Tutorial 11 : Quadratic Curves (cntd.)

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Course of Essentials of Analytical Geometry and Linear Algebra I

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Last weeks' topics

- Quadratic Curves
  - Parabolas
  - Circles
  - Ellipses

Content

- Quadratic Curves
  - Hyperbolas
  - Rotation of axes of conic sections

Conic Sections

Conic sections are the curves obtained by intersecting a plane and a right circular cone.

- ❖ A plane perpendicular to the cone's axis cuts out a circle;
- ❖ A plane parallel to a side of the cone produces a parabola;
- ❖ A plane at an arbitrary angle to the axis of the cone forms an ellipse;
- ❖ A plane parallel to the axis cuts out a hyperbola.

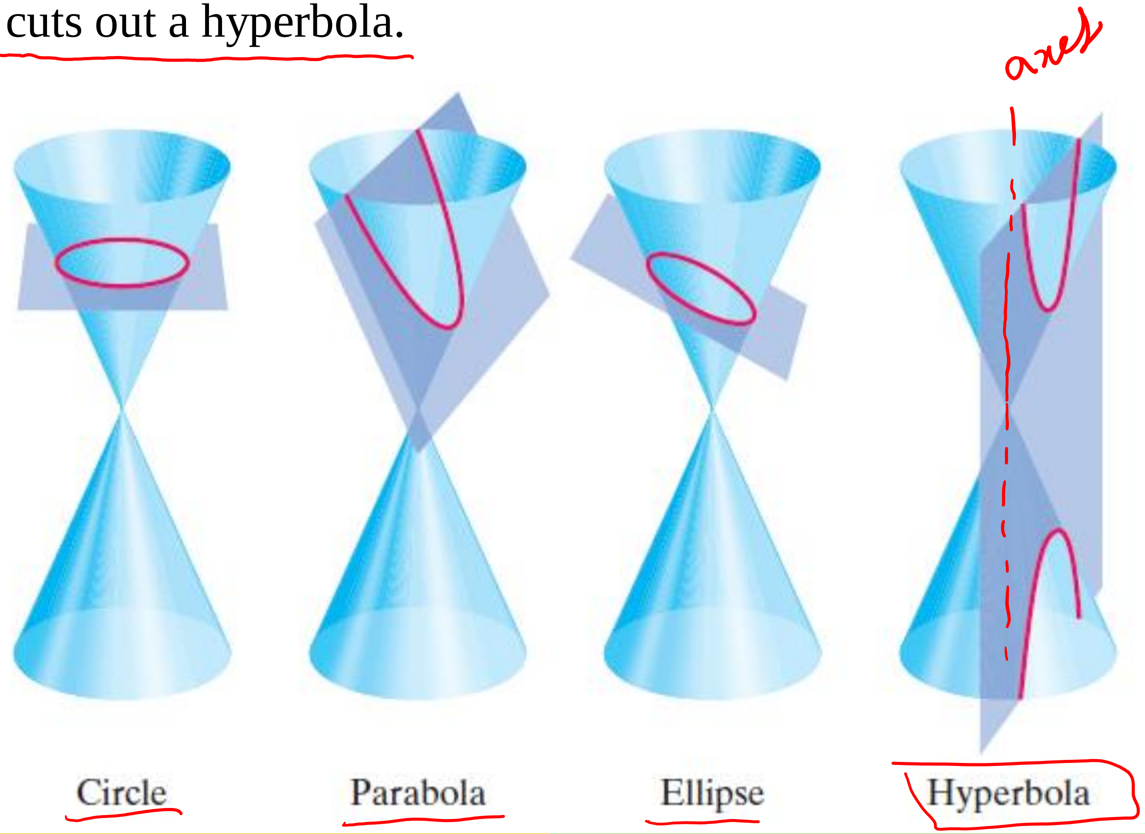
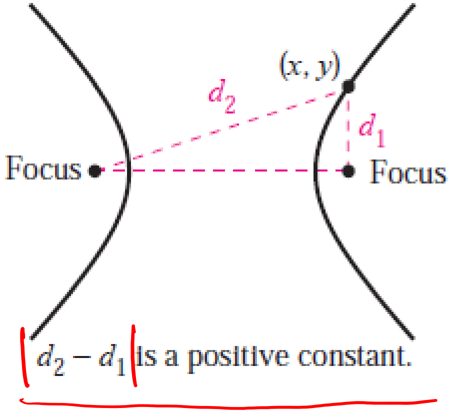
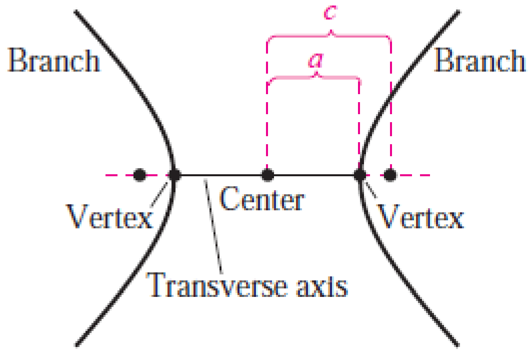


