Discrete Mathematics and Logic
Lecture 4

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Additional chapter

Logic

Negation	Conjunction	Disjunction
$\neg P$	$P_1 \,\&\, P_2$	$P_1 \vee P_2$

Implication	Equivalence	
$P_1 \to P_2$	$P_1 \leftrightarrow P_2$	

P, P_1, P_2 are propositions (either true, or false).

Additional chapter

Predicates

A predicate $P(x_1, \ldots, x_n)$ is a proposition with parameters $x_1, \ldots, x_n \in \mathbf{U}$ (its truth depends on $x_1, \ldots, x_n$).

- $P_1(x) : x + 2 = 2$,
- $P_2(x, y) : x + y = 0$,
- $P_3(x, y)$: A man x is friend of a man y

Additional chapter

Quantifiers

The universal quantifier	The existential quantifier
"for Any", "for All"	"Exists"
$\forall x\, P(x)$	$\exists x\, P(x)$

Additional chapter

The universal quantifier

$$\text{for any } x, P(x) \text{ holds}$$

$$\forall x\, P(x)$$

- "for any x, $2x$ is even",
- "for any x, y, $x + 1 = y$"
- "for any man x, x has one million dollars"

True or not?

Additional chapter

The universal quantifier

for any x, $P(x)$ holds

$$\forall x\, P(x)$$

- "for any x, $2x$ is even",
- "for any x, y, $x + 1 = y$"
- "for any man x, x has one million dollars"

What is the universe?

Additional chapter

The existential quantifier

$$\text{there exists } x, P(x) \text{ holds}$$

$$\exists x P(x)$$

- "there exists x such that x is even",
- "there exist x, y such that $x - y = 0$"
- "there exists x such that x has one billion dollars"

True or not?

Additional chapter

Example

If the first line is parallel to the third line and the second line is parallel to the third line, then the first and the second lines are parallel.

$P(l, l')$ = "a line l is parallel to a line k"

$$\forall l_1, l_2, l_3\, [(P(l_1, l_3)\, \&\, P(l_2, l_3)) \to P(l_1, l_2)]$$

Properties/laws

De Morgan's laws

$\neg(a \,\&\, b) = \neg a \vee \neg b$	$\neg(a \vee b) = \neg a \,\&\, \neg b$
$\neg\forall x\, P(x) = \exists x \neg P(x)$	$\neg\exists x\, P(x) = \forall x \neg P(x)$

Examples

- $\neg(\forall x\, P(x) \,\&\, \exists y\, R(y)) = \neg\forall x\, P(x) \vee \neg\exists y\, R(y) =$
  $= \exists x \neg P(x) \vee \forall y \neg R(y)$

Properties/laws

De Morgan's laws

$\neg\forall x\, P(x) = \exists x \neg P(x)$

Examples

- $\mathbf{U} = \mathbb{N}$, $P(x) =$ "x is even"

$$\forall x\, P(x) = \text{ "any natural number is even"}$$

$\neg\forall x\, P(x) = \exists x \neg P(x) =$ "there exists a natural number which is not even"

Properties/laws

De Morgan's laws

$\neg\forall x\, P(x) = \exists x \neg P(x)$

Examples

- $\mathbf{U} = \mathbb{Z}$, $P(x) =$ "x is a natural number"

$$\forall x\, P(x) = \text{ "any integer number is natural"}$$

$\neg\forall x\, P(x) = \exists x \neg P(x) =$ "there exists an integer number which is not natural"

Properties/laws

De Morgan's laws

$\neg\forall x\, P(x) = \exists x \neg P(x)$

Examples

- **U** is the set of all people, $P(x) =$ "x is a woman"

$$\forall x\, P(x) = \text{"everyone is a woman"}$$

$\neg\forall x\, P(x) = \exists x \neg P(x) =$ "there is a person who is not a woman"

Properties/laws

De Morgan's laws

$\neg\exists x\, P(x) = \forall x \neg P(x)$

Examples

- $\mathbf{U} = \mathbb{N}$, $P(x) =$ "x is even"

$$\exists x\, P(x) = \text{"there is a natural number that is even"}$$

$$\neg\exists x\, P(x) = \forall x \neg P(x) = \text{"every natural number is not even"}$$

