Tutorial 6: Lines in Space

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Last weeks' topics

- Vectors
- Matrices

Content

- Lines in Space
  - Parametric equations for a line
  - Symmetric equations for a line
  - Relationships between lines in space
- Distances in Space
  - The distance between a point and a line

Reference:
Materials ofthis Tutorial are taken with modifications from: Linear algelira, vector algebra and analytical geometry. V.V. Konev. 2009
https://opentextbc.ca/calculusv3openstax/chapter/equations-of-lines-and-planes-in-space/

Equations of Lines

A direction vector of a straight line is a vector parallel to the line.

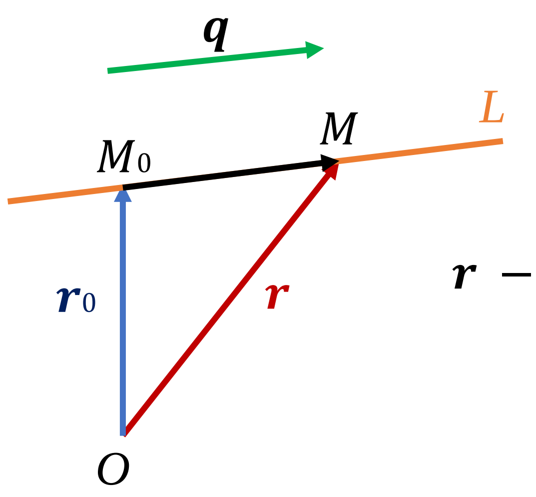
According to the postulates of geometry, a point M_0 and a direction vector **q** determine the straight line L.
Let M be an arbitrary point on the line. The difference $\vec{\boldsymbol{r}} - \vec{\boldsymbol{r}_0}$ between the radius-vectors of the points M and M_0 is a vector in the line, that is,

$$\boldsymbol{r} - \boldsymbol{r}_0 \,||\, \boldsymbol{q}$$

Two parallel vectors are proportional:

$$\boldsymbol{r} - \boldsymbol{r}_0 = t\,\boldsymbol{q}$$

This vector equality is called the **vector equation of the line**. An arbitrary number t is said to be a *parameter*.



$$\boldsymbol{r} - \boldsymbol{r}_0 = \overrightarrow{M_0M}$$

Lines in Space (2/2)

Assume that a rectangular Cartesian coordinate system is chosen.
Vectors $\boldsymbol{r}, \boldsymbol{r}_0$ and $\boldsymbol{q}$ are represented by their coordinates:

$$\boldsymbol{r} - \boldsymbol{r}_0 = \{x - x_0, y - y_0, z - z_0\},$$
$$\boldsymbol{q} = \{q_x, q_y, q_z\}.$$

Then **vector equation of the line** can be written in the coordinate form as the system of three linear equations:

$$\begin{cases} x = x_0 + q_x t \\ y = y_0 + q_y t \\ z = z_0 + q_z t \end{cases} \qquad (*)$$

where *x*, *y* and *z* are running coordinates of a point on the line.

Equations of a line in coordinate form (*) are called the **parametric equations** of a line.

Solving system (*) by elimination of the parameter *t*, we obtain the **canonical equations of a line**:

$$\frac{x - x_0}{q_x} = \frac{y - y_0}{q_y} = \frac{z - z_0}{q_z}$$

If $M_0(x_0, y_0, z_0)$ and $M_1(x_1, y_1, z_1)$ are two given points on a line then the vector joining these points serves as a direction vector of the line.

$$\boldsymbol{q} = \{x_1 - x_0, y_1 - y_0, z_1 - z_0\},$$

Therefore, we get the following equations of a line passing through <u>two given points</u>:

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}$$

Example 1

Let L be a line passing through the points $M_1(1, 0, 2)$ and $M_2(3,1,-2)$.

Check whether the point $A(7, 3, -10)$ lies on the line L.

Solution:

Using $\frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0} = \frac{z-z_0}{z_1-z_0}$ we get the equations of L:

$$\frac{x-1}{2} = \frac{y}{1} = \frac{z-2}{-4}$$

The coordinates of the point A satisfy the equation:

$$\frac{7-1}{2} = \frac{3}{1} = \frac{-10-2}{-4}$$

and so A is a point of the line L.

Example 2

Write down the canonical equations of the line passing through the point $A(2, 3, 4)$ and being parallel to the vector $\mathbf{q} = \{5, 0, -1\}$.

