8. Quadratic Curves

8.1. Circles

A **circle** is a set of points in a plane that are equidistant from a fixed point. The fixed point is called the **center**. A line segment that joins the center with any point of the circle is called the **radius**.

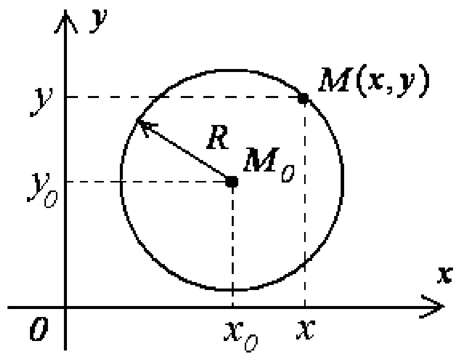
In the *x,y*–plane, the distance between two points $M(x, y)$ and $M_0(x_0, y_0)$ equals

$$\sqrt{(x-x_0)^2+(y-y_0)^2},$$

and so the circle is described by the equation

$$(x-x_0)^2+(y-y_0)^2=R^2, \qquad (1)$$

where x_0 and y_0 are the coordinates of the center, and *R* is the radius.



Equation of a circle centered at the origin

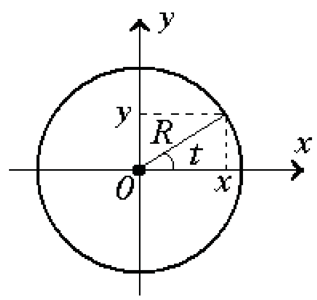
$$x^2+y^2=R^2 \qquad (2)$$

is known as the canonical equation of the circle.

If *t* is a real parameter, then

$$\begin{cases} x = R\cos t \\ y = R\sin t \end{cases}$$

are the parametric equations of the circle centered at the origin with radius *R*.



By elimination of the parameter *t*, we return to canonical equation (2):

$$\begin{cases} x^2 = R^2\cos^2 t \\ y^2 = R^2\sin^2 t \end{cases} \quad \Rightarrow \quad x^2 + y^2 = R^2 .$$

Likewise,

$$\begin{cases} x = x_0 + R\cos t \\ y = y_0 + R\sin t \end{cases}$$

are the parametric equations of the circle centered at the point $M_0(x_0, y_0)$ with radius R.

Examples:

1) The circle is given by the equation

$$x^2 - 4x + y^2 + 6y = 12 .$$

Find the radius and the coordinates of the center.

Solution: Transform the quadratic polynomial on the left-hand side of the equation by adding and subtracting the corresponding constants to complete the perfect squares:

$$x^2 - 4x = (x^2 - 4x + 4) - 4 = (x-2)^2 - 4$$

$$y^2 + 6y = (y^2 + 6y + 9) - 9 = (y+3)^2 - 9 .$$

Then the given equation is reduced to the form

$$(x-2)^2 + (y+3)^2 = 5^2 ,$$

which describes the circle centered at the point $M_0(2,-3)$ with radius 5.

2) Let

$$x^2 + 2x + y^2 - 8y + 17 = 0 .$$

Find the canonical equation of the circle.

Solution:

$$x^2 + 2x + y^2 - 8y + 17 = 0 \quad \Rightarrow \quad (x^2 + 2x + 1) + (y^2 - 8y + 16) = 0 \quad \Rightarrow$$

$$(x+1)^2 + (y-4)^2 = 0 .$$

The radius of the circle equals zero, that means the given equation corresponds to a null point circle.

3) The equation

$$x^2 + 2x + y^2 + 5 = 0$$

can be reduced to the form

$$(x+1)^2 + y^2 = -4 ,$$

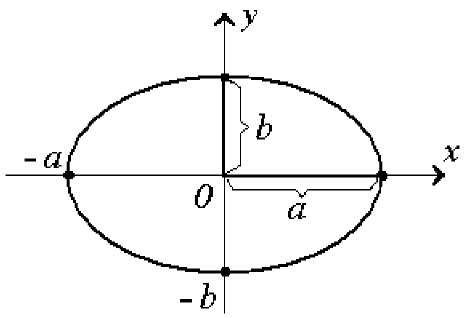
which has no solutions. In this case they say that the equation describes an imaginary circle.

