Tutorial 7: Planes in Space

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Course of Essentials of Analytical Geometry and Linear Algebra I

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Last weeks' topics

- Lines in Space
  - Parametric equations for a line
  - Symmetric equations for a line
  - Relationships between lines in space
- Distances in Space
  - The distance between a point and a line

Content

- Planes in Space
  - General Equation of a Plane
  - Equation of a Plane Passing through Three Points
  - Other Forms of Equations of a Plane
  - Angle Between Two Planes
  - Distance From a Point to a Plane
  - Relative Position of Planes
  - Relative Position of a Plane and a Line

Reference:
Materials ofthis Tutorial are taken with modifications from: Linear algelira, vector algebra and analytical geometry. V.V. Konev, 2009
https://opentextbc.ca/calculusv3openstax/chapter/equations-of-lines-and-planes-in-space/

General Equation of a Plane (1/2)

The general equation of a plane in a rectangular Cartesian coordinate system has the following form:

$$Ax + By + Cz + D = 0, \qquad (1)$$

where x, y and z are running coordinates of a point in the plane.
Let $M_1(x_1, y_1, z_1)$ be a point in the plane, that is,

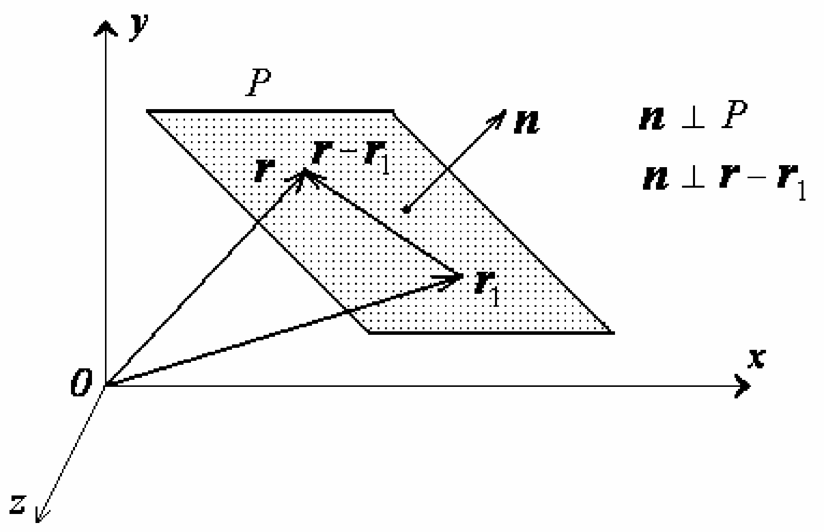
$$Ax_1 + By_1 + Cz_1 + D = 0. \qquad (2)$$

Subtracting identity (2) from equation (1) we obtain another form of the general equation of a plane:

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0\,. \qquad (3)$$

Assume that A, B and C are the coordinates of some vector $\boldsymbol{n}$.
Then the left hand side of equation (3) is the scalar product of the vectors $\boldsymbol{n}$ and $\boldsymbol{r} - \boldsymbol{r}_1 = \{x - x_1, y - y_1, z - z_1\}$:

$$\boxed{(\boldsymbol{r} - \boldsymbol{r}_1) \cdot \boldsymbol{n} = 0\,.}$$



By the properties of the scalar product this equality implies that the vector $\boldsymbol{n}$ is perpendicular to the vector $\boldsymbol{r} - \boldsymbol{r}_1$. Since $\boldsymbol{r} - \boldsymbol{r}_1$ is an arbitrary vector in the plane P, $\boldsymbol{n}$ is a normal vector to the plane P.

Thus, equation (3) describes a plane that passes through the point $M_1(x_1, y_1, z_1)$. The coefficients A, B and C can be interpreted as the coordinates of a normal vector to the plane.

General Equation of a Plane (2/2)

Consider a few particular cases of equation (1).

1) If $D = 0$ then the plane

$$Ax + By + Cz = 0$$

passes through the origin.

2) If $C = 0$ then the plane

$$Ax + By + D = 0$$

is parallel to the z–axis.

3) If $B = 0$ then the plane

$$Ax + Cz + D = 0$$

is parallel to the y–axis.