Solution:

By equation $\frac{x-x_0}{q_x} = \frac{y-y_0}{q_y} = \frac{z-z_0}{q_z}$, we obtain

$$\frac{x-2}{5} = \frac{y-3}{0} = \frac{z-4}{-1}$$

Note that a symbolical notation $\frac{y-3}{0}$ means the equation $y = 3$.

On the x, y–plane, a line is described by the linear equation

$$Ax + By + C = 0\,. \qquad (*)$$

If $M_0(x_0, y_0)$ is a point on the line then

$$Ax_0 + By_0 + C = 0\,. \qquad (**)$$

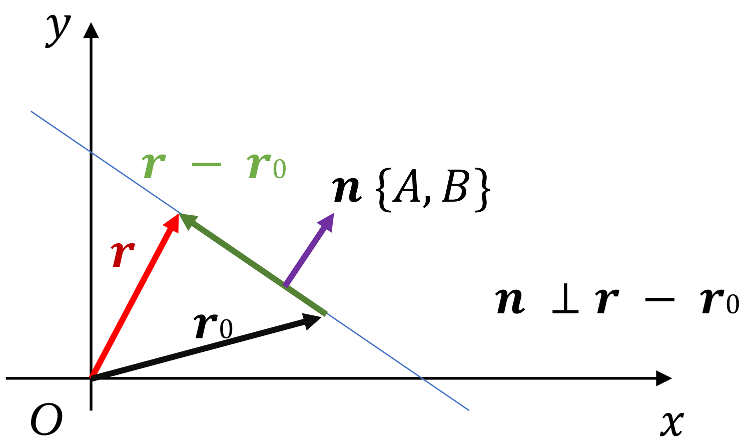
Subtracting identity (**) from equation (*) we obtain the equation of a line passing through the point $M_0(x_0, y_0)$:

$$A(x - x_0) + B(y - y_0) = 0\,.$$

The expression on the left hand side has a form of the scalar product of the vectors $\boldsymbol{n} = \{A, B\}$ and $\boldsymbol{r} - \boldsymbol{r}_0 = \{x - x_0, y - y_0\}$:

$$\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r}_0) = 0\,.$$

Therefore, the coefficients A and B can be interpreted geometrically as the coordinates of a vector in the x, y–plane, being perpendicular to the line.



Lines in a Plane (2/3)

The canonical equation of a line in the *x*, *y*–plane has a form

$$\frac{x - x_0}{q_x} = \frac{y - y_0}{q_y}$$

where $\boldsymbol{q} = \{q_x, q_y\}$ is a direction vector of the line.

In the *x*, *y*–plane, an equation of a line passing through two given points, $M_0(x_0, y_0)$ and $M_1(x_1, y_1)$, is written as follows

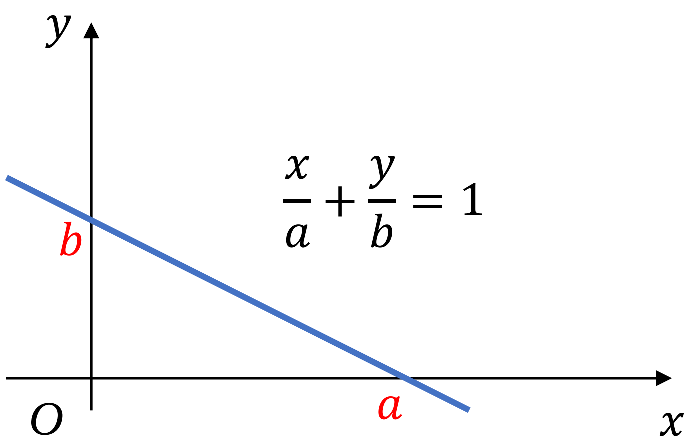
$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0}$$

Sometimes it is helpful to express a straight-line equation in the *x*, *y*–plane as

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad (*)$$

In this case, $y = 0$ implies $x = a$, and $x = 0$ implies $y = b$.

Therefore, the quantities *a* and *b* are, respectively, the *x*-intercept and the *y* intercept of a graph of the line. Equation (*) is called an equation of a line in the intercept form.



Lines in a Plane (3/3)

A line on the *x*,*y*–plane may be also given by the equation in the slope intercept form

$$y = kx + b,$$

where b is the y-intercept of a graph of the line, and k is the slope of the line.

