

Mathematical Analysis. Assignment 6.

Using Derivatives for Exploring Functions

1. Find the asymptotes of the following functions:

(a) $y = \sqrt{x^4 + x^3} - \sqrt{x^4 - x^3}$;

(b) $y = x \tanh x$;

(c) $y = \ln(1 + e^x)$;

(d) $y = \frac{1}{\operatorname{arccot} \frac{1}{x}}$;

(e) $y = x + \sqrt{4x^2 + 1}$;

(f) $y = \frac{(x-3)^2}{4x-4}$;

(g) $x = \frac{t^2}{t-1}, y = \frac{t}{t^2-1}$.

Answer: (a) $y = x$; (b) $y = x, y = -x$; (c) $y = 0, y = x$; (d) $x = 0, y = \frac{2}{\pi}$; (e) $y = -x, y = 3x$; (f) $x = 1, y = \frac{x-5}{4}$; (g) $x = \frac{1}{2}, y = 0, y = \frac{2x-3}{4}$.

2. Find the equations of tangent lines and normal lines to the following curves at the indicated points:

(a) $x^3 + y^2 + 2x - 6 = 0, M(-1; 3)$;

(b) $x = \frac{2t-1}{t^2}, y = \frac{3t^2-1}{t^3}, M(1; 2)$.

Answer: (a) $5x + 6y - 13 = 0$ & $6x - 5y + 21 = 0$; (b) $3x - y - 1 = 0$ & $x + 3y - 7 = 0$.

3. Find the angles between the graphs of the functions at their intersection points:

(a) $f(x) = \ln x$ and $g(x) = \frac{x^2}{2e}$;

(b) $f(x) = 4x^2 + 2x - 8$ and $g(x) = x^3 - x + 10$.

Answer: (a) the intersection point is $(\sqrt{e}; \frac{1}{2})$, the angle is 0; (b) two intersection points $(3; 34)$ and $(-2; 4)$, the angles are equal to 0 and $\arctan \frac{25}{153}$ respectively.

4. Let us consider a function $f(x)$ continuous on an open interval $(a; b)$ such that its derivative is positive on this interval but for a finite number of points. Prove that $f(x)$ is strictly increasing on $(a; b)$.

5. Let us consider $f(x) = \begin{cases} x^2 (2 + \cos \frac{1}{x}), & x \neq 0, \\ 0, & x = 0. \end{cases}$. Prove that

(a) $f(x)$ has a strict minimum at $x = 0$;

(b) for any $\delta > 0$ function is not decreasing on $(-\delta; 0)$ and is not increasing on $(0; \delta)$.

6. Two functions are given:

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0, \\ 0, & x = 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} xe^{-\frac{1}{x^2}}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Prove that

(a) $f^{(n)}(0) = g^{(n)}(0) = 0, n \in \mathbb{N}$;

(b) $f(x)$ has a strict minimum at $x = 0$ whereas $x = 0$ yields neither minimum nor maximum for $g(x)$.

7. Find the inflection points of the following functions:

(a) $f(x) = x^4 - 12x^3 + 48x^2$;

(b) $f(x) = x^3 e^{-4x}$.

Answer: (a) $x = 2, x = 4$; (b) $x = e^{-1.5}$.

8. Let $f(x) = \begin{cases} x^3 \left(2 + \cos \frac{1}{x^2}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$ Prove that

(a) there exists a tangent line to the graph of $f(x)$ at the origin;

(b) when x passes through 0, the graph passes from one side of the tangent line to the other side of the tangent line;

(c) $x = 0$ is not an inflection point of $f(x)$.

9. Draw the graphs of the following functions¹:

(a) $y = (x - 1)^2(x + 2)$;

(b) $y = \frac{(x-1)^2}{(x+1)^3}$;

(c) $y = \frac{x^3+2x^2}{(x-1)^2}$;

(d) $y = \frac{(x-5)^3}{(x-7)^2}$;

(e) $y = x - \sqrt{x^2 - 2x}$;

(f) $y = \frac{3x-2}{\sqrt{x^2-1}}$;

(g) $y = \sqrt[3]{x^2(3-x)}$;

(h) $y = \frac{x}{\sqrt[3]{(x-2)^2}}$;

(i) $y = \sqrt[3]{x^2|2-x|}$;

(j) $y = (x^2 - 2)e^{-2x}$;

(k) $y = \frac{\ln x}{\sqrt{x}}$.

Answer:

(a) $(1; 0)$ is a local minimum, $(-1; 4)$ is a local maximum, $(0; 2)$ is an inflection point.

(b) The asymptotes are $x = -1$ and $y = 0$; $(1; 0)$ is a local minimum, $(5; \frac{2}{27})$ is a local maximum; there are two inflection points at $x = 5 \pm 2\sqrt{3}$.

(c) The asymptotes are $y = x + 4$ and $x = 1$; there are two local minima $(0; 0)$, $(4; \frac{32}{3})$ and one local maximum $(-1; \frac{1}{4})$. $(-\frac{2}{7}; \frac{16}{189})$ is an inflection point.

(d) The asymptotes are $x = 7$ and $y = x - 1$. $(11; 13.5)$ is a local minimum and $(5; 0)$ is an inflection point.

(e) The asymptotes are $y = 0$ and $y = 2x$. There are no local maxima, local minima or inflection points.

(f) The asymptotes are $y = 3$, $y = -3$, $x = 1$, $x = -1$. $(\frac{3}{2}; \sqrt{5})$ is a local minimum, $(2; \frac{4}{\sqrt{3}})$ is an inflection point.

(g) The asymptote is $y = 1 - x$; $(0; 0)$ is a local minimum, $(2; \sqrt[3]{4})$ is a local maximum, $(3; 0)$ is an inflection point.

¹You have to find asymptotes, minima, maxima and inflection points.

- (h) $x = 2$ is an asymptote, $\left(6; \frac{3}{\sqrt[3]{2}}\right)$ is a local minimum, $\left(12; \frac{12}{\sqrt[3]{100}}\right)$ is an inflection point.
- (i) The asymptotes are $y = -x + \frac{2}{3}$ and $y = x - \frac{2}{3}$; $(0; 0)$ and $(2; 0)$ are local minima, $\left(\frac{4}{3}; \frac{2\sqrt[3]{4}}{3}\right)$ is a local maximum.
- (j) $y = 0$ is an asymptote; $(-1; -e^2)$ is a local minimum, $(2; e^{-4})$ is a local maximum; there are two inflection points at $x = 1 \pm \frac{\sqrt{10}}{2}$.
- (k) The asymptotes are $y = 0$ and $x = 0$; $(e^2; \frac{2}{e})$ is a local maximum, $\left(e^{\frac{8}{3}}; \frac{8}{3e^{\frac{4}{3}}}\right)$ is an inflection point.

10. Draw the curves given by parametric equations²:

- (a) $x = (t - 1)^2(t - 2)$, $y = (t - 1)^2(t - 3)$;
- (b) $x = \frac{1}{t^3 - t^2}$, $y = \frac{1}{t^2 - t}$;
- (c) $x = t^3 - 3t$, $y = \left(\frac{t-1}{t}\right)^2$;
- (d) $x = \frac{(t+1)^2}{t}$, $y = \frac{t+1}{t+2}$.

²You have to find asymptotes, minima and maxima, inflection points and cusps.