*Figure from internet.

Hyperbola (1/2)

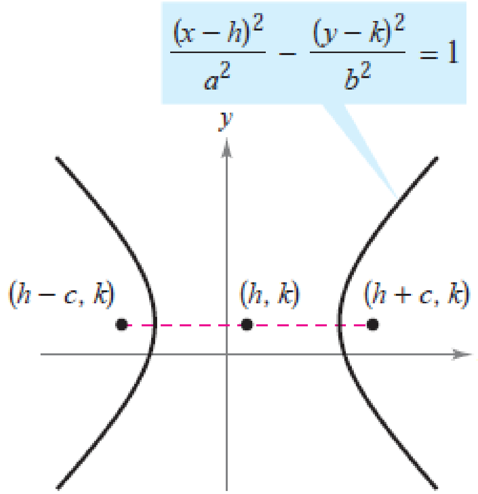
A **hyperbola** is the set of points in a plane such that the absolute value of the difference of the distance of each point from two fixed points is constant. Each fixed point is called a *focus*, and the point midway between the foci is called the *center*. The line containing the foci is the **transverse axis**. The graph is made up of two parts called **branches**. Each branch intersects the transverse axis at a point called the *vertex*.



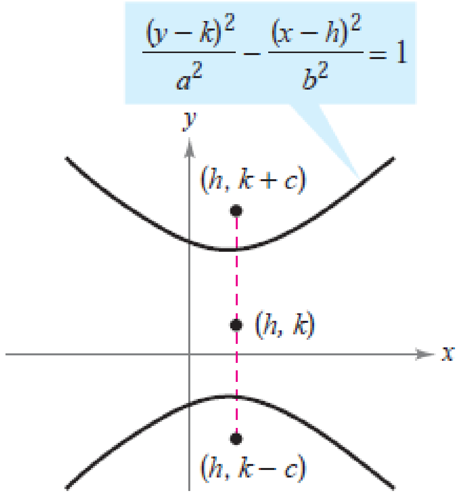


The standard form of the equation
of a hyperbola with center at (h, k) can be seen in figure.

The vertices are a units from the center, and the foci are c units from the center. Moreover, $c^2 = a^2 + b^2$.



Transverse axis is horizontal.



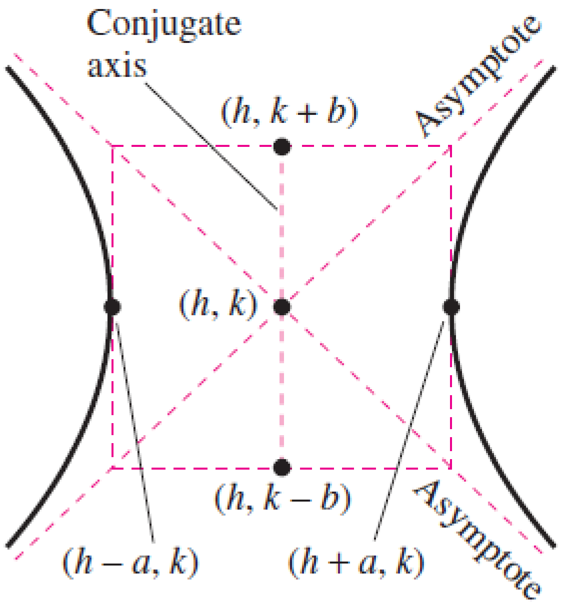
Transverse axis is vertical.

