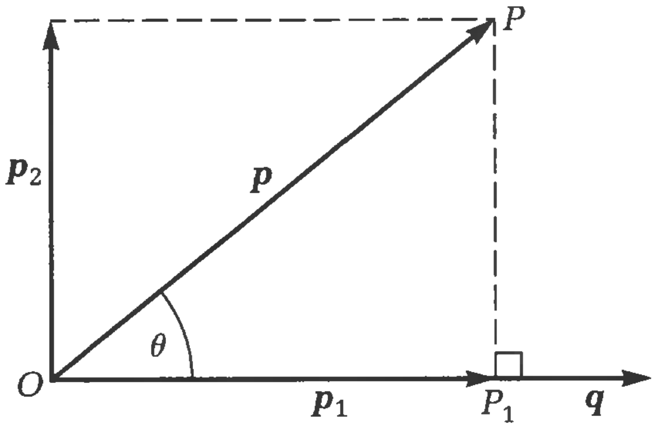
Problem 7

- Decompose the vector **p** = (2, -3, 1) into components parallel and perpendicular to the vector **q** = (12, 3, 4).



Solution

The parallel component is

$$\mathbf{p}_1 = \frac{\mathbf{p}\cdot\mathbf{q}}{|\mathbf{q}|^2}\mathbf{q} = \frac{24-9+4}{12^2+3^2+4^2}(12,3,4) = \frac{19}{169}(12,3,4)$$

And the perpendicular component is

$$\mathbf{p}_2 = \mathbf{p} - \mathbf{p}_1 = (2,-3,1) - \frac{19}{169}(12,3,4) = \left(\frac{110}{169}, -\frac{564}{169}, \frac{93}{169}\right).$$

Unit Vectors

A vector $\mathbf{v}$ of length 1 is called a unit vector.

The standard unit vectors are

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Any vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ can be written as a linear combination of the standard unit vectors as follows:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ v_3 \end{bmatrix}$$

$$= v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = v_1 \, \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$$

Whenever $\mathbf{v} \neq \mathbf{0}$, its length $|\mathbf{v}|$ is not zero and

$$\left| \frac{1}{|\mathbf{v}|} \mathbf{v} \right| = \frac{1}{|\mathbf{v}|} |\mathbf{v}| = 1$$

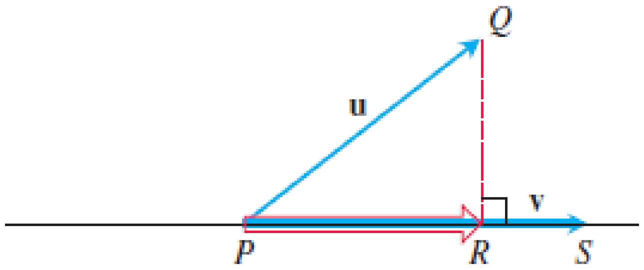
That is, $\frac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector in the direction of $\mathbf{v}$, called the *direction* of the nonzero vector $\mathbf{v}$.

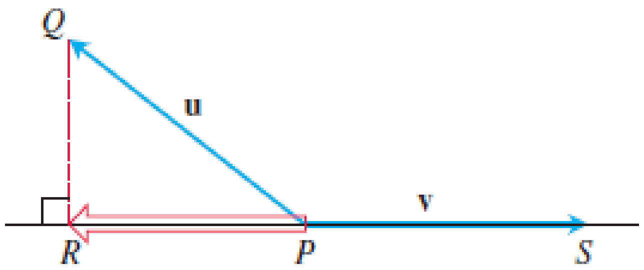
Projection

The vector projection of $\mathbf{u} = \overrightarrow{PQ}$ onto a nonzero vector $\mathbf{v} = \overrightarrow{PS}$ is the vector $\overrightarrow{PR}$ determined by dropping a perpendicular from Q to the line PS.

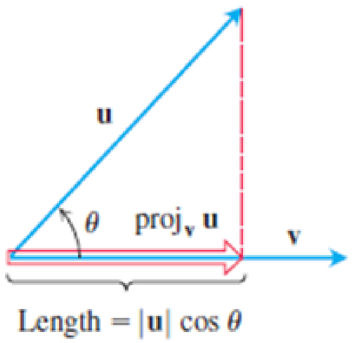
The notation for this vector is

$$proj_{\mathbf{v}}\mathbf{u} \qquad \text{(the vector projection of } \mathbf{u} \text{ onto } \mathbf{v}\text{).}$$

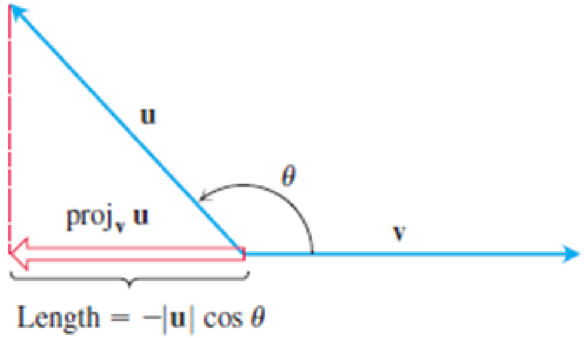




If the angle θ between $\mathbf{u}$ and $\mathbf{v}$ is acute, $proj_{\mathbf{v}}\mathbf{u}$ has length $|\mathbf{u}|\cos\theta$ and direction $\frac{\mathbf{v}}{|\mathbf{v}|}$. If θ is obtuse, $\cos\theta < 0$ and $proj_{\mathbf{v}}\mathbf{u}$ has length $-|\mathbf{u}|\cos\theta$ and direction $-\frac{\mathbf{v}}{|\mathbf{v}|}$.

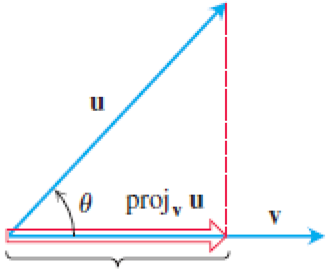


(a)

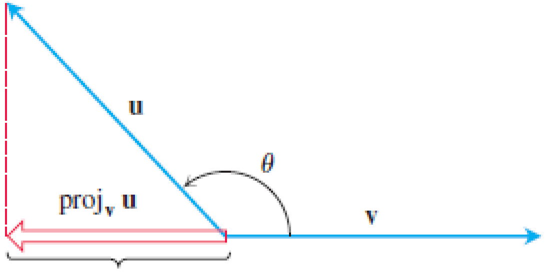


(b)

Projection



(a)



(b)

In both cases,

$$proj_{\mathbf{v}}\mathbf{u} = (|\mathbf{u}|\cos\theta)\frac{\mathbf{v}}{|\mathbf{v}|}$$

$$\boxed{|\mathbf{u}|\cos\theta = \frac{|\mathbf{u}||\mathbf{v}|\cos\theta}{|\mathbf{v}|} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}}$$

$$= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|}$$

$$\boxed{= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}}$$

The number $|\mathbf{u}|\cos\theta$ is called the *scalar component of* $\mathbf{u}$ *in the direction of* $\mathbf{v}$.

Projection

- To summarize,

The vector projection of **u** onto **v** is the vector

$$proj_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$

The scalar component of **u** in the direction of **v** is the scalar

$$|\mathbf{u}|\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = \mathbf{u}\cdot\frac{\mathbf{v}}{|\mathbf{v}|}$$