Tutorial 2: Vectors

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Course of Essentials of Analytical Geometry and Linear Algebra I

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Review (Last week topics)

Points and Vectors

- Three-Dimensional Coordinate Systems (Geometry of Space)
  - Geometric Interpretations of Equations
  - Geometric Interpretations of Inequalities and Equations
  - Inequalities to Describe Sets of Points
  - Distance
  - Spheres
- Vectors
  - Definition
  - Operations
  - Vector spaces

Content

Vectors

- Span, Linear Dependence and Independence
- Vector Spaces and Subspaces
- Dot Product
- Outer Product
- Vector Cross Product

Problem 1

- In the following exercises, list five vectors in Span $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$. For each vector, show the weights on v_1 and v_2, used to generate the vector and list the three entries of the vector. Do not make a sketch.

$$v_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}, \qquad v_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$$

Solution

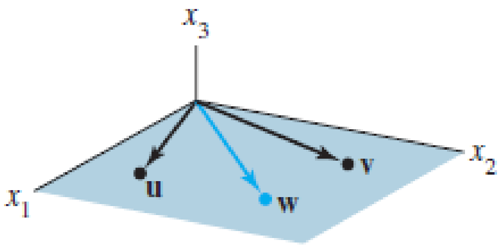
- Non integer weights are acceptable, of course, but some simple choices are $0 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2 = \mathbf{0}$, and

$$1 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, 0 \cdot \mathbf{v}_1 + 1 \cdot \mathbf{v}_2 = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}, 1 \cdot \mathbf{v}_1 + 1 \cdot \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, 1 \cdot \mathbf{v}_1 - 1 \cdot \mathbf{v}_2 = \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix}$$

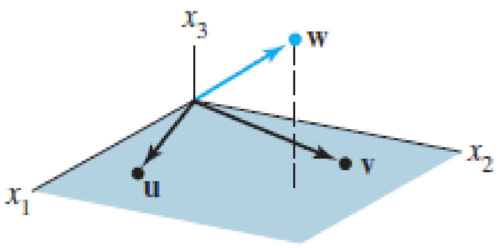
- Some likely choices are $0 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2 = \mathbf{0}$, and

$$1 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, 0 \cdot \mathbf{v}_1 + 1 \cdot \mathbf{v}_2 = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, 1 \cdot \mathbf{v}_1 + 1 \cdot \mathbf{v}_2 = \begin{bmatrix} -1 \\ 4 \\ -2 \end{bmatrix}, 1 \cdot \mathbf{v}_1 - 1 \cdot \mathbf{v}_2 = \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix}$$

Theorem 1



Linearly dependent,
w in Span{**u**, **v**}



Linearly independent,
w *not* in Span{**u**, **v**}

Characterization of Linearly Dependent Sets

- An indexed set $S = \{v_1, \ldots, v_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$, (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$

- If a set contains more vectors than there are entries in vector, then the set is linearly dependent. That is, any set $S = \{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

$$\begin{array}{cc} & p \\ n & \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \end{array}$$

- If a set $S = \{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

Problem 2

- Determine if the vectors are linearly independent. Justify each answer.

$$\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}$$

Solution

1. Use an augmented matrix to study the solution set of $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{0}$ $(*)$, where **u**, **v**, and **w** are the three given vectors. Since $\begin{bmatrix} 5 & 7 & 9 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & -6 & -8 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & 7 & 9 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$ there are no free variables. So the homogeneous equation (*) has only the trivial solution. The vectors are linearly independent.

Problem 3

- Find the value(s) of h for which the vectors are linearly *dependent*. Justify each answer.

$$\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ h \end{bmatrix}$$

Solution

To study the linear dependence of three vectors, say $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, row reduce the augmented matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{0}]$:

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ -2 & -6 & 2 & 0 \\ 4 & 7 & h & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & -2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & -1 & h+4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & -2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & h+4 & 0 \end{bmatrix}$$

The equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has a nontrivial solution if and only if $h + 4 = 0$ (which corresponds to x_3 being a free variable).

Thus, the vectors are linearly dependent if and only if $h = -4$.