8.2. Ellipses

An **ellipse** is a plane curve, which is represented by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (3)$$

in some Cartesian coordinate system.
Equation (3) is called the **canonical equation of an ellipse**, or the equation of an ellipse in the canonical system of coordinates. The positive quantities $2a$ and $2b$ are called the axes of the ellipse. One of them is said to be the major axis, while the other is the minor axis.
In the canonical system, the coordinate axes are the axes of symmetry, that means if a point (x, y) belongs to the ellipse, then the points $(-x, y)$, $(x,-y)$ and $(-x,-y)$ also belong to the ellipse.
The intersection points of the ellipse with the axes of symmetry are called the **vertices** of the ellipse. Hence, the points $(\pm a, 0)$ and $(0, \pm b)$ are the vertices of ellipse (3).



If $a = b = R$ then equation (3) is reduced to equation (2) of a circle. Thus, one can consider a circle as a specific ellipse.
The parametric equations of the ellipse have the following form:

$$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$$

One can easily eliminate the parameter t to obtain the canonical equation of the ellipse:

$$\begin{cases} \dfrac{x^2}{a^2} = \cos^2 t \\ \dfrac{y^2}{b^2} = \sin^2 t \end{cases} \quad \Rightarrow \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The equation

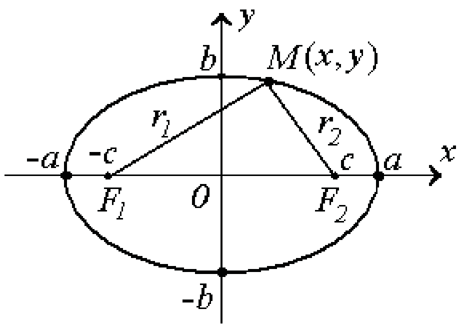
$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_2)^2}{b^2}=1$$

corresponds to the ellipse with the center at the point $M_0(x_0, y_0)$. The axes of symmetry of this ellipse pass through M_0, being parallel to the coordinate axes.

8.2.1. Properties of Ellipses

Consider an ellipse, which is given by equation (3) with the major axis $2a$. Two fixed points, $F_1(-c,0)$ and $F_2(c,0)$, are called the **focuses** of the ellipse, if equality $c^2 = a^2 - b^2$ is satisfied.
Correspondingly, the distances r_1 and r_2 from any point $M(x, y)$ of the ellipse to the points F_1 and F_2 are called the **focal distances**.

The ratio $\frac{c}{a} = \varepsilon$ is called the **eccentricity** of ellipse.
Note that $0 < \varepsilon < 1$.



1) Let x be the abscissa of a point of ellipse (3). Then the focal distances of the point can be expressed as follows:

$$r_1 = a + x\varepsilon, \tag{4a}$$

$$r_2 = a - x\varepsilon. \tag{4b}$$

Proof: By the definition, the distance between two points, $M(x, y)$ and $F_1(-c,0)$, is

$$r_1 = \sqrt{(x+c)^2 + y^2}\,.$$

Consider the expression under the sign of the radical.
By substituting

$$y^2 = (a^2 - x^2)\frac{b^2}{a^2},$$

$$c = a\varepsilon \quad \text{and} \quad b^2 = a^2 - c^2 = a^2(1-\varepsilon^2)$$

in r_1^2, we obtain

$$r_1^2 = (x+c)^2 + y^2 = x^2 + 2cx + c^2 + y^2$$
$$= x^2 + 2ax\varepsilon + a^2\varepsilon^2 + (a^2 - x^2)(1-\varepsilon^2),$$

which results in

$$r_1^2 = a^2 + 2a\,x\,\varepsilon + x^2\varepsilon^2 = (a + x\,\varepsilon)^2 .$$

