Tutorial 1: Vectors. Definition and Operations.

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Course of Essentials of Analytical Geometry and Linear Algebra I

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Content

Points and Vectors

- Three-Dimensional Coordinate Systems (Geometry of Space)
  - Geometric Interpretations of Equations
  - Geometric Interpretations of Inequalities and Equations
  - Inequalities to Describe Sets of Points
  - Distance
  - Spheres
- Vectors
  - Definition
  - Operations
  - Vector spaces

Example

Find the center and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

Recall

The Standard Equation for the Sphere of Radius a and Center (x_0, y_0, z_0)

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$

Example

Find the center and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

Solution

- Complete the squares on the *x*-, *y*-, and *z*- terms as necessary and write each quadratic as a squared linear expression.
- From the equation in standard form, read off the center and radius.

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$(x^2 + 3x) + y^2 + (z^2 - 4z) = -1$$

$$\left(x^2 + 3x + \left(\frac{3}{2}\right)^2\right) + y^2 + \left(z^2 - 4z + \left(\frac{-4}{2}\right)^2\right) = -1 + \left(\frac{3}{2}\right)^2 + \left(\frac{-4}{2}\right)^2$$

$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z - 2)^2 = -1 + \frac{9}{4} + 4 = \frac{21}{4}.$$

The center is (-3/2, 0, 2). The radius is $\sqrt{21}/2$.

Problem 1

- Find a formula for the distance from the point $P(x, y, z)$ to the

a. x-axis. **b.** y-axis. **c.** z-axis.

Solution

a. The distance from point $P(x, y, z)$ to the x-axis .... [x-axis means $(x,0,0)$]

$$\sqrt{(x-x)^2+(y-0)^2+(z-0)^2}=\sqrt{y^2+z^2}$$

b. The distance from point $P(x, y, z)$ to the y-axis .... [y-axis means $(0,y,0)$]

$$\sqrt{(x-0)^2+(y-y)^2+(z-0)^2}=\sqrt{x^2+z^2}$$

c. The distance from point $P(x, y, z)$ to the z-axis .... [z-axis means $(0,0,z)$]

$$\sqrt{(x-0)^2+(y-0)^2+(z-z)^2}=\sqrt{x^2+y^2}$$

Problem 2

- Find a formula for the closet distance from the point *P(x, y, z)* to the

a. *xy*-plane. **b.** *yz*-plane. **c.** *xz*-plane.

Solution

The distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

a. The distance from point *P*(*x*, *y*, *z*) to the *xy*-plane is the distance from *P*(*x*, *y*, *z*) to (*x*,*y*,0) since (*x*,*y*,0) is the point on the *xy*-plane that is closet to *P*. Therefore by the distance formula we have

$$\sqrt{(x-x)^2 + (y-y)^2 + (z-0)^2} = \sqrt{z^2} = |z|$$

b. The distance from point *P*(*x*, *y*, *z*) to the *yz*-plane is the distance from *P*(*x*, *y*, *z*) to (*0*,*y*,z) since (*0*,*y*,z) is the point on the *xy*-plane that is closet to *P*. Therefore by the distance formula we have

$$\sqrt{(x-0)^2 + (y-y)^2 + (z-z)^2} = \sqrt{x^2} = |x|$$

c. The distance from point *P*(*x*, *y*, *z*) to the *xz*-plane is the distance from *P*(*x*, *y*, *z*) to (*x*,*0*,z) since (*x*,*0*,z) is the point on the *xy*-plane that is closet to *P*. Therefore by the distance formula we have

$$\sqrt{(x-x)^2 + (y-0)^2 + (z-z)^2} = \sqrt{y^2} = |y|$$

Problem 3

➢ Find the perimeter of the triangle with vertices $A(-1, 2, 1)$, $B(1, -1, 3)$, and $C(3, 4, 5)$.

Solution

To find the perimeter of the triangle, we should calculate the distance of $|AB|$, $|BC|$ and $|CA|$ by using the distance formula

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$\text{Primeter of the triangle} = |AB| + |BC| + |CA|$$

$$|AB| = \sqrt{(1+1)^2 + (-1-2)^2 + (3-1)^2} = \sqrt{17}$$

$$|BC| = \sqrt{(3-1)^2 + (4-1)^2 + (5-3)^2} = \sqrt{33}$$

$$|CA| = \sqrt{(-1-3)^2 + (2-4)^2 + (1-5)^2} = 6$$

$$\text{Primeter of the triangle} = |AB| + |BC| + |CA|$$

$$= \sqrt{17} + \sqrt{33} + 6 \simeq 15.87$$

Problem 4

- Show that the point $P(3, 1, 2)$ is equidistant from the points $A(2, -1, 3)$ and $B(4, 3, 1)$.

Hint:

i.e. show that the distance from $P(3,1,2)$ to $A(2,-1,3)$ is equal to the distance from $P(3,1,2)$ to $B(4,3,1)$.

Solution

The distance formula is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The distances are

$$|PA| = \sqrt{(3-2)^2 + (1+1)^2 + (2-3)^2} = \sqrt{6}$$

$$|PB| = \sqrt{(3-4)^2 + (1-3)^2 + (2-1)^2} = \sqrt{6}$$

Hence, the point P is equidistant from the points A and B.

Problem 5

- Find an equation for the set of all points equidistant from the planes y = 3 and y = -1.

Solution

It asks to find an equation for the set of all point. whereas, the distance between this equation and the first plane equal the distance between the same equation and the second plane... (use the distance formula)

$$\sqrt{(x-x)^2+(y-(-1))^2+(z-z)^2}=\sqrt{(x-x)^2+(y-3)^2+(z-z)^2}$$

$$\sqrt{0+(y+1)^2+0}=\sqrt{0+(y-3)^2+0}$$

Square both sides of an equation and solve

$$(y+1)^2=(y-3)^2$$

$$y^2+2y+1=y^2-6y+9$$

$$8y=8\rightarrow y=1$$

Problem 6

- Find an equation for the set of all points equidistant from the point (0, 0, 2) and the *xy*-plane.

