Discrete Math

Lab 6 – October, 27

Agenda

- HW discussion
- Midterm discussion
- Basic counting principles
- Permutations and combinations
- Homework

The PRODUCT rule:

- **Product rule:** Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure

- **In terms of sets:** If $A_1, A_2, \ldots, A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements in each set. $|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_m|$.

- **Example:** A new company with two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees? 12*11 = 132

The SUM rule:

- **Sum rule:** If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task

- **In terms of sets:** If $A_1, A_2, \ldots, A_m$ are pairwise disjoint finite sets, then the number of elements in the union of these sets is the sum of the numbers of elements in the sets.
  $|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$ when $A_i \cap A_j = \emptyset$ for all i, j.

- **Example:** A student can choose a project from one of three lists with 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from? 23+15+19 = 57

The SUBTRACTION rule (principle of inclusion–exclusion)

- **Subtraction rule:** If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways

- **In terms of sets**: If A_1 and A_2 are finite sets, the number of ways to select an element from A_1 or from A_2 is the sum of the number of ways to select an element from A_1 and the number of ways to select an element from A_2, minus the number of ways to select an element that is in both A_1 and A_2.
$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$.

- **Example**: How many bit strings of length eight either start with a 1 bit or end with the two bits 00? Answer: $2^7 + 2^6 – 2^5 = 128+64-32=160$

Principle of inclusion-exclusion generalized

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$|A_1 \cup \cdots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j \le n} |A_i \cap A_j \cap A_k| - \cdots +$$

$$+(-1)^{n+1} |A_1 \cap A_2 \ldots \cap A_n|$$

The DIVISION rule

- **Division rule:** There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.
- **In terms of sets**: If the finite set A is the union of n pairwise disjoint subsets each with d elements, then n = |A|/d.
- **Example**: How many different ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left neighbor and the same right neighbor?
- **Solution**: there are 4! = 24 ways to order the given four people for these seats. However, each of the four choices for seat 1 leads to the same arrangement. By the division rule there are 24/4 = 6 different arrangements

Permutations

- A k-**permutation** of a set with n elements, written P_n^k or P(n,k), is an <u>ordered</u> arrangement of k <u>distinct</u> elements of a set. Order matters.

- $P(n,k) = \frac{n!}{(n-k)!} = n(n-1)(n-2)\cdots(n-k+1)$
- $P(n,n) = n!$

- Example: In how many ways the letters of the word MATH can be arranged?
- Solution: P(4,4) = 4! = 24

Permutations with repetitions

- A k-**permutation with repetitions (replacement)** of a set with n elements is an ordered arrangement of k not necessarily distinct elements of a set. Order matters.

- $P(n,k) = n^k$

- Example: How many 3-digit numbers can you create form a set of numbers {0,…,9}.
- Solution: $10^3 = 1000$

Combinations

- A **k-combination** of a set of n elements, written $C_n^k, C(n,k)$ or $\binom{n}{k}$ ("n choose k") is a subset of *k* distinct elements. Order makes no difference

- $C(n,k) = \frac{n!}{k!(n-k)!}$

- Example: In how many ways the 3 prizes (1st, 2nd and 3rd place) can be distributed among 6 teams?
- Solution: $C(6,3) = \frac{6!}{3!3!} = \frac{4*5*6}{2*3} = 20$

Combinations with repetition

- A **k-combination with repetitions** of a set of n elements, written $\widetilde{C_n^k}$, $\tilde{C}(n,k)$ is any subset of k not necessarily distinct elements from n elements. Order makes no difference.

- $\tilde{C}(n,k) = C(n+k-1,k) = \frac{(n+k-1)!}{k!(n-1)!}$

- Example: There are 5 different types of ice-cream. A father would like to buy 15 caps of ice-cream for his family. In how many ways he can do it?
- Solution: $\tilde{C}(5{,}15) = C(5+15-1{,}15) = \frac{(5+15-1)!}{15!(5-1)!} = \frac{19!}{15!4!} = \frac{16*17*18*19}{2*3*4} = 3876$

Multinomial coefficients

We use $\binom{n}{r_1, \ldots, r_m}$ to denote the number of arrangements of $n = r_1 + \cdots + r_m$ objects, where for each i $(1 \leq i \leq m)$ we have r_i indistinguishable objects of type i.

$$\binom{n}{r_1, \ldots, r_m} = \frac{n!}{r_1!\, r_2! \ldots, r_m!}$$

Example: How many permutations of the word “Cocoa” are there?