Likewise,

$$r_2 = \sqrt{(x-c)^2 + y^2} \quad \Rightarrow \quad r_2^2 = a^2 - 2a\,x\,\varepsilon + x^2\varepsilon^2 = (a - x\,\varepsilon)^2 .$$

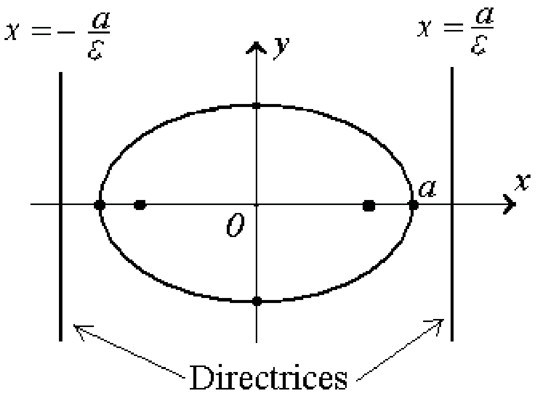
Since $a \pm x\,\varepsilon > 0$, the above formulas give the desired statement.

2) For any point of ellipse (3), the sum of the focal distances is the constant quantity $2a$:

$$r_1 + r_2 = 2a . \tag{5}$$

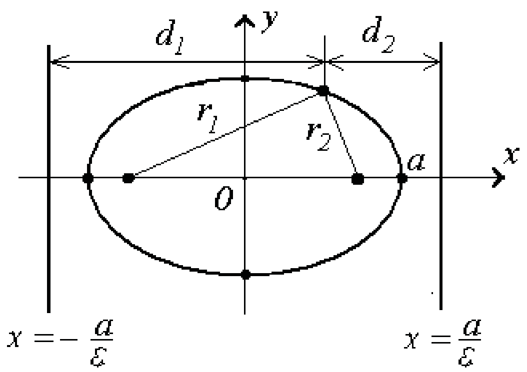
This property follows from formulas (4a) and (4b).

Two symmetric lines passing at the distance $\frac{a}{\varepsilon}$ from the center of an ellipse and being perpendicular to the major axis are called the **directrices**.



3) For any point of ellipse (3), the ratio of the focal distance to the distance from the corresponding directrix is equal to the eccentricity of the ellipse:

$$\frac{r_1}{d_1} = \frac{r_2}{d_2} = \varepsilon \tag{6}$$

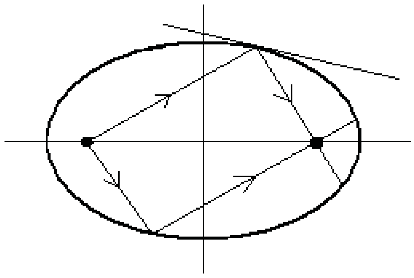


Proof: By Property 1 and in view of the fact that

$$d_1 = \frac{a}{\varepsilon} + x \quad \text{and} \quad d_1 = \frac{a}{\varepsilon} - x,$$

we obtain the desired results.

4) Assume that the curve of an ellipse has the mirror reflection property. If a point light source is located at a focus of the ellipse, then rays of light meet at the other focus after being reflected.



In other words, at any point of an ellipse, the tangent line forms equal angles with the focal radiuses.

5) The orbital path of a planet around the sun is an ellipse such that the sun is located at a focus.

Example: Reduce the equation

$$2x^2 + 4x + 3y^2 - 12y = 1$$

to the canonical form. Give the detailed description of the curve.

Solution: Complete the perfect squares.

$$2x^2 + 4x + 3y^2 - 12y = 1 \quad \Rightarrow$$

$$2(x^2 + 2x + 1) + 3(y^2 - 4y + 4) = 15 \quad \Rightarrow$$

$$2(x+1)^2 + 3(y-2)^2 = 15 \quad \Rightarrow$$

$$\frac{(x+1)^2}{15/2} + \frac{(y-2)^2}{5} = 1.$$

Thus, the given equation describes the ellipse with the center at the point (–1, 2).

