Discrete Math

Lab 5 – October, 13

Agenda

- HW discussion
- Binary Relations

Theory of Relations

- A **relation** R over the sets $X_1, \dots, X_k$ is a subset of their Cartesian product, written $R \subseteq X_1 \times \dots \times X_k$
- **Arity of a function** or operation is the number of arguments or operands that the function takes.
- The **arity of a relation** (or predicate) is the dimension of the domain in the corresponding Cartesian product.

Binary Relations

Binary Relation between two sets A and B is a set of ordered pairs (a, b) consisting of elements a of A and elements b of B.

Binary Relation is a subset of the Cartesian product $A \times B$. It encodes the information of relation: an element a is related to an element b if and only if the pair (a, b) belongs to the set.

The statement $(a, b) \in R$ is read "a is R-related to b", and is denoted by aRb.

The **support** of R is the set of all a such that aRb for at least one b. The **image** of R is the set of all b such that aRb for at least one a.

Properties of Binary Relations

Property	Definition	MR
Reflexive	$\forall x \in S: (xRx)$	Only 1s on the main diagonal
Anti-reflexive (irreflexive)	$\forall x \in S: \neg(xRx)$	Only 0s on the main diagonal
Symmetric	$\forall x, y \in S: (xRy \Rightarrow yRx)$	Symmetric with respect to the main diagonal The transpose of MR is equal to its original MR
Anti-symmetric	$\forall x, y \in S: (xRy \wedge yRx \Rightarrow x = y)$	No two distinct elements of it, that are symmetric with respect to the main diagonal, are both 0 or both 1
Asymmetric	$\forall x, y \in S: (xRy \Rightarrow \neg(yRx))$	Antisymmetric & irreflexive
Transitive	$\forall x, y, z \in S: (xRy \wedge yRz \Rightarrow xRz)$	the squared matrix has no nonzero entry where the original had a zero
Connex	$\forall x, y \in S: (xRy \vee yRx)$	

Exercise 1

Determine the properties of the following relations:

a) line x crossing line y (on a set of lines on the plane)
b) number x is more than number y by 2 (on a set of natural numbers)
c) number x is divisible by number y (on a set of natural numbers)

Exercise 2

Find the support set and the image set of the relation *R.* Is the relation reflexive, symmetric, antisymmetric and transitive?

$$R \subseteq \boldsymbol{Z}^2, (x, y) \in R \Leftrightarrow x^2 + y^2 = 1$$

Equivalence Relation

1. Reflexive (a~a)
2. Symmetric (a~b ⇒ b~a)
3. Transitive (a~b ∧ b~c ⇒ a~c)

Examples:

"Is equal to" on the set of numbers.

"Has the same birthday as" on the set of all people.

"Is similar to" on the set of all triangles.

"Is congruent to, modulo n" on the integers.

"Has the same image under a function" on the elements of the domain of the function.

An **equivalence class** is the name that we give to the subset of S which includes **all elements that are equivalent to each other**

Exercise 3

Check if the relation “two lines are parallel” is the relation of equivalence on a plane

Exercise 4

On the set of real numbers ***R*** the relation S is set: $aSb \Leftrightarrow a^2 + a = b^2 + b$. Prove that S is the relation of equivalence. How many elements are in a class of equivalence?

Combining relations

Because relations from A to B are subsets of A × B, two relations from A to B can be combined in any way two sets can be combined

Composite of relations

Let R be a relation from a set A to a set B and S a relation from B to a set C. The **composite** of R and S is the relation consisting of ordered pairs (a, c), where a ∈ A, c ∈ C, and for which there exists an element b ∈ B such that (a, b) ∈ R and (b, c) ∈ S. We denote the composite of R and S by S∘R

Inverse relation

Let R be a relation from A to B. The **inverse** relation R^{-1} from B to A is defined as follows: $R^{-1} = \{(y, x) \in (B\text{x}A) | (x, y) \in R\}$

Complementary relation

- Let R be a relation from a set A to a set B. The **complementary relation** R is the set of ordered pairs {(a, b) | (a, b) ∉ R}.

Exercise 5

The relation R on set $X = \{a, b, c, d\}$ is defined by the matrix below. What are the properties of the relation? What are the matrices for the relations R^{-1}, $R \circ R$?

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Closure

Definition: The *closure* of a relation R with respect to property P is the relation obtained by adding the minimum number of ordered pairs to R to obtain property P.

In terms of the digraph representation of R

- To find the reflexive closure – add loops
- To find the symmetric closure – add arcs in the opposite direction
- To fins the transitive closure – if there is a path from a to b, add an arc from a to b.