Solution

It asks to find an equation for the set of all point. whereas, the distance between this equation and the point equal the distance between the same equation and *xy*-plane (*xy*-plane means $z = 0$).... (use the distance formula)

$$\sqrt{(x-0)^2+(y-0)^2+(z-2)^2} = \sqrt{(x-x)^2+(y-y)^2+(z-0)^2}$$

$$\sqrt{x^2+y^2+(z-2)^2} = \sqrt{0+0+z^2}$$

square both sides of an equation and solve

$$x^2+y^2+(z-2)^2 = z^2$$

$$x^2+y^2+z^2-4z+4 = z^2$$

$$x^2+y^2-4z+4=0$$

$$z = \frac{x^2}{4}+\frac{y^2}{4}+1$$

Problem 7

- Find the point on the sphere $x^2 + (y-3)^2 + (z+5)^2 = 4$ nearest

  a. the *xy*-plane.

  b. the point (0, 7, -5).

Solution

a. The distance from the *xy* plane is the positive value of *Z*. The *xy* plane can also be represented by *Z*=0. (If this is hard to understand think back to 2D coordinate system when every *Z* is essentially 0.) So we are looking for the point on the sphere with the *Z* value as close to *0* as possible. Since the radius is 2, any point 2 units from the center point of (0, 3, -5) would lie on the sphere. so if we "use“ those 2 units to " travel" towards *xy* plane we would get the closest point. At this point you can add 2 to the *Z* value of the center point for final answer. (if the sphere had been above the *xy* plane then you would subtract 2 from the *Z* value)

$$\boxed{(0,\ 3,\ -3)}$$

b. This one is a little harder to grasp, although it uses very similar reasoning to part A. As you may have noticed the *X* and *Z* of the center point and the out point are the same. so again we want to "move" 2 units towards the point from the center point of the sphere. which would be moving in the *Y* direction since you can not get any closer in the *X* and *Z* direction so you simply add 2 to the *Y* value of the center.

$$\boxed{(0,\ 5,\ -5)}$$

Problem 8

- Find the point equidistant from the points (0, 0, 0), (0, 4, 0), (3, 0, 0), and (2, 2, -3).

Solution

You are looking for the point equadistant to 4 points so a good place to start is the distance formula. All the equations are based off of the distance formula, since you are equating distance formula you can ignore all the square roots.(Think of this as squaring all sides). We will be using these equations to solve for the *x y* and *z* of the point one at a time.

$$x^2+y^2+z^2=x^2+(y-4)^2+z=(x-3)^2+y^2+z^2=(x-2)^2+(y-2)^2+(z+3)^2$$

Using the distance formula from the point (0,0,0) will most likely be the easiest, so in this step; I set the other 3 point's distance formula equal to that of the origin to the unknown point.

$$x^2+y^2+z^2=x^2+(y-4)^2+z$$

$$x^2+y^2+z^2=(x-3)^2+y^2+z^2$$

$$x^2+y^2+z^2=(x-2)^2+(y-2)^2+(z+3)^2$$

Taking the first two equations from part 2 and doing algebra to find both the values of *x* and *y* for the unknown point.

$$y^2=(y-4)^2\rightarrow 8y=16\rightarrow y=2$$

$$x^2=(x-3)^2\rightarrow 6x=9\rightarrow x=1.5$$

Taking the third equation from part two and the *x* and *y* values from part three to find the value of *z* using algebra.

$$x^2+y^2+z^2=(x-2)^2+(y-2)^2+(z+3)^2\xrightarrow[y=2]{x=1.5}0.25+0+6z+9=2.25+4\rightarrow 6z=-3\rightarrow z=-0.5$$

Problem 9 (1)

- Write inequalities to describe the sets in the following exercises

1) The slab bounded by the planes $z = 0$ and $z = 1$ (planes included).

$$\boxed{0 \leq z \leq 1}$$

2) The solid cube in the first octant bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$.

$$\boxed{0 \leq x \leq 2} \qquad \boxed{0 \leq y \leq 2} \qquad \boxed{0 \leq z \leq 2}$$

3) The half-space consisting of the points on and below the xy-plane.

$$\boxed{z \leq 0}$$

Problem 9 (2)

4) The upper hemisphere of the sphere of radius 1 centered at the origin.

The equation for a standard sphere centered at the origin with a radius of one:

$$x^2 + y^2 + z^2 = 1$$

The upper half of the sphere would include all of the area above the xy-plane which can be established by setting a parameter for *z*.

$$\boxed{x^2 + y^2 + z^2 = 1,\ z \geq 0}$$

5) The **(a)** interior and **(b)** exterior of the sphere of radius 1 centered at the point (1, 1, 1).

a. a sphere of radius 1 and centered around (1.1.1). to know that is less than you can try plugging in a point that you know is within or you know is outside the sphere. plugging in the center point of (1.1.1) gives the answer 0 which is less than one so you know that if you want the interior you want to have the less than sign so that 0<1 is true.

$$\boxed{(x-1)^2 + (y-1)^2 + (z-1)^2 < 1}$$

b. You know that it is the opposite sign as part a)

$$\boxed{(x-1)^2 + (y-1)^2 + (z-1)^2 > 1}$$

Problem 9 (3)

6) The closed region bounded by the spheres of radius 1 and radius 2 centered at the origin.

