Tutorial 10 : Quadratic Curves

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Course of Essentials of Analytical Geometry and Linear Algebra I

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Last weeks' topics

- Vectors
- Matrices
- Lines in Space
- Planes in Space

Content

- Quadratic Curves
  - Parabolas
  - Circles
  - Ellipses

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

Materials are taken from (Conic Sections Geometry Circle Ellipse) with modifications.

Conic Sections (1/2)

Conic sections are the curves obtained by intersecting a plane and a right circular cone.

- ❖ A plane perpendicular to the cone's axis cuts out a circle;
- ❖ A plane parallel to a side of the cone produces a parabola;
- ❖ A plane at an arbitrary angle to the axis of the cone forms an ellipse;
- ❖ A plane parallel to the axis cuts out a hyperbola.

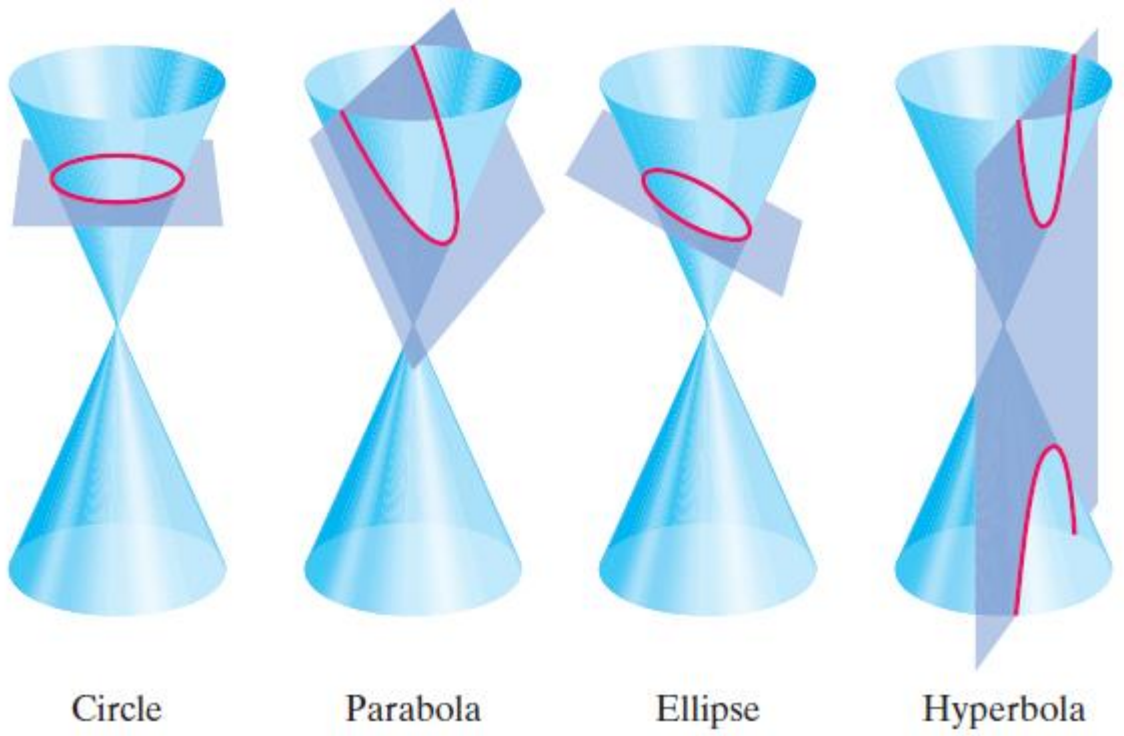


*Figure from internet.

When the plane does pass through the vertex, the resulting figure is a **degenerate conic**.

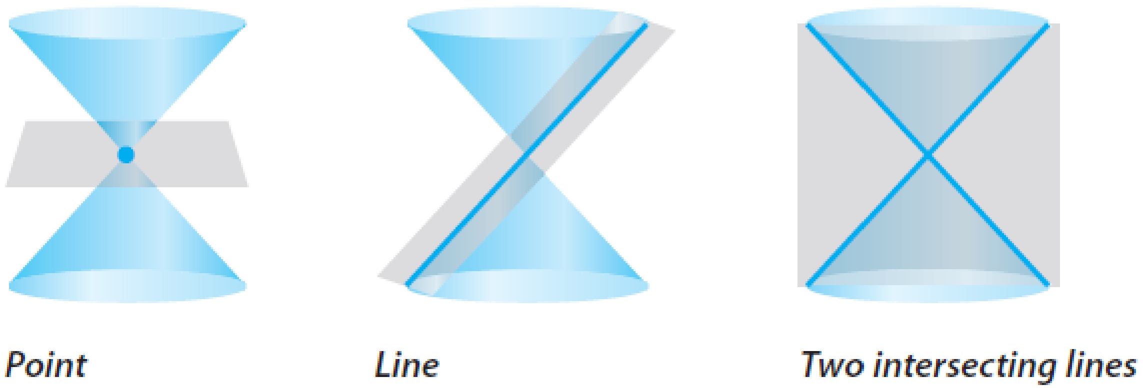


*Figure from internet.

