Discrete Mathematics and Logic
Lecture 6

Andrey Frolov

Innopolis University

Functions

Definitions

A set $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$ is called a function, if

$$\forall x \, \exists^{\leq 1} y \, (x, y) \in X \times Y.$$

The set $\{x \mid f(x) \text{ is defined}\}$ is called the support.

The set $f(X) = \{f(x) \mid x \in X\}$ is called the image.

Functions

Definitions

Let $f : X \to Y$ be a function.

- f is called total, if the support equals to the domain, i.e.,

$$\forall x \, \exists! y \, (x, y) \in X \times Y,$$

- f is called surjective, if the range equals to the image, i.e.,

$$\forall y \in Y \; \exists x \in X \; f(x) = y,$$

- f is called injective, if

$$\forall x_1, x_2 \, (x_1 \neq x_2 \to f(x_1) \neq f(x_2)),$$

- f is called bijection, if it is a total surjective injection.

Cardinality

Definition

$\forall A, B$ (A and B has the same cardinality $\leftrightarrow |A| = |B|$), if there is a bijection $f : A \to B$ (total surjective injection).

Definition

$\forall A, B$ ($|A| \leq |B|$), if there is a injection $f : A \to B$.

Example

$$0 < 1 < 2 < 3 < 4 < \cdots < |\mathbb{N}| = \omega < 2^{\omega}$$

Cardinality

Proposition

- $|A \cup B| = |A| + |B| - |A \cap B|$
- $|A \cap B| \leq \min\{|A|, |B|\}$
- $|A \setminus B| = |A| - |A \cap B| \geq |A| - |B|$
- $|\overline{A}| = |\mathbf{U}| - |A|$
- $|A \times B| = |A| \cdot |B|$

Cardinality

Theorem

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| -$$

$$- \sum_{i,j=1 (i<j)}^{n} |A_i \cap A_j| +$$

$$+ \sum_{i,j,k=1 (i<j<k)}^{n} |A_i \cap A_j \cap A_k| - \cdots$$

$$\cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots A_n|$$

Cardinality

The illustration for $n = 3$.

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| -$$

$$-|A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| +$$

$$+|A_1 \cap A_2 \cap A_3|$$

Cardinality

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example

There are 100 students in a group. 75 students like Discrete Math, 45 students like Math Analysis. How many students like the both courses?

$|A \cup B| = 100$

$|A| = 75$

$|B| = 45$

$|A \cap B| = |A| + |B| - |A \cup B| = 75 + 45 - 100 = 20$

Combinatorics

- Combinatorics is the study of collections of objects.
- Combinatorial proof

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- Combinatorial proof

$$|A \cup B| = |A| + |B| \text{ (if } |A \cap B| = \emptyset)$$

$$|A \times B| = |A| \cdot |B|$$

Combinatorics

Two events are mutually exclusive, i.e., they cannot be done at the same time.

Theorem (Sum Rule)

If

- an event e_1 can be done in n_1 ways,
- an event e_2 can be done in n_2 ways,
- e_1 and e_2 are mutually exclusive,

then

- the number of ways of the both events occurring is $n_1 + n_2$

Sum Rule

Example

In cafe there are the 3 varieties of coffee: Espresso, Latte, Cappuccino; and the 3 varieties of tee: Green tea, Black tee, Flower tee.

The cafe has $3 + 3 = 6$ different varieties.

Combinatorics

Two events are not mutually exclusive.

Theorem (Product Rule)

If

- an event e_1 can be done in n_1 ways,
- an event e_2 can be done in n_2 ways,
- e_1 and e_2 are not mutually exclusive,

then

- the number of ways of the both events occurring is $n_1 \cdot n_2$

Combinatorics

Example

In cafe there are the 3 varieties of coffee: Espresso, Latte, Cappuccino; and the 3 sizes: Small, Medium, Large.

The cafe has $3 \cdot 3 = 9$ different varieties.

	Small	Medium	Large
Espresso	Small Espresso	Medium Espresso	Large Espresso
Latte	Small Latte	Medium Latte	Large Latte
Capp.	Small Cappuccino	Medium Cappuccino	Large Cappuccino

Combinatorics

Example

```
begin
  for i:=1 to n do func1(i);
  for j:=1 to m do func2(j);
end.
```

Combinatorics

Example

```
begin
  for i:=1 to n do func1(i);
  for j:=1 to m do func2(j);
end.
```

$$n + m$$

Combinatorics

Example

```
begin
  for i:=1 to n do
    for j:=1 to m do func(i,j);
end.
```

Combinatorics

Example

```
begin
  for i:=1 to n do
    for j:=1 to m do func(i,j);
end.
```

$$n \cdot m$$

Combinatorics

Exercises

1) Count all of the numbers that have exactly 3 digits; and the numbers that have at most 2 digits.
2) How many possible outcomes are there from a game cube; two game cubes?