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To study the linear dependence of three vectors, say $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, row reduce the augmented matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{0}]$:

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ -2 & -6 & 2 & 0 \\ 4 & 7 & h & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & -2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & -1 & h+4 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & -2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & h+4 & 0 \end{bmatrix}$$

The equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has a nontrivial solution if and only if $h + 4 = 0$ (which corresponds to x_3 being a free variable).

Thus, the vectors are linearly dependent if and only if $h = -4$.

Problem 4 (1)

- Each statement in the following exercises is either true (in all cases) or false (for at least one example). If false, construct a specific example to show that the statement is not always true.

1. If $\mathbf{v}_1, \dots, \mathbf{v}_4$ are in $\mathbb{R}^4$ and $\mathbf{v}_3 = 2\mathbf{v}_1 + \mathbf{v}_2$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent.
   - True, by Theorem in slide 5.
2. If $\mathbf{v}_1, \mathbf{v}_2$ are in $\mathbb{R}^4$ and $\mathbf{v}_2$ is not a scalar multiple of $\mathbf{v}_1$, then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.
   - False. The vector $\mathbf{v}_i$ could be the zero vector.
3. If $\mathbf{v}_1, \dots, \mathbf{v}_5$ are in $\mathbb{R}^5$ and $\mathbf{v}_3 = 0$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ is linearly dependent.
   - True, by Theorem in slide 5.
4. If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are in $\mathbb{R}^3$ and $\mathbf{v}_3$, is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.
5. If $\mathbf{v}_1, \dots, \mathbf{v}_4$ are in $\mathbb{R}^4$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is also linearly dependent

Problem 4 (2)

- Each statement in the following exercises is either true (in all cases) or false (for at least one example). If false, construct a specific example to show that the statement is not always true.

4. If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are in $\mathbb{R}^3$ and $\mathbf{v}_3$, is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.
   - False. Counterexample: Take $\mathbf{v}_1$ and $\mathbf{v}_2$ to be multiples of one vector. Take $\mathbf{v}_3$ to be not a multiple of that vector. For example,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

5. If $\mathbf{v}_1, \dots, \mathbf{v}_4$ are in $\mathbb{R}^4$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is also linearly dependent.
   - True. A linear dependence relation among $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ may be extended to a linear dependence relation among $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, by placing a zero weight on $\mathbf{v}_4$.

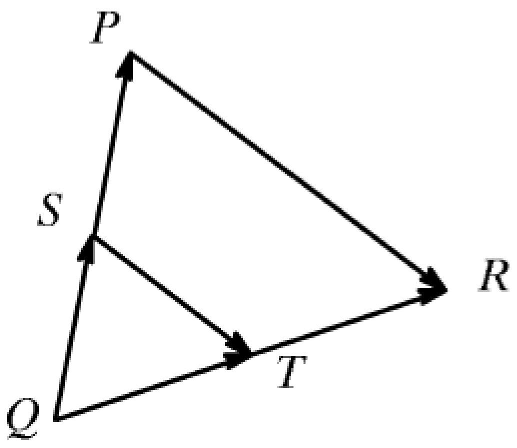
Problem 5

- Let P, Q, and R be the vertices of a triangle in $\mathbb{R}^2$ or $\mathbb{R}^3$. Use vectors to show that the line segment joining the midpoints of any two sides of the triangle is parallel to and one-half the length of the third side. (Note: two vectors are parallel if and only if one is a scalar multiple of the other.)

Solution

Let S be the midpoint of $\overrightarrow{QP}$ and T the midpoint of $\overrightarrow{QR}$. Then

$$\overrightarrow{ST} = \overrightarrow{QT} - \overrightarrow{QS} = \frac{1}{2}\overrightarrow{QR} - \frac{1}{2}\overrightarrow{QP} = \frac{1}{2}(\overrightarrow{QR} - \overrightarrow{QP}) = \frac{1}{2}\overrightarrow{PR}.$$



