For a non-rotated coordinate system, a conic takes on the form of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

A conic in a rotated coordinate system takes on the form of $A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$, where the prime notation represents the rotated axes and associated coefficients.

If the conic isn't rotated then B = 0.

For a non-rotated conic:

A. Parabola if A or C = 0 therefore **AC = 0**
B. Ellipse or Circle if A & C are the same sign therefore **AC > 0**
C. Hyperbola if A & C are different sign therefore **AC < 0**

Take the discriminate $B^2 - 4AC$ for both old and new coordinates:

We know:
$A' = Acos^2\theta + Bcos\theta sin\theta + Csin^2\theta$
$B' = 2(C-A)sin\theta cos\theta + B(cos^2\theta - sin^2\theta)$
$C' = Acos^2\theta - Bcos\theta sin\theta + Csin^2\theta$

So:
$B'^2 - 4A'C' =$
$(2(C-A)sin\theta cos\theta + B(cos^2\theta - sin^2\theta))^2 - 4(Acos^2\theta + Bcos\theta sin\theta + Csin^2\theta)(Acos^2\theta + Bcos\theta sin\theta + Csin^2\theta) =$
$(2Csin\theta cos\theta - 2Asin\theta cos\theta + Bcos^2\theta - Bsin^2\theta)^2 + (-4Acos^2\theta - 4Bcos\theta sin\theta - 4Csin^2\theta)(Acos^2\theta - Bcos\theta sin\theta + Csin^2\theta) =$
$4C^2sin^2\theta cos^2\theta - 4ACsin^2\theta cos^2\theta + 2BCsin\theta cos^3\theta - 2BCsin^3\theta cos\theta - 4ACsin^2\theta cos^2\theta + 4A^2sin^2\theta cos^2\theta - 2ABsin\theta cos^3\theta + 2ABsin^3\theta cos\theta + 2BCsin\theta cos^3\theta - 2ABsin\theta cos^3\theta + B^2cos^4\theta - B^2sin^2\theta cos^2\theta - 2BCsin^3\theta cos\theta + 2ABsin^3\theta cos\theta - B^2sin^2\theta cos^2\theta + B^2sin^4\theta - 4A^2sin^2\theta cos^2\theta + 4ABsin\theta cos^3\theta - 4ACcos^4\theta - 4ABsin^3\theta cos\theta + 4B^2sin^2\theta cos^2\theta - 4BCsin\theta cos^3\theta - 4ACsin^4\theta + 4BCsin^3\theta cos\theta - 4C^2sin^2\theta cos^2\theta =$
$-4ACsin^2\theta cos^2\theta - 4ACsin^2\theta cos^2\theta + B^2cos^4\theta - B^2sin^2\theta cos^2\theta - B^2sin^2\theta cos^2\theta + B^2sin^4\theta - 4ACcos^4\theta + 4B^2sin^2\theta cos^2\theta - 4ACsin^4\theta =$
$-8ACsin^2\theta cos^2\theta + 2B^2sin^2\theta cos^2\theta + B^2sin^4\theta - 4ACsin^4\theta + B^2cos^4\theta - 4ACcos^4\theta =$
$2sin^2\theta cos^2\theta(B^2 - 4AC) + sin^4\theta(B^2 - 4AC) + cos^4\theta(B^2 - 4AC) =$
$(B^2 - 4AC)(sin^4\theta + 2sin^2\theta cos^2\theta + cos^4\theta) =$
$(B^2 - 4AC)(sin^2\theta + cos^2\theta)^2 =$
$(B^2 - 4AC)1^2 = (B^2 - 4AC)$

The Discriminate is invariant under Rotation of Axes

Thus $B'^2 - 4A'C' = B^2 - 4AC$

Now: if we rotate axes by $\boldsymbol{\theta = \left(\frac{1}{2}\right) tan^{-1}(\frac{B}{A-C})}$, we would eliminate rotation, ie B'=0

So

$$\left(\boldsymbol{B^2 - 4AC}\right) = \boldsymbol{-4A'C'}$$

(Discriminate of rotated conic in old coordinate system = discriminate of non-rotated conic in the new coordinate system where B'=0)

A. Parabola in new coordinate system- axes line up with conic thus $\boldsymbol{A'C' = 0}$
$(B^2 - 4AC) = -4A'C' = 0$, thus $(B^2 - 4AC) = 0$

B. Ellipse or Circle in new coordinate system- axes line up with conic thus $\boldsymbol{A'C' > 0}$
$(B^2 - 4AC) = -4A'C' < 0$, thus $(B^2 - 4AC) < 0$

C. Hyperbola in new coordinate system- axes line up with conic thus $\boldsymbol{A'C' < 0}$
$(B^2 - 4AC) = -4A'C' > 0$, thus $(B^2 - 4AC) > 0$

Therefore,

$\boldsymbol{(B^2 - 4AC) < 0}$ **Ellipse (circle)**

$\boldsymbol{(B^2 - 4AC)} = 0$ **Parabola**

$\boldsymbol{(B^2 - 4AC)} > 0$ **Hyperbola**

QED