Discrete Mathematics and Logic
Tutorial 5

Andrey Frolov

Innopolis University

Relations

Definition

Any set $R \subseteq X_1 \times \cdots \times X_n$ is called a relation on $X_1, \ldots, X_n$.

If $X_1 = \cdots = X_n$, then $R \subseteq X^n$ is an n-arity relation on X.

Relations

Definition

Any set $R \subseteq X_1 \times \cdots \times X_n$ is called a relation on $X_1, \ldots, X_n$.

If $n = 1$ then R is a unary relation on X.

Examples

- $R(x) \Leftrightarrow x$ is positive.
- $R(\text{a person}) \Leftrightarrow$ the person is a woman.

Relations

Definition

Any set $R \subseteq X_1 \times \cdots \times X_n$ is called a relation on $X_1, \ldots, X_n$.
If $X_1 = \cdots = X_n$, then $R \subseteq X^n$ is an n-arity relation on X.

If $n = 2$ then R is a binary relation on X

Examples

- $R(x, y) \Leftrightarrow x < y$
- R(a man, a woman) $\Leftrightarrow$ these man and woman are married.

Relations

Definition

Any set $R \subseteq X_1 \times \cdots \times X_n$ is called a relation on $X_1, \ldots, X_n$.
If $X_1 = \cdots = X_n$, then $R \subseteq X^n$ is an n-arity relation on X.

If $n = 3$ then R is a binary relation on X.

Examples

- $R(x, y, z) \Leftrightarrow x + y = z$
- R(a man, a woman, a child) $\Leftrightarrow$ a child is a son of these man and woman.

Binary relations

Definition

Any set $R \subseteq X \times Y$ is called a binary relation on X and Y.

If $X = Y$, then $R \subseteq X^2$ is a binary relation on X.

$(x, y) \in R \leftrightharpoons xRy$

Example

$x \leq y$

Binary relations

Definitions

A binary relation R on a set X is called

- reflexive if $\forall x \in X \; xRx$,
- irreflexive if $\forall x \in X \; \neg(xRx)$,

Examples

- $x \leq y$,
- $x < y$,
- $R(\text{a person } x, \text{a person } y) \leftrightharpoons x \text{ likes } y$.

Binary relations

Definitions

A binary relation R on a set X is called

- symmetric if $\forall x, y \in X\ (xRy \rightarrow yRx)$,
- asymmetric if $\forall x, y \in X\ \neg(xRy \rightarrow yRx)$,
- antisymmetric if $\forall x, y \in X\ (xRy \,\&\, yRx \rightarrow x = y)$,

Examples

- $R(\Phi_1, \Phi_2) \leftrightharpoons$ the formula Φ_1 is equal to the formula Φ_2.
- $x < y$,
- $x \leq y$
  If $x \leq y \,\&\, y \leq x \rightarrow x = y$.

Binary relations

Definitions

A binary relation R on a set X is called

- transitive if $\forall x, y, z \in X\ (xRy \,\&\, yRz \to xRz)$.

Examples

- $R(\Phi_1, \Phi_2) \leftrightharpoons$ the formula Φ_1 is equal to the formula Φ_2,
- R(a city c_1, a city c_2) $\leftrightharpoons$ there is way from c_1 to c_2,
- $R(x, y) \leftrightharpoons x + y = 0$,
  if $x + y = 0$ and $y + z = 0$ then $x - z = 0$.
- R(a person x, a person y) $\leftrightharpoons$ x is a friend of y.

Binary relations

Definitions

A binary relation R on a set X is called

- reflexive if $\forall x \in X\ xRx$,
- irreflexive if $\forall x \in X\ \neg(xRx)$,
- symmetric if $\forall x, y \in X\ (xRy \to yRx)$,
- asymmetric if $\forall x, y \in X\ \neg(xRy \to yRx)$,
- antisymmetric if $\forall x, y \in X\ (xRy \,\&\, yRx \to x = y)$,
- transitive if $\forall x, y, z \in X\ (xRy \,\&\, yRz \to xRz)$.

Strict order relations

Definition

A binary relation R on a set X is called a strict order, if it is irreflexive, asymmetric and transitive, i.e.,

- $\forall x \in X \ \neg(xRx)$ (irreflexive),
- $\forall x, y \in X \ \neg(xRy \rightarrow yRx)$ (asymmetric),
- $\forall x, y, z \in X \ (xRy \,\&\, yRz \rightarrow xRz)$ (transitive).

Examples

- $x < y$,
- a man x is higher than a man y.

Non-strict order relations

Definition

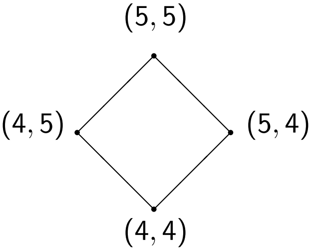
A binary relation R on a set X is called a non-strict order, if it is reflexive, antisymmetric and transitive, i.e.,

- $\forall x \in X \ xRx$ (reflexive),
- $\forall x, y \in X \ (xRy \,\&\, yRx \rightarrow x = y)$ (antisymmetric),
- $\forall x, y, z \in X \ (xRy \,\&\, yRz \rightarrow xRz)$ (transitive).

Examples

- $x \leq y$,
- a man x is older than a man y.

Partial order



Example

$(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n) \Leftrightarrow x_1 \leq y_1 \,\&\, \ldots \,\&\, x_n \leq y_n.$

Linear orders

Definition

An order R is called linear if $\forall x \neq y(xRy \vee yRx)$.

Examples

- $x \leq y$
- $x < y$
- a man x is higher than a man y.
- a man x is older than a man y.

Equivalence relations

Definition

A binary relation R on a set X is called equivalence, if it is reflexive, symmetric and transitive.

- $\forall x \in X \; xRx$ (reflexive),
- $\forall x, y \in X \; (xRy \to yRx)$ (symmetric),
- $\forall x, y, z \in X \; (xRy \,\&\, yRz \to xRz)$ (transitive).

$=, \sim, \simeq, \cong, \equiv$

Examples

- $x = y$,
- $|A| = |B|$,
- a man x and a man y have the same age.

Equivalence relations

Definition

A binary relation R on a set X is called equivalence, if it is reflexive, symmetric and transitive.

An equivalence class is a set such that $x \sim y$ for any x, y form the class.

The intersection of two different equivalence classes is empty.

Binary relations

Definitions

A binary relation R on a set X is called

- $\forall x \in X \; xRx$ (reflexive),
- $\forall x \in X \; \neg(xRx)$ (irreflexive),
- $\forall x, y \in X \; (xRy \to yRx)$ (symmetric),
- $\forall x, y \in X \; \neg(xRy \to yRx)$ (asymmetric),
- $\forall x, y \in X \; (xRy \,\&\, yRx \to x = y)$ (antisymmetric),
- $\forall x, y, z \in X \; (xRy \,\&\, yRz \to xRz)$ (transitive).

Examples

- $X = \{1, 2, 3, 4\}$, $R = \{(1, 2), (1, 3), (2, 4), (4, 4)\}$,
- $X =$ "all humans", $R(x, y) \leftrightharpoons x$ is a father of y,
- $X = \{1, 2, 3, 4\}$, $R = \{(x, y) \mid x + y > 2\}$,
- $X = \mathbb{N}$, $R = \{(x, y) \mid\}$

Thank you for your attention!