Discrete Mathematics
Tutorial 2

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Induction

We need to prove

$$P(n), \text{ for any } n \geq 1$$

Initial step $n = 1$

Prove $P(1)$

Inductive hypothesis

Suppose that

$$P(1), P(2), \ldots, P(k)$$

Inductive step

Prove $P(k+1)$

Induction

Fibonacci numbers

$$f_0 = 0, \quad f_1 = 1, \quad f_n = f_{n-1} + f_{n-2}$$

$$0 \quad 1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad 21 \quad \ldots$$

$$f_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

Induction

Fibonacci numbers

$f_0 = 0, \quad f_1 = 1, \quad f_n = f_{n-1} + f_{n-2}$

Initial step. 1) $n = 0$

$$f_0 = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^0 - \left(\frac{1-\sqrt{5}}{2}\right)^0}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0$$

Induction

Fibonacci numbers

$f_0 = 0, \quad f_1 = 1, \quad f_n = f_{n-1} + f_{n-2}$

Initial step. 2) $n = 1$

$$f_1 = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1}{\sqrt{5}} =$$

$$= \frac{1+\sqrt{5}-(1-\sqrt{5})}{2\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1$$

Induction

Fibonacci numbers

$$f_0 = 0, \quad f_1 = 1, \quad f_n = f_{n-1} + f_{n-2}$$

Induction hypothesis. Suppose that there is k such that, for any $i \leq k$,

$$f_i = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^i - \left(\frac{1-\sqrt{5}}{2}\right)^i}{\sqrt{5}}$$

Induction

Fibonacci numbers

$$f_0 = 0, \quad f_1 = 1, \quad f_n = f_{n-1} + f_{n-2}$$

Inductive step. We need to prove

$$f_{k+1} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\sqrt{5}}$$

Induction

Fibonacci numbers

$f_0 = 0, \quad f_1 = 1, \quad f_n = f_{n-1} + f_{n-2}$

$$f_{k+1} = f_k + f_{k-1}$$

$$f_k = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k}{\sqrt{5}}$$

$$f_{k-1} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}}$$

Induction

$$f_{k+1} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k}{\sqrt{5}} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}} =$$

$$= \frac{\left(\frac{1+\sqrt{5}}{2}+1\right)\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}+1\right)\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}} =$$

$$= \frac{\left(\frac{3+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{3-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}} =$$

Induction

$$f_{k+1} = \frac{\left(\frac{3+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{3-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}} =$$

$$= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{2}\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{2}\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}}{\sqrt{5}} =$$

$$= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\sqrt{5}}$$

$$\left(\frac{1+\sqrt{5}}{2}\right)^{2} = \frac{1+2\sqrt{5}+5}{4} = \frac{3+\sqrt{5}}{2} \qquad \left(\frac{1-\sqrt{5}}{2}\right)^{2} = \frac{1-2\sqrt{5}+5}{4} = \frac{3-\sqrt{5}}{2}$$

Logic

Truth Table

P	$\neg P$
0	1
1	0

P_1	P_2	$P_1 \mathbin{\&} P_2$	$P_1 \vee P_2$	$P_1 \to P_2$	$P_1 \leftrightarrow P_2$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

Logic

P_1	P_2	$P_1 \,\&\, P_2$	$P_1 \vee P_2$	$P_1 \to P_2$	$P_1 \leftrightarrow P_2$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

$$0_{10} = 00_2$$

$$1_{10} = 01_2$$

$$2_{10} = 10_2$$

$$3_{10} = 11_2$$

Logic

P_1	P_2	$P_1 \& P_2$	$P_1 \vee P_2$	$P_1 \to P_2$	$P_1 \leftrightarrow P_2$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	0
1	1	1	1	1	1

$$(\neg a \to b) \leftrightarrow (a \,\&\, b)$$

a	b	$\neg a$	$\neg a \to b$	$a \,\&\, b$	$(\neg a \to b) \leftrightarrow (a \,\&\, b)$
0	0	1	0	0	1
0	1	1	1	0	0
1	0	0	1	0	0
1	1	0	1	1	1

Logic

$$(\neg a \to b) \leftrightarrow (\neg b \,\&\, c)$$

a	b	c	$\neg a$	$\neg b$	$(\neg a \to b)$	$\neg b \,\&\, c$	$(\neg a \to b) \leftrightarrow (\neg b \,\&\, c)$
0	0	0	1	1	0	0	1
0	0	1	1	1	0	1	0
0	1	0	1	0	1	0	0
0	1	1	1	0	1	0	0
1	0	0	0	1	1	0	0
1	0	1	0	1	1	1	1
1	1	0	0	0	1	0	0
1	1	1	0	0	1	0	0

Logic

Truth Table

$$0_{10} = 000_2$$

$$1_{10} = 001_2$$

$$2_{10} = 010_2$$

$$3_{10} = 011_2$$

$$4_{10} = 100_2$$

$$5_{10} = 101_2$$

$$6_{10} = 110_2$$

$$7_{10} = 111_2$$

For n variables, the truth table contains 2^n rows.

Logic

Let $n = 100$.

How long does it take to compute on a computer?

Logic

Let $n = 100$.

How long does it take to compute on a computer?

1 GHz $=$ 1.000.000.000 Hertz $= 10^9$ elem.operations/sec.

10 GHz $=$ 10.000.000.000 Hertz $= 10^{10}$ elem.operations/sec.

Logic

Let $n = 100$.