Asymptotes of a Hyperbola

Each hyperbola has two asymptotes that intersect at the center of the hyperbola.

The asymptotes pass through the corners of a rectangle of dimensions $2a$ by $2b$, with its center at (h, k), as shown in figure.

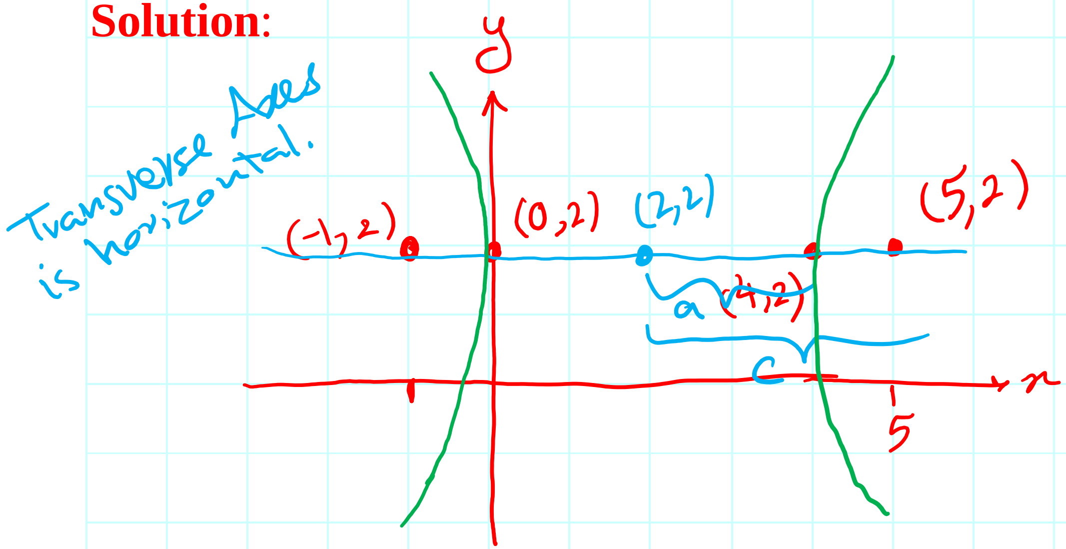
Equations of Asymptotes of a Hyperbola	
Asymptotes for horizontal transverse axis	Asymptotes for vertical transverse axis
$y = k \pm \frac{b}{a}(x - h)$	$y = k \pm \frac{a}{b}(x - h)$



The conjugate axis of a hyperbola is the line segment of length $2b$ joining $(h, k + b)$ and $(h, k - b)$ if the transverse axis is horizontal, and the line segment of length 2b joining $(h + b, k)$ and $(h - b, k)$ if the transverse axis is vertical.

➢ Find the standard form of the equation of the hyperbola with foci (-1, 2) and (5, 2) and vertices (0, 2) and (4, 2).

Solution:



By midpoint Formula we can obtain the center of the hyperbola

$(h, k) = (2, 2)$

$c = 5 - 2 = 3$

$a = 4 - 2 = 2$

$c^2 = a^2 + b^2 \rightarrow b = \sqrt{c^2 - a^2} = \sqrt{9 - 4} = \sqrt{5}$

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \Rightarrow \boxed{\frac{(x-2)^2}{4} - \frac{(y-2)^2}{5} = 1}$$

Example 2

- Sketch the hyperbola whose equation is $4x^2 - y^2 = 16$.