Parabolas (1/2)

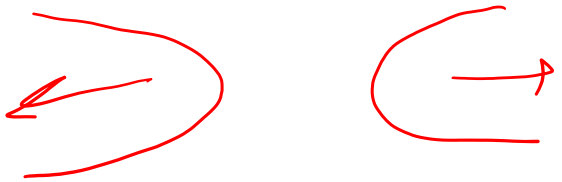
Definition of a Parabola

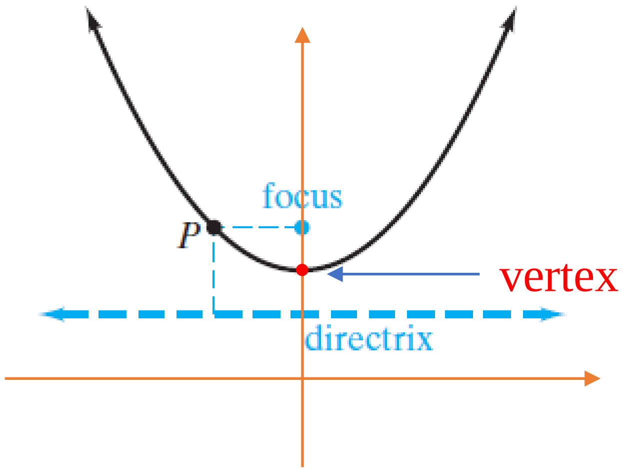
A parabola is the set of all points (x, y) in a plane that are equidistant from a fixed line, the **directrix**, and a fixed point, the **focus**, not on the line. As shown in the figure, the midpoint between the focus and the directrix is the **vertex**, and the line passing through the focus and the vertex is the **axis** of the parabola.

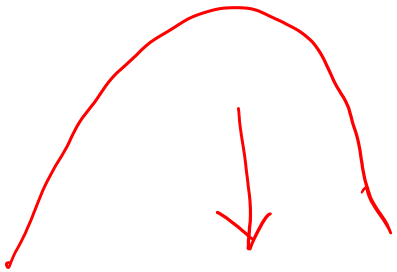
We know that the graph of the quadratic function

$$f(x) = ax^2 + bx + c$$

is a parabola that opens upward or downward.







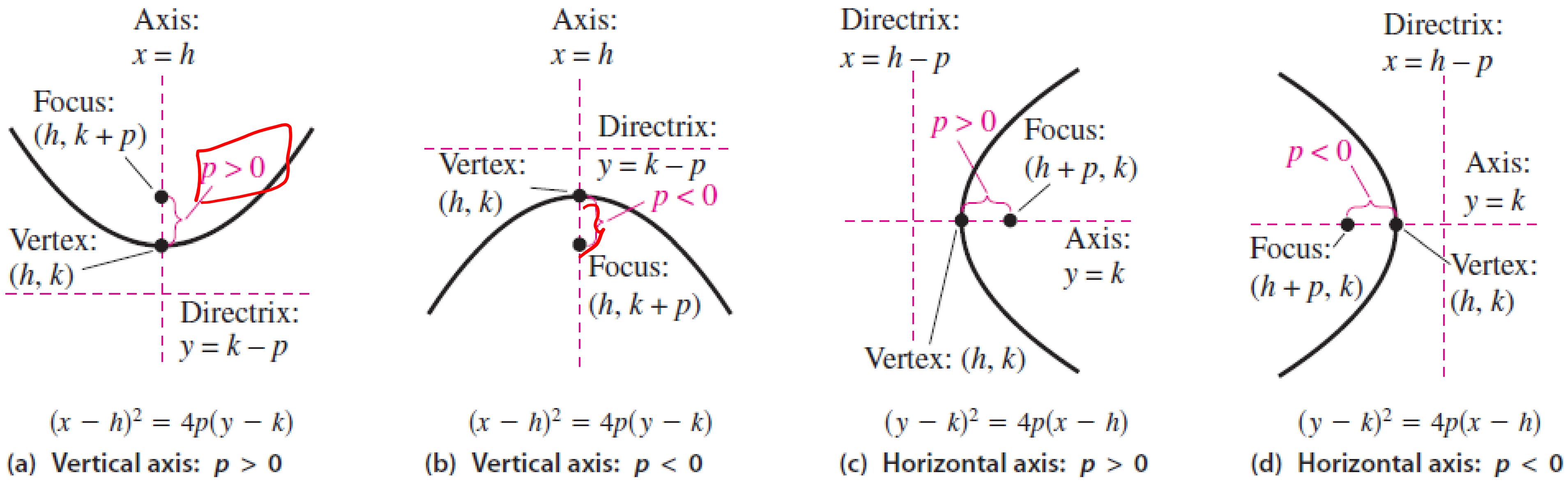
Standard Equation of a Parabola with vertex at (h, k)

