1 Changing Basis and Coordinates

Suppose we have two different coordinate systems. The first (so-called "old" coordinate system[1]) is given by origin O and basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$; the second (the "new" one) is given by O', $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$. Let point N have coordinates $\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^T$ in the old coordinate system and coordinates $\begin{pmatrix} x'_1 & x'_2 & x'_3 \end{pmatrix}^T$ in the new one. It means that

$$\begin{aligned} \overrightarrow{ON} &= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3; \\ \overrightarrow{O'N} &= x'_1\mathbf{e}'_1 + x'_2\mathbf{e}'_2 + x'_3\mathbf{e}'_3. \end{aligned} \tag{1}$$

Let us say that we know how the coordinate systems are related with each other. That is, we can express new basis vectors via the old ones and we know the coordinates of the new origin in the old basis. In detail,

$$\begin{aligned} \mathbf{e}'_1 &= \alpha_{11}\mathbf{e}_1 + \alpha_{21}\mathbf{e}_2 + \alpha_{31}\mathbf{e}_3 \\ \mathbf{e}'_2 &= \alpha_{12}\mathbf{e}_1 + \alpha_{22}\mathbf{e}_2 + \alpha_{32}\mathbf{e}_3 \\ \mathbf{e}'_3 &= \alpha_{13}\mathbf{e}_1 + \alpha_{23}\mathbf{e}_2 + \alpha_{33}\mathbf{e}_3 \\ \overrightarrow{OO'} &= b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 \end{aligned}$$

As $\overrightarrow{ON} = \overrightarrow{OO'} + \overrightarrow{O'N}$, we get

$$\begin{aligned} \overrightarrow{ON} = {} & b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 + \\ & x'_1\left(\alpha_{11}\mathbf{e}_1 + \alpha_{21}\mathbf{e}_2 + \alpha_{31}\mathbf{e}_3\right) + x'_2\left(\alpha_{12}\mathbf{e}_1 + \alpha_{22}\mathbf{e}_2 + \alpha_{32}\mathbf{e}_3\right) + x'_3\left(\alpha_{13}\mathbf{e}_1 + \alpha_{23}\mathbf{e}_2 + \alpha_{33}\mathbf{e}_3\right) = \\ & \left(b_1 + \alpha_{11}x'_1 + \alpha_{12}x'_2 + \alpha_{13}x'_3\right)\mathbf{e}_1 + \left(b_2 + \alpha_{21}x'_1 + \alpha_{22}x'_2 + \alpha_{23}x'_3\right)\mathbf{e}_2 + \left(b_3 + \alpha_{31}x'_1 + \alpha_{32}x'_2 + \alpha_{33}x'_3\right)\mathbf{e}_3. \end{aligned}$$

Taking (1) into account yields

$$\begin{aligned} x_1 &= b_1 + \alpha_{11}x'_1 + \alpha_{12}x'_2 + \alpha_{13}x'_3, \\ x_2 &= b_2 + \alpha_{21}x'_1 + \alpha_{22}x'_2 + \alpha_{23}x'_3, \\ x_3 &= b_3 + \alpha_{31}x'_1 + \alpha_{32}x'_2 + \alpha_{33}x'_3, \end{aligned}$$

or, using matrix notation,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix}.$$

Thus knowing how new basis depends on the old one enables us to immediately express the old coordinates through the new ones.

Matrix $A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$ is called a *transition matrix from the old basis to the new basis.* Using matrix notation, one can easily derive that basis vectors satisfy the equality

$$\begin{pmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 & \mathbf{e}'_3 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix} A.$$

As for coordinates,

$$\mathbf{x} = \mathbf{b} + A\mathbf{x}'.$$

[1]In order not to get confused we will refer to a basis, coordinates etc. without primes as to "old" ones and to those with primes as to "new" ones.