Discrete Mathematics and Logic
Lecture 2

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Logic

Propositions

True or False

- "$2 + 2 = 4$"
- "All apples are brown"
- "There are prime even numbers greater than 2"
- "Oh, great!" is not!

TRUE	FALSE
T	F
1	0

Logic

Operations

1. Negation (logical "not")

A proposition P is true iff the negation of P is false.

Notions $\overline{P}$ $\neg P$

P	$\neg P$
• $2+2=4$ • "All apples are brown" • "There are prime even numbers greater than 2"	• $2+2\neq 4$ • "Not all apples are brown" • "There are no prime even numbers greater than 2"

Logic

Operations

1. Negation (logical "not")

Truth Table

P	$\neg P$
0	1
1	0

* 0 = F(alse), 1 = T(rue)

Logic

Operations

2. Conjunction (logical "and")

The conjunction of propositions P_1 and P_2 is true iff the both P_1 and P_2 are true.

Notions $P_1 \wedge P_2$ $P_1 \cdot P_2$ $P_1 \,\&\, P_2$

Truth Table

P_1	P_2	$P_1 \,\&\, P_2$
0	0	0
0	1	0
1	0	0
1	1	1

Logic

Operations

3. Disjunction (logical “or”)

The disjunction of propositions P_1 and P_2 is true iff at least one of P_1 and P_2 are true.

Notions $P_1 \vee P_2$

Truth Table

P_1	P_2	$P_1 \vee P_2$
0	0	0
0	1	1
1	0	1
1	1	1

Logic

Operations

4. Implication (logical “if ..., then ...”)

"True implies only true.“

Notions $P_1 \Rightarrow P_2$ $P_1 \rightarrow P_2$

Truth Table

P_1	P_2	$P_1 \rightarrow P_2$
0	0	1
0	1	1
1	0	0
1	1	1

Logic

Operations

4. Implication (logical “if ..., then ...”)

Example

If x is dividable by 4, then x is dividable by 2

$x = 8$

P_1	P_2	$P_1 \rightarrow P_2$
0	0	1
0	1	1
1	0	0
1	1	1

Logic

Operations

4. Implication (logical “if ..., then ...”)

Example

If x is dividable by 4, then x is dividable by 2

$x = 6$

P_1	P_2	$P_1 \to P_2$
0	0	1
0	1	1
1	0	0
1	1	1

Logic

Operations

4. Implication (logical “if ..., then ...”)

Example

If x is dividable by 4, then x is dividable by 2

$x = 5$

P_1	P_2	$P_1 \rightarrow P_2$
0	0	1
0	1	1
1	0	0
1	1	1

Logic

Operations

4. Implication (logical “if ..., then ...”)

Example

If x is dividable by 4, then x is dividable by 2

There is no x such that x is dividable by 4 and not dividable by 2

P_1	P_2	$P_1 \rightarrow P_2$
0	0	1
0	1	1
1	0	0
1	1	1

Logic

Operations

5. Equivalence (logical "... if and only if ...")

P_1 is true iff P_2 is true

Notions $P_1 \Leftrightarrow P_2$ $P_1 \leftrightarrow P_2$

Truth Table

P_1	P_2	$P_1 \leftrightarrow P_2$
0	0	1
0	1	0
1	0	0
1	1	1

Logic

Definition (by induction)

1) Any proposition is a formula (with 0 operations).
2) Suppose that Φ, Φ_1, Φ_2 are formulas (with n operations). Then the following are formulas (with $n+1$ operations):

- $(\neg\Phi)$
- $(\Phi_1 \mathbin{\&} \Phi_2)$
- $(\Phi_1 \vee \Phi_2)$
- $(\Phi_1 \rightarrow \Phi_2)$
- $(\Phi_1 \leftrightarrow \Phi_2)$

Logic

Example

If a line l_1 is parallel to a line l_3 and a line l_2 is parallel to l_3, then l_1 is parallel to l_2.

Let $A = l_1 \parallel l_3$, $B = l_2 \parallel l_3$, $C = l_1 \parallel l_2$

$$(A \,\&\, B) \rightarrow C$$

Logic

Example

$$(A \,\&\, B) \rightarrow C$$

- A
- B
- C
- $(A \,\&\, B)$
- $((A \,\&\, B) \rightarrow C)$

Logic

Truth table

A	B	C	$A \& B$	$(A \& B) \to C$
0	0	0	0	1
0	0	1	0	1
0	1	0	0	1
0	1	1	0	1
1	0	0	0	1
1	0	1	0	1
1	1	0	1	0
1	1	1	1	1

The naive set theory
Operations

1. Complement

$$x \in \overline{A} \text{ iff } \neg(x \in A)$$

2. Intersection

$$x \in A_1 \cap A_2 \text{ iff } (x \in A_1) \,\&\, (x \in A_2)$$

3. Union

$$x \in A_1 \cup A_2 \text{ iff } (x \in A_1) \vee (x \in A_2)$$

The naive set theory
Operations

5. Subset

$$A \subseteq B \text{ iff, for any } x, [(x \in A) \rightarrow (x \in B)]$$

6. Equivalence

$$A = B \text{ iff, for any } x, [(x \in A) \leftrightarrow (x \in B)]$$

Logic

Properties

$$a \,\&\, 0 = 0 \qquad a \vee 1 = 1$$
$$a \,\&\, 1 = a \qquad a \vee 0 = a$$
$$a \to b = \neg a \vee b \qquad a \leftrightarrow b = (a \to b) \,\&\, (b \to a)$$

