Mathematical Analysis. Assignment 8.
Definite Integral & Its Applications

1. Calculate the integral $\int\limits_0^{\pi/2} \sin x\,dx$ as a limit of integral sums.

2. Prove that Dirichlet function $D(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q} \end{cases}$ is not integrable on any interval on the real line.

3. Compare the integrals:

   (a) $\int\limits_0^{\pi/2} \frac{\sin x}{x}\,dx$ and $\int\limits_0^{\pi} \frac{\sin x}{x}\,dx$; (b) $\int\limits_1^2 \frac{dx}{\sqrt{1+x^2}}$ and $\int\limits_1^2 \frac{dx}{x}$.

4. Prove that changing the value of a function at one point of the interval does not change the value of the integral of this function over this interval.

5. Find the following derivatives: (a) $\frac{d}{dx}\int\limits_a^b \sin x^2\,dx$; (b) $\frac{d}{da}\int\limits_a^b \sin x^2\,dx$; (c) $\frac{d}{db}\int\limits_a^b \sin x^2\,dx$; (d) $\frac{d}{dx}\int\limits_{x^2}^{x^3} \frac{dt}{\sqrt{1+t^4}}$.

   Answer: (a) 0; (b) $-\sin a^2$; (c) $\sin b^2$; (d) $\frac{3x^2}{\sqrt{1+x^{12}}} - \frac{2x}{\sqrt{1+x^8}}$.

6. Let $f(x)$ be a continuous function on $[-l; l]$. Prove that

   (a) if $f(x)$ is odd then $\int\limits_{-l}^{l} f(x)\,dx = 0$;

   (b) if $f(x)$ is even then $\int\limits_{-l}^{l} f(x)\,dx = 2\int\limits_0^{l} f(x)\,dx$.

7. Prove that for any function $f(x)$ continuous on $[0; 1]$ the following equalities hold:

   (a) $\int\limits_0^{\pi/2} f(\sin x)\,dx = \int\limits_0^{\pi/2} f(\cos x)\,dx$; (b) $\int\limits_0^{\pi} xf(\sin x)\,dx = \frac{\pi}{2}\int\limits_0^{\pi} f(\sin x)\,dx$.

8. Let $f(x)$ be a periodic function with period T, continuous on the whole real line. Prove that $\int\limits_a^{a+T} f(x)\,dx = \int\limits_0^{T} f(x)\,dx$ for any real value of a.

9. Prove that $\int\limits_0^1 (1-x)^m x^n\,dx = \frac{m!n!}{(m+n+1)!}$, $m \in \mathbb{N}$, $n \in \mathbb{N}$.

10. Explain why the following equalities are incorrect:

    (a) $\int\limits_{-1}^{1} \frac{d}{dx}\left(\arctan\frac{1}{x}\right)dx = \arctan\frac{1}{x}\Big|_{-1}^{1} = \frac{\pi}{2}$;

    (b) $\int\limits_0^{2\pi} \frac{dx}{\cos^2 x(2+\tan^2 x)} = \frac{1}{\sqrt{2}}\arctan\frac{\tan^2 x}{\sqrt{2}}\Big|_{-1}^{1} = 0$.

11. Prove that (a) $\sin 1 < \int\limits_{-1}^{1} \frac{\cos x}{x^2+1}\,dx < 2\sin 1$; (b) $\frac{2}{\pi}\ln\frac{\pi+2}{2} < \int\limits_0^{\pi/2} \frac{\sin x}{x(x+1)}\,dx < \ln\frac{\pi+2}{2}$.

12. Calculate the following integrals: (a) $\int\limits_0^1 \frac{x^2\,dx}{x^6+1}$; (b) $\int\limits_{0.75}^{2} \frac{dx}{\sqrt{2+3x-2x^2}}$; (c) $\int\limits_0^{0.5} \arcsin x\,dx$; (d) $\int\limits_0^{e} \sin\ln x\,dx$; (e) $\int\limits_{-\pi}^{\pi} e^{x^2}\sin x\,dx$; (f) $\int\limits_{-\pi/2}^{\pi/2} (\cos^2 x + x^4\sin x)\,dx$; (g) $\int\limits_{\pi/3}^{\pi/2} \frac{dx}{3+\cos x}$; (h) $\int\limits_0^3 \arcsin\sqrt{\frac{x}{x+1}}\,dx$; (i) $\int\limits_0^{e} x^2\ln^2 x\,dx$.