The smaller sphere's equation is

$$x^2 + y^2 + z^2 = 1$$

The larger sphere's equation is

$$x^2 + y^2 + z^2 = 4$$

and a double inequality will ensure that the values stay between 1 and 4 including both 1 and 4.

$$\boxed{1 \le x^2 + y^2 + z^2 \le 4}$$

Section II

Definition 1

The set of all ordered pairs of real numbers is called the *vector* space $\mathbb{R}^2$ and the ordered pairs vectors (or coordinate vectors) if, for any $(p_1, p_2), (q_1, q_2)$ and any scalar c, we have the operations of vector addition and multiplication of vectors by scalars defined by

$$(p_1, p_2) + (q_1, q_2) = (p_1 + q_1, p_2 + q_2),$$

and

$$c(p_1, p_2) = (c\, p_1, c\, p_2).$$

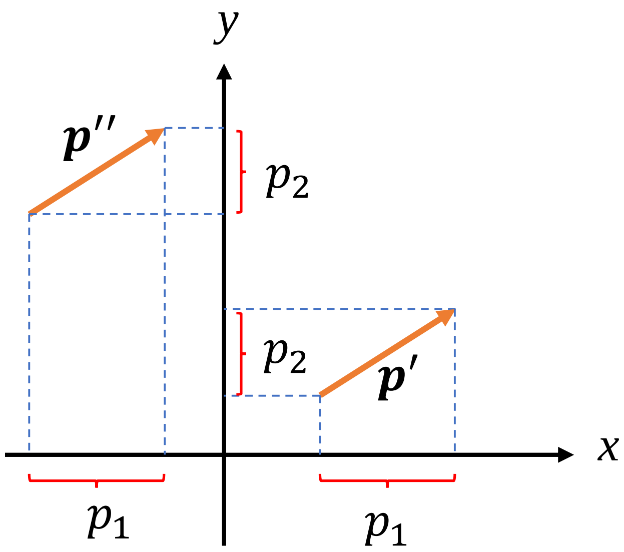
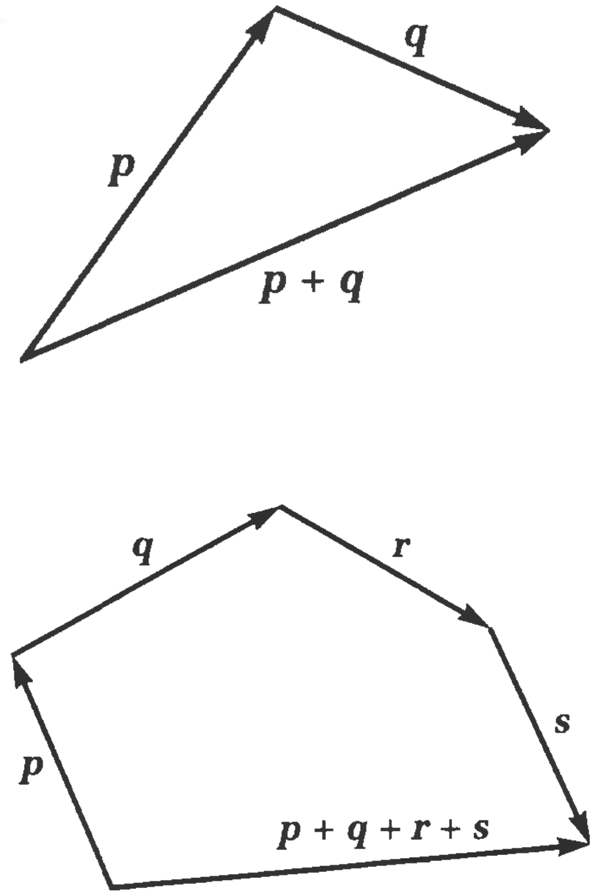
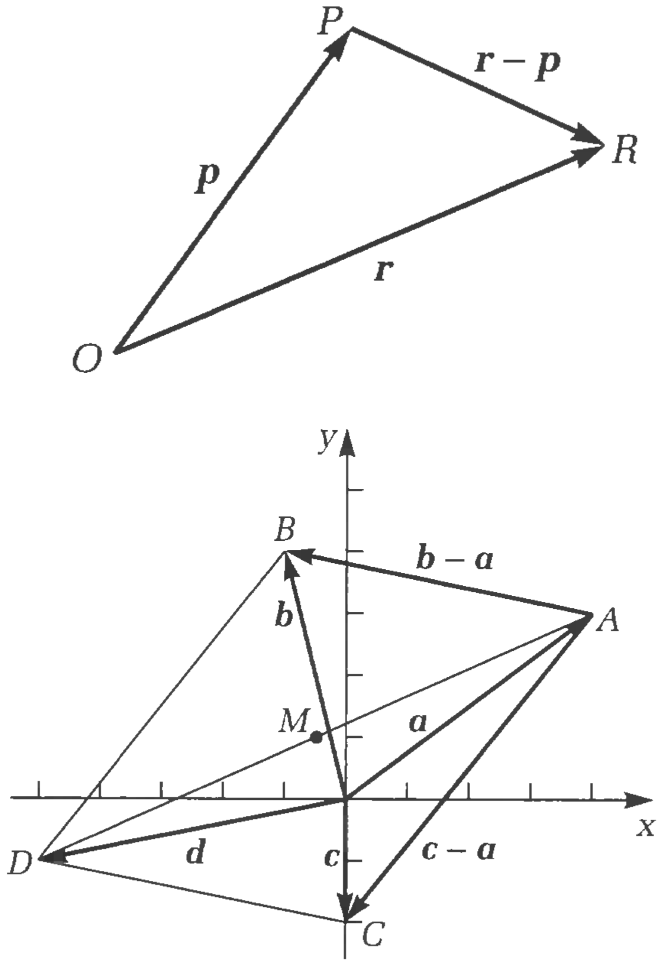
The scalars p_1 and p_2 are called the *components* of the vector $\boldsymbol{p} = (p_1, p_2)$.