- $(x - h)^2 = 4p\,(y - k)$, where the focus is $(h, k + p)$ and the directrix is $y = k - p$ and $p \neq 0.$ Vertical axis
- $(y - k)^2 = 4p\,(x - h)$, where the focus is $(h + p, k)$ and the directrix is $x = h - p$ and $p \neq 0.$

The focus lies on the axis p units (directed distance) from the vertex.

Another form of the equation of a parabola with vertex at (h, k)

- $y = a(x - h)^2 + k$, where a, h and k are real numbers.



Example 1

➢ Find the focus of the parabola given by $y = -\frac{1}{2}x^2 - x + \frac{1}{2}$. → convert to the standard form by completing the square.

Solution:

$$y = -\frac{1}{2}x^2 - x + \frac{1}{2}$$

$\xrightarrow{\times -2}$

$$-2y = x^2 + 2x - 1$$

$\xrightarrow{+(+1)}$

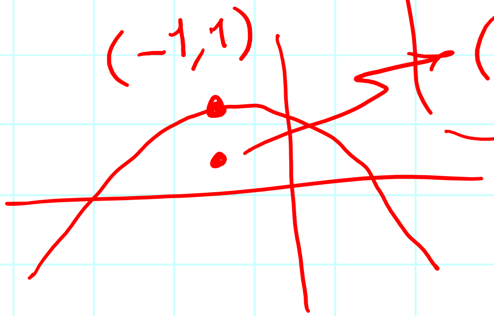
$$+1+1-2y = x^2 + 2x + 1$$

$$+1+1-2y = x^2 + 2x + 1$$

$$2 - 2y = x^2 + 2x + 1$$

$$\begin{cases} -2(y-1) = (x+1)^2 & \text{standard eq.} \\ (x-h)^2 = 4p(y-k) \end{cases}$$

$\Rightarrow$ $h = -1$, $k = 1$, $p = -\frac{1}{2} < 1 \rightarrow$ downward



Focus $\left(-1, \frac{1}{2}\right)$

Example 2

- Find the standard form of the equation of the parabola with the vertex $\overset{(h,k)}{(1,0)}$ and focus at $(2,0)$.

Solution:

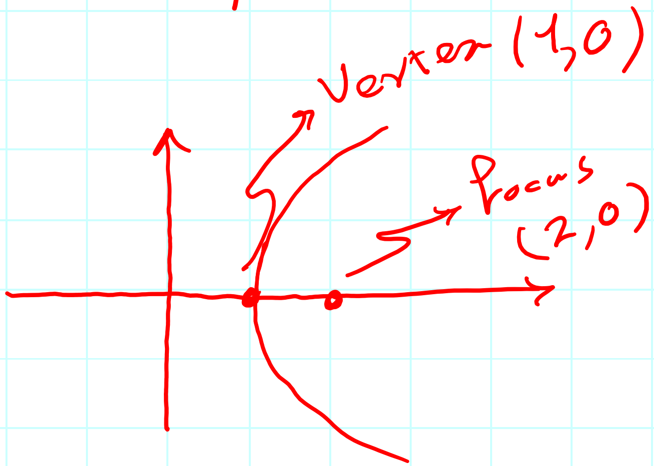
axis of parabola is horizontal

$$(y-k)^2 = 4p(x-h)$$

$$h=1,\ k=0 \quad \& \quad p = 2-1 = 1$$

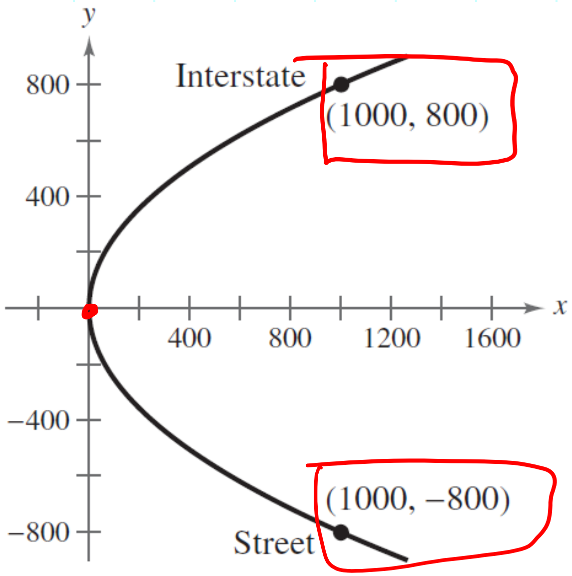
$$\left.\begin{array}{l}(y-k)^2 = 4p(x-h) \\ h=1,\ k=0 \ \& \ p=2-1=1\end{array}\right\} \longrightarrow (y-0)^2 = 4(1)(x-1) \Longrightarrow$$

$$y^2 = 4(x-1)$$



Example 3

- **Highway Design** Highway engineers design a parabolic curve for an entrance ramp from a straight street to an interstate highway (see figure). Find an equation of the parabola.



Solution:

Vertex (0,0) / horizontal axis

$\hookrightarrow y^2 = 4px \longrightarrow 800^2 = 4p(1000) \Rightarrow p = 160$

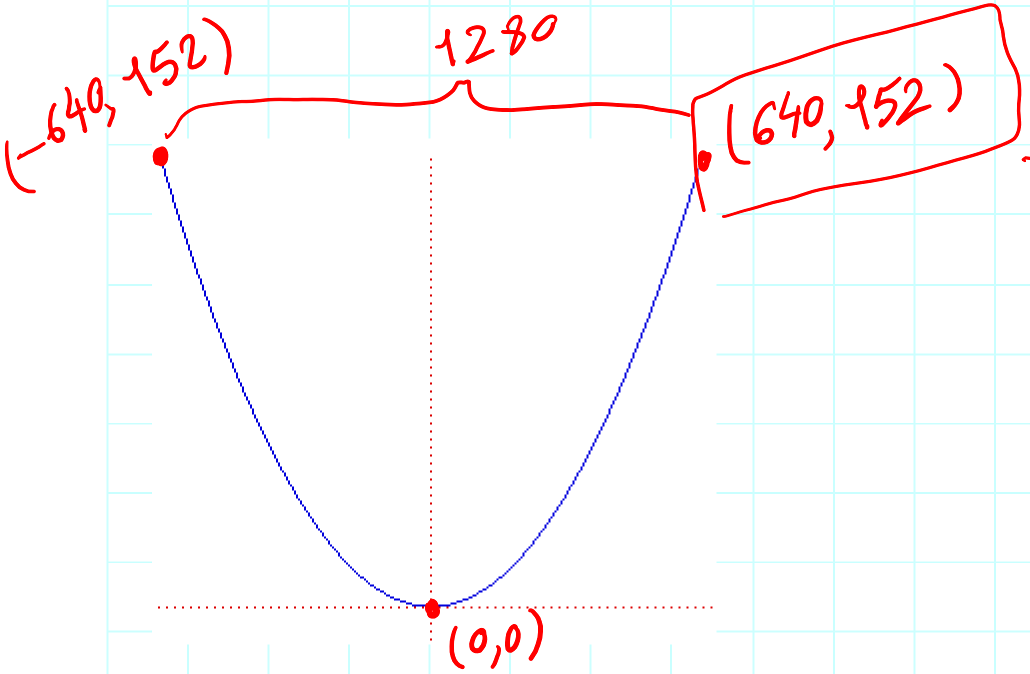
Point (1000, 800)

$$\boxed{y^2 = 640x}$$

Example 4

- Each cable of the Golden Gate Bridge is suspended (in the shape of a parabola) between two towers that are 1280 meters apart. The top of each tower is 152 meters above the roadway. The cables touch the roadway midway between the towers.
    a) Draw a sketch of the bridge. Locate the origin of a rectangular coordinate system at the center of the roadway. Label the coordinates of the known points.
    b) Write an equation that models the cables.