Solution: $\frac{5!}{2!2!1!} = \frac{120}{4} = 30$

Just remember it!

	Without repetitions	With repetitions
Order matters: permutation	$\mathrm{P}(n,k) = \frac{n!}{(n-k)!}$ $\mathrm{P}(n,n) = n!$ $\binom{n}{r_1,\dots,r_m} = \frac{n!}{r_1!\,r_2!\dots,r_m!}$	n^k
Order does not matter: combination	$C(n,k) = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$	$\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!\,(n-1)!}$

Exercises

1. How many options are there to select 6 objects out of 36?
2. How many options are there to select 6 objects out of 36 in the same order?
3. How many positive integers of 5 digits may be made from the characters 1,2,3,4,5, if each character may be used just once? How many of them will begin with 5? How many of them will be even?
4. There are 4 Russian and 3 English books on the bookshelf. Russian books should be placed on the left side of the bookshelf and English books on the right side of the bookshelf. How many ways are there to arrange the books?
5. A password of 6 digits is made of digits 926002. How many possible passwords are there?
6. How many different words can be created by rearranging the letters in SELFIESTICK

Exercises - II

7. You are going to an amusement park. There are four attractions, (haunted house, roller coaster, a carousel, water ride). You buy 25 tokens. Each attraction cost 3 tokens each ride, except the roller coaster that costs 5. Obviously, you want to ride each ride at least once, but the order of the rides does not matter. In how many ways can you spend your tokens? You may have some remaining tokens in the end of the day.
8. How many words can you create of length 6, from the letters a, b, c and d if
    - you must include each letter at least once, **and**
    - “a” must appear exactly once
9. How many bit strings of length 10 over the alphabet {a, b, c} have either exactly three a’s or exactly four b’s

Exercises - III

10. There are five people of different height. In how many ways can they stand in a line, so that there is no 3 consecutive people with increasing height?
11. We have a smorgasbord, with 50 dishes, — 5 countries are represented, and there are 10 dishes from each. We want to make a plate with 8 dishes (no duplicates), but make sure that no country is missing. In how many ways we can do it?

Homework 8

1. Read textbook Kenneth H. Rosen "Discrete Mathematics and Its Applications, 7th Edition“ – paragraphs 6.3 – 6.5

2. Solve exercises 1-7 and submit by 10 pm on Friday October, 30.

Homework

1. How many full houses are there in poker? The deck has 52 cards of 4 denominations. A full house has 5 cards, 3 of one kind and 2 of another. E.g.: 3 5's and 2 Kings.
2. How many ways are there of choosing k numbers from $\{1, . . . , n\}$ if 1 and 2 can't both be chosen? (Suppose $n, k \geq 2$, $n \geq k$)
3. How many positive integers not exceeding 100 are divisible either by 4 or by 6?
4. A multiple-choice test contains 10 questions. There are four possible answers for each question. In how many ways can a student answer the questions on the test if:

    a) the student answers every question?

    b) the student can leave answers blank?

Homework continued

5. How many integer solutions does the equation $x_1 + x_2 + x_3 + x_4 = 15$ have, if we require that $x_1 \geq 2$, $x_2 \geq 3$, $x_3 \geq 10$ and $x_4 \geq -3$?
6. We go to a pizza party, and there are 5 types of pizza. We have starved for days, so we can eat 13 slices, but we want to sample each type at least once. In how many ways can we do this? Order does not matter
7. There are 8 types(the same type can be used several times) of cookies available in a store. Count the number of ways

   (a) to pick 6 of them and arrange them in a line.

   (b) to pick 6 of them and place them into lines named A and B, with 3 in each.

   (c) to pick 6 of them and place them into two equal-sized unlabeled lines.