Solution:

$\xrightarrow{\div 16}$ $\frac{x^2}{2^2} - \frac{y^2}{4^2} = 1$

$a = 2$ $h = 0$

$b = 4$ $k = 0$

Transverse axes is horizontal.

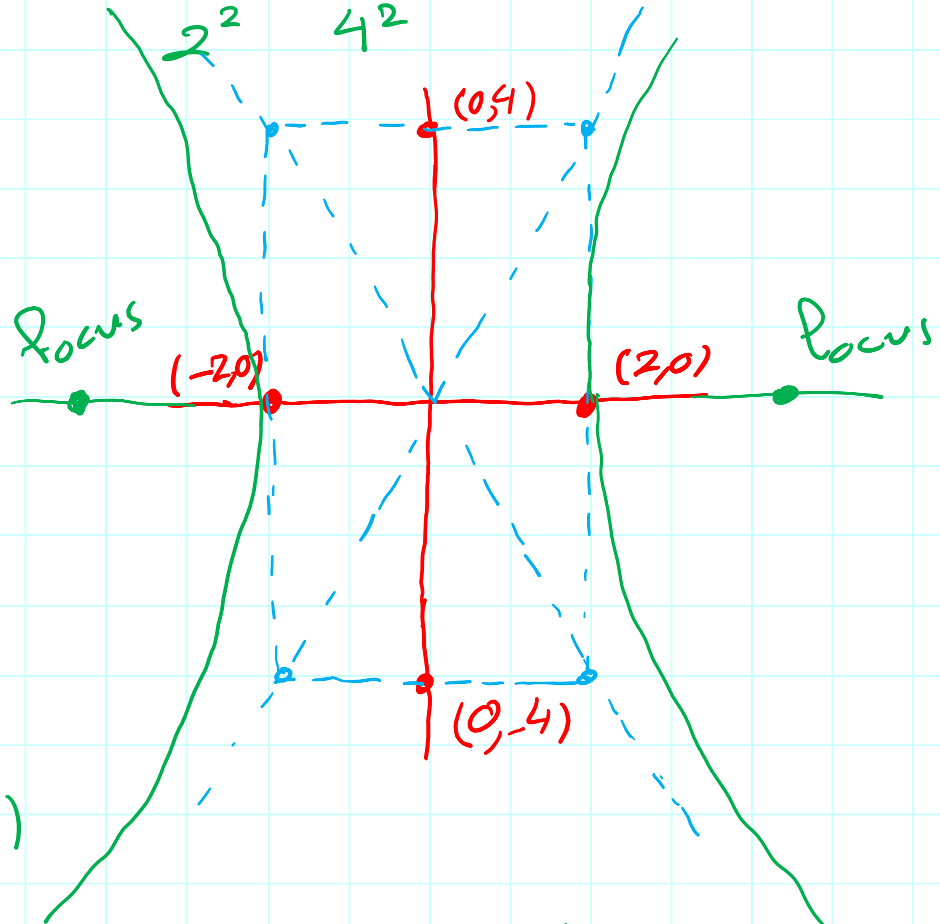
Vertices $\Rightarrow$ $\begin{cases} (h+a, k) = (2,0) \\ (h-a, k) = (-2,0) \end{cases}$

end point of conjugate axes

$\begin{cases} (h, k+b) = (0,4) \\ (h, k-b) = (0,-4) \end{cases}$

Foci $\begin{cases} (h+c, k) \rightarrow (4.47, 0) \\ (h-c, k) \rightarrow (-4.47, 0) \end{cases}$

$c^2 = a^2 + b^2 \rightarrow c = \sqrt{4+16} = 2\sqrt{5} \approx 4.47$



➢ Sketch the hyperbola given by $4x^2 - 3y^2 + 8x + 16 = 0$ and find the equations of its asymptotes.