Properties/laws

De Morgan's laws

$\neg \exists x\, P(x) = \forall x \neg P(x)$

Examples

- $\mathbf{U} = \mathbb{Z}$, $P(x) =$ "x is a natural number"

$$\exists x\, P(x) = \text{"there is an integer number that is not natural"}$$

$$\neg \exists x\, P(x) = \forall x \neg P(x) = \text{"any integer number is not natural"}$$

Properties/laws

De Morgan's laws

$\neg \exists x\, P(x) = \forall x \neg P(x)$

Examples

- **U** is the set of all people, $P(x) =$ "x is a woman"

$$\exists x\, P(x) = \text{"a woman exists"}$$

$$\neg \exists x\, P(x) = \forall x \neg P(x) = \text{"everyone is not a woman"}$$

Properties/laws

$$\exists x\, \exists y\, P(x, y) = \exists y\, \exists x\, P(x, y)$$

$$\forall x\, \forall y\, P(x, y) = \forall y\, \forall x\, P(x, y)$$

Properties/laws

$$\forall x\,(P_1(x)\,\&\,P_2(x)) = \forall x\,P_1(x)\,\&\,\forall x\,P_2(x)$$

$$\forall x\,(P_1(x) \vee P_2(x)) \neq \forall x\,P_1(x) \vee \forall x\,P_2(x)$$

$$\exists x\,(P_1(x)\,\&\,P_2(x)) \neq \exists x\,P_1(x)\,\&\,\exists x\,P_2(x)$$

$$\exists x\,(P_1(x) \vee P_2(x)) = \exists x\,P_1(x) \vee \exists x\,P_2(x)$$

Properties/laws

$$\forall x\,(P_1(x)\,\&\,P_2(x)) = \forall x\,P_1(x)\,\&\,\forall x\,P_2(x)$$

$$\forall x\,(P_1(x) \vee P_2(x)) \neq \forall x\,P_1(x) \vee \forall x\,P_2(x)$$

Counterexample

Let $P_1(x) = x$ "is even" and $P_2(x) = x$ "is odd".

$\forall x\,(P_1(x) \vee P_2(x)) = 1$

$\forall x\,P_1(x) \vee \forall x\,P_2(x) = 0 \vee 0 = 0$

Properties/laws

$$\forall x\,(P_1(x) \vee P_2(x)) \neq \forall x\, P_1(x) \vee \forall x\, P_2(x)$$

Other way

$$\forall x\, P_1(x) \vee \forall x\, P_2(x) = \forall x\, P_1(x) \vee \forall y\, P_2(y) =$$

$$= \forall x \forall y\,(P_1(x) \vee P_2(y))$$

Properties/laws

$$\exists x\, (P_1(x)\,\&\, P_2(x)) \neq \exists x\, P_1(x)\,\&\, \exists x\, P_2(x)$$

$$\exists x\, (P_1(x) \vee P_2(x)) = \exists x\, P_1(x) \vee \exists x\, P_2(x)$$

Counterexample

Let $P_1(x) = x$ "is even" and $P_2(x) = x$ "is odd".

$\exists x(P_1(x)\,\&\, P_2(x)) = 0$

$\exists x\, P_1(x)\,\&\, \exists x\, P_2(x) = 1\,\&\,1 = 1$

Properties/laws

$$\exists x\,(P_1(x)\,\&\,P_2(x)) \neq \exists x\,P_1(x)\,\&\,\exists x\,P_2(x)$$

$$\exists x\,(P_1(x) \vee P_2(x)) = \exists x\,P_1(x) \vee \exists x\,P_2(x)$$

Other way

$\exists x\,P_1(x)\,\&\,\exists x\,P_2(x) = \exists x\,P_1(x)\,\&\,\exists y\,P_2(y) =$