Idempotency

$$a \,\&\, a = a \qquad a \vee a = a$$

Commutativity

$$a \,\&\, b = b \,\&\, a \qquad a \vee b = b \vee a$$

Associativity

$$a \,\&\, (b \,\&\, c) = (a \,\&\, b) \,\&\, c \qquad a \vee (b \vee c) = (a \vee b) \vee c$$

Distributivity

$$a \,\&\, (b \vee c) = (a \,\&\, b) \vee (a \,\&\, c) \qquad a \vee (b \,\&\, c) = (a \vee b) \,\&\, (a \vee c)$$

Logic

Negation Properties

$\neg(\neg a) = a$

$a \,\&\, \neg a = 0 \quad a \vee \neg a = 1$

De Morgan's laws

$\neg(a \,\&\, b) = \neg a \vee \neg b \quad \neg(a \vee b) = \neg a \,\&\, \neg b$

The naive set theory

Properties

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Proof

$$x \in A \cup (B \cap C) \Leftrightarrow (x \in A) \vee (x \in B \,\&\, x \in C)$$

Since $a \vee (b \,\&\, c) = (a \vee b) \,\&\, (a \vee c)$,

$$(x \in A) \vee (x \in B \,\&\, x \in C) \Leftrightarrow (x \in A \vee x \in B) \,\&\, (x \in A \vee x \in C) \Leftrightarrow$$

$$\Leftrightarrow x \in (A \cup B) \cap (A \cup C)$$

Exercise

- Write the rest properties/laws
- Prove all of them

Logic

Definition

A conjunctive term is a conjunction of literals, where each literal is either a variable, or its negation.

Definition

A disjunctive normal form (DNF) is a disjunction of conjunctive terms.

DNF	Not!
$(a \,\&\, \neg b) \vee (\neg a \,\&\, c)$	$(a \,\&\, \neg b) \vee \neg(a \,\&\, c)$
$(\neg a \,\&\, b) \vee d$	$\neg a \,\&\, (b \vee d)$
$a \vee c$	$a \,\&\, (b \vee (c \,\&\, d))$
$\neg b \,\&\, d$	
a	

Logic

Definition

A disjunctive term is a disjunction of literals, where each literal is either a variable, or its negation.

Definition

A conjunctive normal form (CNF) is a conjunction of disjunctive terms.

CNF	Not!
$(a \vee \neg b) \,\&\, (\neg a \vee c)$	$(a \vee \neg b) \,\&\, \neg(a \vee c)$
$(\neg a \vee b) \,\&\, d$	$\neg a \vee (b \,\&\, d)$
$a \,\&\, c$	$a \,\&\, (b \vee (c \,\&\, d))$
$\neg b \vee d$	
a	

Logic

Theorem

Any formula has both a DNF and a CNF.

Example

$$(A \,\&\, B) \to C = \neg(A \,\&\, B) \vee C =$$

$$= \neg A \vee \neg B \vee C$$

Logic

Theorem

Any formula has both a DNF and a CNF.

Proof by induction

Initial step. If Φ_0 is a formula with 0 operations, then Φ_0 is equal to a variable. So, Φ_0 itself is a DNF and a CNF.

Induction hypothesis. Suppose that any formula with k operations has both a DNF and a CNF.

Induction step. Let Φ be a formula with $k+1$ operations. Then, by definition, there are formulas Φ', Φ_1, Φ_2 such that every of them has k operations and either $\Phi = \neg(\Phi')$, or $\Phi = \Phi_1 \,\&\, \Phi_2$, or $\Phi = \Phi_1 \vee \Phi_2$, or $\Phi = \Phi_1 \to \Phi_2$, or $\Phi = \Phi_1 \leftrightarrow \Phi_2$.

Logic

Induction step. 1) Suppose that $\Phi = \neg(\Phi')$, where Φ' has k operations. By induction hypothesis, Φ' has a CNF, i.e.,

$$\Phi' = (l_1^1 \vee \ldots \vee l_{i_1}^1) \,\&\, \ldots \,\&\, (l_1^m \vee \ldots \vee l_{i_m}^m)$$

Hence,

$$\Phi = \neg[(l_1^1 \vee \ldots \vee l_{i_1}^1) \,\&\, \ldots \,\&\, (l_1^m \vee \ldots \vee l_{i_m}^m)] =$$

$$= \neg(l_1^1 \vee \ldots \vee l_{i_1}^1) \vee \ldots \vee \neg(l_1^m \vee \ldots \vee l_{i_m}^m) =$$

$$= (\neg l_1^1 \,\&\, \ldots \,\&\, \neg l_{i_1}^1) \vee \ldots \vee (\neg l_1^m \,\&\, \ldots \,\&\, \neg l_{i_m}^m)$$

De Morgan's laws

$\neg(a \,\&\, b) = \neg a \vee \neg b \quad \neg(a \vee b) = \neg a \,\&\, \neg b$

Logic

Proof by induction

So,

$$\Phi = (\neg l_1^1 \,\&\, \ldots \,\&\, \neg l_{i_1}^1) \vee \ldots \vee (\neg l_1^m \,\&\, \ldots \,\&\, \neg l_{i_m}^m)$$

Since each l_j^i is a literal, $\neg l_j^i$ is also a literal. (Recall $\neg(\neg a) = a$.)