Solution:

$$4x^2 - 3y^2 + 8x + 16 = 0$$

$$4(x^2 + 2x + 1) - 3y^2 = -16 + 4$$

$$4(x+1)^2 - 3y^2 = -12$$

$$\frac{y^2}{2^2} - \frac{(x+1)^2}{(\sqrt{3})^2} = 1$$ standard form

center $(h,k) = (-1,0)$

$a = 2$

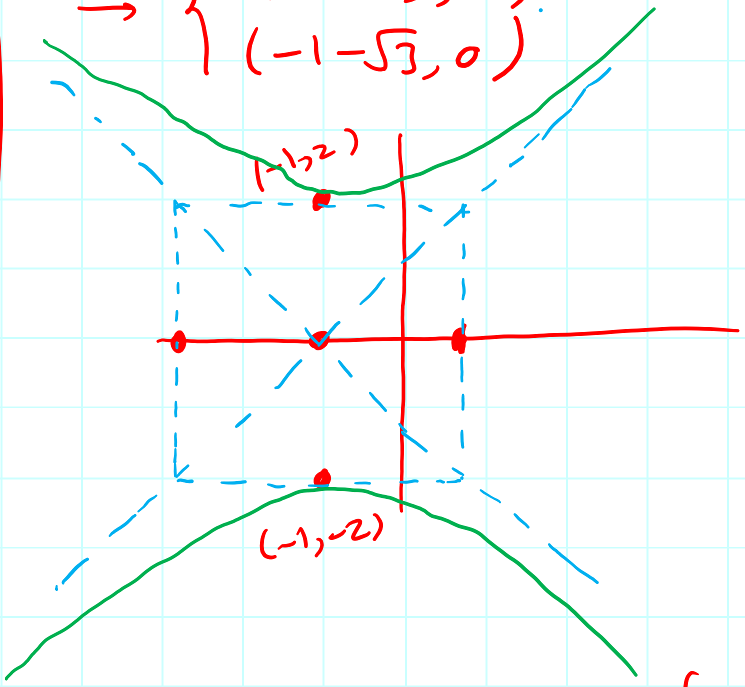
$b = \sqrt{3} \approx 1.73$

hyperbola has vertical transverse axes

vertices : $\begin{cases} (-1,2) & (h,k+a) \\ (-1,-2) & (h,k-a) \end{cases}$

endpoint of the conjugate axes $\begin{cases} (h+b,k) \\ (h-b,k) \end{cases}$

$\rightarrow \begin{cases} (-1+\sqrt{3},0) \\ (-1-\sqrt{3},0) \end{cases}$

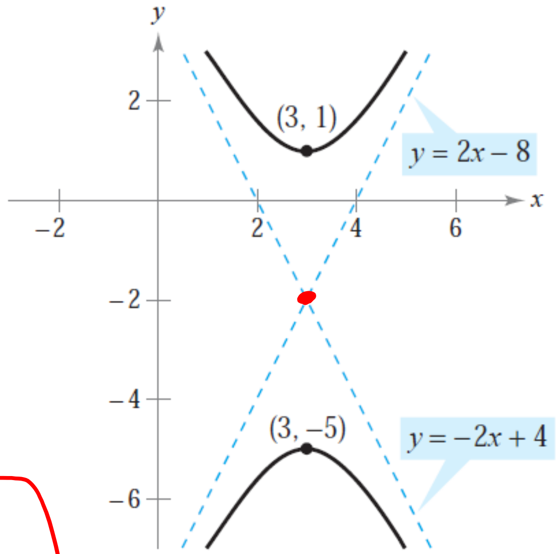


$y = k \pm \frac{a}{b}(x-h) \longrightarrow \begin{cases} y = \frac{2}{\sqrt{3}}(x+1) \\ y = -\frac{2}{\sqrt{3}}(x+1) \end{cases}$

- Find the standard form of the equation of the hyperbola having vertices (3, -5) and (3, 1) and having asymptotes

$$\underbrace{y = 2x - 8}_{1} \text{ and } \underbrace{y = -2x + 4}_{2}$$

as shown in figure.