Permutations

$$\{a, b, c, d\} = \{d, b, c, a\}$$

$$(a, b, c, d) \neq (d, b, c, a)$$

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$$\{a, b, c, d\} = \{d, b, c, a\}$$

$$(a, b, c, d) \neq (d, b, c, a)$$

$$\{x, x, x, x\} = \{x\}$$

$$(x, x, x, x) \neq (x, x, x) \neq (x, x) \neq x$$

Permutations

$$\{a, b, c, d\} = \{d, b, c, a\}$$

$$(a, b, c, d) \neq (d, b, c, a)$$

$$\{x, x, x, x\} = \{x\}$$

$$(x, x, x, x) \neq (x, x, x) \neq (x, x) \neq x$$

Example

Let $A = \{a, b, c\}$.
$A^2 = \{(a, a), (a, b), (b, a), (b, b)\}$
Without repetitions: $\{(a, b), (b, a)\}$ – permutations.

Permutations

Definition

A permutation of a set of distinct objects is an ordered arrangement of these objects.

Theorem

The number of k permutations (k-permutations) of a set with n distinct objects is

$$P(n,k) = n(n-1)(n-2)\cdots(n-k+1) = \prod_{i=0}^{k-1}(n-i)$$

Permutations

Theorem

The number of k permutations of a set with n distinct objects is

$$P(n,k) = n(n-1)(n-2)\cdots(n-k+1) = \prod_{i=0}^{k-1}(n-i)$$

Example

Let $A = \{a, b, c\}$.

2-permutations: $(a,b),(a,c),(b,a),(b,c),(c,a),(c,b)$

$P(3,2) = 3 \cdot 2 = 6$

Permutations

Theorem

The number of k permutations of a set with n distinct objects is

$$P(n,k) = n(n-1)(n-2)\cdots(n-k+1) = \prod_{i=0}^{k-1}(n-i)$$

Example

How many 3-letter words can you form from the letters of the word "HOW"?

3-permutations: HOW, HWO, OHW, OWH, WHO, WOH.

$P(3,1) = 3 \cdot 2 \cdot 1 = 6$

Permutations

Theorem

The number of k permutations of a set with n distinct objects is

$$P(n,k) = n(n-1)(n-2)\cdots(n-k+1) = \prod_{i=0}^{k-1}(n-i)$$

Proof by induction (by k)

Let $A = \{a_1, \ldots, a_n\}$

1) $k = 1$. $P(n,1) = n$
1-permutations: $(a_1), (a_2), \ldots, (a_n)$.

Permutations

Proof by induction

2) Suppose that there is k_0 such that
$P(n, k_0) = n(n-1)(n-2)\cdots(n-k_0+1)$.

3) We need to prove the theorem for $k_0 + 1$.

(k_0+1)-permutations: $(\underbrace{a_{i_1}, a_{i_2}, \ldots, a_{i_{k_0}}}_{k_0-\text{permutations}}, a)$

$a \neq a_{i_j}$ for $1 \leq j \leq k_0$

So, for any k_0-permutation there are $n - k_0$ many elements a

Permutations

Proof by induction

Therefore,

$$P(n, k_0 + 1) = P(n, k_0) \cdot (n - k_0) =$$

$$= n(n-1)(n-2)\cdots(n-k_0+1)(n-k_0) =$$

$$= n(n-1)(n-2)\cdots(n-k_0+1)(n-(k_0+1)+1) =$$

$$= \prod_{i=0}^{(k_0+1)-1} (n-i)$$

Permutations

Notation

We use $n!$ (n factorial) to denote the number of permutations of n objects.

$$n! = n(n-1)(n-2)\cdots 2 \cdot 1$$

For convenient, $0! = 1$

Permutations

Definition (by induction)

1. Let $0! = 1$.
2. Suppose that $k!$ is defined.
3. Let $(k+1)! = (k+1) \cdot k!$

Permutations

Intuitively, why $0! = 1$

$$(k+1)! = \frac{(k+2)!}{k+2}$$

$$k! = \frac{(k+1)!}{k+1}$$

$$\cdots$$

$$2! = \frac{3!}{3}$$

$$1! = \frac{2!}{2}$$

$$0! = \frac{1!}{1} = 1$$

Permutations

Notation

$$P(n,k) = n(n-1)(n-2)\cdots(n-k+1) =$$

$$= n(n-1)(n-2)\cdots(n-k+1)\frac{(n-k)(n-k-1)\cdots 2\cdot 1}{(n-k)(n-k-1)\cdots 2\cdot 1} = \frac{n!}{(n-k)!}$$

$$P(n,k) = \frac{n!}{(n-k)!}$$

$$P(n,n) = n!$$

Permutations

Example

```
begin
 for i:=1 to n do
  for j:=1 to n do
   if (i<>j) then func(i,j);
end.
```

Permutations

Example

```
begin
  for i:=1 to n do
    for j:=1 to n do
      if (i<>j) then func(i,j);
end.
```

$i=1$	$i=2$	$i=3$	$\cdots$	$i=n-1$	$i=n$
2	1	1	$\cdots$	1	1
3	3	2	$\cdots$	2	2
4	4	4	$\cdots$	3	3
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$n-1$	$n-1$	$n-1$	$\cdots$	$n-3$	$n-2$
n	n	n	$\cdots$	n	$n-1$

$$n \times (n-1)$$

Permutations

Exercise

```
begin
  for i:=1 to n do
    for j:=i to m do func(i,j);
end.
```

Permutations

Exercise

```
begin
  for i:=1 to n do
    for j:=i to m do func(i,j);
end.
```

How many times the function ”func“ works?

Thank you for your attention!