4) If $A = 0$ then the plane

$$By + Cz + D = 0$$

is parallel to the x–axis.

5) If $A = B = 0$ then the plane

$$Cz + D = 0$$

is parallel to the x, y–plane, that is, the plane is perpendicular to the z–axis.

Example 1

- Let $M_1\ (1, -2, 3)$ be a point in a plane, and $\boldsymbol{n}$ = {4, 5,−6} be a normal vector to the plane. Determine the equation of the plane.

Solution:

Using the equation (3)

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0,$$

the plane is described by

$$4(x - 1) + 5(y + 2) - 6(z - 3) = 0 \Rightarrow 4x + 5y - 6z + 24 = 0.$$

Example 2

- A plane is given by the equation $x - 2y + 3z - 6 = 0$.
  a) Find a unit normal vector **u** to the plane,
  b) Find any two points in the plane.

Solution:

a) Since $\boldsymbol{n} = \{1, -2, 3\}$ and $|\boldsymbol{n}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$, then

$$\boldsymbol{u} = \frac{\boldsymbol{n}}{|\boldsymbol{n}|} = \frac{1}{\sqrt{14}}(\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k})$$

b) Setting $x = y = 0$, we obtain $z = 2$.

Likewise, if $x = z = 0$, then $y = -3$.

Therefore, $M_1(0, 0, 2)$ and $M_2(0, -3, 0)$ are the points in the given plane.

Equation of a Plane Passing Through Three Points

Let $M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$ and $M_3(x_3, y_3, z_3)$ be three given points in a plane P, and $M(x, y, z)$ be an arbitrary point in P.

Consider three vectors,

$$\overrightarrow{M_1\ M} = \boldsymbol{r} - \boldsymbol{r}_1 = \{x - x_1, y - y_1, z - z_1\}$$

$$\overrightarrow{M_1\ M_2} = \boldsymbol{r_2} - \boldsymbol{r}_1 = \{x_2 - x_1, y_2 - y_1, z_2 - z_1\}$$

and

$$\overrightarrow{M_1\ M_3} = \boldsymbol{r_3} - \boldsymbol{r}_1 = \{\{x_3 - x_1, y_3 - y_1, z_3 - z_1\}$$

They all lie in the plane P, and so their scalar triple product is equal to zero:

$$(\boldsymbol{r} - \boldsymbol{r}_1)(\boldsymbol{r_2} - \boldsymbol{r}_1)(\boldsymbol{r_3} - \boldsymbol{r}_1) = 0 \Rightarrow$$

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0 \qquad (4)$$

Equation (4) describes a plane passing through three given points.

Example 3

- Let $M_1(2,5,-1)$, $M_2(2,-3,3)$ and$M_3(4,5,0)$ be points in a plane.

  Find an equation of that plane.

Solution:

By equation (4), we have

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = \begin{vmatrix} x-2 & y-5 & z-(-1) \\ 2-2 & -3-5 & 3-(-1) \\ 4-2 & 5-5 & 0-(-1) \end{vmatrix} = \begin{vmatrix} x-2 & y-5 & z+1 \\ 0 & -8 & 4 \\ 2 & 0 & 1 \end{vmatrix} = 0 \Rightarrow$$

$$-8(x-2)+8(y-5)+16(z+1)=0 \Rightarrow$$

$$-8x+8y+16z-8=0 \Rightarrow$$

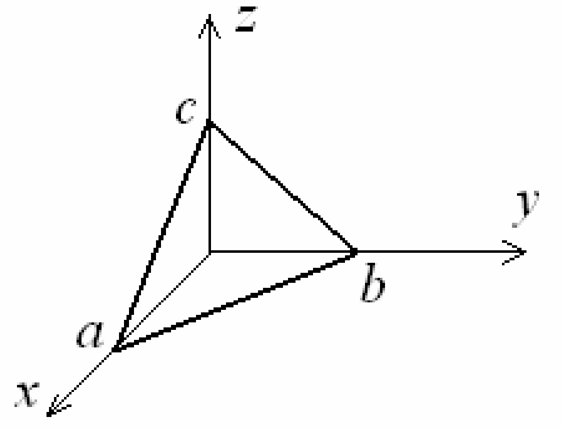
$$\underline{x-y-2z+1=0}$$

Assume that the general equation of a plane is expressed in the form of the following equality:

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad (5)$$

Then

$$\begin{aligned} y = z = 0 &\Rightarrow x = a\,, \\ x = z = 0 &\Rightarrow y = b\,, \\ x = y = 0 &\Rightarrow z = c\,. \end{aligned}$$



Therefore, the quantities a, b and c are, respectively, the *x*-intercept, *y* intercept and *z*-intercept of the plane.