Solution:

By midpoint formula $\Rightarrow$ center $(3,-2)$

Vertical transverse axis

$a=3$

1 $\Rightarrow$ $m_1 = 2 = \frac{a}{b}$ $\xrightarrow{a=3}$ $b = 3/2$

2 $\Rightarrow$ $m_2 = -2 = -\frac{a}{b}$

$$\boxed{\frac{(y+2)^2}{3^2} - \frac{(x-3)^2}{(3/2)^2} = 1}$$

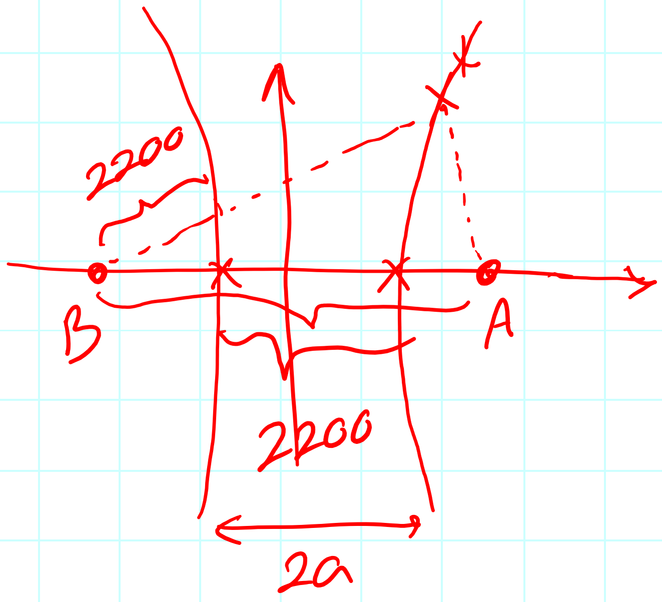
Example 5

- Two microphones, 1 mile apart, record an explosion. Microphone A receives the sound 2 seconds before microphone B. Where did the explosion occur? (Assume sound travels at 1100 feet per second.)

Solution:

1 mile = 5280 feet

$(1100)(2) = 2200$



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$2a = 2200 \longrightarrow a = 1100$$

$$2c = 5280 \longrightarrow c = 2640$$

$$b^2 = c^2 - a^2 = 5759600$$

$$\boxed{\frac{x^2}{(1100)^2} - \frac{y^2}{5759600} = 1}$$

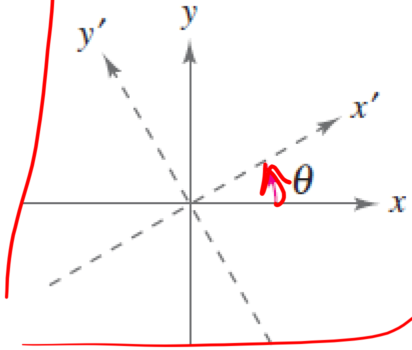
Rotation (1/2)

It is known that the equation of a conic with axes parallel to the coordinate axes has a standard form that can be written in the general form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0. \quad \text{Horizontal or vertical axes}$$

In this section, you will study the equations of conics whose axes are rotated so that they are not parallel to either the *x*-axis or the *y*-axis. The general equation for such conics contains an xy-term.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad \text{Equation in } xy - \text{plane}$$



To eliminate this *xy*-term, you can use a procedure called **rotation of axes**. The objective is to rotate the *x*- and *y*-axes until they are parallel to the axes of the conic. The rotated axes are denoted as the x'-axis and the y'-axis, as shown in the figure.

Rotation of Axes to Eliminate an *xy*-Term

The general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

can be rewritten as

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$$

by rotating the coordinate axes through an angle θ, where $\cot 2\theta = \frac{A-C}{B}$ The coefficients of the new equation are obtained by making the substitutions

$$x = x'\cos\theta - y'\sin\theta \text{ and } y = x'\sin\theta + y'\cos\theta\,.$$

Rotation Invariants

The rotation of the coordinate axes through an angle θ that transforms the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ into the form $A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$ has the following rotation invariants.

