Discrete Mathematics and Logic
Lecture 5

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Quantifiers

Recall

The universal quantifier	The existential quantifier
$\forall x\, P(x)$	$\exists x\, P(x)$

$\forall\, n \leq 5\ \exists! m\ \ P(n, m)$

Quantifiers

Special cases

$$\forall n\,(R(n))\ P(n)$$

Examples

- $\forall n \leq 5\ P(n)$
- $\forall \epsilon > 0\ P(\epsilon)$
- $\forall i\,(1 \leq i \leq 10)\ P(i) = \forall i \in \{1, \ldots, 10\}\ P(i)$

Quantifiers

Special cases

$$\forall n\,(R(n))\ P(n) = \forall n\,(R(n) \to P(n))$$

Examples

- $\forall n \leq 5\ P(n) = \forall n\,(n \leq 5 \to P(n))$
- $\forall \epsilon > 0\ P(\epsilon) = \forall \epsilon\,(\epsilon > 0 \to P(\epsilon))$
- $\forall i \in \{1, \ldots, 10\}\ P(i) = \forall i\,(i \in \{1, \ldots, 10\} \to P(i))$

Quantifiers

Special cases

$$\exists n\,(R(n))\ P(n)$$

Examples

- $\exists n \geq 7\ P(n)$
- $\exists \delta > 0\ P(\delta)$
- $\exists i \in \mathbb{N}\ P(i)$

Quantifiers

Special cases

$$\exists n\,(R(n))\;\; P(n) = \exists n\,(R(n)\,\&\,P(n))$$

Examples

- $\exists n \geq 7\;\; P(n) = \exists n\,(n \geq 7\,\&\,P(n))$
- $\exists \delta < 2^{100}\;\; P(\delta) = \exists \delta\,(\delta < 2^{100}\,\&\,P(\delta))$
- $\exists i \in \mathbb{N}\;\; P(i) = \exists i\,(i \in \mathbb{N}\,\&\,P(i))$

Quantifiers

Special cases

$$\exists! n\, P(n) \leftrightharpoons \text{"there exists a unique } n \text{ such that } P(n)\text{"}$$

Examples

- $\exists! x > 0\, (x^2 = 16)$
- $\exists! n \in \mathbb{N}\, (n < x \,\&\, x \in \mathbb{R})$

Quantifiers

Special cases

$$\exists! n\, P(n) = \exists n\, [P(n) \,\&\, \forall m\, (P(m) \to n = m)]$$

$$\exists! n\, P(n) = \exists n\, [P(n) \,\&\, \forall m \neq n\ \neg P(m)]$$

Examples

- $\exists! x > 0\, (x^2 = 16) = \exists x\, (x > 0 \,\&\, x^2 = 16 \,\&\, \forall y \neq x\ y^2 \neq 16)$
- $\exists! n \in \mathbb{N}\, (n < x \,\&\, x \in \mathbb{R}) = \exists n\, [n \in \mathbb{N} \,\&\, n < x \,\&\, x \in \mathbb{R} \,\&$
  $\&\, \forall m\, (m \in \mathbb{N} \,\&\, m < x \to m \neq n)]$

Quantifiers

Special cases

$$\exists^R n\, P(n)$$

Examples

- $\exists^2 x \in \mathbb{R}\,(x^2 = 16) \leftrightharpoons$ "there are exactly 2 numbers $x \in \mathbb{R}$ such that $x^2 = 16$"
- $\exists^{\leq 2} x \in \mathbb{R}\,(ax^2 + bx + c = 0) \leftrightharpoons$ "there are at most 2 numbers $x \in \mathbb{R}$ such that $ax^2 + bx + c = 0$"
- $\exists^{\infty} n\,(n$ is a prime number$) \leftrightharpoons$ "there are infinitely many prime numbers"

Quantifiers

Examples

- $\exists^n P(x) =$

$$\exists x_1 \ldots, \exists x_n \, (P(x_1) \,\&\, \ldots \,\&\, P(x_n) \,\&\, \forall i, j \, (i \neq j) \, x_i \neq x_j)$$

- $\exists^{\leq n} P(x) = \exists i \leq n \; \exists^i P(x)$
- $\exists^{\infty} x \, P(x) = \forall n \in \mathbb{N} \; \exists^n P(x)$

Functions

Definitions

A set $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$ is called a function, if

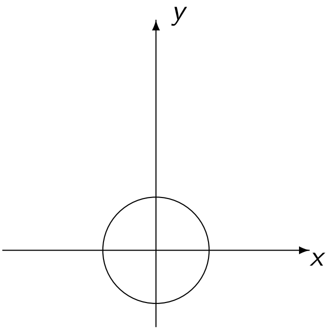
$$\forall x \, \exists^{\leq 1} y \, (x, y) \in X \times Y.$$

Define the function f as the following

$$f(x) = y \Leftrightarrow (x, y) \in X \times Y.$$

Write $f : X \to Y$.