Equation (5) is called the equation of a plane in the intercept form.

For instance, the equation

$$\frac{x}{2}+\frac{y}{-5}+\frac{z}{4}=1$$

describes the plane with the *x*–, *y*–, *z*–intercepts equal 2, –5 and 4, respectively.

Example 4

- Find the equation of a plane which passes through the point $M_1(5,4,-8)$ and orthogonal to the line given by $P = (3,-2,1) + t(2,1,-3)$.

Solution:

Direction vector of the line is the normal vector to the plane.

$$\boldsymbol{n} = \{2,1,-3\}$$

Then we take

$$\boldsymbol{r}_1 = \{5,4,-8\}$$

Therefore we can use

$$(\boldsymbol{r} - \boldsymbol{r}_1) \cdot \boldsymbol{n} = 0 \qquad \text{(From slide 4)}$$

Which yields to

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{r}_1 \cdot \boldsymbol{n} \Rightarrow$$

$$(x,y,z) \cdot (2,1,-3) = (5,4,-8) \cdot (2,1,-3) = 38 \Rightarrow$$

$$\underline{2x + y - 3z = 38}$$

Angle Between Two Planes

The angle θ between two planes equals the angle between their normal vectors $\boldsymbol{n}$ and $\boldsymbol{m}$:

$$\cos\theta = \frac{\boldsymbol{n}\cdot\boldsymbol{m}}{|\boldsymbol{n}|\cdot|\boldsymbol{m}|}$$

If the planes are given by equations in the general form

$$A_1x_1 + B_1y_1 + C_1z_1 + D_1 = 0$$
$$A_2x_2 + B_2y_2 + C_2z_2 + D_2 = 0$$

then

$$\cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{{A_1}^2 + {B_1}^2 + {C_1}^2}\sqrt{{A_2}^2 + {B_2}^2 + {C_2}^2}}$$

If two planes are perpendicular to each other then their normal vectors are also perpendicular:

$$\boldsymbol{n}\cdot\boldsymbol{m} = A_1A_2 + B_1B_2 + C_1C_2 = 0$$

If two planes are parallel then the normal vectors are proportional:

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$$

Note that the vector product of two non-parallel vectors in a plane gives a normal vector to the plane. In particular, if a plane is given by three points $M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$ and $M_3(x_3, y_3, z_3)$ then a normal vector to the plane is

$$\boldsymbol{n} = \overrightarrow{M_1M_2} \times \overrightarrow{M_1M_3} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix}$$

Example 5

- Find the angle between two planes P_1 and P_2 , if P_1 passes through the points $M_1(-2,2,2), M_2(0,5,3)$ and $M_3(-2,3,4)$, and P_2 is given by the equation $3x - 4y + z + 5 = 0$.

Solution:

A normal vector to the plane P_1 is determined by

$$\boldsymbol{m} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 3 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 5\boldsymbol{i} - 4\boldsymbol{j} + 2\boldsymbol{k}$$

A normal vector to the plane P_2 is $\boldsymbol{n} = \{3, -4, 1\}$.