? $B' = 0$

1) $F = F'$

2) $A + C = A' + C'$

3) $B^2 - 4AC = (B')^2 - 4A'C'$

Note that because $B' = 0$, the invariant $B^2 - 4AC$ reduces to

$$\boxed{B^2 - 4AC = -4A'C'}$$

Discriminant

Classification of Conics by the Discriminant

The graph of the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is, <u>except in degenerate cases</u>, determined by its discriminant as follows.

1. Ellipse or circle: $B^2 - 4AC < 0$
2. Parabola: $B^2 - 4AC = 0$
3. Hyperbola: $B^2 - 4AC > 0$

$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$

- Rotate the axes to eliminate the *xy*-term in the equation $xy - 1 = 0$. Then write the equation in standard form and sketch its graph.

Solution:

$A=0,\ B=1,\ C=0$

$\cot 2\theta = \frac{A-C}{B} = 0 \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$

① $x = x'\cos\frac{\pi}{4} - y'\sin\frac{\pi}{4} \rightarrow x = \frac{x'-y'}{\sqrt{2}}$

and

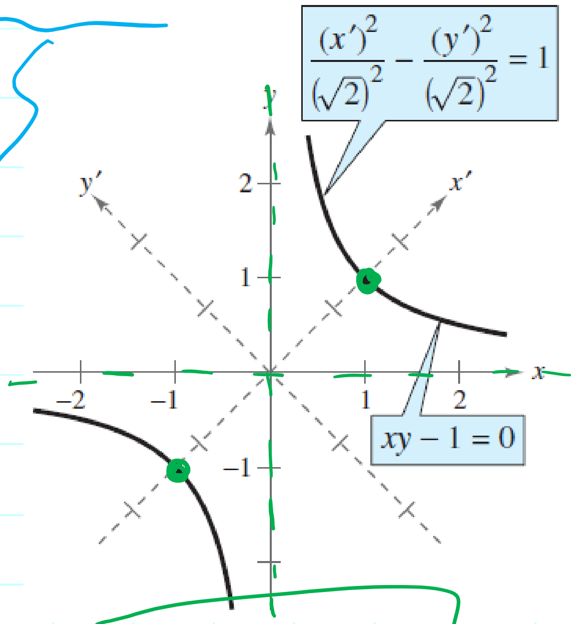
② $y = x'\sin\frac{\pi}{4} + y'\cos\frac{\pi}{4} \rightarrow y = \frac{x'+y'}{\sqrt{2}}$

$xy-1=0 \xrightarrow{①②} \left(\frac{x'-y'}{\sqrt{2}}\right)\left(\frac{x'+y'}{\sqrt{2}}\right) - 1 = 0$

$\frac{(x')^2-(y')^2}{2} - 1 = 0 \Rightarrow \boxed{\frac{(x')^2}{(\sqrt{2})^2} - \frac{(y')^2}{(\sqrt{2})^2} = 1}$ in $x'y'$ system

$x'y'$-sys: vertices at $(\pm\sqrt{2}, 0)$, center $(0,0)$

xy-sys: $x = \frac{x'-y'}{\sqrt{2}}$, $y = \frac{x'+y'}{\sqrt{2}}$ $\Rightarrow$ $(1,1)$, $(-1,-1)$ in xy-system



$\boxed{y' = \pm x'}$

Asymptotes in $x'y'$-sys.

Example 7

➢ Rotate the axes to eliminate the xy-term in the equation

$$7x^2 - 6\sqrt{3}\,xy + 13y^2 - 16 = 0.$$

Then write the equation in standard form and sketch its graph.

