# Mathematical Analysis. Assignment 2.
# Limits of functions. Continuity

1. (a) It is known that functions $f(x)$ and $g(x)$ do not have a limit as $x \to x_0$. Does it imply that functions $f(x)+g(x)$ and $f(x)g(x)$ do not have a limit as $x \to x_0$?

   (b) It is known that function $f(x)$ has a finite limit and $g(x)$ does not have a limit as $x \to x_0$. Does it imply that functions $f(x)+g(x)$ and $f(x)g(x)$ do not have a limit as $x \to x_0$?

2. Formulate the following statements using $\delta - \epsilon$ notation:

   (a) $\lim\limits_{x\to x_0^-} f(x) = a$;

   (b) $\lim\limits_{x\to x_0^+} f(x) = +\infty$;

   (c) $\lim\limits_{x\to -\infty} f(x) = \infty$;

   (d) function $f(x)$ has a finite limit as $x \to x_0$;

   (e) function $f(x)$ does not have a finite limit as $x \to x_0$.

3. Using the Cauchy definition of a limit prove that $\lim\limits_{x\to 4} x^3 = 64$.

4. Find the following limits:

   (a) $\lim\limits_{x\to -2} \frac{x^3+3x^2+2x}{x^2-x-6}$;

   (b) $\lim\limits_{x\to 3} \frac{x^3-5x^2+3x+9}{x^3-8x^2+21x-18}$;

   (c) $\lim\limits_{x\to 1} \frac{x^4-2x+1}{x^8-2x+1}$;

   (d) $\lim\limits_{x\to 1} \left(\frac{x^2-4x+6}{x^2-5x+4} + \frac{x-4}{3x^2-9x+6}\right)$;

   (e) $\lim\limits_{x\to \infty} \frac{(x+1)^2(3-7x)^2}{(2x-1)^4}$;

   (f) $\lim\limits_{x\to \infty} \frac{\left(1+x^{11}+7x^{13}\right)^3}{(1+x^4)^{10}}$;

   (g) $\lim\limits_{x\to \infty} \left(\frac{x^2}{2x+1} + \frac{x^3+4x^2-2}{1-2x^2}\right)$;

   (h) $\lim\limits_{x\to 3} \frac{\sqrt{x^2-2x+6}-\sqrt{x^2+2x-6}}{x^2-4x+3}$;

   (i) $\lim\limits_{x\to 0} \frac{\sqrt[3]{x+8}-2}{\sqrt{1+2x}-1}$;

   (j) $\lim\limits_{x\to \infty} \left(\sqrt{x^4+2x^2-1} - \sqrt{x^4-2x^2-1}\right)$;

   (k) $\lim\limits_{x\to -\infty} \left(\sqrt{x^2+8x+3} - \sqrt{x^2+4x+3}\right)$.

   **Answer:** (a) $-\frac{2}{5}$; (b) 4; (c) $\frac{1}{3}$; (d) 1; (e) $\frac{49}{16}$; (f) 0; (g) $-\frac{9}{4}$; (h) $-\frac{1}{3}$; (i) $\frac{1}{12}$; (j) 2; (k) $-2$.

5. Find the following limits:

   (a) $\lim\limits_{x\to 0} \frac{\sin 3x}{x}$;

   (b) $\lim\limits_{x\to \infty} x \sin\frac{\pi}{x}$;

   (c) $\lim\limits_{x\to 0} \left(\frac{2}{\sin 2x \sin x} - \frac{1}{\sin^2 x}\right)$;

   (d) $\lim\limits_{x\to 1} \frac{\sin 7\pi x}{\sin 2\pi x}$;

(e) $\lim\limits_{x\to\infty} x^2\left(\cos\frac{1}{x}-\cos\frac{3}{x}\right)$;

(f) $\lim\limits_{x\to 0} \frac{\sqrt{1+2\sin 3x}-\sqrt{1-4\sin 5x}}{\sin 6x}$;

(g) $\lim\limits_{x\to\infty} \left(\sin\sqrt{x^2+1}-\sin\sqrt{x^2-1}\right)$;

(h) $\lim\limits_{x\to\infty} \left(\frac{x}{2x+1}\right)^x$;

(i) $\lim\limits_{x\to\infty} \left(\frac{x}{x+1}\right)^x$;

(j) $\lim\limits_{x\to 0^+} \frac{\arccos(1-x)}{\sqrt{x}}$;

(k) $\lim\limits_{x\to -1^+} \frac{\sqrt{\pi}-\sqrt{\arccos x}}{\sqrt{x+1}}$.

**Answer:** (a) 3; (b) $\pi$; (c) $\frac{1}{2}$; (d) $-\frac{7}{2}$; (e) 4; (f) $\frac{13}{6}$; (g) 0; (h) the limit does not exist; (i) $\frac{1}{e}$; (j) $\sqrt{2}$; (k) $\frac{1}{\sqrt{2\pi}}$.

6. Which of the following statements are true? Justify your answer.

   (a) $x^3 = o(x)$, $x\to 0$;

   (b) $x^3 = o(x)$, $x\to\infty$;

   (c) $x = o(x^3)$, $x\to 0$;

   (d) $x = o(x^3)$, $x\to\infty$.

   **Answer:** (a) and (d).

7. Let $x\to 0$, $n\in\mathbb{N}$, $k\in\mathbb{N}$, $n\geqslant k$. Show that

   (a) $o\left(x^k\right)+o\left(x^n\right)=o\left(x^k\right)$;

   (b) $o\left(x^k\right)\cdot o\left(x^n\right)=o\left(x^{n+k}\right)$.

8. Prove that function $f(x)=\cos x$, $x\in\mathbb{R}$ is continuous[1].

9. Find the discontinuities of the following functions and determine their type:

   (a) $y=\frac{1-\sqrt{x}}{x^2-1}$;

   (b) $y=2^{\frac{1}{x}}$;

   (c) $y=\lg\left(x^2+3x\right)$.

10. Give an example of a function continuous on some open interval and (a) not bounded on this interval; (b) bounded on this interval but reaching neither its infimum nor its supremum.

11. Give an example of a discontinuous function determined on a closed interval and such that its range is also a closed interval.

12. Prove that the equation $x^5-3x=1$ has at least three real roots, and that at least one of the roots belongs to $(1;2)$.

13. Prove that any polynomial of an odd degree has at least one real root. Does this statement hold for polynomials of even degrees?

14. Let $f(x)$ be a continuous function determined on $[a;+\infty)$. Prove that if there exists a finite $\lim\limits_{x\to+\infty} f(x)$ then $f(x)$ is bounded on $[a;+\infty)$. Is the statement going to be true if an interval $[a;+\infty)$ is replaced with $(a;+\infty)$?

[1] You might need the inequality $|\sin x|\leqslant|x|$, $x\in\mathbb{R}$.