If $M_0(x_0, y_0)$ is a point on the line, that is, $y_0 = kx_0 + b$, then the point–slope equation of the line is

$$y - y_0 = k(x - x_0).$$

Example 3

A line on the x, y–plane is given by the equation

$$2x - 3y + 24 = 0\,.$$

Find:

1) any two points on the line,
2) the slope of the line,
3) the x– and y–intercepts.

Solution:

1) Setting $x = 0$ we obtain $y = 8$.

If $x = 3$ then $y = 10$.

Therefore, the points $P(0,8)$ and $Q(3,10)$ lie on the line.

2) $2x - 3y + 24 = 0 \Rightarrow y = \frac{2}{3}x + 8$. Therefore, the slope of the line is $k = \frac{2}{3}$.

3) The y–intercept equals 8. The x–intercept is the solution of the equation $y = 0$, that is, $x = -12$.

Example 4

In the x, y–plane, find the equation of the line passing through the point $M_1(5,3)$ and being perpendicular to the vector $\boldsymbol{n} = \{2,-1\}$.

Solution:

Using equation $A(x-x_0) + B(y-y_0) = 0$ we obtain

$$2(x-5) - (y-3) = 0 \Rightarrow y = 2x - 7\,.$$

Example 5

Let $M_1(-2,4)$ and $M_2(1,6)$ be the points on a line.

Which of the following points, $A(-3,1)$, $B(0,-3)$ and $C(3,\ 6)$, are the points on the line?

Solution:

The equation of a line passing through two given points:

$$\frac{x+2}{1+2} = \frac{y-4}{6-4} \Rightarrow \frac{x+2}{3} = \frac{y-4}{2} \Rightarrow 2x - 3y + 16 = 0\,.$$

Substituting the coordinates of the points results in

$A(-3,1)$ is <u>not</u> a point on the line, since

$$2\cdot(-3) - 3\cdot 1 + 16 = 6 \neq 0;$$

$B(0,-3)$ is <u>not</u> a point on the line, since

$$2\cdot 0 + 3\cdot 3 + 16 = 24 \neq 0\,;$$

$C(3,\ 6)$ <u>is not</u> a point on the line, since

$$2\cdot 3 - 3\cdot 6 + 16 = 4 \neq 0.$$

Angle Between Two Lines (1/3)

The angle between two lines is the angle between direction vectors of the lines.

If $\boldsymbol{p} = \{p_x, p_y, p_z\}$ and $\boldsymbol{q} = \{q_x, q_y, q_z\}$ are direction vectors of lines, then the cosine of the angle between the lines is given by the following formula:

$$\cos\theta = \frac{\boldsymbol{p} \cdot \boldsymbol{q}}{|\boldsymbol{p}| \cdot |\boldsymbol{q}|} = \frac{p_x q_x + p_y q_y + p_z q_z}{\sqrt{{p_x}^2 + {p_y}^2 + {p_z}^2}\,\sqrt{{q_x}^2 + {q_y}^2 + {q_z}^2}}$$

If two lines are perpendicular to each other then their direction vectors are also perpendicular. This means that the scalar product of the direction vectors is equal to zero:

$$\boldsymbol{p} \cdot \boldsymbol{q} = p_x q_x + p_y q_y + p_z q_z = 0$$

If two lines are parallel then their direction vectors are proportional:

$$\boldsymbol{p} = c\boldsymbol{q},$$

where c is a number.

In the coordinate form, this condition looks like

$$\frac{p_x}{q_x} = \frac{p_y}{q_y} = \frac{p_z}{q_z}$$

Angle Between Two Lines (2/3)

We need direction vectors of lines to find the angle between the lines.

Consider a few particular cases.

1) Let a line be given by two points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$.

Then

$$\boldsymbol{p} = \{x_2 - x_1, y_2 - y_1, z_2 - z_1\},$$

is a direction vector of the line.

2) If a line in the *x*, *y*–plane is given by the equation

$$Ax + By + C = 0,$$

then we can easily find two points on the line. For instance, $M_1(0, -\frac{C}{B})$and $M_2(-\frac{C}{A}, 0)$ are two points on the line.