Solution:

$A = 7, \quad B = -6\sqrt{3}, \quad C = 13$

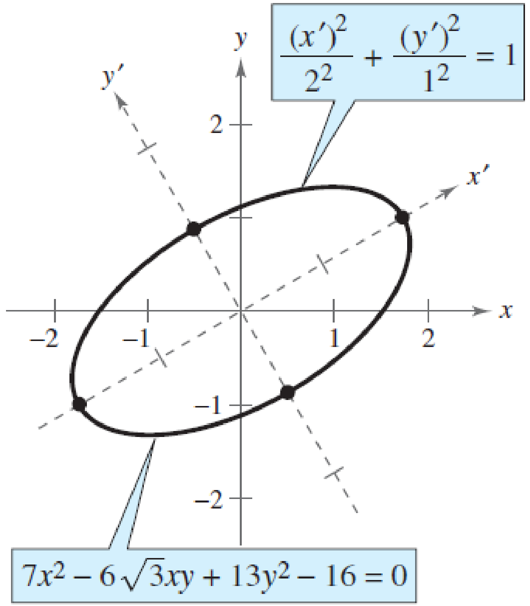
$\cot 2\theta = \dfrac{A-C}{B} = \dfrac{1}{\sqrt{3}} \Rightarrow \theta = \pi/6$

$x = x'\cos\frac{\pi}{6} - y'\sin\frac{\pi}{6} \rightarrow x = \dfrac{\sqrt{3}x' - y'}{2} \quad *$

$y = x'\sin\frac{\pi}{6} + y'\cos\frac{\pi}{6} \rightarrow y = \dfrac{x' + \sqrt{3}y'}{2} \quad **$

$7\left(\dfrac{\sqrt{3}x' - y'}{2}\right)^2 - 6\sqrt{3}\left(\dfrac{\sqrt{3}x' - y'}{2}\right)\left(\dfrac{x' + \sqrt{3}y'}{2}\right) + 13\left(\dfrac{x' + \sqrt{3}y'}{2}\right)^2 - 16 = 0$

$\Rightarrow \boxed{\dfrac{(x')^2}{2^2} + \dfrac{(y')^2}{1} = 1}$ center $(0, 0)$ vertices $(\pm 2, 0)$



Example 8

- Rotate the axes to eliminate the xy-term in the equation

$$(0)\quad x^2 - 4xy + 4y^2 + 5\sqrt{5}y + 1 = 0.$$

Then write the equation in standard form and sketch its graph.

Solution:

$A=1,\ B=-4,\ C=4$

$\cot 2\theta = \dfrac{A-C}{B} \Longrightarrow \theta \cong 26.6^\circ$

$\sin\theta = \dfrac{1}{\sqrt{5}} \qquad \cos\theta = 2/\sqrt{5}$

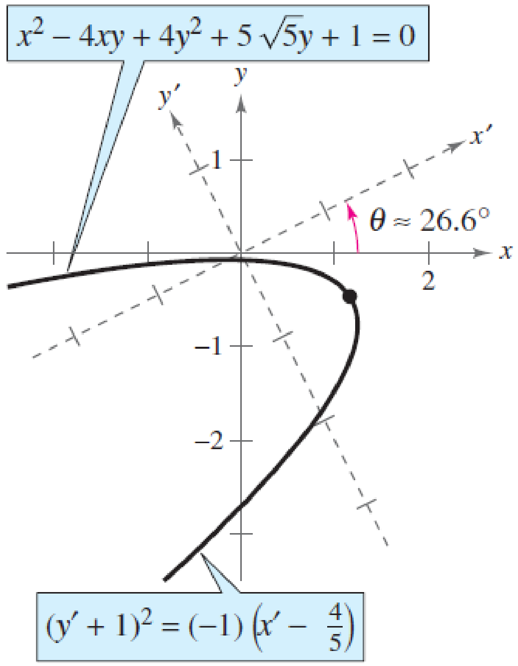
$(1)\quad x = x'\cos\theta - y'\sin\theta = \dfrac{2x'-y'}{\sqrt{5}}$

$(2)\quad y = x'\sin\theta + y'\cos\theta = \dfrac{x'+2y'}{\sqrt{5}}$

Subs. (1) & (2) into (0) $\Longrightarrow$

$$\boxed{(y'+1)^2 = (-1)(x' - 4/3)}$$

vertex $x'y'$ – $(4/3, -1)$



- Quadratic Surfaces

Good Luck