Theorem 1

- For all vectors $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$ in $\mathbb{R}^2$ and all scalars a, b we have:
  1. $\boldsymbol{p} + \boldsymbol{q} = \boldsymbol{q} + \boldsymbol{p}$ (commutativity of addition)
  2. $(\boldsymbol{p} + \boldsymbol{q}) + \boldsymbol{r} = \boldsymbol{p} + (\boldsymbol{q} + \boldsymbol{r})$ (associativity of addition)
  3. There is a vector $\mathbf{0}$ such that $\boldsymbol{p} + \mathbf{0} = \boldsymbol{p}$ (existence of zero vector)
  4. $1\boldsymbol{p} = \boldsymbol{p}$ (rule of multiplication by 1)
  5. $a(b\boldsymbol{p}) = (ab)\boldsymbol{p}$ (associativity of multiplication by scalars)
  6. $(a + b)\boldsymbol{p} = a\boldsymbol{p} + b\boldsymbol{p}$ (first distributive law)
  7. $a(\boldsymbol{p} + \boldsymbol{q}) = a\boldsymbol{p} + a\boldsymbol{q}$ (second distributive law)

- For all vectors $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{x}$ in $\mathbb{R}^2$ and all scalars c and d we have
  1. $0\boldsymbol{p} = \mathbf{0}$
  2. $c\mathbf{0} = \mathbf{0}$
  3. $\boldsymbol{p} + \boldsymbol{x} = \boldsymbol{q}$ if and only if $\boldsymbol{x} = \boldsymbol{q} - \boldsymbol{p}$
  4. If $c\boldsymbol{p} = \mathbf{0}$ then either $c = 0$ or $\boldsymbol{p} = 0$ or both
  5. $(-c)\boldsymbol{p} = c(-\boldsymbol{p}) = -(c\boldsymbol{p})$
  6. $c(\boldsymbol{p} - \boldsymbol{q}) = c\boldsymbol{p} - c\boldsymbol{q}$
  7. $(c - d)\boldsymbol{p} = c\boldsymbol{p} - d\boldsymbol{p}$

Definition 2

Free vector	Vector addition	Vector subtraction
		

Problem 1

- Given the points P = (1, 2, 3) and Q = (-1, 6, 5), find the midpoint M of the line segment PQ.

Solution

Just as in two dimensions, we can write $\overrightarrow{PQ} = \boldsymbol{q} - \boldsymbol{p} = (-2, 4, 2)$ and the position vector $\boldsymbol{m}$ of the point M as

$$\boldsymbol{m} = \boldsymbol{p} + \frac{1}{2}\overrightarrow{PQ} = (1, 2, 3) + \frac{1}{2}(-2, 4, 2) = (0, 4, 4).$$

Notice that in the problem above we could also have written

$$\boldsymbol{m} = \boldsymbol{p} + \frac{1}{2}(\boldsymbol{q} - \boldsymbol{p}) = \boldsymbol{p} + \frac{1}{2}\boldsymbol{q} - \frac{1}{2}\boldsymbol{p} = \frac{1}{2}(\boldsymbol{p} + \boldsymbol{q}),$$

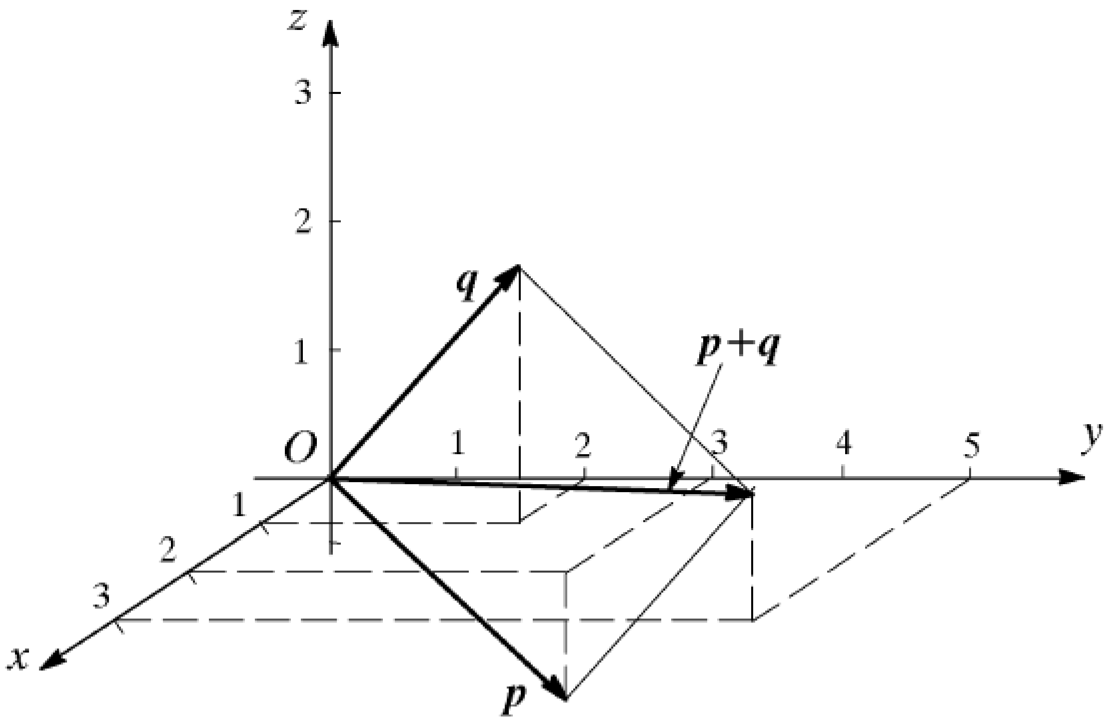
which gives a general formula for the midpoint of a line segment.

Problem 2

- Let $\boldsymbol{p} = (2, 3, -1)$ and $\boldsymbol{q} = (1, 2, 2)$ be two vectors in $\mathbb{R}^3$.
  Find $\boldsymbol{p} + \boldsymbol{q}$, and draw all three vectors from the origin in the *xyz* coordinate system to illustrate the parallelogram law in three dimensions.

Solution

$$\boldsymbol{p} + \boldsymbol{q} = (2,3,-1) + (1,2,2) = (3,5,1)$$



Problem 3

- Given n point masses m_i, at the points with position vectors $\boldsymbol{r}_i$, in either two or three dimensions, their center of mass is defined as the point with position vector

$$\boldsymbol{r} = \frac{1}{M}\sum_{i=1}^{n} m_i\, \boldsymbol{r}_i \text{ where } M = \sum_{i=1}^{n} m_i \text{ is the total mass.}$$

If three mass points are given with
$m_1 = 2, m_2 = 3,\ m_3 = 5,$
$\boldsymbol{r}_1 = (2,-1,4), \boldsymbol{r}_2 = (1,5,-6)$, and $\boldsymbol{r}_3 = (-2,-5,4)$,
then find $\boldsymbol{r}$.