$= \exists x \exists y\,(P_1(x)\,\&\,P_2(y))$

Properties/laws

$$\forall x\,(P_1(x)\,\&\,P_2(x)) = \forall x\,P_1(x)\,\&\,\forall x\,P_2(x)$$

$$\exists x\,(P_1(x) \vee P_2(x)) = \exists x\,P_1(x) \vee \exists x\,P_2(x)$$

Proof

$$\exists x\,(P_1(x) \vee P_2(x)) = \neg\neg[\exists x\,P_1(x) \vee \exists x\,P_2(x)] =^{*}$$

$$=^{*} \neg[\neg(\exists x\,P_1(x))\,\&\,\neg(\exists x\,P_2(x))] =$$

$*\neg(A \vee B) = \neg A\,\&\,\neg B$

Properties/laws

$$\forall x\,(P_1(x)\,\&\,P_2(x)) = \forall x\,P_1(x)\,\&\,\forall x\,P_2(x)$$

$$\exists x\,(P_1(x) \vee P_2(x)) = \exists x\,P_1(x) \vee \exists x\,P_2(x)$$

Proof

$$= \neg[\neg(\exists x\,P_1(x))\,\&\,\neg(\exists x\,P_2(x))] =^*$$

$$=^* \neg[(\forall x \neg P_1(x))\,\&\,(\forall x \neg P_2(x))] =$$

$*\neg\exists x\,P(x) = \forall x \neg P(x)$

Properties/laws

$$*\forall x\,(P_1(x)\,\&\,P_2(x)) = \forall x\,P_1(x)\,\&\,\forall x\,P_2(x)$$

$$\exists x\,(P_1(x) \vee P_2(x)) = \exists x\,P_1(x) \vee \exists x\,P_2(x)$$

Proof

$$= \neg[(\forall x \neg P_1(x))\,\&\,(\forall x \neg P_2(x))] =^*$$

$$=^* \neg[\forall x\,(\neg P_1(x)\,\&\,\neg P_2(x))] =$$

Properties/laws

$$\forall x\,(P_1(x)\,\&\,P_2(x)) = \forall x\,P_1(x)\,\&\,\forall x\,P_2(x)$$

$$\exists x\,(P_1(x) \vee P_2(x)) = \exists x\,P_1(x) \vee \exists x\,P_2(x)$$

Proof

$= \neg[\forall x\,(\neg P_1(x)\,\&\,\neg P_2(x))] =^*$

$=^* \exists x\,\neg(\neg P_1(x)\,\&\,\neg P_2(x)) =$

$*\neg\forall x\,P(x) = \exists x\neg P(x)$

Properties/laws

$$\forall x\,(P_1(x)\,\&\,P_2(x)) = \forall x\,P_1(x)\,\&\,\forall x\,P_2(x)$$

$$\exists x\,(P_1(x)\vee P_2(x)) = \exists x\,P_1(x)\vee\exists x\,P_2(x)$$

Proof

$$= \exists x\,\neg(\neg P_1(x)\,\&\,\neg P_2(x)) =^*$$

$$=^* \exists x\,(\neg\neg P_1(x)\vee\neg\neg P_2(x)) = \exists x\,(P_1(x)\vee P_2(x))$$

$*\neg(A\,\&\,B) = \neg A\vee\neg B$

Rotation of quantifiers

$$\exists x \, \forall y \, P(x, y) \neq \forall y \, \exists x \, P(x, y)$$

Rotation of quantifiers

$$\exists x\, \forall y\, P(x,y) \neq \forall y\, \exists x\, P(x,y)$$

Let $\mathbf{U} = \mathbb{N}$, $P(x,y) \leftrightharpoons$ "$x = y$".

$\exists x\, \forall y\, P(x,y) = 0$

$\forall y\, \exists x\, P(x,y) = 1$

Rotation of quantifiers

$$\exists x\, \forall y\, P(x,y) \neq \forall y\, \exists x\, P(x,y)$$

Let $\mathbf{U}$ = "all families", $\mathbf{U}_1$ = "all husbands", $\mathbf{U}_2$ = "all wives",

$P(x,y) \leftrightharpoons$ "x and y married".

$\forall y \in \mathbf{U}_2\, \exists x \in \mathbf{U}_1\, P(x,y) = 1$

$\exists x \in \mathbf{U}_1\, \forall y \in \mathbf{U}_2\, P(x,y) = 0$

Rotation of quantifiers

$$\forall t_7 \, \exists t_6 \, \forall t_5 \, \exists t_4 \, \forall t_3 \, \exists t_2 \, \forall t_1 \, \exists t_0 \; R(t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7)$$

Thank you for your attention!