Solution:



$$(x-h)^2 = 4p(y-k) \qquad (h,k) = (0,0)$$

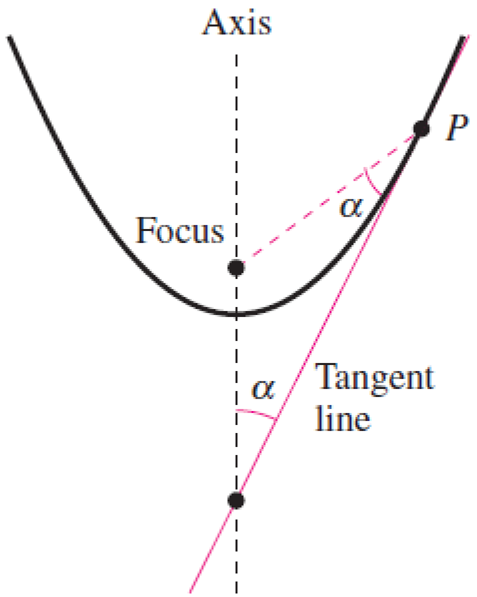
$$(640-0)^2 = 4p(152-0) \Longrightarrow p = \frac{12800}{19}$$

$$x^2 = 4\left(\frac{12800}{19}\right)y \to \boxed{x^2 = \frac{51200}{19}y}$$

Reflective Property of a Parabola

The tangent line to a parabola at a point *P* makes equal angles with the following two lines (shown in figure).

1. The line passing through P and the focus
2. The axis of the parabola



➢ Find the equation of the tangent line to the parabola given by $y = x^2$ at the point $(1, 1)$.

Solution:

$(h,k) = (0,0)$, $P = \frac{1}{4}$

focus $(0, 1/4)$

$d_1 = d_2$

$d_1 = \frac{1}{4} - b$

$d_2 = \sqrt{(1-0)^2 + (1-1/4)^2} = 5/4$

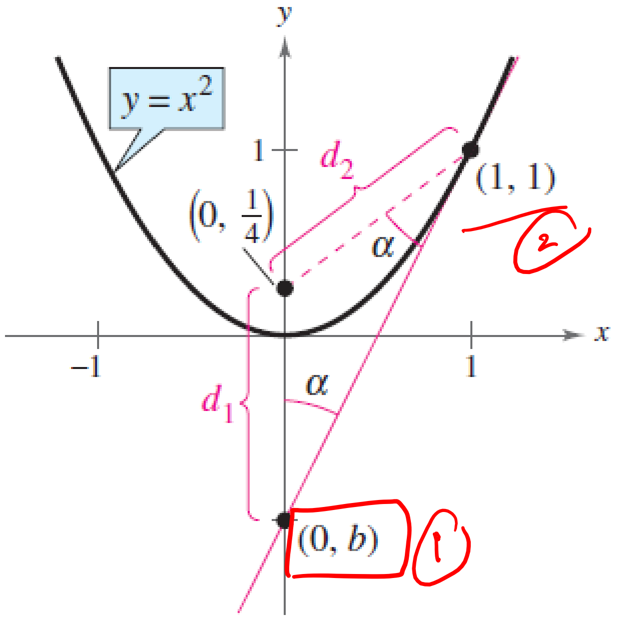
$\frac{1}{4} - b = \frac{5}{4} \Rightarrow b = -1$

slope of the tangent line

$m = \frac{1-(-1)}{1-0} = 2 \Rightarrow y = mx + b$

$y = 2x + b \longrightarrow \boxed{b = -1}$

$\boxed{y = 2x - 1}$ ✓



Definition of a Circle

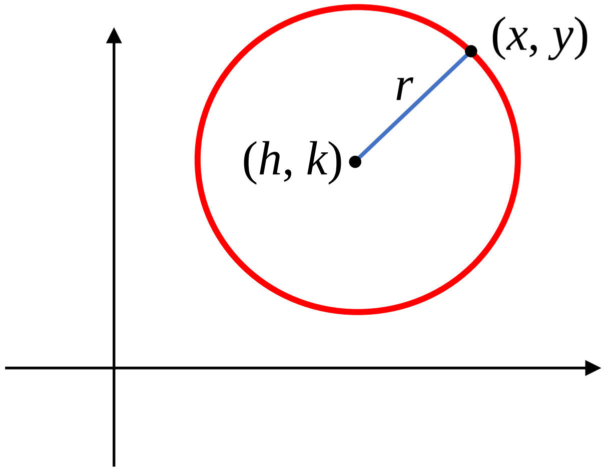
A circle is the set of all points (x, y) in a plane that are equidistant from a fixed point (h, k), called the center of the circle. As shown in the figure, the distance r between the center and any point (x, y) on the circle is the radius.