Solution

$$M = \sum_{i=1}^{3} m_i = 2+3+5=10$$

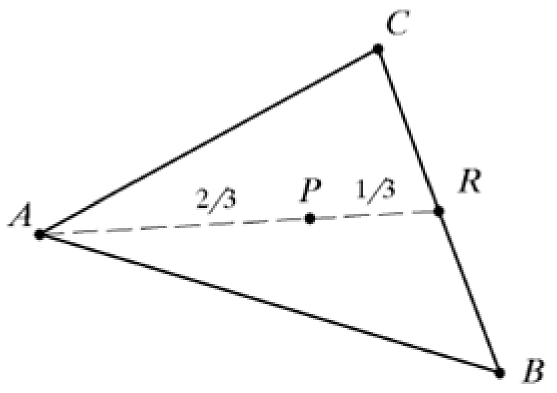
$$\mathbf{r} = \frac{1}{M}\sum_{i=1}^{3} m_i \mathbf{r}_i = \frac{1}{10}[2(2,-1,4)+3(1,5,-6)+5(-2,-5,4)] = (-\frac{3}{10},-\frac{6}{5},1)$$

- Let $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ be the position vectors of the vertices of a triangle. The point given by $\boldsymbol{p} = \frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$ is the triangle's centroid.
  - Show that it lies one third of the way from the midpoint of any side to the opposite vertex on the line joining these points. (Such a line is called a *median* of the triangle.)
  - Draw an illustration.

Solution

The center R of the side BC has position vector $\boldsymbol{r} = \frac{1}{2}(\boldsymbol{b} + \boldsymbol{c})$ and the vector from R to A is $\boldsymbol{a} - \boldsymbol{r}$. Thus the point P, 1/3 of the way from R to A, has position vector

$$\mathbf{p} = \frac{1}{2}(\mathbf{b}+\mathbf{c}) + \frac{1}{3}(\mathbf{a}-\mathbf{r}) = \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c} + \frac{1}{3}\mathbf{a} - \frac{1}{3}\frac{1}{2}(\mathbf{b}+\mathbf{c}) = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$$



Problem 5

➢ In the following exercises, write a system of equations that is equivalent to the given vector equation.

$$x_1\begin{bmatrix}3\\-2\\8\end{bmatrix}+x_2\begin{bmatrix}5\\0\\-9\end{bmatrix}=\begin{bmatrix}2\\-3\\8\end{bmatrix},\qquad x_1\begin{bmatrix}3\\-2\end{bmatrix}+x_2\begin{bmatrix}7\\3\end{bmatrix}+x_3\begin{bmatrix}-2\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.$$

Solution

$$x_1\begin{bmatrix}3\\-2\\8\end{bmatrix}+x_2\begin{bmatrix}5\\0\\-9\end{bmatrix}=\begin{bmatrix}2\\-3\\8\end{bmatrix},$$

$$\begin{bmatrix}3x_1\\-2x_1\\8x_1\end{bmatrix}+\begin{bmatrix}5x_2\\0\\-9x_2\end{bmatrix}=\begin{bmatrix}2\\-3\\8\end{bmatrix},\quad \begin{bmatrix}3x_1+5x_2\\-2x_1\\8x_1-9x_2\end{bmatrix}=\begin{bmatrix}2\\-3\\8\end{bmatrix},$$

$$\begin{aligned}3x_1+5x_2&=2\\-2x_1&=-3\\8x_1-9x_2&=8\end{aligned}$$

$$x_1\begin{bmatrix}3\\-2\end{bmatrix}+x_2\begin{bmatrix}7\\3\end{bmatrix}+x_3\begin{bmatrix}-2\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},$$

$$\begin{bmatrix}3x_1\\-2x_1\end{bmatrix}+\begin{bmatrix}7x_2\\3x_2\end{bmatrix}+\begin{bmatrix}-2x_3\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},\quad \begin{bmatrix}3x_1+7x_2+-2x_3\\-2x_1+3x_2+x_3\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix},$$

$$\begin{aligned}3x_1+7x_2+-2x_3&=0\\-2x_1+3x_2+x_3&=0\end{aligned}$$

Usually the intermediate steps are not displayed.

- In the following exercises, write a vector equation that is equivalent to the given system of equations.

$$\begin{cases} x_1 + 5x_2 = 0 \\ 4x_1 + 6x_2 - x_3 = 0 \\ -x_1 + 3x_2 - 8x_3 = 0 \end{cases}, \qquad \begin{cases} 3x_1 - 2x_2 + 4x_3 = 3 \\ -2x_1 - 7x_2 + 5x_3 = 1 \\ 5x_1 + 4x_2 - 3x_3 = 2 \end{cases}$$

Solution

$$\begin{cases} x_2 + 5x_3 = 0 \\ 4x_1 + 6x_2 - x_3 = 0 \\ -x_1 + 3x_2 - 8x_3 = 0 \end{cases},$$

$$\begin{bmatrix} x_2 + 5x_3 \\ 4x_1 + 6x_2 - x_3 \\ -x_1 + 3x_2 - 8x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 4x_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 6x_2 \\ 3x_2 \end{bmatrix} + \begin{bmatrix} 5x_3 \\ -x_3 \\ -8x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow$$

$$x_1 \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -1 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{cases} 3x_1 - 2x_2 + 4x_3 = 3 \\ -2x_1 - 7x_2 + 5x_3 = 1, \\ 5x_1 + 4x_2 - 3x_3 = 2 \end{cases}$$

$$\begin{bmatrix} 3x_1 - 2x_2 + 4x_3 \\ -2x_1 - 7x_2 + 5x_3 \\ 5x_1 + 4x_2 - 3x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 3x_1 \\ -2x_1 \\ 5x_1 \end{bmatrix} + \begin{bmatrix} -2x_2 \\ -7x_2 \\ 4x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ 5x_3 \\ -3x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \rightarrow$$

$$x_1 \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -7 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

Usually the intermediate steps are not displayed.