The major semi-axis equals $\sqrt{15/2}$, and the minor semi-axis is $\sqrt{5}$. The focuses are located on the horizontal line $y = 2$. The distance between each focus and the center is

$$c = \sqrt{a^2 - b^2} = \sqrt{\frac{15}{2} - 5} = \sqrt{\frac{5}{2}} = \frac{\sqrt{10}}{2}.$$

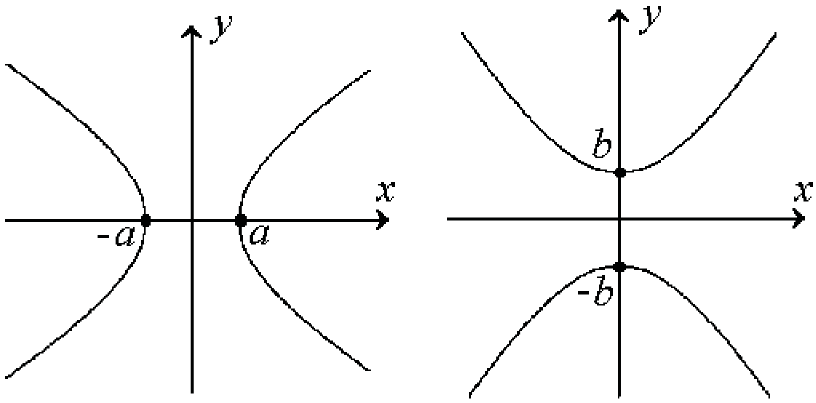
The eccentricity equals

$$\varepsilon = \frac{c}{a} = \frac{\sqrt{5}}{\sqrt{15}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

8.3. Hyperbolas

A **hyperbola** is a plane curve, which can be represented in some Cartesian coordinate system by one of the below equations

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad\qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \qquad (7)$$



Equations (7) are called the **canonical equations of a hyperbola**. The corresponding coordinate system is said to be the canonical system. In this coordinate system, the coordinate axes are axes of symmetry, that is, if a point (x, y) belongs to the hyperbola then the points $(-x, y)$, $(x, -y)$ and $(-x, -y)$ also belong to the hyperbola.

The intersection points of the hyperbola with the axis of symmetry are called the **vertices** of the hyperbola. Any hyperbola has two vertices.

If $a = b$ then the hyperbola is called an equilateral hyperbola.

The equations

$$\frac{(x - x_0)^2}{a^2} - \frac{(y - y_2)^2}{b^2} = \pm 1$$

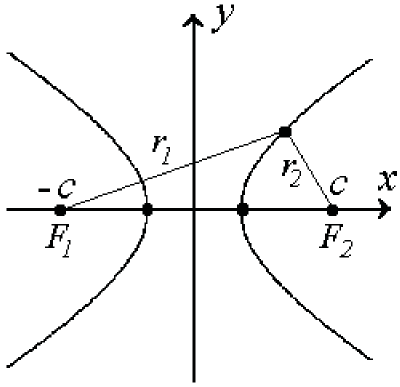
describe hyperbolas with the center at the point $M_0(x_0, y_0)$. The axes of symmetry of the hyperbolas pass through M_0, being parallel to the coordinate axes.

Consider a hyperbola, which is given by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \qquad (8)$$

Two fixed points, $F_1(-c, 0)$ and $F_2(c, 0)$, are called the **focuses** of the hyperbola, if the equality $c^2 = a^2 + b^2$ is satisfied.

Correspondingly, the distances r_1 and r_2 from any point $M(x, y)$ of the hyperbola to the points F_1 and F_2 are called the **focal distances**.



The ratio $\frac{c}{a} = \varepsilon$ is called the **eccentricity** of hyperbola.
Note that $\varepsilon > 1$.

8.3.1. Properties of Hyperbolas

1) Let x be the abscissa of a point of hyperbola (8). Then the focal distances of the point are the following:

$$r_1 = \pm(x\varepsilon + a), \qquad (9a)$$

$$r_2 = \pm(x\varepsilon - a). \qquad (9b)$$

In the above formulas we have to apply the sign '+' for the points on the right half-hyperbola, while the sign '–' is used for the points on the left half-hyperbola.