Standard Form of the Equation of a Circle

The standard form of the equation of a circle is

$$(x - h)^2 + (y - k)^2 = r^2.$$

The point (h, k) is the center of the circle, and the positive number r is the radius of the circle.



The standard form of the equation of a circle whose center is the origin, $(h, k) = (0, 0)$, is

$$x^2 + y^2 = r^2$$

Example 6

- **Earthquake** An earthquake was felt up to 81 miles from its epicenter. You were located 60 miles west and 45 miles south of the epicenter.

a) Let the epicenter be at the point (0, 0). Find the standard equation that describes the outer boundary of the earthquake.

b) Would you have felt the earthquake?

c) Verify your answer to part (b) by graphing the equation of the outer boundary of the earthquake and plotting your location. How far were you from the outer boundary of the earthquake?

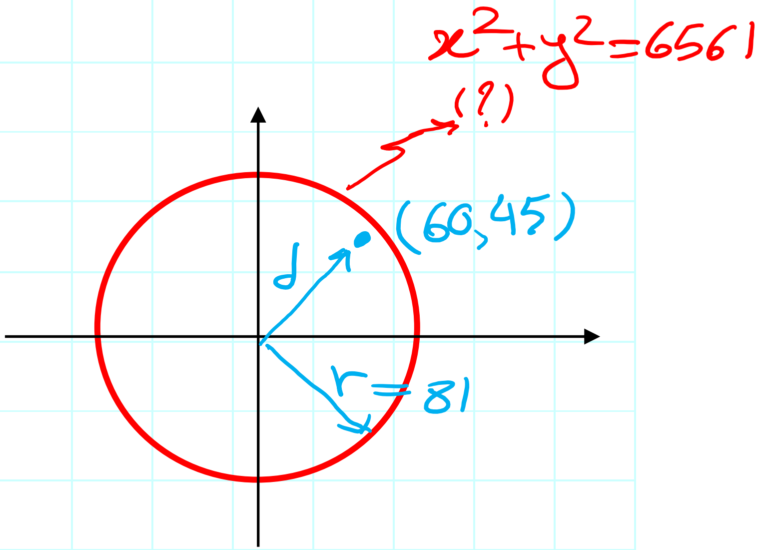
Solution:

a) Radius: 81 → $x^2+y^2=6561$

Center: $(0,0)$

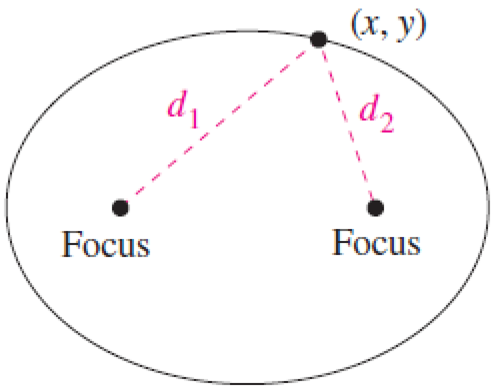
b) $d=\sqrt{60^2+45^2}=\sqrt{5625}=75$ miles <81 miles

c) $81-75=6$ miles

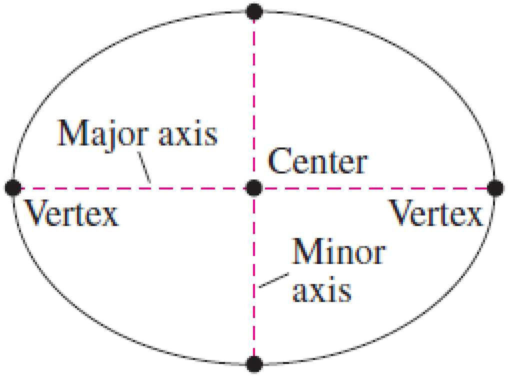


Ellipses (1/3)