Answer: (a) $\frac{\pi}{12}$; (b) $\frac{\pi}{2\sqrt{2}}$; (c) $\frac{\pi}{12}+\frac{\sqrt{3}}{2}-1$; (d) $\frac{1}{2}e(\sin 1-\cos 1)$; (e) 0; (f) $\frac{\pi}{2}$; (g) $\frac{1}{\sqrt{2}}\left(\arctan\frac{1}{\sqrt{2}}-\arctan\frac{1}{\sqrt{6}}\right)$; (h) $\frac{4\pi}{3}-\sqrt{3}$; (i) $\frac{5e^3}{27}$.

13. Find the area of the domain bounded by lines $y=\ln(1+x)$, $y=-xe^x$ and $x=1$.

    Answer: $\ln 4-\frac{2}{e}$.

14. Find the area of the domain bounded by parabola $y=x^2+4x+9$ and two lines tangent to this parabola at points $x_1=-3$ and $x_2=0$.

    Answer: $\frac{9}{4}$.

15. A chord parallel to y-axis is drawn through a focus of a curve L (with a positive abscissa) given by (a) $\frac{x^2}{2}+y^2=1$, (b) $\frac{x^2}{16}-\frac{y^2}{9}=1$. Find the area of a segment cut off by this chord.

    Answer: (a) $\frac{\pi-2}{2\sqrt{2}}$; (b) $\frac{45}{4}-12\ln 2$.

16. Find the lengths of the following curves[1]: (a) $y=\frac{x\sqrt{x+12}}{6}$, $-11\leqslant x\leqslant -3$; (b) $y=\cosh x$, $0\leqslant x\leqslant a$; (c) $y=\frac{x^2}{2}-\ln\frac{x}{4}$, $1\leqslant x\leqslant 3$; (d) $x=(t^2-2)\sin t+2t\cos t$, $y=(t^2-2)\cos t-2t\sin t$, $0\leqslant t\leqslant\pi$; (e) $x=a\cosh t$, $y=b\sinh t$, $z=at$, $0\leqslant t\leqslant t_0$.

    Answer: (a) $\frac{25}{3}$; (b) $\sinh a$; (c) $4+\frac{\ln 3}{4}$; (d) $\frac{\pi^3}{3}$; (e) $\sqrt{a^2+b^2}\sinh t_0$.

17. Find the volume of a solid obtained by rotating figure F around x-axis if

    (a) F is bounded by $y=\sqrt{x}e^{-x}$, $y=0$, $x=a$;
    (b) F is given by inequalities $0\leqslant y\leqslant\sqrt[4]{1+e^{2x}}$, $\frac{1}{2}\ln 3\leqslant x\leqslant\frac{3}{2}\ln 2$;
    (c) F is bounded by $y=x$, $y=\frac{1}{x}$, $y=0$, $x=2$.

    Answer: (a) $\frac{\pi}{4}\left(1-e^{-2a}(1+2a)\right)$; (b) $\pi+\frac{\pi}{2}\ln\frac{3}{2}$; (c) $\frac{5\pi}{6}$.

18. Find the volume of a solid obtained by rotating a figure bounded by $y=2x-x^2$ and $y=0$ around y-axis.

    Answer: $\frac{8\pi}{3}$.

19. Find the area of a surface formed by rotating a curve around x-axis: (a) $y=\sqrt{x}$, $\frac{5}{4}\leqslant x\leqslant\frac{21}{4}$; (b) $y=\sqrt{x^2+1}$, $0\leqslant x\leqslant\frac{1}{4}$.

    Answer: (a) $\frac{98\pi}{3}$; (b) $\sqrt{2}\pi\frac{3+4\ln 2}{16}$.

20. Find the area of a surface formed by rotating a curve around y-axis: (a) $3x=4\cos y$, $-\frac{\pi}{2}\leqslant y\leqslant 0$; (b) $x=a\arcsin\sqrt{\frac{y}{a}}+\sqrt{y(a-y)}$, $\frac{a}{4}\leqslant y\leqslant\frac{3a}{4}$.

    Answer: (a) $\frac{\pi}{9}(20+9\ln 3)$; (b) $\frac{\pi a^2}{6}\left(11-9\sqrt{3}+2\pi\left(2\sqrt{3}-1\right)\right)$.

[1] All parameters in this and further problems are positive.