Essentials of Analytical Geometry and Linear Algebra 1

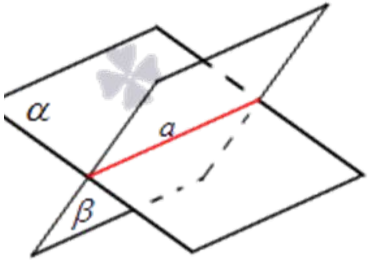
Plane
Line in space

What elements do we know

Line in plane

Plane

Line in space



Plane (equations)

$$-(\bar{n}, \bar{r}_0) = (-Ax_0 - By_0 - Cz_0)$$

1) General $\quad Ax + By + Cz + D = 0$

2) Vector $\quad \bar{r} \cdot \bar{n} + D = 0$; $\quad \bar{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$

3) in sections $\quad \dfrac{x}{\underbrace{-\frac{D}{A}}_{a}} + \dfrac{y}{\underbrace{-\frac{D}{B}}_{b}} + \dfrac{z}{\underbrace{-\frac{D}{C}}_{c}} = 1$

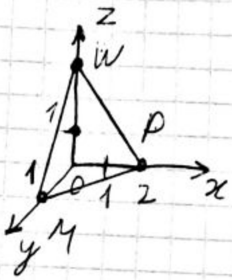
4) Param $\quad r = r_0 + au + bv$

Plane (Task)

- Write down all forms of the plane
- Draw this plane

3) $\frac{x}{2} + \frac{y}{1} + \frac{z}{3} = 1$

Plane (Answer)



3) $\frac{x}{2}+\frac{y}{1}+\frac{z}{3}=1$

1) $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = \begin{vmatrix} x-2 & y & z \\ 0-2 & 1 & 0 \\ \cancel{0-2} & \cancel{0} & \cancel{3} \end{vmatrix} = 3x+6y+2z-6=0$

A, B, C, D

$M_0 = V_0$; $\overline{MP}=(2;-1;0)$

$\cancel{MW}=(0;-1;3)$

4) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}\alpha + \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}\beta$

2) $\begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} - 6 = 0$

Task 1

1. Find the equation of the plane passing through the point $(2, -3, 4)$ and parallel to the plane $2x - 5y - 7z + 15 = 0$.

Line in space (equations)

1) Canonical $\quad \dfrac{x - x_0}{a_x} = \dfrac{y - y_0}{a_y} = \dfrac{z - z_0}{a_z} = \tau$

2) Parametrical $\quad \begin{cases} x = x_0 + a_x \tau \\ y = y_0 + a_y \tau \\ z = z_0 + a_z \tau \end{cases}$

3) 2 points $\quad \dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$

4) Vector 1 $\quad [\bar{a}, \bar{r}] = \bar{b} \; ; \; [\bar{a}, \bar{r} - \bar{r}_0] = 0$

5) Vector 2 $\quad \begin{cases} (\bar{n}_1, \bar{r} - \bar{r}_0) = 0 \\ (\bar{n}_2, r - \bar{r}_0) = 0 \end{cases} \quad n = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$

$[n_1 ; n_2] = \bar{a}$

6) General $\quad \begin{cases} A_1 x + B_1 y + C_1 z + D_1 = 0 \\ A_2 x + B_2 y + C_2 z + D_2 = 0 \end{cases}$

Line in space (Task)

- Write down all forms of the line
- Draw this line

$$\begin{cases} 15x + 9y + 0z - 45 = 0 \\ 0x - 8y + 0z = 0 \end{cases}$$

Line in space (Answer) (1)

6) $\begin{cases} 15x + 9y + 0z - 45 = 0 \\ 0x - 8y + 0z = 0 \end{cases}$

$n_1 = \begin{pmatrix} 15 \\ 9 \\ 0 \end{pmatrix}$ $n_2 = \begin{pmatrix} 0 \\ -8 \\ 0 \end{pmatrix}$ $[n_1, n_2] = [n_1 \times] n_2 = \begin{bmatrix} 0 & 0 & 9 \\ 0 & 0 & -15 \\ -9 & 15 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ -8 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 0 \\ -120 \end{pmatrix}$

4) $\underbrace{\begin{pmatrix} 0 \\ 0 \\ -120 \end{pmatrix}}_{\to \bar{d}} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -120y \\ +120x \\ 0 \end{pmatrix} \to \bar{b}$

$\begin{pmatrix} 0 \\ 0 \\ -120 \end{pmatrix} \times \begin{pmatrix} x-3 \\ y \\ z-1 \end{pmatrix} = 0$