Definition 2

A set V is called a (real) vector space and its elements vectors if V is not empty and with each $\boldsymbol{p}, \boldsymbol{q} \in V$ and each real number c a unique sum $\boldsymbol{p} + \boldsymbol{q} \in V$ and a unique product $c\boldsymbol{p} \in V$ are associates satisfying the rules below:

1. $\boldsymbol{p} + \boldsymbol{q} = \boldsymbol{q} + \boldsymbol{p}$ (commutativity of addition)
2. $(\boldsymbol{p} + \boldsymbol{q}) + \boldsymbol{r} = \boldsymbol{p} + (\boldsymbol{q} + \boldsymbol{r})$ (associativity of addition)
3. There is a vector $\mathbf{0}$ such that $\boldsymbol{p} + \mathbf{0} = \boldsymbol{p}$ (existence of zero vector)
4. $1\boldsymbol{p} = \boldsymbol{p}$ (rule of multiplication by 1)
5. $a(b\boldsymbol{p}) = (ab)\boldsymbol{p}$ (associativity of multiplication by scalars)
6. $(a + b)\boldsymbol{p} = a\boldsymbol{p} + b\boldsymbol{p}$ (first distributive law)
7. $a(\boldsymbol{p} + \boldsymbol{q}) = a\boldsymbol{p} + a\boldsymbol{q}$ (second distributive law)

Problem 7

- Is the set of all polynomials of degree two and the zero polynomial a vector space?

Solution

The set of all polynomials of degree two and the zero polynomial is not a vector space.

For example, it is not closed under addition: the sum of $x^2 + x - 1$ and $-x^2$ is $x - 1$, which is neither a polynomial of degree two nor the zero polynomial.

Problem 8

- Is the set of all solutions (x, y) of the equation $2x + 3y = 1$ a vector space?

Solution

The set of all solutions (x, y) of the equation $2x + 3y = 1$ is not a vector space.

For example, $(-1, 1)$ and $(-4, 3)$ are solutions, but their sum $(-1, 1) + (-4, 3) = (-5, 4)$ is not a solution since $2(-5) + 3(4) = 2 \neq 1$.

Thus the set is not closed under addition.

Problem 9

- Is the set of all ordered pairs of real numbers with addition and multiplication by scalars defined by

  $$(p_1, p_2) + (q_1, q_2) = (p_1+q_2, p_1+q_2)$$

  and

  $$c(p_1, p_2) = (cp_1, cp_2).$$

  a vector space?

Solution

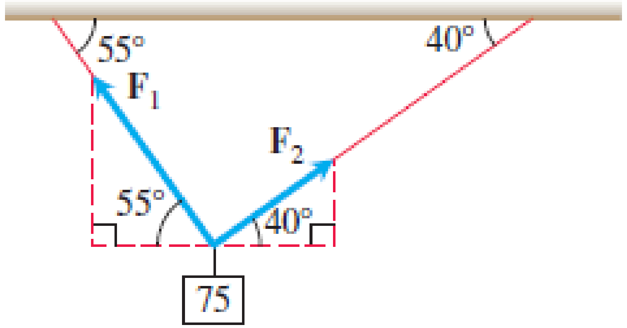
This set is not a vector space, because addition is not <u>commutative</u>:

For example, let $(p_1, p_2) = (1, 2)$ and $(q_1, q_2) = (1, 3)$. Then, by the given addition rule, we have $(1, 2) + (1, 3) = (1 + 3, 2 + 1) = (4, 3)$, but $(1, 3) + (1, 2) = (1 + 2, 3 + 1) = (3, 4)$.

Practice Problems

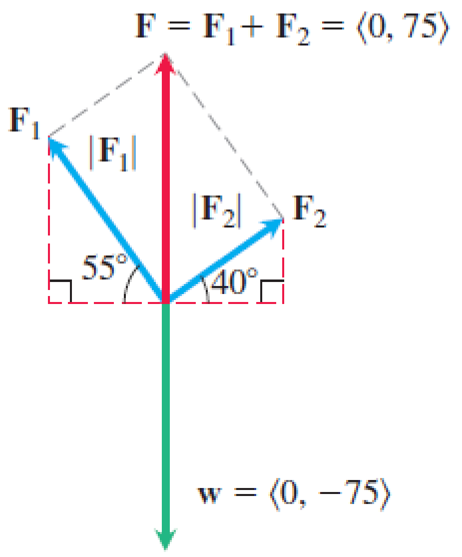
- A 75-N weight is suspended by two wires, as shown in the figure.

  Find the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting in both wires.



Solution

The force vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ have magnitudes $|\mathbf{F}_1|$ and $|\mathbf{F}_2|$ and components that are measured in newtons. The resultant force is the sum $\mathbf{F}_1 + \mathbf{F}_2$ and must be equal in magnitude and acting in the opposite (or upward) direction to the weight vector $\mathbf{w}$. It follows from the figure that



$$\mathbf{F}_1 = \langle -|\mathbf{F}_1|\cos 55°, \ |\mathbf{F}_1|\sin 55° \rangle \quad \text{and} \quad \mathbf{F}_2 = \langle |\mathbf{F}_2|\cos 40°, \ |\mathbf{F}_2|\sin 40° \rangle.$$

$$-|\mathbf{F}_1|\cos 55° + |\mathbf{F}_2|\cos 40° = 0$$
$$|\mathbf{F}_1|\sin 55° + |\mathbf{F}_2|\sin 40° = 75.$$

$$|\mathbf{F}_2| = \frac{|\mathbf{F}_1|\cos 55°}{\cos 40°} \quad \text{and} \quad |\mathbf{F}_1|\sin 55° + \frac{|\mathbf{F}_1|\cos 55°}{\cos 40°}\sin 40° = 75.$$

$$|\mathbf{F}_1| = \frac{75}{\sin 55° + \cos 55° \tan 40°} \approx 57.67 \text{ N},$$