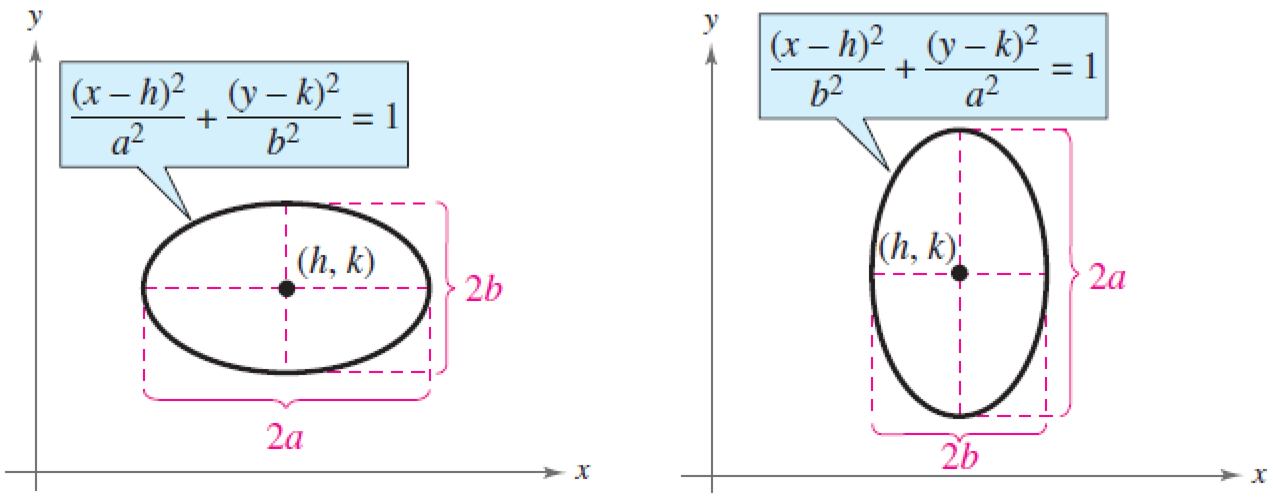
An **ellipse** is the set of points in a plane such that the sum of the distances from each point to two fixed points is constant. Each of the two fixed points is called a **focus** (plural, foci). The line containing the foci intersects the ellipse at points called **vertices** (singular, vertex). The line segment between the vertices is called the **major axis**, and its midpoint is the center of the ellipse. A line perpendicular to the major axis through the center intersects the ellipse at points called the **co-vertices**, and the line segment between the co-vertices is called the **minor axis**.



$d_1 + d_2$ is constant.



Equations of an ellipse with center (h, k) and major and minor axes of length $2a$ and $2b$, respectively, where $0 < b < a$ are shown in figure below for both the vertical and horizontal orientations.



Major axis is horizontal. Major axis is vertical.

The foci lie on the major axis, c units from the center, with $c^2 = a^2 - b^2$.

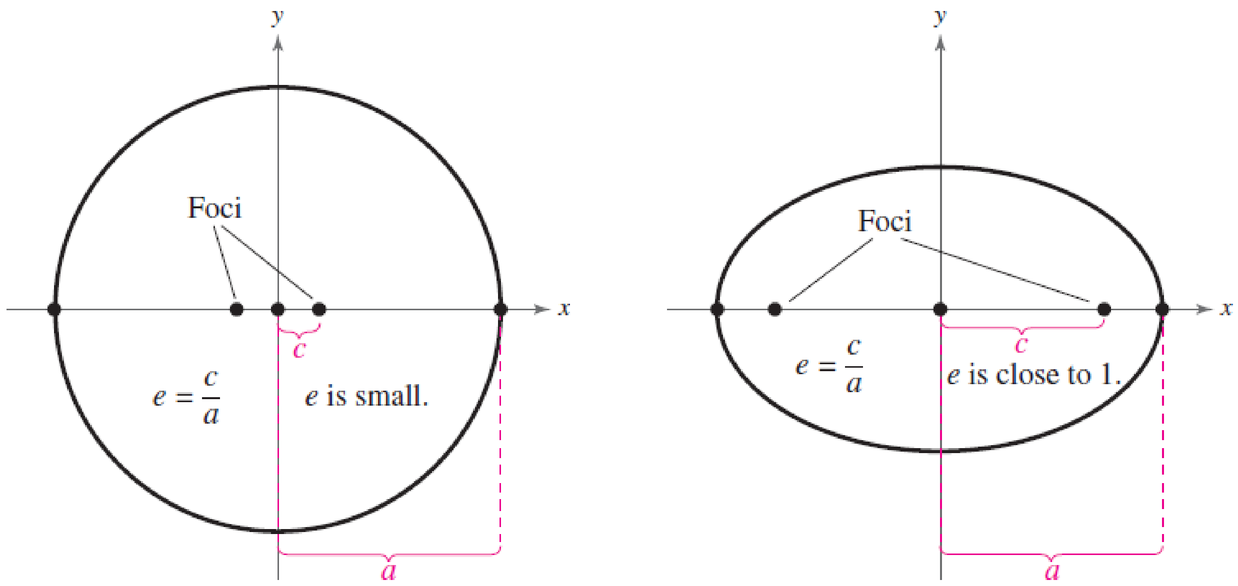
Definition of Eccentricity

The eccentricity e of an ellipse is given by the ratio $e = \frac{c}{a}$.