Functions



$$x^2 + y^2 = 1$$

Functions

Definitions

A set $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$ is called a function, if

$$\forall x \, \exists^{\leq 1} y \, (x, y) \in X \times Y.$$

X is called the domain of f,

Y is called the range (or co-domain) of f.

Functions

Definitions

A set $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$ is called a function, if

$$\forall x\, \exists^{\leq 1} y\, (x, y) \in X \times Y.$$

The set $\{x \mid f(x) \text{ is defined}\}$ is called the support.

The set $f(X) = \{f(x) \mid x \in X\}$ is called the image.

Functions

Definitions

Let $f : X \to Y$ be a function.

- f is called total, if the support equals to the domain, i.e.,

$$\forall x\, \exists! y\, (x, y) \in X \times Y,$$

- f is called surjective, if the range equals to the image, i.e.,

$$\forall y \in Y\ \exists x \in X\ \ f(x) = y,$$

- f is called injective, if

$$\forall x_1, x_2\, (x_1 \neq x_2 \to f(x_1) \neq f(x_2)),$$

- f is called bijection, if it is a total surjective injection.

Functions

Definitions

Let $f : X \to Y$ be a function.

- f is called total, if the support equals to the domain, i.e.,

$$\forall x \, \exists! y \, (x, y) \in X \times Y,$$

$f : \mathbb{R} \to \mathbb{R}$

Examples

- $f(x) = 1 - x$
- $f(x) = \frac{1}{x}$

Total?

The support? The image?

Functions

Definitions

- f is called surjective, if the range equals to the image, i.e.,

$$\forall y \in Y \ \exists x \in X \ f(x) = y,$$

- f is called injective, if

$$\forall x_1, x_2 \, (x_1 \neq x_2 \to f(x_1) \neq f(x_2)),$$

$f : \mathbb{R} \to \mathbb{R}$

Examples

- $y = x^2$
- $y = 1 - x^3$

Functions

Definitions

- f is called bijection, if it is a total surjective injection.

$f : \mathbb{R} \to \mathbb{R}$

Examples

- $y = 1 - x$
- $y = \sqrt{x^3}$

Cardinality

Definition

$\forall A, B$ (A and B has the same cardinality $\leftrightarrow |A| = |B|$), if there is a bijection $f : A \to B$ (total surjective injection).

Example

There exists a bijection $f : \mathbb{Z} \to \mathbb{N}$.

Cardinality

Proposition

If $f : A \to B$ is a bijection then $f^{-1} : B \to A$ is also a bijection.

Where $f^{-1}(y) = x \Leftrightarrow f(x) = y$

Exercise

Prove this.

Functions

Definitions

A set $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$ is called a function, if

$$\forall x \, \exists^{\leq 1} y \, (x, y) \in X \times Y.$$

$X = X_1 \times \cdots \times X_n$

$\bar{x} = (x_1, \ldots, x_n)$

$$f(x_1, \ldots, x_n) = y \Leftrightarrow (x_1, \ldots, x_n, y) \in X_1 \times \cdots \times X_n \times Y.$$

Write $f : X_1 \times \cdots \times X_n \to Y$.

Relations

Definition

A function $f : X_1 \times \cdots \times X_n \to \{T, F\} = \{1, 0\}$ is called a relation.

Example

$x \leq y \Leftrightarrow \leq (x, y) = 1$

Relations

Definition

A function $f : X_1 \times \cdots \times X_n \to \{T, F\} = \{1, 0\}$ is called a relation.

Example

$x \leq y \Leftrightarrow \leq (x, y) = 1 \Leftrightarrow \ (x, y) \in \leq$

Relations

Definition

A function $f : X_1 \times \cdots \times X_n \to \{T, F\} = \{1, 0\}$ is called a relation.

Example

$x \leq y \Leftrightarrow \leq (x, y) = 1 \Leftrightarrow \ (x, y) \in \leq$

Definition

Any set $R \subseteq X_1 \times \cdots \times X_n$ is called a relation on $X_1, \ldots, X_n$.

Thank you for your attention!