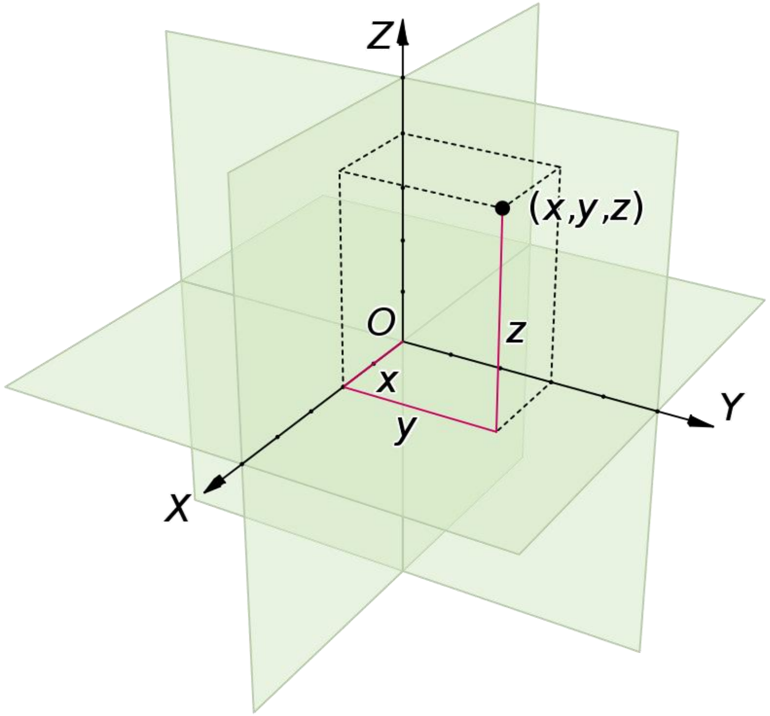
$$|\mathbf{F}_2| = \frac{75\cos 55°}{\sin 55°\cos 40° + \cos 55°\sin 40°} = \frac{75\cos 55°}{\sin(55° + 40°)} \approx 43.18 \text{ N}.$$

The force vectors are then $\mathbf{F}_1 = \langle -33.08, 47.24 \rangle$ and $\mathbf{F}_2 = \langle 33.08, 27.76 \rangle$.

Recall

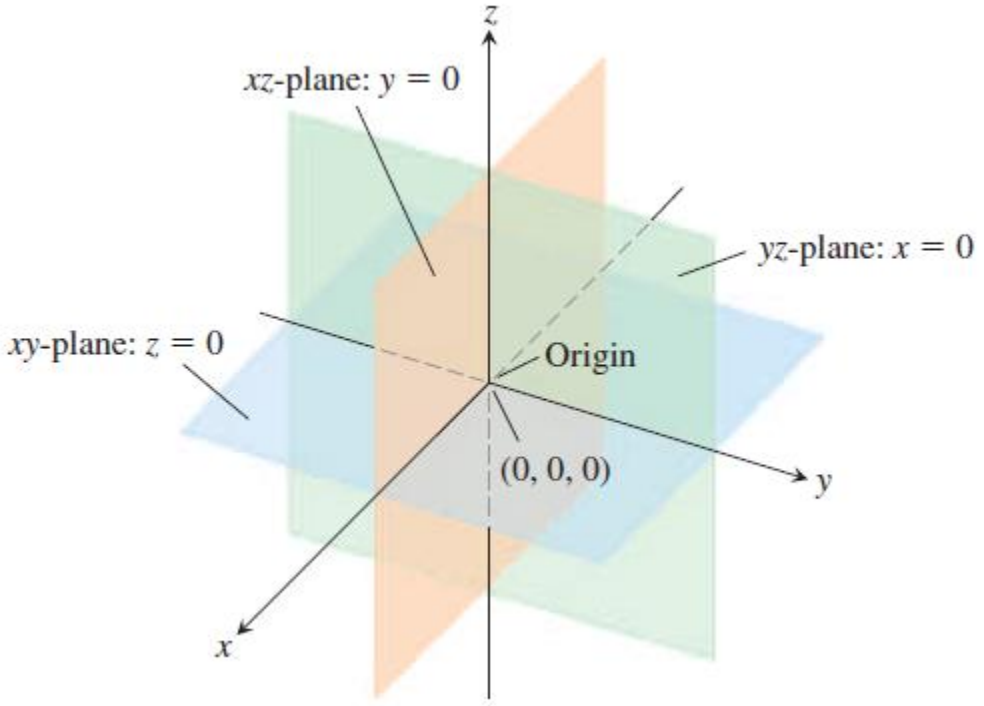
The Cartesian coordinate system

- The Cartesian coordinates (x, y, z) of a point P in space are the values at which the planes through P perpendicular to the axes cut the axes. Cartesian coordinates for space are also called **rectangular coordinates** because the axes that define them meet at right angles.
- Points on the x-axis have y- and z-coordinates equal to zero. That is, they have coordinates of the form $(x, 0, 0)$.
- Similarly, points on the y-axis have coordinates of the form $(0, y, 0)$, and points on the z-axis have coordinates of the form $(0, 0, z)$.