Therefore, the cosine of the angle between the given planes is

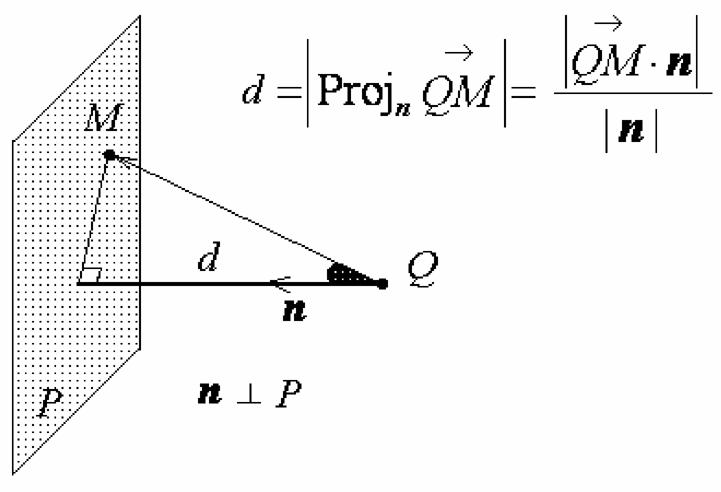
$$\cos\theta = \frac{3 \cdot 5 + (-4) \cdot (-4) + 1 \cdot 2}{\sqrt{3^2 + (-4)^2 + 1^2}\sqrt{5^2 + (-4)^2 + 2^2}} = \frac{11}{\sqrt{130}}$$

$$\theta \approx 16°$$

Assume that a plane P is determined by the equation in the general form:

$$Ax + By + Cz + D = 0. \qquad (6)$$

Let $Q(x_1, y_1, z_1)$, be a given point not in the plane, and $M(x, y, z)$ be an arbitrary point in P. Then the distance d between the point Q and the plane P is equal to the absolute value of the projection of



By equality (6),

$$Ax + By + Cz = -D,$$

and so the distance between point and plane (6) is given by the following formula:

$$d = \left|\frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}\right|$$

Example 6

- Find the distance from the point $Q(8,-7,1)$ to the plane given by the equation $2x + 3y - 4z + 5 = 0$.

Solution:

$$d = \left|\frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}\right| = \left|\frac{2 \cdot 8 + 3 \cdot (-7) + (-4) \cdot 1 + 5}{\sqrt{2^2 + 3^2 + (-4)^2}}\right| = \frac{4}{\sqrt{29}} = \frac{4}{29}\sqrt{29}$$

Relative Position of Planes (1/2)

Let two planes, P_1 and P_2 , be given by their general equations

$$P_1: \qquad A_1x + B_1y + C_1z + D_1 = 0$$

$$P_2: \qquad A_2x + B_2y + C_2z + D_2 = 0$$

Consider the system of two linear equations

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases} \tag{7}$$

1) If system (7) is inconsistent, then the planes are parallel, and so the coordinates of the normal vectors $\boldsymbol{n}_1(A_1, B_1, C_1)$ and $\boldsymbol{n}_2(A_2, B_2, C_2)$ are proportional:

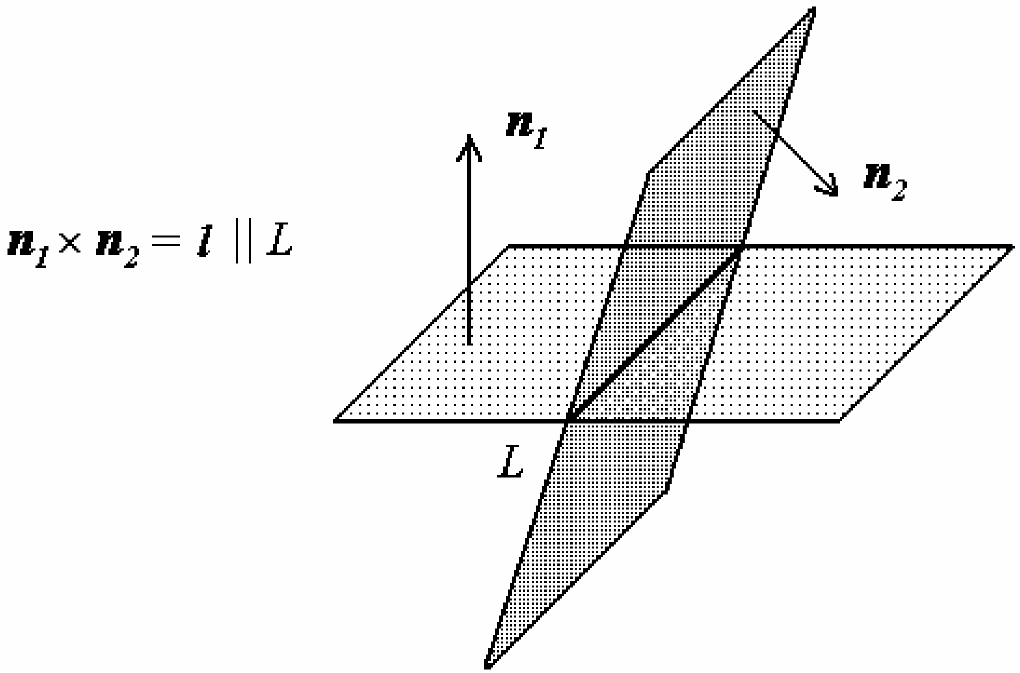
$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} \neq \frac{D_1}{D_2}$$

