LINEAR PROGRAMMING PROJECT

GROUP 2 REPORT FOR OPTIMIZATION F24 COURSE

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1 Introduction

Initial problem is formulated as following:

$$\begin{aligned} &\min_{x' \in \mathbb{R}^p} \|Ax' - y'\|_1 \\ &\text{s.t. } 0 \le x' \le 1 \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{m \times p}$ with $m \ge p$ — message encoding matrix, y' — received encoded (noisy) message, x' — encoded initial message to be find.

2 Notations

Notation	Meaning
e_i	unit vector with 1 at index i and all other zeroes
1_n	vector of n ones
I_n	identity matrix of size $n \times n$
0_n	vector of n zeroes
$0_{m \times n}$	zero matrix of size $m \times n$
x_i (or $(Ax)_i$)	i-th component of vector x (or Ax)

3 Q1: Linear problem formulation

Initial problem (eq. (1)) is not linear as cost function $\|Ax' - y'\|_1 = \sum_{i=1}^{m} |(Ax')_i - y'_i|$, is not linear. However, this objective function is **piecewise linear convex** function. Therefore, each element $|(Ax')_i - y'_i| = \max((Ax')_i - y'_i, y'_i - (Ax')_i)$ can be substituted with new variable z' with the following additional constraints: $z_i \ge (Ax')_i - y'_i$ and $z_i \ge y'_i - (Ax')_i$.

So the following problem is **linear** and is equivalent to the initial one:

$$\begin{aligned} &\min_{x' \in \mathbb{R}^p, z \in \mathbb{R}^m} \sum_{i=1}^{m} z_i \\ \text{s.t. } & x' \ge 0 \\ & x' \le 1 \\ & z_i \ge (Ax')_i - y'_i,\ i = 1 \dots m \\ & z_i \ge y'_i - (Ax')_i,\ i = 1 \dots m \end{aligned} \tag{2}$$

4 Q2: Linear problem in standard form

For the easier and more evident deviation of standard form of Equation (2), linear problem will be firstly rewritten in geometric form, and only then — in standard. The obtained linear optimization problem in standard form will be equivalent to initial problem (eq. (1)).

4.1 Geometric form

The equivalent **geometric** form of Equation (2) is

$$\begin{aligned} &\min_{z' \in \mathbb{R}^{p+m}} \; c^T z' \\ &\text{s.t.} \; \underbrace{\begin{pmatrix} I_p & 0_{p\times m} \\ -I_p & 0_{p\times m} \\ -A & I_m \\ A & I_m \end{pmatrix}}_{A'} z' \geq \underbrace{\begin{pmatrix} 0_p \\ -1_p \\ -y' \\ y' \end{pmatrix}}_{b'}, \end{aligned} \tag{3}$$

where $c = \sum_{i=p+1}^{p+m} e_i \in \mathbb{R}^{(p+m)}$, $b' \in \mathbb{R}^{2p+2m}$ and $A' \in \mathbb{R}^{(2p+2m)\times(p+m)}$.

The first p components of z' correspond to the components of x', and the next m components correspond to the components of z from Equation (2). Rows and columns of A' representation in Equation (3) are separated in blocks for clarity: the vertical separation is for x' and z correspondingly, and the horizontal separations denote corresponding constraints from Equation (2).

4.2 Standard form

Note that $z' = (x', z)^T$ from Equation (3) is already non-negative, because $x' \geq 0$ by problem definition and $z \geq 0$ by construction[1]. Therefore, to convert Equation (3) to standard form, only introduction of slack variables is needed to get rid of inequality sign. The equivalent **standard** form of Equation (3) is

$$\begin{aligned} &\min_{\tilde{x} \in \mathbb{R}^{2p+3m}} \; c^T \tilde{x} \\ &\text{s.t.} \; \underbrace{\begin{pmatrix} -I_p & 0_{p\times m} & -S^{1,p} \\ -A & I_m & -S^{p+1,p+m} \\ A & I_m & -S^{p+m+1,p+2m} \end{pmatrix}}_{A'} z' = \underbrace{\begin{pmatrix} -1_p \\ -y' \\ y' \end{pmatrix}}_{b'}, \\ &\tilde{x} \geq 0, \end{aligned} \tag{4}$$

where $c = \sum_{i=p+1}^{p+m} e_i \in \mathbb{R}^{(2p+3m)}$, $b' \in \mathbb{R}^{p+2m}$, $A' \in \mathbb{R}^{(p+2m)\times(2p+3m)}$, and $S^{a,b}$ — slack variable matrix of size $(b-a+1)\times(p+2m)$ with rows $S_i^{a,b} = e_{a+i-1}$, which represents necessary slack variables.

The first p components of $\tilde{x}$ correspond to the components of x', the next m components correspond to the components of z from Equation (2) and the last $(p+2m)$ components correspond to slack variables s. Rows and columns of A' representation in Equation (4) are again separated in blocks for clarity: the vertical separations are for x', z and s correspondingly, and the horizontal separations are related to corresponding constraints from Equation (3) (except first one, as non-negativity in standard from is separate constraint).

[1] Intuitively, z substitutes the absolute value, so is non-negative. Formally, from Equation (2), $z \geq t$ and $z \geq -t$ for some t. So if $t \geq 0$, then $z \geq t \geq 0$, and if $t \leq 0$ then $z \geq -t \geq 0$