Problem 6

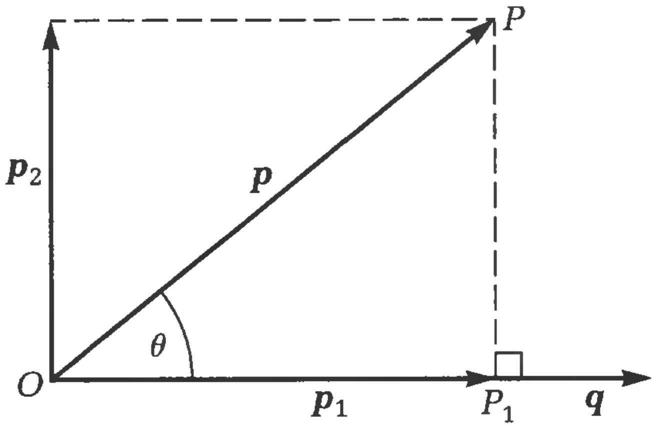
➢ Find the angle between the vectors $\mathbf{p} = (1, -2, 4)$ and $\mathbf{q} = (3, 5, 2)$.

Solution

$$\cos\theta = \frac{\mathbf{p}\cdot\mathbf{q}}{|\mathbf{p}||\mathbf{q}|} = \frac{1\cdot 3 + (-2)\cdot 5 + 4\cdot 2}{\sqrt{21}\cdot\sqrt{38}} = \frac{1}{\sqrt{798}} \approx 0.0354.$$

Hence $\theta \approx 1.535$ radians $\approx 87.9\,°$

- Decompose the vector **p** = (2, -3, 1) into components parallel and perpendicular to the vector **q** = (12, 3, 4).



Solution

The parallel component is

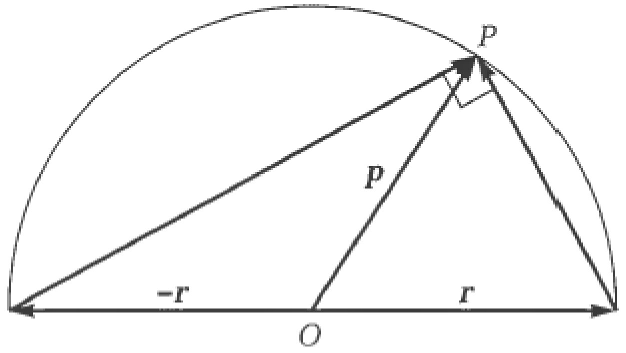
$$\mathbf{p}_1 = \frac{\mathbf{p}\cdot\mathbf{q}}{|\mathbf{q}|^2}\mathbf{q} = \frac{24-9+4}{12^2+3^2+4^2}(12,3,4) = \frac{19}{169}(12,3,4)$$

And the perpendicular component is

$$\mathbf{p}_2 = \mathbf{p}-\mathbf{p}_1 = (2,-3,1) - \frac{19}{169}(12,3,4) = \left(\frac{110}{169}, -\frac{564}{169}, \frac{93}{169}\right).$$

Problem 8

- Using dot products, prove the Theorem of Thales: If we take a point P on a circle and form a triangle by joining it to the opposite ends of an arbitrary diameter, then the angle at P is a right angle.



Solution

Call the end points of the given diameter A and B. Then, with the notation of on the figure,

$$\overrightarrow{AP} \cdot \overrightarrow{BP} = (\mathbf{p} - \mathbf{r}) \cdot (\mathbf{p} + \mathbf{r}) = \mathbf{p} \cdot \mathbf{p} - \mathbf{r} \cdot \mathbf{r} = |\mathbf{p}|^2 - |\mathbf{r}|^2 = 0.$$

The last equality holds because **p** and **r** are both radius vectors, and so their lengths are equal.

Problem 9 (1)

- Let $\mathbf{u} = 5\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{j} - 5\mathbf{k}$, $\mathbf{w} = -15\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$. Which vectors, if any, are (a) perpendicular? (b) Parallel? Give reasons for your answers.

