Discrete Mathematics and Logic
Lecture 7

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Combinatorics

$$|A \cup B| = |A| + |B| \text{ (if } |A \cap B| = \emptyset)$$

$$|A \times B| = |A| \cdot |B|$$

Theorem

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| -$$

$$- \sum_{i,j=1 (i<j)}^{n} |A_i \cap A_j| + \sum_{i,j,k=1 (i<j<k)}^{n} |A_i \cap A_j \cap A_k| - \cdots$$

$$\cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

Ordered arrangements

Let $A = \{a_1, \ldots, a_n\}$. How many ordered arrangements $(a_{i_1}, \ldots, a_{i_k})$?

With repetitions

$$|A^k| = |A|^k$$

Without repetitions (permutations)

$$P(n,k) = n(n-1)(n-2)\cdots(n-k+1) = \prod_{i=0}^{k-1}(n-i)$$

Unordered arrangements

Definition

Suppose that we have n distinct objects. An r-**combination** of the n objects is a subset consisting of r of the objects.

Example

There are 30 students in a group. We need to choose:

a) 2 students as volunteers,

b) 2 students as a group leader and his assistant.

Unordered arrangements

Definition

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Example

There are 30 students in a group. We need to choose:

a) 2 students as volunteers,

b) 2 students as a group leader and his assistant.

a) unordered arrangements

b) ordered arrangements
$(R(30, 2) = 30 \cdot (30 - 1) = 30 \cdot 29 = 870)$

Unordered arrangements

Example

There are 30 students in a group. We need to choose:

a) 2 students as volunteers,

$(a, b) \neq (b, a)$, but $\{a, b\} = \{b, a\}$

Unordered arrangements

Example

There are 30 students in a group. We need to choose:

a) 2 students as volunteers,

$(a, b) \neq (b, a)$, but $\{a, b\} = \{b, a\}$

So, $\frac{870}{2} = 435$.

Unordered arrangements

Example

There are 4 students in a group. We need to choose 3 students as volunteers.

$A = \{a, b, c, d\}$

Unordered	Ordered
$\{a, b, c\}$	$(a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, a, b), (c, b, a)$
$\{a, b, d\}$	$(a, b, d), (a, d, b), (b, a, d), (b, d, a), (d, a, b), (d, b, a)$
$\{a, c, d\}$	$(a, c, d), (a, d, c), (c, a, d), (c, d, a), (d, a, c), (d, c, a)$
$\{b, c, d\}$	$(b, c, d), (b, d, c), (c, b, d), (c, d, b), (d, b, c), (d, c, b)$

Unordered arrangements

Unordered	Ordered
$\{a, b, c\}$	$(a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, a, b), (c, b, a)$
$\{a, b, d\}$	$(a, b, d), (a, d, b), (b, a, d), (b, d, a), (d, a, b), (d, b, a)$
$\{a, c, d\}$	$(a, c, d), (a, d, c), (c, a, d), (c, d, a), (d, a, c), (d, c, a)$
$\{b, c, d\}$	$(b, c, d), (b, d, c), (c, b, d), (c, d, b), (d, b, c), (d, c, b)$

$P(4, 3) = 4 \cdot 3 \cdot 2 = 24$

The number of 3-combinations of 4 objects is

$$\frac{P(4,3)}{6} = \frac{P(4,3)}{3!} = \frac{P(4,3)}{P(3,3)}$$

Unordered arrangements

Theorem

The number of r-combinations of n objects is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

(We read $\binom{n}{r}$ as "n choose r")

Proof

There are $P(n,k) = \frac{n!}{(n-r)!}$ r-permutations of n objects.

For each r-combinations there are $r!$ ways in which we could order the elements (i.e. $r!$ permutations).

Unordered arrangements

Theorem

The number of r-combinations of n objects is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Proof

Therefore, the number of r-combinations of n objects is $\frac{P(n,k)}{r!} = \frac{n!}{r!(n-r)!}$.

I.e., $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

Unordered arrangements

Example

How many possible committees of 5 people can be chosen from 20 men and 12 women if

1) exactly 3 men must be on each committee?
2) at least 4 women must be on each committee?

1)

Unordered arrangements

Example

How many possible committees of 5 people can be chosen from 20 men and 12 women if

1) exactly 3 men must be on each committee?
2) at least 4 women must be on each committee?

1) $\binom{20}{3}\binom{12}{2}$

Unordered arrangements

Example

How many possible committees of 5 people can be chosen from 20 men and 12 women if

1) exactly 3 men must be on each committee?
2) at least 4 women must be on each committee?

2)

Unordered arrangements

Example

How many possible committees of 5 people can be chosen from 20 men and 12 women if

1) exactly 3 men must be on each committee?
2) at least 4 women must be on each committee?

2) $\binom{20}{1}\binom{12}{4}$

Unordered arrangements

Example

How many possible committees of 5 people can be chosen from 20 men and 12 women if

1) exactly 3 men must be on each committee?
2) at least 4 women must be on each committee?

