Discrete Mathematics and Logic
Lecture 3

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The naive set theory

The naive set theory	Logic
Complement $\overline{A}$	Negation $\neg P$
Intersection $A_1 \cap A_2$	Conjunction $P_1 \,\&\, P_2$
Union $A_1 \cup A_2$	Disjunction $P_1 \vee P_2$

The naive set theory

The naive set theory	Logic
$A \cap A = A \quad A \cup A = A$ $A \cap B = B \cap A \quad A \cup B = B \cup A$	$a \,\&\, a = a \quad a \vee a = a$ $a \,\&\, b = b \,\&\, a \quad a \vee b = b \vee a$
$A \cap (B \cap C) = (A \cap B) \cap C$ $A \cup (B \cup C) = (A \cup B) \cup C$	$a \,\&\, (b \,\&\, c) = (a \,\&\, b) \,\&\, c$ $a \vee (b \vee c) = (a \vee b) \vee c$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$a \,\&\, (b \vee c) = (a \,\&\, b) \vee (a \,\&\, c)$ $a \vee (b \,\&\, c) = (a \vee b) \,\&\, (a \vee c)$
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\neg(a \,\&\, b) = \neg a \vee \neg b$ $\neg(a \vee b) = \neg a \,\&\, \neg b$

Universe

The naive set theory	Logic
The empty set $\emptyset$	False F | 0
The universe $\mathbf{U}$	True T | 1

Example of Universe

$\{x \in \mathbb{Z} \mid x < 0\}$

Complement

$\overline{A} = \{x \in \mathbf{U} \mid x \notin A\}$

Universe

Properties

The naive set theory		Logic	
$A \cap \emptyset = \emptyset$	$A \cup \mathbf{U} = \mathbf{U}$	$a \,\&\, 0 = 0$	$a \vee 1 = 1$
$A \cap \mathbf{U} = A$	$A \cup \emptyset = A$	$a \,\&\, 1 = a$	$a \vee 0 = a$
$A \cap \overline{A} = \emptyset$	$A \cup \overline{A} = \mathbf{U}$	$a \,\&\, \neg a = 0$	$a \vee \neg a = 1$
$\overline{\overline{A}} = A$		$\neg(\neg a) = a$	

for any A, $\emptyset \subseteq A$ & $A \subseteq \mathbf{U}$

Multiplication

Definition

$X \times Y = \{(x, y) \mid x \in X \,\&\, y \in Y\}$

Example

If $A = \{1, 2, 3\}$ and $B = \{a, b\}$, then

$$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

Multiplication

Definition

$$X_1 \times \cdots \times X_n = \{(x_1, \ldots, x_n) \mid x_1 \in X_1 \,\&\, \ldots \,\&\, x_n \in X_n\}$$

Example

$$\underbrace{X \times \cdots \times X}_{n \text{ times}} = X^n$$

$\mathbb{R}^3$

Multiplication

Properties

$$A \times A \neq A$$

$$A \times B \neq B \times A$$

$$A \times \emptyset = \emptyset \times A = \emptyset$$

$$A \times (B \times C) = (A \times B) \times C$$

*$(x, (y, z)) = ((x, y), z) = (x, y, z)$

Multiplication

Properties

$$(A \cup B) \times C = (A \times C) \cup (B \times C)$$
$$A \times (B \cup C) = (A \times C) \cup (A \times C)$$

$$(A \cap B) \times C = (A \times C) \cap (B \times C)$$
$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

$$(A \cap B) \times (X \cap Y) = (A \times X) \cap (B \times Y)$$

Multiplication

Properties

$(A \cup B) \times C = (A \times C) \cup (B \times C)$

Proof

$(x, y) \in (A \cup B) \times C \Leftrightarrow (x \in A \cup B) \,\&\, (y \in C) \Leftrightarrow$

$\Leftrightarrow (x \in A \vee x \in B) \,\&\, (y \in C) \Leftrightarrow {}^{*}$

$\Leftrightarrow {}^{*}(x \in A \,\&\, y \in C) \vee (x \in B \,\&\, y \in C) \Leftrightarrow$

$\Leftrightarrow ((x, y) \in A \times C) \vee ((x, y) \in B \times C) \Leftrightarrow (x, y) \in (A \times C) \cup (B \times C)$