If two lines in the *x*, *y*–plane are given by the equations

$$A_1x + B_1y + C_1 = 0 \text{ and } A_2x + B_2y + C_2 = 0$$

then the angle between the lines is equal to the angle between perpendicular vectors $\boldsymbol{n}_1 = \{A_1, B_1\}$ and $\boldsymbol{n}_2 = \{A_2, B_2\}$ to the lines:

$$\cos\theta = \frac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1| \cdot |\boldsymbol{n}_2|}$$

Note that a perpendicular vector to a line is also called a **normal vector** to the line.

Angle Between Two Lines (3/3)

3) If a line in the x, y–plane is given by the equation

$$\frac{x}{a} + \frac{y}{b} = 1$$

then $M_1(0, b)$ and $M_2(a, 0)$ are two points on the line, and so $\boldsymbol{p} = \{a, -b\}$ is a direction vector of the line.

4) If two lines in the x, y–plane are given by the equations in the slope intercept form $y = k_1 x + b_1$ and $y = k_2 x + b_2$, and θ is the angle between the lines, then

$$\tan\theta = \frac{k_2 - k_1}{1 + k_1 k_2}$$

The lines are parallel, if

$$k_2 = k_1$$

The lines are perpendicular, if

$$k_1 k_2 = -1$$

Example 6

Find the angle θ between two lines in the x, y–plane, if they are given by the following equations:

$3x - 4y + 1 = 0$ and $2x + y - 5 = 0$.

Solution:

Normal vectors to the lines are, respectively, $\boldsymbol{n}_1 = \{3, -4\}$ and $\boldsymbol{n}_2 = \{2, 1\}$. Therefore,

$$\cos\theta = \frac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1| \cdot |\boldsymbol{n}_2|}$$

$$\cos\theta = \frac{3 \cdot 2 + (-4) \cdot 1}{\sqrt{3^2 + (-4)^2}\,\sqrt{2^2 + 1^2}} = \frac{2}{5\sqrt{5}} = \frac{2}{25}\sqrt{5}$$

$$\theta \approx 80°$$

Example 7

Find the angle θ between two lines in the x, y–plane, if they are given by the equations in the slope-intercept form:

$y = -\sqrt{3}x + 1$ and $y = \frac{\sqrt{3}}{3}x + 5.$

Solution:

We have $k_1 = -\sqrt{3}$ and $k_2 = \sqrt{3}/3.$

Since

$$k_1 k_2 = -\frac{\sqrt{3}\sqrt{3}}{3} = -1,$$

the lines are orthogonal: $\theta = 90°$

Example 8

Let $A = \{2, -1\}, B = \{4, 4\}$ and $C = \{9, 7\}$ be the vertices of a triangle. Find the equation of the altitude from the vertex A, and write down the equation in the intercept form.

Solution:

If $D = \{x, y\}$ is an arbitrary point on the altitude from the vertex A, then the vectors $\overrightarrow{AD} = \{x - 2, y + 1\}$ and $\overrightarrow{BC} = \{5, 3\}$ are orthogonal.

Therefore, the scalar product of *AD* and BC is equal to zero, and we obtain the desired equation:

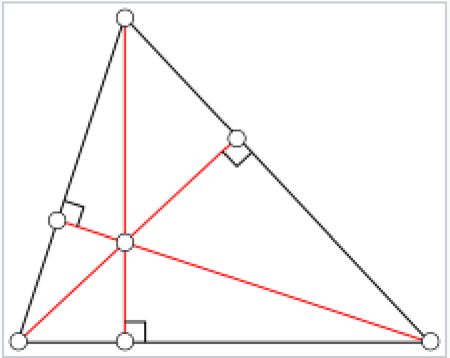
$$\overrightarrow{AD} \cdot \overrightarrow{BC} = 5(x - 2) + 3(y + 1) = 0 \Rightarrow$$

$$5x + 3y - 7 = 0 \Rightarrow$$

$$\frac{x}{7/5} + \frac{y}{7/3} = 1$$

Altitude (triangle) – from wikipedia

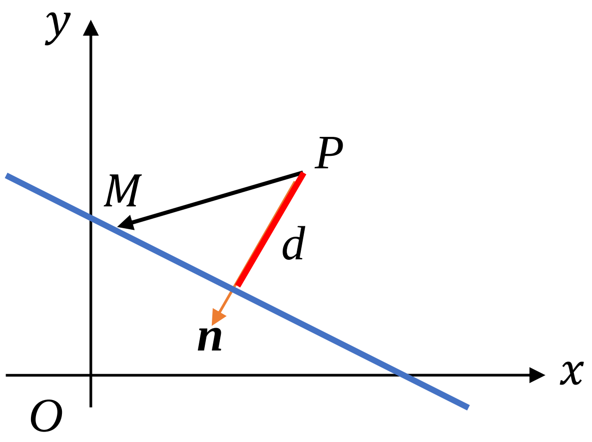
In geometry, an altitude of a triangle is a line segment through a vertex and perpendicular to (i.e., forming a right angle with) a line containing the base (the side opposite the vertex).



