Mathematical Analysis. Assignment 1.
Sequences. Limits of sequences

1. Find the formula of a general term of a sequence

   (a) $x_1 = \frac{1}{2}$, $x_{n+1} = \frac{2}{3-x_n}$, $n \in \mathbb{N}$;

   (b) $x_1 = 0$, $x_2 = 1$, $x_{n+2} = \frac{3x_{n+1}-x_n}{2}$, $n \in \mathbb{N}$.

   Answer: (a) $x_n = \frac{3\cdot 2^{n-1}-2}{3\cdot 2^{n-1}-1}$; (b) $x_n = 2 - 2^{2-n}$.

2. Prove that the following sequences are bounded:

   (a) $x_n = \frac{5n^6+6}{(n^4+1)(n^2-2)}$;

   (b) $x_n = \sum\limits_{k=1}^{n} \frac{1}{k(k+1)}$;

   (c) $x_n = \sum\limits_{k=1}^{n} \frac{k}{(2k-1)(2k+1)(2k+3)}$;

   (d) $x_n = \sum\limits_{k=1}^{n} \frac{1}{k^2}$;

   (e) $x_n \sum\limits_{k=1}^{n} \frac{1}{n+k}$;

   (f) $x_n = n\left(\sqrt{n^4+n} - \sqrt{n^4-n}\right)$;

   (g) $x_n = \left(1+\frac{1}{n}\right)^n$;

   (h) $x_n = \sqrt[n]{n}$.

3. Prove that the following sequences are unbounded:

   (a) $x_n = \frac{3^n-2^n}{2^n+1}$;

   (b) $x_n = \frac{2^n}{n^2}$;

   (c) $x_1 = x_2 = 1$, $x_{n+2} == x_{n+1} + \frac{3}{4}x_n$.

4. Prove that the following sequences are monotone starting from some term:

   (a) $x_n = \frac{n^3}{n^2-3}$;

   (b) $x_n = \sqrt{n^2+n} - n$;

   (c) $x_n = \ln\left(n^2+9n\right) - 2\ln n$;

   (d) $x_n = \frac{100^n}{n!}$.

5. Using the definition of a limit of a sequence prove that

   (a) $\lim\limits_{k\to\infty} \frac{1}{k} = 0$;

   (b) $\lim\limits_{k\to\infty} \frac{3k}{2k-1} = \frac{3}{2}$;

   (c) $\lim\limits_{k\to\infty} (4\sqrt{n} - n) = -\infty$.

6. Using the definition of a limit of a sequence prove that the following sequences are infinitesimal:

   (a) $x_k = \frac{2+(-1)^k}{k}$;

   (b) $x_k = q^k$ if $|q| < 1$.

7. Find such sequences x_n and y_n that $\lim\limits_{n\to\infty} x_n = +\infty$, $\lim\limits_{n\to\infty} y_n = +\infty$, and besides that

   (a) $\lim\limits_{n\to\infty} (x_n - y_n) = +\infty$;
   (b) $\lim\limits_{n\to\infty} (x_n - y_n) = -\infty$;
   (c) $\lim\limits_{n\to\infty} (x_n - y_n) = -\lg 13$;
   (d) sequence $x_n - y_n$ has neither a finite nor an infinite limit.

8. Find such sequences x_n and y_n that $\lim\limits_{n\to\infty} x_n = 0$, $\lim\limits_{n\to\infty} y_n = +\infty$, and besides that

   (a) $\lim\limits_{n\to\infty} (x_n y_n) = 0$;
   (b) $\lim\limits_{n\to\infty} (x_n y_n) = 19$;
   (c) $\lim\limits_{n\to\infty} (x_n y_n) = -\infty$;
   (d) sequence $x_n y_n$ has neither a finite nor an infinite limit.

9. Prove that the definitions of a limit point of a sequence below are equivalent to each other.

   (a) A is a limit point of sequence x_n if any neighborhood of this point contains infinitely many terms of a sequence.
   (b) A is a limit point of sequence x_n if any deleted neighborhood of this point contains at least one term of a sequence.
   (c) A is a limit point of sequence x_n if A is a limit of some subsequence of x_n.

10. Give an example of such a sequence x_n that its set of limit points is $\mathbb{N}$.

11. Prove that sequence $x_n = \frac{n \cos \pi n - 1}{2n}$ diverges using Cauchy convergence criterion.

12. (Bernoulli's inequality) Prove that $(1+x)^k > 1 + kx$ for any integer $k > 1$ and for any $x > -1$, $x \neq 0$.

13. Justify the following statements without using continuity of elementary functions (i.e. it has not yet been proved that $x_n \to a$, $n \to \infty$ implies that $f(x_n) \to f(a)$, $n \to \infty$):

    (a) $\lim\limits_{k\to\infty} \sqrt[k]{a} = 1$, $a > 0$;
    (b) $\lim\limits_{k\to\infty} \sqrt[k]{k} = 1$;
    (c) $\lim\limits_{k\to\infty} \frac{k^\alpha}{b^k} = 0$, $b > 1$;
    (d) $\lim\limits_{k\to\infty} \frac{a^k}{k!} = 0$.

14. Find limits of the following sequences:

    (a) $x_n = \frac{n^2+1}{2n+1} - \frac{3n^2+1}{6n+1}$;
    (b) $x_n = \frac{(n+1)^4 - (n-1)^4}{(n^2+1)^2 - (n^2-1)^2}$;
    (c) $x_n = \frac{\ln\left(n^2-n+1\right)}{\ln(n^{10}+n+1)}$;
    (d) $x_n = \frac{1}{\sqrt{n}} \sum\limits_{k=1}^{n} \frac{1}{\sqrt{k+1}+\sqrt{k-1}}$;
    (e) $x_n = \frac{\sqrt{n^2+1}-n}{\sqrt{n^3+1}-n\sqrt{n}}$;
    (f) $x_n = n\sqrt{n}\left(\sqrt{n+1} + \sqrt{n-1} - 2\sqrt{n}\right)$;

(g) $x_1 = 13,\ x_{n+1} = \sqrt{12 + x_n}$;

(h) $x_n = \left(\frac{2n+2}{2n-1}\right)^n$;

(i) $x_n = \left(\frac{n^2-n+1}{n^2+n+1}\right)^n$;

(j) $x_n = \frac{1}{n^3} \sum\limits_{k=1}^{n} (2k-1)$;

(k) $x_n = \sum\limits_{k=1}^{n} \left(\frac{k}{n}\right)^2 - \frac{n}{3}$;

(l) $x_1 = a > 0,\ x_{k+1} = \frac{1}{3}\left(2x_k + \frac{125}{x_k^2}\right)$.

Answer: (a) $-\frac{1}{6}$; (b) $+\infty$; (c) $\frac{1}{5}$; (d) $\frac{1}{\sqrt{2}}$; (e) $+\infty$; (f) $-\frac{1}{4}$; (g) 4; (h) e^2; (i) e^{-2}; (j) 0; (k) $\frac{1}{2}$; (l) 5.

15. Give an example of a sequence that diverges and such that for any positive integer p

$$\lim_{k\to\infty} |x_{k+p} - x_k| = 0.$$