$^{*}(P_1 \vee P_2) \,\&\, P_3 = (P_1 \,\&\, P_3) \vee (P_2 \,\&\, P_3)$

Multiplication

Properties

$$(A \cup B) \times C = (A \times C) \cup (B \times C)$$
$$A \times (B \cup C) = (A \times C) \cup (A \times C)$$

$$(A \cap B) \times C = (A \times C) \cap (B \times C)$$
$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

$$(A \cap B) \times (X \cap Y) = (A \times X) \cap (B \times Y)$$

Exercise

Prove all properties

Power of a set

Definition

For a set A, the power of A is the set $2^A = \mathcal{P}(A) = \{B \mid B \subseteq A\}$

Examples

1) If $A = \{a\}$ then $\mathcal{P}(A) = \{\emptyset, \{a\}\}$
2) If $A = \{a, b\}$ then $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

Cardinality

Definition

Intuitively, the cardinality of a set A, denotes by $|A|$, is the number of elements of A.

Examples

1. $|\emptyset| = 0$
2. if $A = \{2\}$ then $|A| = 1$
2. if $A = \{1, 2, 3\}$ then $|A| = 3$
3. $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| = \omega$
4. $|\mathbb{R}| = 2^{\omega}$

Cardinality

Properties

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B| = |A| + |B|, \text{ if } A \cap B = \emptyset$$

$$|A \times B| = |A| \cdot |B|$$

Cardinality

Properties

$$|A \cup B| = |A| + |B|, \text{ if } A \cap B = \emptyset$$

Proof

Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$.

Since $A \cap B = \emptyset$, $A \cup B = \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m\}$.

Therefore, $|A \cup B| = |A| + |B|$

Cardinality

Properties

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Proof

Define $A \setminus B = \{x \in A \mid x \notin B\} = A \cap \overline{B}$.

$$\frac{\begin{array}{cc} A = (A \setminus B) \cup (A \cap B) & B = (B \setminus A) \cup (A \cap B) \\ (A \setminus B) \cap (A \cap B) = \emptyset & (B \setminus A) \cap (A \cap B) = \emptyset \end{array}}{\begin{array}{cc} |A| = |A \setminus B| + |A \cap B| & |B| = |B \setminus A| + |A \cap B| \end{array}}$$

$$A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$$

$$|A \cup B| = |A \setminus B| + |A \cap B| + |B \setminus A|$$

Cardinality

Properties

$$|A \times B| = |A| \cdot |B|$$

Example

Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_m\}$.

	a_1	a_2	$\ldots$	a_n
b_1	(a_1, b_1)	(a_2, b_1)	$\ldots$	(a_n, b_1)
b_2	(a_1, b_2)	(a_2, b_2)	$\ldots$	(a_n, b_2)
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
b_n	(a_1, b_n)	(a_2, b_n)	$\ldots$	(a_n, b_n)

Obviously, the table contains $n \times m$ elements.

Cardinality

Properties

$$|A \times B| = |A| \cdot |B|$$

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_n|$$

$$|A^n| = |A|^n$$

Cardinality

$$|2^A| = 2^{|A|}$$

$A = \{a_1, a_2, \dots, a_{n-1}, a_{n-2}, a_n\}$

a_1	a_2	$\dots$	a_{n-1}	a_n	*Subsets*
0	0	$\dots$	0	0	$\emptyset$
0	0	$\dots$	0	1	$\{a_n\}$
0	0	$\dots$	1	0	$\{a_{n-1}\}$
0	0	$\dots$	1	1	$\{a_{n-1}, a_n\}$
		$\dots$		$\dots$	
1	1	$\dots$	0	0	$\{a_1, a_2, \dots, a_{n-2}\}$
1	1	$\dots$	0	1	$\{a_1, a_2, \dots, a_{n-2}, a_n\}$
1	1	$\dots$	1	0	$\{a_1, a_2, \dots, a_{n-2}, a_{n-1}\}$
1	1	$\dots$	1	1	$A = \{a_1, a_2, \dots, a_{n-2}, a_{n-1}, a_n\}$

*$0 = $ "$a_i \notin A$" & $1 = $ "$a_i \in A$"

Cardinality

Exercise

Prove by induction that the number of rows of truth table for n columns is 2^n.

Thus,

$$|2^A| = 2^{|A|}$$

Thank you for your attention!