This property is similar to the corresponding one of ellipses.
The distance between two points $M(x, y)$ and $F_1(-c, 0)$ is

$$r_1 = \sqrt{(x+c)^2 + y^2},$$

where

$$y^2 = (x^2 - a^2)\frac{b^2}{a^2},$$

$$c = a\varepsilon \quad \text{and} \quad b^2 = c^2 - a^2 = a^2(\varepsilon^2 - 1).$$

Therefore,

$$\begin{aligned} r_1^2 &= (x+c)^2 + y^2 = x^2 + 2cx + c^2 + y^2 \\ &= x^2 + 2ax\varepsilon + a^2\varepsilon^2 + (x^2 - a^2)(\varepsilon^2 - 1) \\ &= x^2\varepsilon^2 + 2ax\varepsilon + a^2 = (x\varepsilon + a)^2. \end{aligned}$$

Likewise,

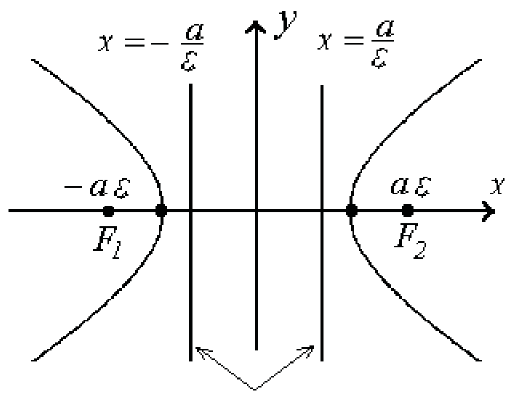
$$r_2 = \sqrt{(x-c)^2 + y^2} \quad \Rightarrow \quad r_2^2 = (x\varepsilon - a)^2.$$

Since $x\varepsilon \pm a > 0$ for the points of the right half-hyperbola, and $x\varepsilon \pm a < 0$ for points of the left half-hyperbola, we have got the desired results.

2) For any point of hyperbola (8), the difference between the focal distances is the constant quantity $(\pm 2a)$:
$$r_1 - r_2 = \pm 2a. \qquad (10)$$
The sign depends on whether the point lies on the right or left half-hyperbola.

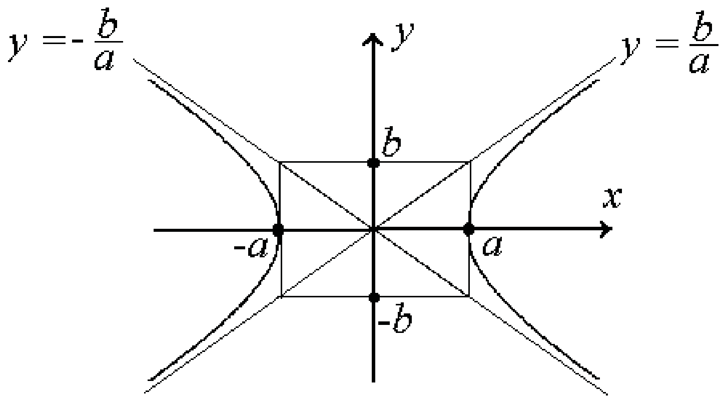
The proof is straightforward. We only need to apply Property 1.

The **directrices** of hyperbola (8) are two vertical lines $x = \pm\frac{a}{\varepsilon}$.



3) For any point of hyperbola (8) the ratio of the focal distance to the distance from the corresponding directrix is equal to the eccentricity of the hyperbola.
$$\frac{r_1}{d_1} = \frac{r_2}{d_2} = \varepsilon \qquad (11)$$

4) Two straight lines $y = \pm\frac{b}{a}$ are the asymptotes of hyperbola (8).