2) If system (7) is consistent and the equations are proportional to each other, then P_1 is just the same plane as P_2 :

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = \frac{D_1}{D_2}$$

Relative Position of Planes (2/2)

3) If system (7) is consistent, and the rank of the coefficient matrix equals 2, then P_1 and P_2 are intersecting planes. The locus of these distinct intersecting planes is exactly one line L. The vector product of normal vectors to the planes P_1 and P_2 is the vector, which is perpendicular to the normal vectors, and so it lies in both planes. Therefore, $\boldsymbol{n}_1 \times \boldsymbol{n}_2$ is a direction vector $\boldsymbol{l}$ of the intersection line L.



Example 7 (1/2)

- Suppose U and V are two planes, with equations $x_1 + 2x_2 - x_3 = 4$ and $2x_1 + x_2 - 5x_3 = 2$, respectively. Find all (x_1, x_2, x_3) which satisfy both equations.

Solution:

To manipulate the equations efficiently, we write them as follows:

$$x_1 + 2x_2 - x_3 = 4$$
$$2x_1 + x_2 - 5x_3 = 2$$

Subtract the first equation 2 times from the second (and leave the first equation as it is):

$$x_1 + 2x_2 - x_3 = 4$$
$$-3x_2 - 3x_3 = -6$$

Divide the resulting second equation by -3 to obtain:

$$x_1 + 2x_2 - x_3 = 4$$
$$x_2 + x_3 = 2$$

Example 7 (2/2)

Now get rid of x_2 in the first equation by subtracting 2 times of the new second equation from the first equations:

$$x_1 - 3x_3 = 0$$

$$x_2 + x_3 = 2$$

In this stage, both x_1 and x_2 can be expressed in terms of x_3 :

$$x_1 = 3x_3$$

$$x_2 = -x_3 + 2$$

Assign an arbitrary value λ to x_3, then we get

$$(x_1, x_2, x_3) = (3\lambda, -\lambda + 2, \lambda) = (0, 2, 0) + \lambda(3, -1, 1)$$

the parametric description of a line with direction vector (3,−1, 1).

Example 8

➢ State the relative positions (coincident, parallel, or intersecting) of the following planes.

1) $\begin{cases} x + 2y - 3 = 0 \\ -2x - 4y + 7 = 0 \end{cases}$

2) $\begin{cases} 3x - 12y + 6z - 3 = 0 \\ -x + 4y - 2z + 1 = 0 \end{cases}$

Solution:

1) Since the coordinates of the normal vectors $\boldsymbol{n}_1(A_1, B_1)$ and $\boldsymbol{n}_2(A_2, B_2)$ are proportional:

$$\frac{1}{2} = \frac{2}{4} \neq \frac{3}{7}$$

then the planes are parallel.

2) Since the system is consistent and the equations are proportional to each other, then P_1 is just the same plane as P_2:

$$\frac{3}{1} = \frac{12}{4} = \frac{6}{2} = \frac{3}{1}$$

Relative Position of a Plane and a Line

Let a plane P be given by the equation in the general form

$$Ax + By + Cz + D = 0,$$

and a line L be determined by the system of two linear equations

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

To investigate the relative positions of the line and the plane, consider the integrated system of equations:

$$\begin{cases} Ax + By + Cz + D = 0 \\ A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases} \qquad (8)$$

There are three possible cases:

1) If the rank of the coefficient matrix equals 3, then the system is consistent and has a unique solution $\{x_0, y_0, z_0\}$. It means that $M_0(x_0, y_0, z_0)$ is the point of intersection of the plane and the line.

