = 1,\ f_2 = 1,\ f_3 = 2 \Rightarrow f_1^2 + f_2^2 = 1 + 1 = 2 = f_2 \cdot f_3 = 1 \cdot 2$

it is true

Induction step: Let's suppose that $f_1^2 + f_2^2 + \ldots + f_n^2 = f_n f_{n+1}$ is true for $n \geq 2$. Let's prove that then $f_1^2 + f_2^2 \ldots + f_{n+1}^2 = f_{n+1} \cdot f_{n+2}$ is also true.

$$f_1^2 + f_2^2 + \ldots + f_n^2 + f_{n+1}^2 =$$

$$= f_n \cdot f_{n+1} + f_{n+1}^2 = f_{n+1}(f_n + f_{n+1})$$

From the recurrent equation: $f_{n+2} = f_{n+1} + f_n$. So,

$$f_{n+1}(f_n + f_{n+1}) = f_{n+1} \cdot f_{n+2}. \quad \text{Q.E.D.}$$

(2) $x^5+7x^3+5x \geq x^4+x^2+8$, $x \geq 0$ $x \in R$

Proof by Contradiction

Let's suppose that $x^5+7x^3+5x \geq x^4+x^2+8$ and $x<0$, $x \in R$. For any $x<0$ we can find such integer $n>0$, such as $x=(-1)\cdot n$. Substituting to inequality:

$$(-1)^5 n^5 + (-1)^3\cdot(7n)^3 + (-1)\cdot(5n) \geq (-1)^4 n^4 + (-1)^2 n^2 + 8$$

$$(-1)n^5 + (-1)(7n)^3 + (-1)\cdot(5n) \geq n^4+n^2+8$$

$$-(n^5+(7n)^3+5n) \geq n^4+n^2+8$$

As n is an integer $n>0$, $n^4+n^2+8>0$, $\forall n>0$

$$n^5+(7n)^3+5n>0, \forall n>0$$

$$-(n^5+(7n)^3+5n)<0, \forall n>0$$

To sum it all up:

$$\begin{cases} n^4+n^2+8>0 \\ -(n^5+(7n)^3+5n)<0 \\ \cancel{-(n^5+(7n)^3+5n) \geq n^4+n^2+8} \end{cases} \Rightarrow \boxed{n^4+n^2+8 > -(n^5+(7n)^3+5n)}$$

It is a contradiction. So, our suggestion is false. Therefore, if $x^5+7x^3+5x \geq x^4+x^2+8$, then $x \geq 0$, $x \in R$

Q.E.D.