Discrete Mathematics and Logic
Lecture 1

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About the course

Course points

Type	Points
Labs classes	20
Interim performance assessment	30
Exams	50

Labs points (only my recommendations for TAs):

- In-class participation 1 point for each individual contribution in a class but not more than 1 point a week (i.e. 14 points in total for 14 study weeks),
- overall course contribution (to accumulate extra-class activities valuable to the course progress, e.g. a short presentation, book review, very active in-class participation, etc.) up to 6 points.

About the course

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Interim performance assessment:

Each of 3 in-class tests costs 10 points.

- Section 1. Basic elements and the naive set theory
- Section 2. Relations, functions and enumerating combinatorics
- Section 3. Graph theory

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Interim performance assessment:

- Mid-term exam and final examination costs up to 25 points each (i.e. 50 points for both).

About the course

Grades range

Grade	Range
A. Excellent	80-100
B. Good	70-79
C. Satisfactory	60-69
D. Poor	0-59

About the course

Discrete Mathematics and Logic

- Basic elements and the naive set theory
- Relations, functions and enumerating combinatorics
- Graph theory

Introduction

Basic objects of Mathematics

- Numbers
- Sets
- Functions
- Relations
- Structures

Natural numbers

Natural numbers

$$\mathbb{N}$$

$$1, \ 2, \ 3, \ 4, \ 5, \ \ldots$$

Natural numbers

Natural numbers

$$\mathbb{N} \text{ or } \mathbb{N}^*$$

$$0?,\ 1,\ 2,\ 3,\ 4,\ 5,\ \ldots$$

Natural numbers

Natural numbers

$$\mathbb{N} = \{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ \dots\}$$

Integer numbers

Integer numbers

$$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, \quad 0, \quad 1, \quad 2, \quad 3, \quad 4, \quad \ldots\}$$

Integer numbers

Integer numbers

$$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, \quad 0, \quad 1, \quad 2, \quad 3, \quad 4, \quad \ldots\}$$

$$\ldots, \quad 8, \quad 6, \quad 4, \quad 2, \quad 0, \quad 1, \quad 3, \quad 5, \quad 7, \quad \ldots$$

Integer numbers

Definition

An object is called countable, if there exists its enumeration by natural numbers.

So, the set of all integer numbers is countable.

Remark

The proof technique above is called by construction.

Rational numbers

Rational numbers

$$\mathbb{Q} = \left\{ \frac{n}{m} \mid n, m \text{ are integer numbers } \& \, m \neq 0 \right\}$$

Rational numbers

Positive rational numbers

$$
\begin{array}{ccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\
\frac{2}{1} & \frac{2}{3} & \frac{2}{5} & \frac{2}{7} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{4} & \frac{3}{5} & \cdots \\
\frac{4}{1} & \frac{4}{3} & \frac{4}{5} & \frac{4}{7} & \cdots
\end{array}
$$

Rational numbers

Positive rational numbers

$$
\begin{array}{ccccc}
\frac{1}{1}\,{\color{red}1} & \frac{1}{2}\,{\color{red}2} & \frac{1}{3}\,{\color{red}4} & \frac{1}{4}\,{\color{red}7} & \cdots \\[2ex]
\frac{2}{1}\,{\color{red}3} & \frac{2}{3}\,{\color{red}5} & \frac{2}{5}\,{\color{red}8} & \frac{2}{7}\,{\color{red}12} & \cdots \\[2ex]
\frac{3}{1}\,{\color{red}6} & \frac{3}{2}\,{\color{red}9} & \frac{3}{4}\,{\color{red}13} & \frac{3}{5} & \cdots \\[2ex]
\frac{4}{1}\,{\color{red}10} & \frac{4}{3}\,{\color{red}14} & \frac{4}{5} & \frac{4}{7} & \cdots
\end{array}
$$

Rational numbers

So, using the proof technique by construction, we proof

Proposition

The set of all rational numbers is countable.

Real numbers

Real numbers

$$\mathbb{R} = \{X, a_0 a_1 a_2 a_3 \ldots \mid X \text{ is integer }, \text{ all } a_i \text{ are digit } \}$$

Any rational number is real.

For example,

$$\frac{4}{3} = 1,33333\ldots$$

Is there irrational numbers?

Real numbers

Proposition

$\sqrt{2}$ is irrational.

Proof by contradiction.

Suppose for a contradiction that $\sqrt{2}$ is rational number.
Let $\sqrt{2} = \frac{n}{m}$ be an irreducible fraction. Then,

$$2 = \frac{n^2}{m^2}$$

$$n^2 = 2m^2$$

Real numbers

Proof by contradiction.

Suppose for a contradiction that $\sqrt{2}$ is rational number.
Let $\sqrt{2} = \frac{n}{m}$ be an irreducible fraction. Then,

$$2 = \frac{n^2}{m^2}$$

$$n^2 = 2m^2$$

Therefore, n is even, i.e., $n = 2k$. Then

$$(2k)^2 = 2m^2$$

$$2k^2 = m^2$$

Therefore, n and m are both even. Since $\sqrt{2} = \frac{n}{m}$ is irreducible, this is a contradiction.

Real numbers

Proposition

The set of all real number is uncountable.

Real numbers

Proposition

The set of all real number is uncountable.

Proof by contradiction.