Closure

Reflexive Closure

The reflexive closure of a relation R on A is obtained by adding (a,a) to R for each a ∈ A

Symmetric Closure

The symmetric closure of a relation R on A is obtained by adding (b,a) to R for each (a,b) ∈ R

Transitive Closure

The transitive closure of a relation R on A is obtained by repeatedly adding (a,c) to R for each (a,b) ∈ R and (b,c) ∈ R

Transitive Closure

Transitive closure of a binary relation R on a set X is the smallest relation on X that contains R and is transitive.

$$R^+ = \bigcup_{i \in \{1, 2, 3, \ldots\}} R^i, \quad \textit{where } R^1 = R,\ R^{i+1} = R \circ R^i$$

Examples:

- X is a set of people ever lived on Earth. R is the relation “is a parent of”. Then the transitive closure R^+ is the relation “is an ancestor of”.
- X is a set of integers $\boldsymbol{Z}$. R is the relation “next number”. Then the transitive closure R^+ is the relation “is more than”.

Relation of (Partial) Order

1. Reflexive: $\forall x \in R$: (xRx)
2. Anti-symmetric: $\forall x, y \in R$: $(xRy \wedge yRx \Rightarrow x = y)$
3. Transitive: $\forall x, y, z \in R$: $(xRy \wedge yRz \Rightarrow xRy)$
4. *(linear/total order) $\forall x, y \in R$: $(xRy \vee yRx)$

   (That is, every element is related with every element one way or the other)

Examples:

- “more or equal” or “less or equal” on a set of real numbers ***R***
- “is divisible by” on a set of integers ***Z***
- The letters of the alphabet ordered by the standard dictionary order

Relation of (Partial) Order

The ordered pair <A, R> is called a poset (partially ordered set) when R is a partial order.

Let < A, $\preccurlyeq$ > be a poset, where $\preccurlyeq$ represents an arbitrary partial order.

- An element $b \in A$ is a **minimal element** of A if there is no element $a \in A$ that satisfies $a \preccurlyeq b$.
- An element $b \in A$ is a **maximal element** of A if there is no element $a \in A$ that satisfies $b \preccurlyeq a$.

Homework

1. Read textbook Kenneth H. Rosen "Discrete Mathematics and Its Applications, 7th Edition“ – paragraphs 9.2, 9.3, 9.4
2. Complete exercises 6-7, 9-19 (ex 15-16 and graphs in ex 17 are optional) and submit on Moodle by 10pm on Friday 16 October.

Exercise 6

Prove that for any binary relations R_1 and R_2 the following holds:

$(R_1 \circ R_2)^{-1} = R_2^{-1} \circ R_1^{-1}$

Exercise 7

Prove that if R and S are antisymmetric, then $R \cap S$ is antisymmetric as well.

Exercise 9

For the set $X = \{1, 2, 3, 6\}$ and the relation $R = \{(x, y) \mid x, y \in X, x \text{ is a divisor of } y\}$ show that the relation is the relation of order. Are there minimal and maximal elements in set X?

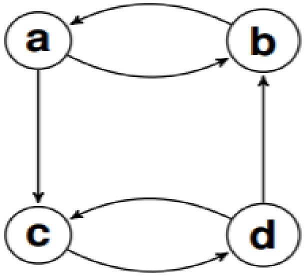
Exercise 10

Let R be the relation on the set {0, 1, 2, 3} containing the ordered pairs (0, 1),(1, 1),(1, 2),(2, 0),(2, 2),(3, 0). Find:

a) reflexive closure of R
b) symmetric closure of R

Exercise 11

For this directed graph



a) Find the reflexive closure (draw a graph)
b) Find the symmetric closure (draw a graph)

Exercise 12

Let S and T be binary relations on some set. Prove that:

a) $(S \cup T)^{-1} = (S^{-1}) \cup (T^{-1})$

b) $(S \circ T)^{-1} = (T^{-1})\circ(S^{-1})$

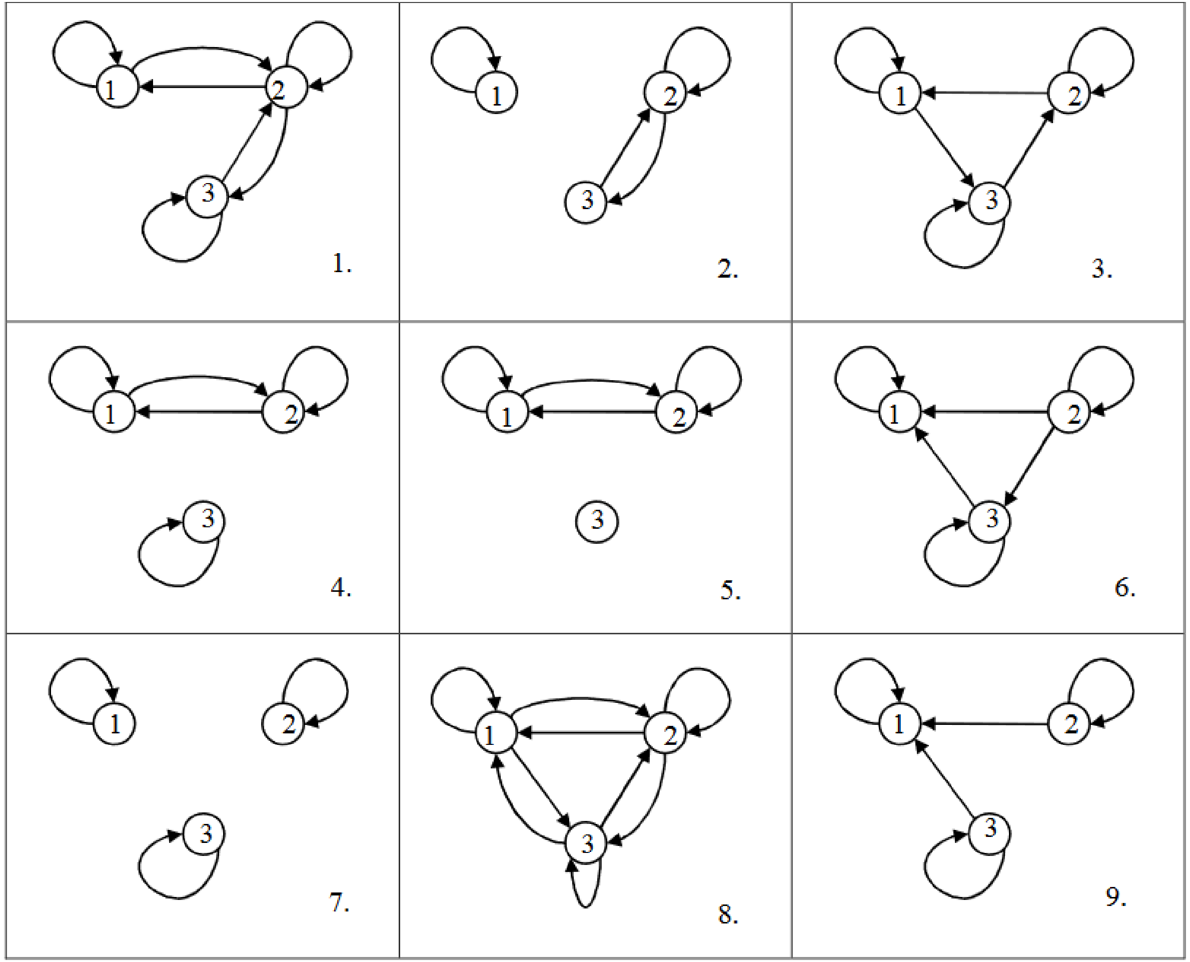
Exercise 13

Determine whether the relation R on the set of all people is reflexive, symmetric, antisymmetric, and/or transitive, where $(a, b) \in R$ if and only if

A) a is taller than b.
B) a and b were born on the same day.
C) a has the same first name as b.
D) a and b have a common grandparent

Exercise 14

Which of the following relations are reflexive, symmetric, transitive?



Exercise 15 (optional)

Let us fix a finite set with $n>0$ elements and represent a binary relation R on this set by boolean matrices R[i,j]= if iRj then True else False.
Write boolean expressions for elements of matrices for

1. $(X \cup Y)$,
2. $(X \circ Y)$,
3. $(X)^{-1}$,

assuming that matrices for binary relations X and Y are given;
Describe algorithm that computes matrix for (1),(2),(3).

Exercise 16 (optional)

Let P be any property (e.g. reflexivity, symmetry, transitivity, etc.) of binary relations on a set. For any binary relation R on a set let P-closure of R be the smallest (the least) binary relation S that contains R. For a given binary relation R express in terms of R and operations on binary relations (including the inverse):

- symmetric closure of R,
- reflexive closure of R,
- transitive closure of R

Exercise 17

Are the following relations on N reflexive, transitive, (a/anti)symmetric?

$R_1 \colon a\, R_1\, b \ \leftrightarrow |a-b| = 1$

$R_2 \colon a\, R_2\, b \ \leftrightarrow \ 0 < a - b < 3$

$R_3 \colon a\, R_3\, b \ \leftrightarrow \ a + b - \ even$

$R_4 \colon a\, R_4\, b \ \leftrightarrow \ a \geq b^2$

$R_5 \colon a\, R_5\, b \leftrightarrow greatest\ common\ divisor(a, b) = 1$

(optional) Draw graph for:

a) $R_1 \cap R_2$;

b) $R_1 \cup R_2$;

c) R_2^{-1};

d) $R_2 \circ R_4$;

e) $R_4 \circ R_2$;

f) $R_5 \backslash R_4^{-1}$

Exercise 18

How can the matrix for $\bar{R}$, the complement of the relation R , be found from the matrix representing R, when R is a relation on a finite set A?

Exercise 19

What is the symmetric closure of the relation $R = \{(a, b) \mid a > b\}$ on the set of positive integers?