The three altitudes of a triangle intersect at the orthocenter, which for an acute triangle is inside the triangle.

Distance From a Point to a Line

Consider a line in the x, y–plane.
Let $\boldsymbol{n}$ be a normal vector to the line and $M(x_0, y_0)$ be any point on the line. Then the distance d from a point P not on the line is equal to the absolute value of the projection of $\overrightarrow{PM}$ on $\boldsymbol{n}$:



$$d = \left|\mathrm{Proj}_n\, \overrightarrow{PM}\right| = \left|\frac{\overrightarrow{PM} \cdot \boldsymbol{n}}{|\boldsymbol{n}|}\right|$$

In particular, if the line is given by the equation

$$Ax + By + C = 0\,,$$

and the coordinates of the point P are x_1 and y_1, that is,
$\boldsymbol{n} = \{A, B\}$ and $\overrightarrow{PM} = \{x_1 - x_0, y_1 - y_0\}$,
then the distance from the point $P(x_1, y_1)$ to the line is calculated according to the following formula:

$$d = \frac{|A(x_1 - x_0) + B(y_1 - y_0)|}{\sqrt{A^2 + B^2}}$$

Since $M(x_0, y_0)$ is a point on the line,

$$Ax_0 + By_0 + C = 0$$

Therefore, we obtain

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

Relative Position of Lines (1/2)

Let two lines, L_1 and L_2 , be given by their equations, e.g., in the canonical form:

$$L_1: \qquad \frac{x - x_1}{p_x} = \frac{y - y_1}{p_y} = \frac{z - z_1}{p_z}$$

$$L_2: \qquad \frac{x - x_2}{q_x} = \frac{y - y_2}{q_y} = \frac{z - z_2}{q_z}$$

where $\boldsymbol{p} = \{p_x, p_y, p_z\}$ and $\boldsymbol{q} = \{q_x, q_y, q_z\}$ are direction vectors of the lines.

In order to determine the relative position of the lines, it is necessary to consider the equations of both lines as a system of linear equations. Each lines is described by two linear equations, and so we have the following system of four linear equations with three unknowns x, y and z:

$$\begin{cases} \dfrac{x - x_1}{p_x} = \dfrac{y - y_1}{p_y} \\ \dfrac{x - x_1}{p_x} = \dfrac{z - z_1}{p_z} \\ \dfrac{x - x_2}{q_x} = \dfrac{y - y_2}{q_y} \\ \dfrac{x - x_2}{q_x} = \dfrac{z - z_2}{q_z} \end{cases} \qquad (*)$$

Relative Position of Lines (2/2)

Let us analyze all possible cases.

1) Assume that system (*) is inconsistent. Then the lines are either parallel or skew. If the coordinates of the direction vectors **p** and **q** are proportional, that is,

$$\frac{p_x}{q_x} = \frac{p_y}{q_y} = \frac{p_z}{q_z}$$

then the lines are parallel; otherwise, they are skew.

2) Suppose that system (*) is consistent, and the rank of the coefficient matrix equals 3. Then L_1 and L_2 are intersecting lines, that is, they have exactly one point of intersection.

3) If system (*) is consistent, and the rank of the coefficient matrix equals 2, then the lines coincide with each other.

