Mathematical Analysis. Assignment 5. Taylor formula and its applications

Reference Material

Let $f(x)$ be defined in some neighborhood of x_0 and have derivatives of all orders up to $(n-1)$ in this neighborhood. Let also $f^{(n)}(x_0)$ exist. Then

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + o\left((x-x_0)^n\right), \quad x \to x_0.$$

If function $f(x)$ defined in some neighborhood of x_0 has derivatives of all orders up to $(n+1)$ in this neighborhood then

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1},$$

where ξ lies between x and x_0.

$\sum\limits_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$ is called a Taylor polynomial;

$o\left((x-x_0)^n\right)$ is a Peano remainder term;

$\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$ is a Lagrange remainder term.

A special case of Taylor series for $x_0 = 0$ is called Maclaurin series:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k + o(x^n), \quad x \to 0.$$

A list of essential Maclaurin series decompositions is provided below:

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + o(x^n), \quad x \to 0;$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + o(x^{2n+1}), \quad x \to 0;$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!} + o(x^{2n}), \quad x \to 0;$

$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots + \frac{x^{2n}}{(2n)!} + o(x^{2n+1}), \quad x \to 0;$

$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots + \frac{x^{2n-1}}{(2n-1)!} + o(x^{2n}), \quad x \to 0;$

$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + \frac{(-1)^{n-1} x^{2n-1}}{2n-1} + o(x^{2n}), \quad x \to 0;$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^{n-1} x^n}{n!} + o(x^n), \quad x \to 0;$

$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n + o(x^n), \quad x \to 0.$

The coefficients of the latter decomposition are also called binomial coefficients and can be denoted as $\binom{\alpha}{k}$. Namely, $\binom{\alpha}{0} = 1$, $\binom{\alpha}{1} = \alpha$, $\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}$, $k \geqslant 2$.

There are some functions that have more complicated formulae for coefficients, and we just give the first several terms of their decompositions:

$\arcsin x = x + \frac{x^3}{3} + \frac{3x^5}{40} + o(x^6);$

$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + o(x^6);$

$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} + o(x^6).$

Problems

1. Write Maclaurin formulae for the following functions (the remainder should be $o(x^n)$):

   (a) $\sin(2x+3)$;

(b) $(x^2 - x)\, e^x$;

(c) $\frac{x^2+3e^x}{e^{2x}}$;

(d) $\frac{x}{\sqrt[3]{9-6x+x^2}}$;

(e) $\ln\left(6 + 11x + 6x^2 + x^3\right)$.

Answer:

(a) $\sum\limits_{k=0}^{n} \frac{2^k \sin\left(3+\frac{k\pi}{2}\right)}{k!} x^k + o\left(x^n\right)$;

(b) $-x + \sum\limits_{k=2}^{n} \frac{(-1)^k k}{(k-1)!} x^k + o\left(x^n\right)$;

(c) $3 + \sum\limits_{k=1}^{n} \left(3 + k(k-1)2^{k-2}\right) \frac{(-1)^k}{k!} x^k + o\left(x^n\right)$;

(d) $\sum\limits_{k=1}^{n} 3^{\frac{1}{3}-k}(-1)^{k-1} \binom{-\frac{2}{3}}{k-1} x^k + o\left(x^n\right)$

(e) $\ln 6 + \sum\limits_{k=1}^{n} \frac{(-1)^{k-1}}{k} \left(1 + 2^{-k} + 3^{-k}\right) x^k + o\left(x^n\right)$.

2. Write Taylor formula for $\frac{x^2+3x}{x+1}$ in the neighborhood of $x_0 = 1$. (The remainder should be $o\left(x-1\right)^n$).)

Answer: $2 + \frac{3}{2}(x-1) + \sum\limits_{k=2}^{n} (-1)^{k-1} \frac{(x-1)^k}{2^k} + o\left((x-1)^n\right)$.

3. Decompose the following functions using Maclaurin formula:

(a) $e^{\sqrt{1+2x}}$ up to $o\left(x^2\right)$;

(b) $\frac{1-x+x^2}{1+x+x^2}$ up to $o\left(x^3\right)$;

(c) $\sqrt{1+2x-x^2} - \sqrt[3]{1-3x+x^3}$ up to $o\left(x^3\right)$;

(d) $\sqrt[3]{1-3x\cos 2x}$ up to $o\left(x^3\right)$;

(e) $e^{\frac{x}{\sin x}}$ up to $o\left(x^4\right)$;

(f) $\frac{x}{e^x-1}$ up to $o\left(x^4\right)$.