2) If system (8) is consistent, and the rank of the coefficient matrix equals 2, then the line L lies in the plane P.

3) If system (8) is inconsistent then the line L is parallel to the plane P.

Example 9

- State the relative positions of the line and the plane.

$$P:\ x\ +\ y\ +z\ -1\ =\ 0,$$

$$L:\ \begin{cases} 3x-2y-3=\ 0 \\ 2x-y-3z-2=\ 0 \end{cases}$$

Solution:

The coefficient matrix

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 3 & -2 & 0 & -3 \\ 2 & -1 & -3 & 2 \end{bmatrix}$$

System is consistent, and the rank of the coefficient matrix equals 3, then the line L and the plane P are intersecting.

Example 10

- State the relative positions of the line and the plane.

$$P: x + y = 0,$$

$$L: \begin{cases} x - 3 = 0 \\ 2x - z + 7 = 0 \end{cases}$$

Solution:

The coefficient matrix

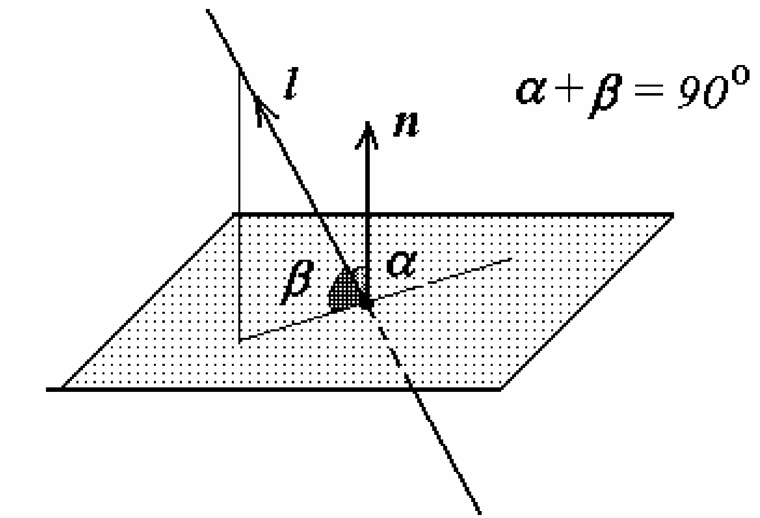
$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -3 \\ 2 & 0 & -1 & 7 \end{bmatrix}$$

System is consistent, and the rank of the coefficient matrix equals 3, then the plane and the line have intersection point.

The Angle Between a Plane and a Line

Let α be the angle between a normal vectors $\boldsymbol{n}$ to a plane and a direction vector $\boldsymbol{l}$ of a line, and β be the angle between the plane and the line.

Then α and β are complementary angles shown in the figure below.



Therefore,

$$\sin\beta = \cos\alpha = \frac{\boldsymbol{n}\cdot\boldsymbol{l}}{|\boldsymbol{n}|\cdot|\boldsymbol{l}|}$$

Example 11

- Compute the angle between the line L with parametric equation $(2,0,3)+s(0,1,1)$ and the plane U with equation $x_1+x_2+2x_3=\sqrt{13}$.

Solution:

Direction vector of the line is

$$\boldsymbol{l}=(0,1,1)$$

a normal vectors $\boldsymbol{n}$ to a plane is

$$\boldsymbol{n}=(1,1,2)$$

Therefore,

$$\sin\beta=\cos\alpha=\frac{\boldsymbol{n}\cdot\boldsymbol{l}}{|\boldsymbol{n}|\cdot|\boldsymbol{l}|}=\frac{3}{\sqrt{6}\sqrt{2}}=\frac{\sqrt{3}}{2}\Rightarrow$$

$$\alpha=30° \text{ and } \beta=60°$$

Next Week Topics

- Quadratic Curves

Good Luck