Note that 0 < e < 1 for every ellipse.

To see how this ratio is used to describe the shape of an ellipse, note that because the foci of an ellipse are located along the major axis between the vertices and the center, it follows that

$$0 < c < a.$$



Example 7

➢ Find the center, vertices, and foci of the ellipse $4x^2 + y^2 - 8x + 4y - 8 = 0$. $\xrightarrow{\text{by completing the square}}$ standard form

Solution:

$$4x^2+y^2-8x+4y-8=0$$

$$4(x^2-2x\boxed{+1})+(y^2+4y\boxed{+4})=8$$

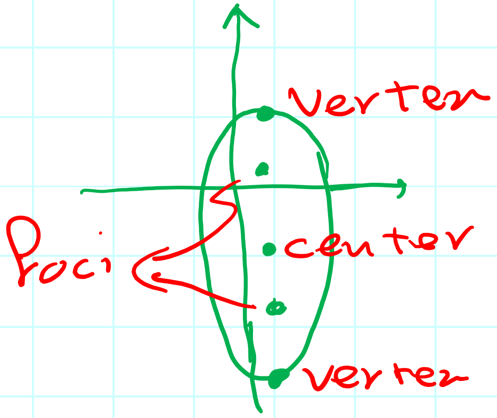
$$4(x^2-2x+1)+(y^2+4y+4)=8+4(1)+4$$

$$4(x-1)^2+(y+2)^2=16$$

$$\frac{(x-1)^2}{4}+\frac{(y+2)^2}{16}=1$$

$$c=\sqrt{a^2-b^2}$$

$$\Rightarrow c=2\sqrt{3}$$



$$\begin{cases} h=1 \\ k=-2 \\ a=4 \\ b=2 \end{cases} \Rightarrow$$

✓center $(1,-2)$

✓Vertices: $(1,-6)$, $(1,2)$

✓foci: $(1,-2-2\sqrt{3})$, $(1,-2+2\sqrt{3})$

Example 8

- Find the standard form of the equation of the ellipse with the given characteristics.
  a) Vertices: $(0, 2), (8, 2)$; minor axis of length 2.
  b) Center: $(3, 2)$; $\boxed{a = 3c}$; foci: $(1, 2), (5, 2)$.

Solution:

a) $(h, k) = \left(\frac{0+8}{2}, \frac{2+2}{2}\right) \to (4, 2)$

$2b = 2 \to b = 1$

$2a = (8-0) = 8 \to a = 4$

$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

$\Rightarrow \boxed{\frac{(x-4)^2}{16} + \frac{(y-2)^2}{1} = 1}$

b) $(h, k) = (3, 2)$

$2c = |5-1| = 4 \Rightarrow c = 2$

$a = 3c \to a = 6$

$c^2 = a^2 - b^2 \to b = \sqrt{a^2 - c^2}$

$\Rightarrow b = 4\sqrt{2}$

$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

$\Rightarrow \boxed{\frac{(x-3)^2}{36} + \frac{(y-2)^2}{32} = 1}$

Example 9

- For the given equation, identify the conic as a circle or an ellipse. Then find the center, radius, vertices, foci, and eccentricity of the conic (if applicable).

$$16x^2 + 25y^2 - 32x + 50y + 16 = 0$$

Solution:

$$16x^2 - 32x + 25y^2 + 50y = -16$$

$$16(x^2 - 2x) + 25(y^2 + 2y) = -16$$

$$16(x^2 - 2x + 1) + 25(y^2 + 2y + 1) = -16 + 16 + 25 \quad 41$$

$$16(x-1)^2 + 25(y+1)^2 = 25$$

$$\frac{16(x-1)^2}{25} + \frac{(y+1)^2}{1} = 1$$

$\frac{25}{16}$

center $(1, -1)$

horizontal ellipse

$a = \frac{5}{4} \quad b = 1$

vertices

$(-\frac{1}{4}, -1)$

$(\frac{9}{4}, -1)$

$c = \sqrt{a^2 - b^2} = \frac{3}{4}$

foci

$(\frac{1}{4}, -1)$ & $(\frac{7}{4}, -1)$

$$e = \frac{c}{a} = \frac{3/4}{5/4} = \frac{3}{5}$$

Next Week Topics

- Quadratic Curves (to be continued)
  - Hyperbolas
  - Rotation
  - Parametric Equations

Good Luck