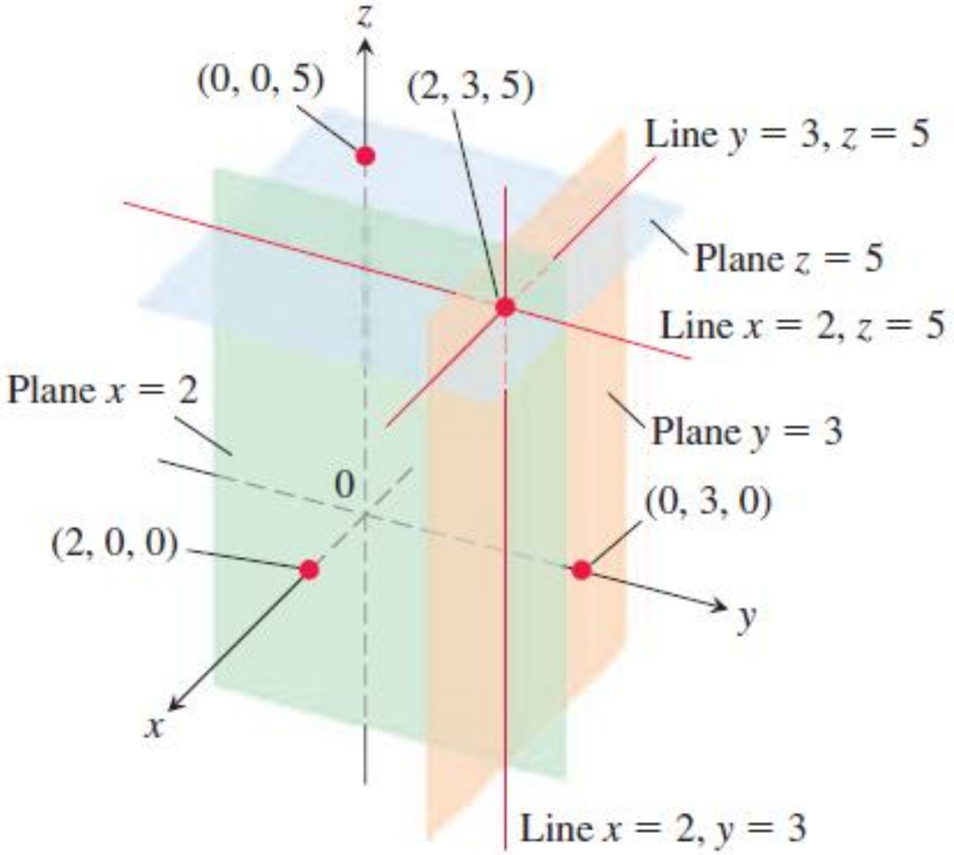
- The planes determined by the coordinates axes are the ***xy*-plane**, whose standard equation is $z = 0$; the ***yz*-plane**, whose standard equation is $x = 0$; and the ***xz*-plane**, whose standard equation is $y = 0$. They meet at the **origin** (0, 0, 0). The origin is also identified by simply 0 or sometimes the letter O.

- The three **coordinate planes** $x = 0$, $y = 0$, and $z = 0$ divide space into eight cells called **octants**. The octant in which the point coordinates are all positive is called the **first octant**; there is no convention for numbering the other seven octants.



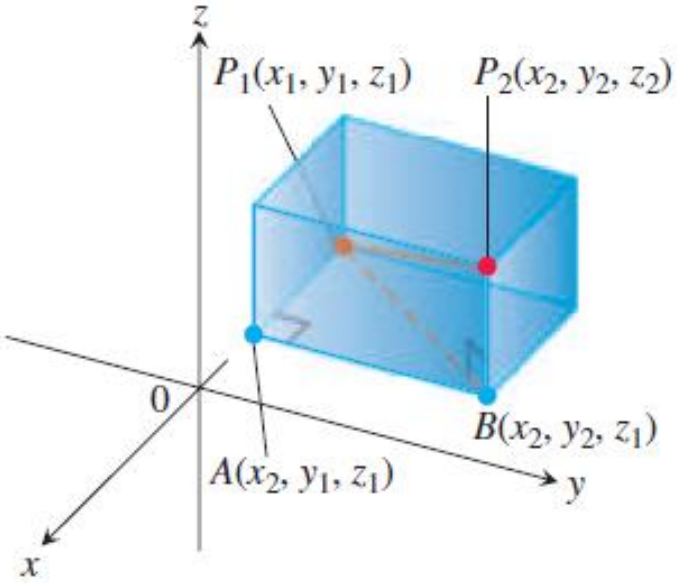
Example

Figure shows the planes $x = 2$, $y = 3$, and $z = 5$, together with their intersection point (2, 3, 5).



- The planes $x = 2$ and $y = 3$ in Figure intersect in a line parallel to the z-axis. This line is described by the *pair* of equations $x = 2$, $y = 3$. A point (x, y, z) lies on the line if and only if $x = 2$ and $y = 3$.
- Similarly, the line of intersection of the planes $y = 3$ and $z = 5$ is described by the equation pair $y = 3$, $z = 5$. This line runs parallel to the x-axis.
- The line of intersection of the planes $x = 2$ and $z = 5$, parallel to the y-axis, is described by the equation pair $x = 2$, $z = 5$.

Distance and Spheres in Space

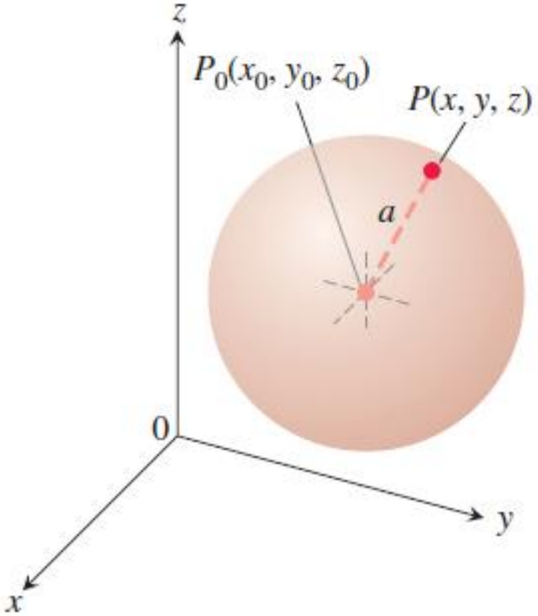


- The Distance Between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example The distance between $P_1(2, 1, 5)$ and $P_2(-2, 3, 0)$ is

$$\begin{aligned} |P_1P_2| &= \sqrt{(-2-2)^2 + (3-1)^2 + (0-5)^2} \\ &= \sqrt{16 + 4 + 25} \\ &= \sqrt{45} \approx 6.708. \end{aligned}$$



- The Standard Equation for the Sphere of Radius a and Center (x_0, y_0, z_0)

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$

Example Geometric interpretations

$x^2 + y^2 + z^2 < 4 \longrightarrow$ The interior of the sphere $x^2 + y^2 + z^2 = 4$