2) $\binom{20}{1}\binom{12}{4} + \binom{20}{0}\binom{12}{5}$

Unordered arrangements

Properties

1) $\begin{pmatrix} n \\ n \end{pmatrix} = \frac{n!}{n!(n-n)!} = 1$

2) $\begin{pmatrix} n \\ 0 \end{pmatrix} = \frac{n!}{0!(n-0)!} = 1$

(Home-work) For any $k \leq n$:

3) $\binom{n}{k} = \binom{n}{n-k}$

4) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Binomial coefficients

$\binom{n}{k}$ is also called a *binomial coefficient*

Binomial Theorem

Let x, y be variables, $n \geq 1$ be a natural. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$$(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \ldots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

Binomial coefficients

Binomial Theorem

Let x, y be variables, $n \geq 1$ be a natural. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

Proof (by induction)

1. **Base** Let $n = 1$. $(x+y)^1 = \binom{1}{0}x + \binom{1}{1}y$

2. **Hypothesis** Suppose that for some k

$$(x+y)^k = \sum_{j=0}^{k} \binom{k}{j} x^{k-j} y^j$$

Binomial coefficients

Proof (by induction)

3. **Inductive step** We need to prove

$$(x+y)^{k+1} = \sum_{j=0}^{k+1} \binom{k+1}{j} x^{k+1-j} y^j$$

$$(x+y)^{k+1} = (x+y)^k (x+y) = (\sum_{j=0}^{k} \tbinom{k}{j} x^{k-j} y^j)(x+y) =$$

$$= \sum_{j=0}^{k} \tbinom{k}{j} x^{k+1-j} y^j + \sum_{j=0}^{k} \tbinom{k}{j} x^{k-j} y^{j+1}$$

Binomial coefficients

Proof (by induction)

$$(x+y)^{k+1} = \sum_{j=0}^{k} \binom{k}{j} x^{k+1-j} y^j + \sum_{j=0}^{k} \binom{k}{j} x^{k-j} y^{j+1} =$$

$$= \sum_{j=0}^{k} \binom{k}{j} x^{k+1-j} y^j + \sum_{j=1}^{k+1} \binom{k}{j-1} x^{k+1-j} y^j =$$

$$= \sum_{j=0}^{k+1} \binom{k+1}{j} x^{k+1-j} y^j$$

Using properties above ($\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ and $\binom{n}{n} = \binom{n}{0} = 1$).

Binomial coefficients

Example

What is the coefficient of the term x^8y^6 in the expansion of $(2x+3y)^{14}$?

$$(2x+3y)^{14} = \sum_{j=0}^{14} \binom{14}{j} (2x)^{14-j}(3y)^j$$

Let $j=6$. We have:

$$\binom{14}{6}(2x)^8(3y)^6 =$$

Hence, the coefficient is $\binom{14}{6}2^8 3^6 = \ldots$

Binomial coefficients

Corollary

1) $\sum_{j=0}^{n} \binom{n}{j} = (1+1)^n = 2^n$,

2) $\sum_{j=0}^{n} \binom{n}{j}(-1)^j = (1+(-1))^n = 0$

Multinomial coefficients

Example

There are 30 students in a group. We need to choose 4 students to work in 2 groups setting by 2 student in each group.

$$\binom{30}{2}\binom{28}{2} = \frac{30!}{2!28!}\frac{28!}{2!26!} = \frac{30!}{2!2!26!}$$

Multinomial coefficients

Definition

We use $\binom{n}{r_1,\ldots,r_m}$ to denote the number of arrangements of $n = r_1 + \cdots + r_m$ objects, where for each i $(1 \leq i \leq m)$ we have r_i indistinguishable objects of type i.

$\binom{n}{r_1,\ldots,r_m} = \frac{n!}{r_1!r_2!\ldots r_m!}$

$\binom{n}{r} = \binom{n}{r,n-r}$

Multinomial coefficients

Theorem

$$\binom{n}{r_1,\ldots,r_m} = \frac{n!}{r_1!r_2!\ldots r_m!}$$

Proof

$$\binom{n}{r_1,\ldots,r_m} = \binom{n}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_1-r_2-\cdots-r_{m-1}}{r_m} =$$

$$= \frac{n!}{r_1!(n-r_1)!}\frac{(n-r_1)!}{r_2!(n-r_1-r_2)!}\cdots\frac{(n-r_1-r_2-\cdots-r_m)!}{r_m!0!} =$$

$$= \frac{n!}{r_1!r_2!\ldots r_m!}$$

Multinomial coefficients

Multinomial Theorem

$$(x_1 + x_2 + \cdots x_m)^n = \sum_{k_1+k_2+\cdots k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} \prod_{1\le r\le m} x_r^{k_r}$$

Multinomial coefficients

Example

How many permutations of the word "Mississippi" are there?

Multinomial coefficients

Example

How many permutations of the word “Mississippi” are there?

$$\frac{11!}{1!4!4!2!}$$

Unordered arrangements with repetitions

Example

There are 4 varieties topics of cakes in cafe: chocolate, cream, nuts, jam. How many ways are there to order 7 cakes?

Let a denote "chocolate", b denote "cream", c denote "nuts", d denote "jam".

Examples: aaabbcd, abbbbccc.

$111|11|1|1$, $1|1111|111|$, where 1 means $\in$, $\emptyset$ means $\notin$

Unordered arrangements with repetitions

Example

There are 4 varieties topics of cakes in cafe: chocolate, cream, nuts, jam. How many ways are there to order 7 cakes?

$111|11|1|1$, $1|1111|111|$, where 1 means $\in$, $\emptyset$ means $\notin$

We have 3 symbols $|$, and 7 "1"s. Hence, the answer is $\binom{3+7}{7} = \binom{10}{7}$

Unordered arrangements with repetitions

Theorem

The number of ways of choosing r objects from n types of objects (with replacement or repetition allowed) is

$$\binom{n+r-1}{r}$$

Proof

Since there are n different types of objects, we need $n-1$ dividing markers to keep them apart.

Since we are choosing r objects, we need r "1"s. Thus, $n+r-1$ positions to be filled.

We choose the r positions in $\binom{n+r-1}{r}$ ways.

Unordered arrangements with repetitions

	without repetitions	with repetitions
order matters	$R(n,k)$	n^k
order doesn't matter	$\binom{n}{k}$	$\binom{n+k-1}{k}$

Thank you for your attention!