Put this stuff in (6) and solve this eq

Transform z to λ

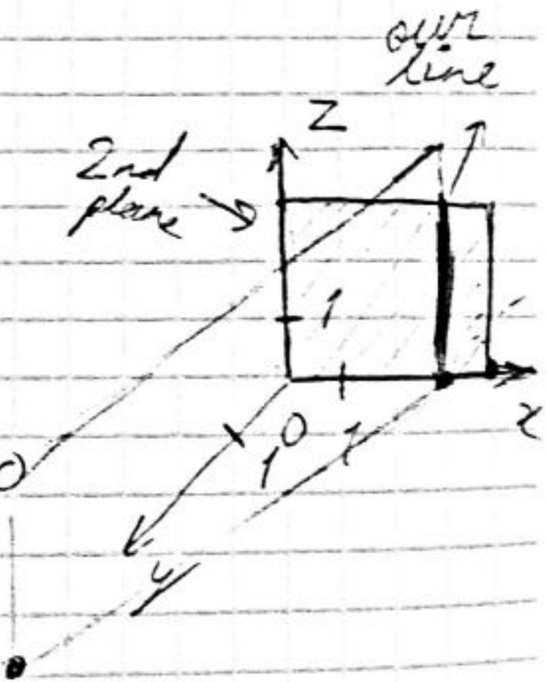
Solving eq $\Rightarrow \begin{cases} x_0 = 3 \\ y_0 = 0 \\ z_0 = \lambda \end{cases} \Rightarrow$ $\lambda = 1$ $\begin{cases} x_0 = 3 \\ y_0 = 0 \\ z_0 = 1 \end{cases}$

Line in space (Answer) (2)

1) $\dfrac{x-3}{0} = \dfrac{y-0}{0} = \dfrac{z-1}{-120}$

2) $\begin{cases} x = 3 \\ y = 0 \\ z = 1 + 120\lambda \end{cases}$

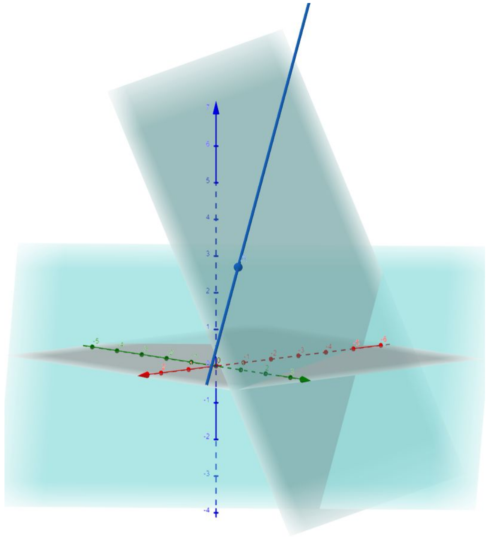
5) $\begin{cases} \left((15,9,d)^T; \bar{v} - (3,0,1)^T\right) = 0 \\ \left((0,-8,0)^T; \bar{v} - (3,0,1)^T\right) = 0 \end{cases}$



Task 2

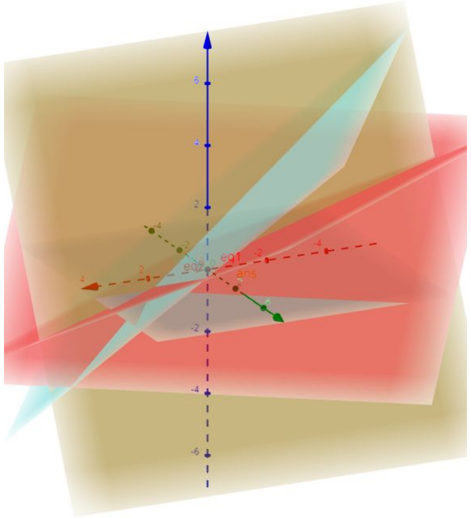
5. Find the equations of the line passing through the point $(1, 2, 3)$ and perpendicular to the planes $x - 2y - z + 5 = 0$ and $x + y + 3z + 6 = 0$.

Perpendicular to the normals of planes, or parallel to the planes



Task 3

4. Find the equation of the plane which passes through the intersection of the planes $2x + 3y + 10z - 8 = 0$, $2x - 3y + 7z - 2 = 0$ and is perpendicular to the plane $3x - 2y + 4z - 5 = 0$.




WHY
SLAVS
WEAR ADIDAS

Formulas of distances

1) Point to line $$d = \left| \frac{[R - r_0, a]}{|a|} \right|$$ Umnov 107

2) Line to line → if collinear $$h = \frac{|[r_2 - r_1, a_1]|}{|a_1|}$$

→ skew $$h = \left| \frac{(r_2 - r_1, a_1, a_2)}{|[a_1, a_2]|} \right|$$ Umnov 117

3) Point to Plane $$h = \left| \frac{(R - r_0, n)}{|n|} \right|$$

Projection (1)

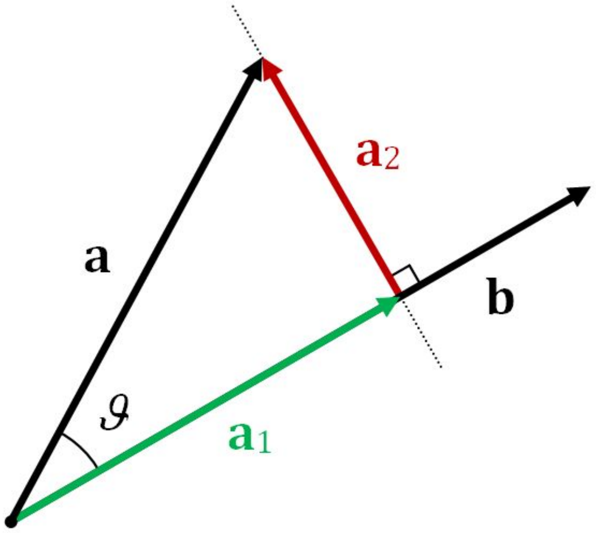
Def: Projection

Where it can be used:

1. Maps
2. Blueprints
3. Fitting algorithms (Least squares)
4. Reduce matrix dimension

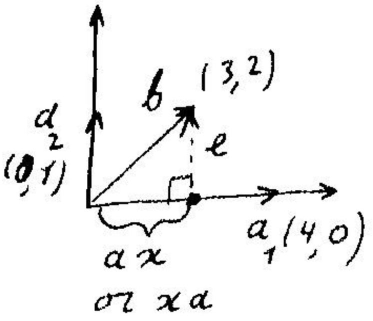
Links:

MIT 15 lecture



Projection 2D case from school

2D



$a \equiv$ basis

$$e = b - ax$$

$$a_1 \cdot (b - a_1 x) = 0$$

$$a_1^T (b - a_1 x) = 0$$

$$a_1^T b = a_1^T a_1 x$$

$$\frac{a_1^T b}{a_1^T a_1} = x$$ — classic formula from school

$$\frac{\begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}}{\begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix}} = \frac{12}{16} = \frac{3}{4}$$

$$\Downarrow$$

projection $a_1 x = \frac{3}{4} \cdot \begin{bmatrix} 4 \\ 0 \end{bmatrix} = 3$

Proofing 2d case formula (we project “b” on “a_1”)

2D case, using Projection matrix

Not so usefull in general case

outer product

$Pb = xa_1$ $\Rightarrow P = \dfrac{a_1 a_1^T}{a_1^T a_1}$ $P = \dfrac{\begin{bmatrix}4\\0\end{bmatrix}\begin{bmatrix}4 & 0\end{bmatrix}}{16} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$

↑ like in affine tr. (matrix)

projection matrix

projection $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix}$

The same task, but using more general approach (projection matrix)

Proj. general case formula + example

$a = [a_1 \ a_2]$ In this case projection will be the same as initial (lying pen on table)

$$P = A(A^TA)^{-1}A^T$$ – general formula

$$A^TAx = A^Tb$$

$$P = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \left(\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} \begin{bmatrix} a_1 & a_2 \end{bmatrix} \right)^{-1} \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$(I-P) \to$ perpendicular projection
In our case it's „0" point $\to$ all val $= 0$

$$P\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

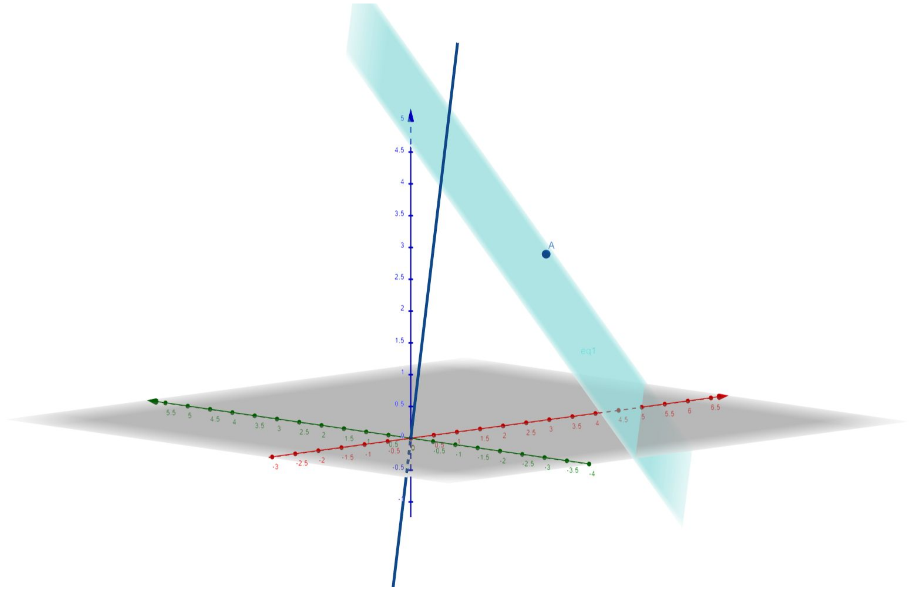
The same picture, $a = [a_1 \ a_2]$, P here is general formula for N space

Task 4

6. Find the perpendicular distance from the point $(1, 3, -1)$ to the line $\frac{x-13}{5} = \frac{y+8}{-8} = \frac{z-31}{1}$

Task 5

8. Find the distance of the point $(1, -2, 3)$ from the plane $x - y + z = 5$ measured parallel to the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$



Deserve “A” grade!

– Oleg Bulichev

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Room 105 (Underground robotics lab)