Answer:

(a) $e + ex + o\left(x^2\right)$;

(b) $1 - 2x + 2x^2 + o\left(x^3\right)$;

(c) $2x + \frac{7x^3}{3} + o\left(x^3\right)$;

(d) $1 - x - x^2 + \frac{x^3}{3} + o\left(x^3\right)$;

(e) $e + \frac{ex^2}{6} + \frac{ex^4}{30} + o\left(x^4\right)$;

(f) $1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + o\left(x^4\right)$.

4. Find constants A and B that satisfy the following equalities:

(a) $Ae^x - \frac{B}{1-x} = -\frac{1}{2}x^2 - \frac{5}{6}x^3 + o\left(x^3\right)$;

(b) $(A + B\cos x)\sin x = x + o\left(x^4\right)$.

Answer: (a) $A = B = 1$; (b) $A = \frac{4}{3}$, $B = -\frac{1}{3}$.

5. Find limits using Maclaurin formula:

(a) $\lim\limits_{x\to 0} \frac{\ln(1+x)-x}{x^2}$;

(b) $\lim\limits_{x\to 0} \frac{\sqrt{1+x}+\sqrt[3]{1+x}-2\sqrt[4]{1-x}}{x}$;

(c) $\lim\limits_{x\to 0} \frac{\arctan x-\arcsin x}{\tan x-\sin x}$;

(d) $\lim\limits_{x\to 0} \frac{\sqrt[3]{1-x^2}-x\cot x}{x\sin x}$;

(e) $\lim\limits_{x\to 0} \frac{(1+x)^{\frac{1}{x}}-e\left(1-\frac{x}{2}\right)}{x^2}$;

(f) $\lim\limits_{x\to 0} \frac{\cos x-\sqrt{1-2x}-x}{x^2\tan x-e^{-x^3}+1}$;

(g) $\lim\limits_{x\to 0} \frac{x+\cosh x-e^{\arcsin x}}{\tan x+\sqrt[3]{1-3x}-2\cos x+1}$;

(h) $\lim\limits_{x\to 0} \left(\frac{\cos x}{\cosh 3x}\right)^{\frac{1}{x^2}}$;

(i) $\lim\limits_{x\to 0} \left(\tan\frac{x}{3}-\sqrt[3]{1+x}+2\right)^{\cot^2 x}$;

(j) $\lim\limits_{x\to 0} \left(\frac{e^{-x}}{1-x}+\frac{1}{2}\left(\ln\sqrt{1+2x}-\tan x\right)\right)^{\frac{1}{x\cos x-x}}$;

(k) $\lim\limits_{x\to 0} \left(\cos x+x^2\sqrt{x+\frac{1}{4}}\right)^{\frac{x+e}{\arcsin x^3}}$;

(l) $\lim\limits_{x\to 0} \left(\frac{6}{\ln\left(1+3\sin^2 x\right)}-\frac{4}{\ln(2-\cos 2x)}\right)^{\frac{1}{x^2}}$;

(m) $\lim\limits_{x\to 1} \frac{e^{\frac{x-1}{x}}-\sqrt[4]{4x-3}}{\cosh(x-1)-\cos(2x-2)}$;

(n) $\lim\limits_{x\to 0} \left(\frac{1}{(x+1)\sinh x}-\frac{\ln(1+x)}{x^2}\right)$;

(o) $\lim\limits_{x\to+\infty} \frac{\sqrt[6]{x^6+x^5}+\sqrt[6]{x^6-x^5}-2x}{x\ln(1+x)-x\ln x-x\sin\frac{1}{x}}$;

(p) $\lim\limits_{x\to+\infty} \left(e^{\frac{1}{x}}(x^2-x+2)-\sqrt{x^4+x^2+1}\right)$.

Answer: (a) $-\frac{1}{2}$; (b) $\frac{4}{3}$; (c) -1; (d) 0; (e) $\frac{11e}{24}$; (f) $\frac{1}{4}$; (g) $\frac{1}{4}$; (h) e^{-5}; (i) $e^{\frac{1}{9}}$; (j) $e^{-\frac{5}{3}}$; (k) e^e; (l) $e^{-\frac{5}{6}}$; (m) $-\frac{2}{5}$; (n) $-\frac{1}{2}$; (o) $\frac{5}{18}$; (p) 1.