Solution

a) The dot product tests for perpendicular vectors. For example, if $\mathbf{u} \cdot \mathbf{v} = 0$. then the vectors are perpendicular.

$$\mathbf{u} \cdot \mathbf{v} = (5, -1, 1) \cdot (0, 1, -5)$$
$$= 5(0) + (-1)(1) + 1(-5)$$
$$= -6$$

$\therefore$ not perpendicular

$$\mathbf{u} \cdot \mathbf{w} = (5, -1, 1) \cdot (-15, 3, -3)$$
$$= 5(-15) + (-1)(3) + 1(-3)$$
$$= -81$$

$\therefore$ not perpendicular

$$\mathbf{v} \cdot \mathbf{w} = (0, 1, -5) \cdot (-15, 3, -3)$$
$$= 0(-15) + 1(3) + (-5)(-3)$$
$$= 18$$

$\therefore$ not perpendicular

Problem 9 (2)

b) The cross product tests for parallel vectors.

If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$. then the vectors are parallel.

If $\mathbf{u} \times \mathbf{w} = \mathbf{0}$. then the vectors are parallel.

If $\mathbf{v} \times \mathbf{w} = \mathbf{0}$. then the vectors are parallel.

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -5 \\ -15 & 3 & -3 \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} 1 & -5 \\ 3 & -3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 0 & -5 \\ -15 & -3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 0 & 1 \\ -15 & 3 \end{vmatrix}$$

$$= \mathbf{i}(-3 + 15) - \mathbf{j}(0 - 75) + \mathbf{k}(0 + 15)$$

$$= (12{,}75{,}15)$$

$$\neq 0$$

$\therefore$ not parallel

$$\mathbf{u} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & 1 \\ -15 & 3 & -3 \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} -1 & 1 \\ 3 & -3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 5 & 1 \\ -15 & -3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 5 & -1 \\ -15 & 3 \end{vmatrix}$$

$$= \mathbf{i}(3 - 3) - \mathbf{j}(-15 + 15) + \mathbf{k}(15 - 15)$$

$$= (0{,}0{,}0)$$

$$= 0$$

$\therefore$ parallel

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & 1 \\ 0 & 1 & -5 \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} -1 & 1 \\ 1 & -5 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 5 & 1 \\ 0 & -5 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 5 & -1 \\ 0 & 1 \end{vmatrix}$$

$$= \mathbf{i}(5 - 1) - \mathbf{j}(-25 - 0) + \mathbf{k}(5 - 0)$$

$$= (4{,}25{,}0)$$

$$\neq 0$$

$\therefore$ not parallel

Problem 10 (1)

- Which of the following are always true, and which are not always true? Give reasons for your answers.

  a. $|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}}$
  b. $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|$
  c. $\mathbf{u}\times\mathbf{0} = \mathbf{0}\times\mathbf{u} = \mathbf{0}$
  d. $\mathbf{u}\times(-\mathbf{u}) = \mathbf{0}$
  e. $\mathbf{u}\times\mathbf{v} = \mathbf{v}\times\mathbf{u}$
  f. $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v} + \mathbf{u}\times\mathbf{w}$
  g. $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = 0$
  h. $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$

a)

Let $\mathbf{u} = \langle u_x, u_y, u_z\rangle$

$$|\mathbf{u}| = \sqrt{u_x^2+u_y^2+u_z^2} \qquad \text{(a-1)}$$

$$\sqrt{\mathbf{u}\cdot\mathbf{u}} = \sqrt{\langle u_x,u_y,u_z\rangle\cdot\langle u_x,u_y,u_z\rangle}$$
$$\Rightarrow\sqrt{\mathbf{u}\cdot\mathbf{u}} = \sqrt{u_x^2+u_y^2+u_z^2} \qquad \text{(a-2)}$$

From (a-1) and (a-2)

$$|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}}$$

without any restrictions, hence **(a)** is always true

b)

$$|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}} \qquad \text{(was proven in (a))}$$
$$\Rightarrow|\mathbf{u}|^2 = \mathbf{u}\cdot\mathbf{u} \qquad \text{(squaring both sides)}$$