Proof: Express the variable y from equality (8) in the explicit form.
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad y^2 = \frac{b^2}{a^2}(x^2 - a^2) \quad \Rightarrow$$

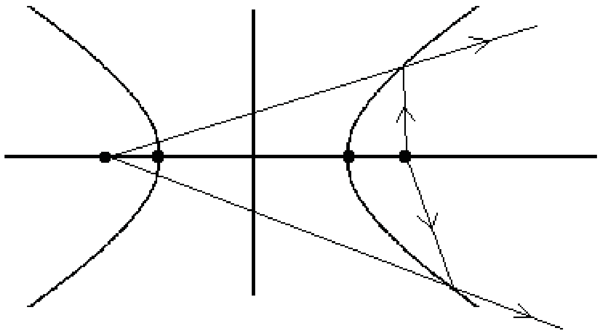
$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2}\,.$$

If x approaches infinity, then constant a^2 is a negligible quantity, that is,

$$y \to \pm \frac{b}{a}x\,.$$

Hence, the property.

5) Assume that the curve of a hyperbola has the mirror reflection property. If a point light source is located at a focus of the hyperbola, then the other focus is the image source of rays that being reflected.



The drawing illustrates that reflected rays form a divergent beam.

Example: Reduce the equation

$$x^2 - 6x + 2y^2 + 8y = 0$$

to the canonical form. Give the detailed description of the curve.

Solution: Complete the perfect squares.

$$x^2 - 6x - 2y^2 - 8y = 7 \qquad \Rightarrow$$

$$(x^2 - 6x + 9) - 2(y^2 + 4y + 4) = 8 \qquad \Rightarrow$$

$$(x-3)^2 - 2(y+2)^2 = 8\,.$$

Dividing both sides by 8 we obtain the equation

$$\frac{(x-3)^2}{8} - \frac{(y+2)^2}{4} = 1\,,$$

which describes the hyperbola with the center at the point (3, –2). The focuses are located on the horizontal line y = –2. The distance between each focus and the center of the hyperbola is

$$c = \sqrt{a^2 + b^2} = \sqrt{8+4} = \sqrt{12} = 3\sqrt{2}\,.$$

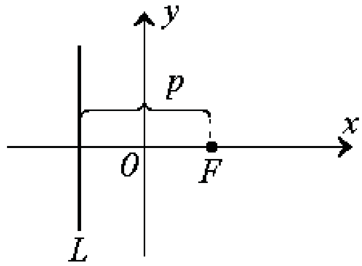
The eccentricity of the hyperbola equals

$$\varepsilon = \frac{c}{a} = \frac{3\sqrt{2}}{\sqrt{8}} = \frac{3}{2}.$$

8.4. Parabolas

A **parabola** is the locus of points, which are equidistant from a given point F and line L. The point F is called the **focus**. The line L is called the **directrix** of the parabola.

Let the focus be on the x-axis and the directrix be parallel to the y-axis at the distance p from the focus as it is shown in the figure below.



Then the focal distance of a point $M(x,y)$ is

$$r = \sqrt{(x - p/2)^2 + y^2}$$

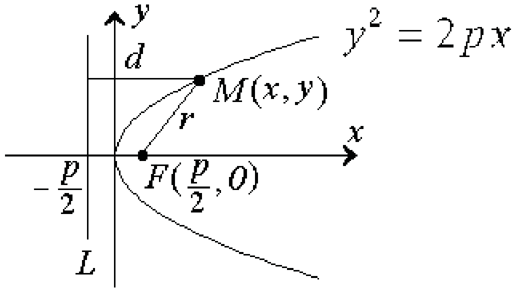
and the distance from M to the directrix is

$$d = x + p/2.$$

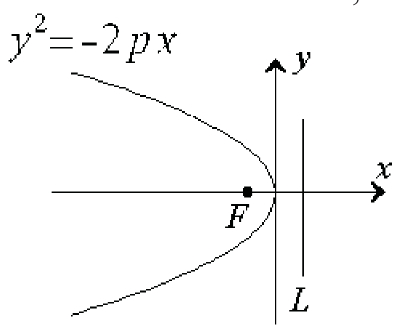
Therefore, due to the transformations

$$r = d \quad \Rightarrow \quad \sqrt{(x-p/2)^2 + y^2} = x + p/2 \quad \Rightarrow$$
$$(x-p/2)^2 + y^2 = (x+p/2)^2,$$