Thus, Φ has a DNF.

Similarly, Φ has a CNF.

Logic

Proof by induction

2) Suppose that $\Phi = \Phi_1 \,\&\, \Phi_2$.

By induction hypothesis, Φ_1 and Φ_2 have both a CNF, i.e.,

$$\Phi_1 = (l_1^1 \vee \ldots \vee l_{i_1}^1) \,\&\, \ldots \,\&\, (l_1^m \vee \ldots \vee l_{i_m}^m)$$

$$\Phi_2 = (t_1^1 \vee \ldots \vee t_{j_1}^1) \,\&\, \ldots \,\&\, (t_1^p \vee \ldots \vee t_{j_p}^p)$$

Then
$\Phi = \Phi_1 \,\&\, \Phi_2 = (l_1^1 \vee \ldots \vee l_{i_1}^1) \,\&\, \ldots \,\&\, (l_1^m \vee \ldots \vee l_{i_m}^m) \,\&\, (t_1^1 \vee \ldots \vee t_{j_1}^1) \,\&\, \ldots \,\&\, (t_1^p \vee \ldots \vee t_{j_p}^p)$ is also CNF.

Logic

Proof by induction

2) Suppose that $\Phi = \Phi_1 \,\&\, \Phi_2$.

By induction hypothesis, Φ_1 and Φ_2 have both a DNF, i.e.,

$$\Phi_1 = (l_1^1 \,\&\, \ldots \,\&\, l_{i_1}^1) \vee \ldots \vee (l_1^m \,\&\, \ldots \,\&\, l_{i_m}^m)$$
$$\Phi_2 = (t_1^1 \,\&\, \ldots \,\&\, t_{j_1}^1) \vee \ldots \vee (t_1^p \,\&\, \ldots \,\&\, t_{j_p}^p)$$

Then

$\Phi_1 \,\&\, \Phi_2 = [(l_1^1 \,\&\, \ldots \,\&\, l_{i_1}^1) \vee \ldots \vee (l_1^m \,\&\, \ldots \,\&\, l_{i_m}^m)] \,\&\, [(t_1^1 \,\&\, \ldots \,\&\, t_{j_1}^1) \vee \ldots \vee (t_1^p \,\&\, \ldots \,\&\, t_{j_p}^p)]$.

Logic

$A \,\&\, (B \vee C) = (A \,\&\, B) \vee (A \,\&\, C)$

$\Phi_1 \,\&\, \Phi_2 = [(l_1^1 \,\&\, \ldots \,\&\, l_{i_1}^1) \vee \ldots \vee$
$(l_1^m \,\&\, \ldots \,\&\, l_{i_m}^m)] \,\&\, [(t_1^1 \,\&\, \ldots \,\&\, t_{j_1}^1) \vee \ldots \vee (t_1^p \,\&\, \ldots \,\&\, t_{j_p}^p)] =$

$= (l_1^1 \,\&\, \ldots \,\&\, l_{i_1}^1 \,\&\, t_1^1 \,\&\, \ldots \,\&\, t_{j_1}^1) \vee (l_1^1 \,\&\, \ldots \,\&\, l_{i_1}^1 \,\&\, t_1^2 \,\&\, \ldots \,\&\, t_{j_2}^2) \vee$
$\ldots \vee (l_1^1 \,\&\, \ldots \,\&\, l_{i_1}^1 \,\&\, t_1^p \,\&\, \ldots \,\&\, t_{j_p}^p) \vee \ldots$

$(l_1^m \,\&\, \ldots \,\&\, l_{i_1}^m \,\&\, t_1^p \,\&\, \ldots \,\&\, t_{j_p}^p) \vee (l_1^m \,\&\, \ldots \,\&\, l_{i_2}^m \,\&\, t_1^p \,\&\, \ldots \,\&\, t_{j_p}^p) \vee$
$\ldots \vee (l_1^m \,\&\, \ldots \,\&\, l_{i_m}^m \,\&\, t_1^p \,\&\, \ldots \,\&\, t_{j_p}^p)$ is a DNF.

Similarly, if $\Phi = \Phi_1 \vee \Phi_2$, or $\Phi = \Phi_1 \to \Phi_2$, or $\Phi = \Phi_1 \leftrightarrow \Phi_2$, then Φ has both a DNF and a CNF.

Exercise

- Finish the proof.

Thank you for your attention!