Example 10

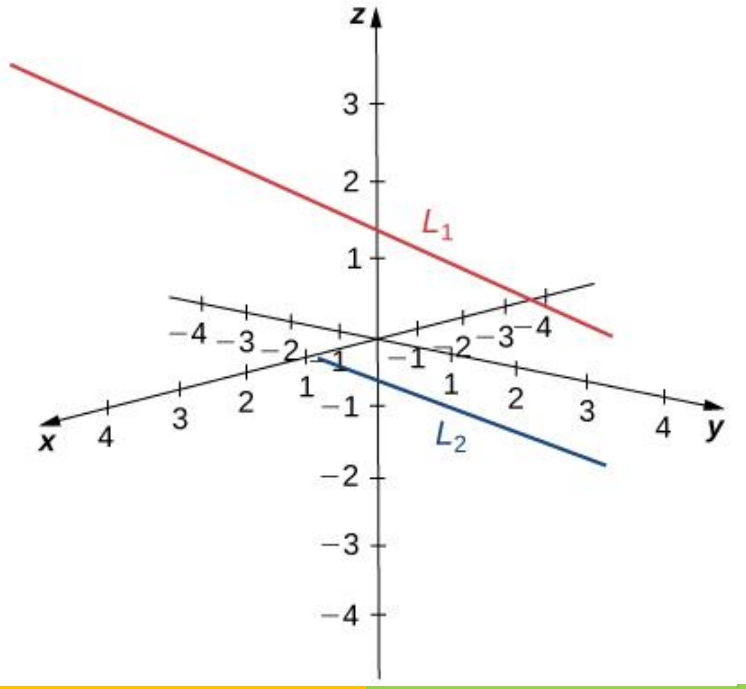
For the pair of lines, determine whether the lines are equal, parallel but not equal, skew, or intersecting.

$$L_1: \quad x = 6s - 1, y = -2s, z = 3s + 1$$

$$L_2: \quad \frac{x-4}{6} = \frac{y+3}{-2} = \frac{z-1}{3}$$

Solution:

Line L_1 and L_2 have equivalent direction vectors: $v\{6, -2, 3\}$. These two lines are parallel.



Example 10

For the pair of lines, determine whether the lines are equal, parallel but not equal, skew, or intersecting.

$$L_1: \quad x = 2s - 1, y = s - 1, z = s - 4$$

$$L_2: \quad x = t - 3, y = 3t + 8, z = 5 - 2t$$

Solution:

Line L_1 has direction vector $\boldsymbol{v}_1\{2, 1, 1\}$; line L_2 has direction vector $\boldsymbol{v}_2\{1, 3, -2\}$. Because the direction vectors are not parallel vectors, the lines are either intersecting or skew. To determine whether the lines intersect, we see if there is a point, (x, y, z), that lies on both lines. To find this point, we use the parametric equations to create a system of equalities:

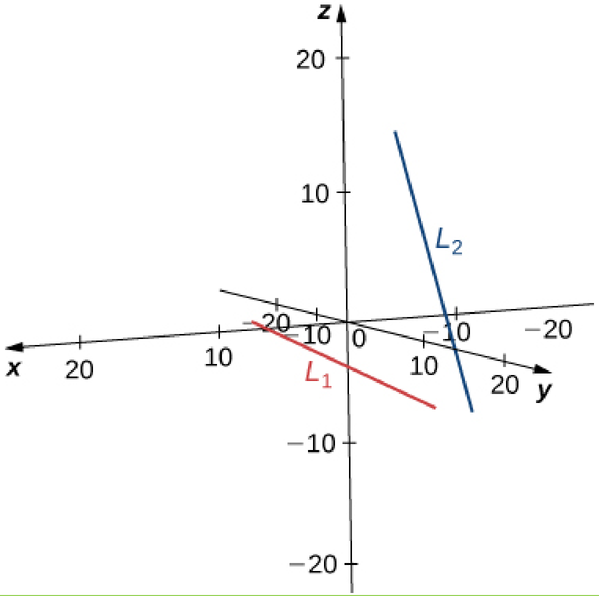
$$2s - 1 = t - 3, s - 1 = 3t + 8, s - 4 = 5 - 2t$$

By the first equation, $2s + 2 = t$ .Substituting into the second equation yields

$$s - 1 = 3(2s + 2) + 8 \Rightarrow s = -3$$

Substitution into the third equation, however, yields a contradiction:

$$s - 4 = 5 - 2(2s + 2) \Rightarrow s = 1$$



There is no single point that satisfies the parametric equations for L_1 and L_2. These lines do not intersect, and the coordinates of the direction vectors are not proportional, so they are skew.

Next Week Topics

- Planes in Space
  - General Equation of a Plane
  - Equation of a Plane Passing Through Three Points
  - Other Forms of Equations of a Plane
  - Angle Between Two Planes
  - Distance From a Point To a Plane
  - Relative Position of Planes
  - Relative Position of a Plane and a Line