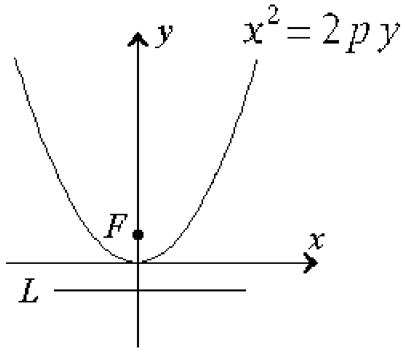
we obtain the following equation of a parabola: $y^2 = 2px$.

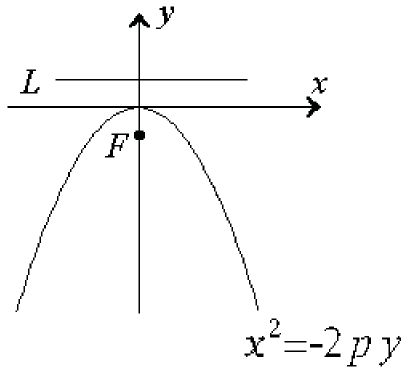


If the focus is located on the left of the directrix, then we obtain



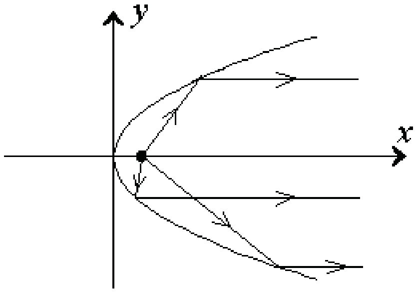
Some more cases are shown in the drawings below.





Parabola properties:

1) Any parabola has the axis of symmetry, which passes through the vertex of the parabola, being perpendicular to the directrix.
2) Let an ellipse be the mirror reflection curve. If a point light source is located at a focus of the ellipse, then rays of light are parallel after being reflected.



The equations

$$(y - y_0)^2 = \pm 2p(x - x_0),$$
$$(x - x_0)^2 = \pm 2p(y - y_0)$$

describe parabolas with the vertex at the point $M_0(x_0, y_0)$.

Example: Reduce the equation

$$x^2 + 4x - 3y = -5.$$

to the canonical form. Give the detailed description of the curve.

Solution:

$$x^2 + 4x - 3y = -5 \quad \Rightarrow$$
$$(x+2)^2 = 3(y - \frac{5}{3}).$$

This equation describes the parabola with the vertex at the point $M_0(-2, 5/3)$. The axis of symmetry is a line $x = -2$ which is parallel to the y–axis.

8.5. Summary

Let F be a point (focus) and L be a line (directrix) of a quadric curve. Consider the locus of points such that the ratio of the distances to the focus and to the directrix is a constant quantity (eccentricity),

$$\frac{r}{d} = \varepsilon . \qquad (12)$$

If $0 < \varepsilon < 1$ then equation (12) describes an ellipse.
If $\varepsilon = 1$ then equation (12) describes a parabola.
If $\varepsilon > 1$ then (12) is the equation of a hyperbola.
Thus, the curves of the second order can be classified by the value of the eccentricity.
From the algebraic point of view, the equation

$$a_1 x^2 + b_1 x + a_2 y^2 + b_2 y + a_3 xy + c = 0$$

describes a curve of the second order in the x,y–plane, provided that at least one of the leading coefficients is non-zero.
The presence of the term xy means that the axes of symmetry of the curve are rotated with respect to the coordinate axes.
The linear term x (or y) means that the center (or vertex) of the curve is shifted along the corresponding axis.

Examples:

1) The equation

$$xy = \text{const}$$

describes a hyperbola, whose axes of symmetry are rotated on the angle $45°$ with respect to the coordinate axes.

2) If $c \to 0$, then a hyperbola

$$x^2 - y^2 = c$$

collapses to the pair of the lines $y = \pm x$.

3) The equation

$$x^2 + 2y^2 = -1$$

has no solutions and corresponds to an imaginary ellipse.