Hence **(b)** is not true, except for the cases where
$|\mathbf{u}|^2 = |\mathbf{u}| \to |\mathbf{u}| = 1$ or $|\mathbf{u}| = 0$

c)

$$\mathbf{u}\times\mathbf{0} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_x & u_y & u_z \\ 0 & 0 & 0 \end{vmatrix}$$

$$\Rightarrow\mathbf{u}\times\mathbf{0} = \begin{vmatrix} u_y & u_z \\ 0 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_x & u_z \\ 0 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_x & u_y \\ 0 & 0 \end{vmatrix}\mathbf{k} = 0\mathbf{i}+0\mathbf{j}+0\mathbf{k} = \mathbf{0} \qquad \text{(c-1)}$$

$$\mathbf{0}\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0 \\ u_x & u_y & u_z \end{vmatrix}$$

$$\Rightarrow\mathbf{0}\times\mathbf{u} = \begin{vmatrix} 0 & 0 \\ u_y & u_z \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 0 \\ u_x & u_z \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 0 \\ u_x & u_y \end{vmatrix}\mathbf{k} = 0\mathbf{i}+0\mathbf{j}+0\mathbf{k} = \mathbf{0} \qquad \text{(c-2)}$$

From (c-1) and (c-2)

$$\mathbf{u}\times\mathbf{0} = \mathbf{0}\times\mathbf{u} = \mathbf{0}$$

Hence **(c)** is always true.

a. $|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}}$

b. $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|$

c. $\mathbf{u}\times\mathbf{0} = \mathbf{0}\times\mathbf{u} = \mathbf{0}$

d. $\mathbf{u}\times(-\mathbf{u}) = \mathbf{0}$

e. $\mathbf{u}\times\mathbf{v} = \mathbf{v}\times\mathbf{u}$

f. $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v} + \mathbf{u}\times\mathbf{w}$

g. $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = 0$

h. $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$

d)

$$\mathbf{u}\times-\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_x & u_y & u_z \\ -u_x & -u_y & -u_z \end{vmatrix}$$

$$\Rightarrow \mathbf{u}\times-\mathbf{u} = \begin{vmatrix} u_y & u_z \\ -u_y & -u_z \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_x & u_z \\ -u_x & -u_z \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_x & u_y \\ -u_x & -u_y \end{vmatrix}\mathbf{k} = 0\mathbf{i}+0\mathbf{j}+0\mathbf{k} = \mathbf{0}$$

Hence **(d)** is always true.

e)

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}$$

$$\Rightarrow \mathbf{u}\times\mathbf{v} = \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_x & u_z \\ v_x & v_z \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}\mathbf{k}$$

$$\Rightarrow \mathbf{u}\times\mathbf{v} = (u_y v_z - u_z v_y)\,\mathbf{i} - (u_x v_z - u_z v_x)\,\mathbf{j} + (u_x v_y - u_y v_x)\,\mathbf{k} \qquad \text{(e-1)}$$

$$\mathbf{v}\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_x & v_y & v_z \\ u_x & u_y & u_z \end{vmatrix}$$

$$\Rightarrow \mathbf{v}\times\mathbf{u} = \begin{vmatrix} v_y & v_z \\ u_y & u_z \end{vmatrix}\mathbf{i} - \begin{vmatrix} v_x & v_z \\ u_x & u_z \end{vmatrix}\mathbf{j} + \begin{vmatrix} v_x & v_y \\ u_x & u_y \end{vmatrix}\mathbf{k}$$

$$\Rightarrow \mathbf{v}\times\mathbf{u} = (v_y u_z - v_z u_y)\,\mathbf{i} - (v_x u_z - v_z u_x)\,\mathbf{j} + (v_x u_y - v_y u_x)\,\mathbf{k}$$

$$\Rightarrow \mathbf{v}\times\mathbf{u} = -\left((v_z u_y - v_y u_z)\,\mathbf{i} - (v_z u_x - v_x u_z)\,\mathbf{j} + (v_y u_x - v_x u_y)\,\mathbf{k}\right) = -\mathbf{u}\times\mathbf{v} \quad \text{(from (e-1))}$$