How long does it take to compute on a computer?

$$1 \text{ GHz } = 1.000.000.000 \text{ Hertz } = 10^9 \text{ elem.operations/sec.}$$

$$10 \text{ GHz } = 10.000.000.000 \text{ Hertz } = 10^{10} \text{ elem.operations/sec.}$$

$$2^{100} = (2^{10})^{10} = (1024)^{10} > (1000)^{10} = (10^3)^{10} = 10^{30}$$

Logic

$n = 100$

$$2^{100} > 10^{30}$$

$$\text{Seconds} = \frac{10^{30}}{10^{10}} = 10^{30-10} = 10^{20}$$

$$60 \times 60 \times 24 = 86400 \text{ seconds in each day} < 100000 = 10^5$$

$$\text{Days} = \frac{10^{20}}{10^5} = 10^{20-5} = 10^{15}$$

Logic

$n = 100$

$$\text{Days} = \frac{10^{20}}{10^5} = 10^{20-5} = 10^{15}$$

$$365 < 500$$

$$\text{Years} = \frac{10^{15}}{500} = 2 \times \frac{10^{15}}{10^3} = 2 \times 10^{15-3} = 2 \times 10^{12} =$$

2.000.000.000.000 years

Logic

$A \leftrightarrow B = (A \to B) \,\&\, (B \to A)$

$A \to B = \neg A \vee B$

$$A \leftrightarrow B = (A \to B) \,\&\, (B \to A) =$$
$$= (\neg A \vee B) \,\&\, (\neg B \vee A) = \color{red}{(\neg A \vee B) \,\&\, (A \vee \neg B)}$$

Logic

$$A \,\&\, (B \vee C) = (A \,\&\, B) \vee (A \,\&\, C)$$

$$A \,\&\, \neg A = 0$$

$$A \leftrightarrow B = (\neg A \vee B) \,\&\, (A \vee \neg B) =$$

$$= (\neg A \,\&\, A) \vee (\neg A \,\&\, \neg B) \vee (B \,\&\, A) \vee (B \,\&\, \neg B) =$$

$$= \color{red}{(A \,\&\, B) \vee (\neg A \,\&\, \neg B)}$$

Logic

$$A \leftrightarrow B = (\neg A \vee B) \,\&\, (A \vee \neg B) = (A \,\&\, B) \vee (\neg A \,\&\, \neg B)$$

$$A \rightarrow B = \neg A \vee B$$

$$\neg(\neg a) = a$$

Example

$$(\neg a \rightarrow b) \leftrightarrow (\neg b \,\&\, c) = (\neg(\neg a) \vee b) \leftrightarrow (\neg b \,\&\, c) =$$

$$= (a \vee b) \leftrightarrow (\neg b \,\&\, c) =$$

Logic

$$A \leftrightarrow B = (\neg A \vee B) \,\&\, (A \vee \neg B) = (A \,\&\, B) \vee (\neg A \,\&\, \neg B)$$

$$\neg(A \vee B) = \neg A \,\&\, \neg B \qquad \neg(A \,\&\, B) = \neg A \vee \neg B$$

Example

$$(\neg a \to b) \leftrightarrow (\neg b \,\&\, c) = (a \vee b) \leftrightarrow (\neg b \,\&\, c) =$$

$$= [(a \vee b) \,\&\, (\neg b \,\&\, c)] \vee [\neg(a \vee b) \,\&\, \neg(\neg b \,\&\, c)] =$$

$$= [(a \vee b) \,\&\, (\neg b \,\&\, c)] \vee [(\neg a \,\&\, \neg b) \,\&\, (b \vee \neg c)] =$$

$$= (a \,\&\, \neg b \,\&\, c) \vee (b \,\&\, \neg b \,\&\, c) \vee (\neg a \,\&\, \neg b \,\&\, b) \vee (\neg a \,\&\, \neg b \,\&\, c) =$$

$$= (a \,\&\, \neg b \,\&\, c) \vee (\neg a \,\&\, \neg b \,\&\, c)$$

DNF/CNF

Algorithm

1. Dispose of $\to$ and $\leftrightarrow$
   - $A \leftrightarrow B = (\neg A \lor B) \,\&\, (A \lor \neg B) = (A \,\&\, B) \lor (\neg A \,\&\, \neg B)$
   - $A \to B = \neg A \lor B$
2. Use De Morgan's laws
   - $\neg(A \lor B) = \neg A \,\&\, \neg B$
   - $\neg(A \,\&\, B) = \neg A \lor \neg B$
3. Use Distributivity laws
   - $A \,\&\, (B \lor C) = (A \,\&\, B) \lor (A \,\&\, C)$ (for DNF)
   - $A \lor (B \,\&\, C) = (A \lor B) \,\&\, (A \lor C)$ (for CNF)

a	b	F
0	0	0
0	1	1
1	0	1
1	1	0

a	b	F
0	0	0
0	1	1
1	0	1
1	1	0

DNF: $F = (\neg a \,\&\, b) \vee (a \,\&\, \neg b)$

a	b	F
0	0	0
0	1	1
1	0	1
1	1	0

DNF: $F = (\neg a \,\&\, b) \vee (a \,\&\, \neg b)$

CNF: $F = (a \vee b) \,\&\, (\neg a \vee \neg b)$

Thank you for your attention!