Suppose for a contradiction that there exists an enumeration of all real numbers from $[0, 1)$.

$$
\begin{aligned}
x_0 &= 0, a_0^0 a_1^0 a_2^0 a_3^0 \ldots \\
x_1 &= 0, a_0^1 a_1^1 a_2^1 a_3^1 \ldots \\
x_2 &= 0, a_0^2 a_1^2 a_2^2 a_3^2 \ldots \\
x_3 &= 0, a_0^3 a_1^3 a_2^3 a_3^3 \ldots
\end{aligned}
$$

$$\ldots$$

Real numbers

Proof by contradiction.

Suppose for a contradiction that there exists an enumeration of all real numbers from $[0, 1)$.

$$x_0 = 0, a_0^0 a_1^0 a_2^0 a_3^0 \dots$$
$$x_1 = 0, a_0^1 a_1^1 a_2^1 a_3^1 \dots$$
$$x_2 = 0, a_0^2 a_1^2 a_2^2 a_3^2 \dots$$
$$x_3 = 0, a_0^3 a_1^3 a_2^3 a_3^3 \dots$$
$$\dots$$

Let $r = 0, (9 - a_0^0)(9 - a_1^1)(9 - a_2^2)(9 - a_3^3) \dots$.

Real numbers

Proof by contradiction.

Suppose for a contradiction that there exists an enumeration of all real numbers from $[0,1)$.

$$x_0 = 0, a_0^0 a_1^0 a_2^0 a_3^0 \dots$$

$$x_1 = 0, a_0^1 a_1^1 a_2^1 a_3^1 \dots$$

$$x_2 = 0, a_0^2 a_1^2 a_2^2 a_3^2 \dots$$

$$x_3 = 0, a_0^3 a_1^3 a_2^3 a_3^3 \dots$$

$$\dots$$

Let $r = 0,(9 - a_0^0)(9 - a_1^1)(9 - a_2^2)(9 - a_3^3)\dots$.
We see that $0 \leq r < 1$ and $r \neq x_i$ for any $i \geq 0$.

Real numbers

Proof by contradiction.

Recall that x_0, x_1, x_2, x_3 is a list of all real numbers from $[0, 1)$.

And we have build the real number r from $[0, 1)$ such that $r \neq x_i$ for any $i \geq 0$.

This is a contradiction.

The proof by contradiction is complete.

The set of all real numbers is uncountable and, hence, is not an object of Discrete Mathematics!

The naive set theory

A set

$$\{x \mid P(x)\}$$

Examples

- $\mathbb{N} = \{x \mid x \text{ is integer and } x \geq 0\}$
- $[0, 1] = \{x \mid x \text{ is real and } 0 \leq x \leq 1\}$
- $\{1, 2, 3, 4\} = \{x \mid x \text{ is integer and } 0 < x \leq 4\}$
- $\emptyset = \{x \mid x > 0 \text{ and } x < 0\}$

$\emptyset$ is called an empty set.

The naive set theory

Let X be a set.

Notions

$x \in X$ means "x is an element of X"

$x \notin X$ means "x is not an element of X"

Examples

- $0 \in \mathbb{N}$
- $-2 \notin \mathbb{N}$
- $-5 \in \mathbb{Z}$
- $\sqrt{2} \notin \mathbb{Q}$

The naive set theory

Let X and Y be sets.

Notions

$X \subseteq Y$ means "X is a subset of Y", i.e., if $x \in X$ then $x \in Y$.

$X \not\subseteq Y$ means "X is not a subset of Y"

Examples

- $\mathbb{N} \subseteq \mathbb{Z}$
- $\mathbb{Z} \subseteq \mathbb{Q}$
- $\mathbb{R} \not\subseteq \mathbb{N}$ ($\sqrt{2} \notin \mathbb{N}$)

The naive set theory

Let X and Y be sets.

Notions

$X = Y$ means "sets X and Y are equal", i.e., $x \in X$ iff $x \in Y$.

*iff = "if and only if"

Examples

- $\{5, 3, 2, 9\} = \{2, 3, 5, 9\}$
- $\mathbb{Z} \neq \mathbb{Q}$

The naive set theory

Properties

- if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$
- $A = B$ iff $A \subseteq B$ and $B \subseteq A$

The naive set theory

Properties

- if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

Direct proof

We need to prove $A \subseteq C$, i.e., for any x, if $x \in A$ then $x \in C$.

Let $x \in A$.

Since $A \subseteq B$, if follows from $x \in A$ that $x \in A$.

Since $B \subseteq C$, if follows from $x \in B$ that $x \in C$.

The naive set theory

Properties

- $A = B$ iff $A \subseteq B$ and $B \subseteq A$

Direct proof

$\Rightarrow$. By definition, $A = B$ means $x \in A$ iff $x \in B$ for any x. It obviously follows from this that $A \subseteq B$ and $B \subseteq A$.

$\Leftarrow$. Since $A \subseteq B$, for any x, if $x \in A$ then $x \in B$.

Since $B \subseteq A$, for any x, if $x \in B$ then $x \in A$.

Therefore, $x \in A$ iff $x \in B$ for any x. Thus, by definition, $A = B$.

The naive set theory

Definition

The empty set $\emptyset$ is a set with no elements, i.e., $x \notin \emptyset$ for any element x.

A property

- $\emptyset \subseteq X$ for any set X

Thank you for your attention!