From the previous proof $\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}$, hence **(e)** will be true iff

$$\mathbf{u}\times\mathbf{v} = \mathbf{0} \text{ because } \mathbf{0} = -\mathbf{0}$$

a. $|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$

b. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|$

c. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

d. $\mathbf{u} \times (-\mathbf{u}) = \mathbf{0}$

e. $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$

f. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$

g. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$

h. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$

f)

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_x & u_y & u_z \\ (v_x + w_x) & (v_y + w_y) & (v_z + w_z) \end{vmatrix}$$

$$\Rightarrow \mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \begin{vmatrix} u_y & u_z \\ (v_y + w_y) & (v_z + w_z) \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_x & u_z \\ (v_x + w_x) & (v_z + w_z) \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_x & u_y \\ (v_x + w_x) & (v_y + w_x) \end{vmatrix} \mathbf{k}$$

$$\Rightarrow \mathbf{u} \times (\mathbf{v} + \mathbf{w})$$
$$= (u_y(v_z + w_z) - u_z(v_y + w_y))\,\mathbf{i} - (u_x(v_z + w_z) - u_z(v_x + w_x))\,\mathbf{j} + (u_x(v_y + w_y) - u_y(v_x + w_x))\,\mathbf{k}$$

$$\Rightarrow \mathbf{u} \times (\mathbf{v} + \mathbf{w})$$
$$= (u_y v_z - u_z v_y)\,\mathbf{i} - (u_x v_z - u_z v_x)\,\mathbf{j} + (u_x v_y - u_y v_x)\,\mathbf{k} + (u_y w_z - u_z w_y)\,\mathbf{i} - (u_x w_z - u_z w_x)\,\mathbf{j} + (u_x w_y - u_y w_x)\,\mathbf{k}$$

$$\Rightarrow \mathbf{u} \times (\mathbf{v} + \mathbf{w})$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_x & u_y & u_z \\ w_x & w_y & w_z \end{vmatrix} = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$$

Hence (f) is always true

g)

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ v_x & v_y & v_z \end{vmatrix}$$

$$\Rightarrow (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = \begin{vmatrix} v_y & v_z \\ v_y & v_z \end{vmatrix} u_x - \begin{vmatrix} v_x & v_z \\ v_x & v_z \end{vmatrix} u_y + \begin{vmatrix} v_x & v_y \\ v_x & v_y \end{vmatrix} u_z$$

$$\Rightarrow (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = (0)\ u_x - (0)\ u_y + (0)\ u_z = 0$$

Hence **(g)** is always true

h)

From the determinant properties, if two rows replace their positions, the determinant value will be negative of the original value

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \xrightarrow{\text{replacing the positions of } r_1 \text{ and } r_2} = - \begin{vmatrix} v_x & v_y & v_z \\ u_x & u_y & u_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$\Rightarrow (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = - \begin{vmatrix} v_x & v_y & v_z \\ u_x & u_y & u_z \\ w_x & w_y & w_z \end{vmatrix} \xrightarrow{\text{replacing the positions of } r_2 \text{ and } r_3} = - \left(- \begin{vmatrix} v_x & v_y & v_z \\ w_x & w_y & w_z \\ u_x & u_y & u_z \end{vmatrix} \right)$$

$$\Rightarrow (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} v_x & v_y & v_z \\ w_x & w_y & w_z \\ u_x & u_y & u_z \end{vmatrix} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

(the scalar product is commutative)

Hence **(g)** is always true

Problem 11

- **a.** Since $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$, show that the inequality $|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}||\mathbf{v}|$ holds for any vectors $\mathbf{u}$ and $\mathbf{v}$.
- **b.** Under what circumstances, if any, does $|\mathbf{u} \cdot \mathbf{v}|$equal $|\mathbf{u}||\mathbf{v}|$? Give reasons for your answer.

(a) Since $|\cos\theta| \leq 1$, we have

$$|u \cdot v| = |u||v||\cos\theta| \leq |u||v|(1) = |u||v|$$

(b) We have equality precisely when $|cos\theta| = 1$ or when one or both of u and v is 0. In the case of nonzero vectors, we have equality when

$\theta = 0$ or π, i.e., when the vectors are parallel.