Discrete Mechanics and Optimal Control (DMOC) for the Cart-Pole and Acrobot

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6.832 Underactuated Robotics: Final Project

Introduction

The inverted pendulum on a cart (also known as the "cart-pole") and the acrobat are commonly used model systems for control algorithms. Numerous strategies have been used to stabilize the cart-pole and acrobot to their unstable equilibrium points where the inverted pendulum stands upright. Some of these strategies include the linear quadratic regulator, energy shaping, and optimal control methods such as shooting and direct collocation.

In this paper, the discrete mechanics and optimal control (DMOC) approach is used to find a solution to the minimum-cost swing-up problem for the cart-pole and acrobot. The solutions are then compared to those from a direct collocation method, and the effect of the initial guess on the solution is examined.

Cart-Pole

The cart-pole is best described as an inverted pendulum on a cart. Taking the cart's position to be x, and the pendulum angle to be ϑ, the position vector $\underline{q}$ and the state vector $\underline{x}$ are chosen to be the following:

$$\underline{q} = [x\ \theta]^T$$

$$\underline{x} = [x\ \theta\ \dot{x}\ \dot{\theta}]^T$$

With $\vartheta = 0$ being defined as the downwards position (i.e. the pendulum's stable equilibrium point), the Lagrangian L is given by the following equation:

$$L = \frac{1}{2}(m_c + m_p)\dot{x}^2 + m_p l \dot{x}\dot{\theta}\cos\theta + \frac{1}{2}m_p l^2 \dot{\theta}^2 + m_p g l \cos\theta$$

Here, m_c is the mass of the cart, m_p is the mass of the pendulum bob, l is the length of the pendulum, and g is gravitational acceleration. Continuing with the Lagrangian method, we find the equations of motion to be the following:

$$(m_c + m_p)\ddot{x} + m_p l \ddot{\theta}\cos\theta - m_p l \dot{\theta}^2 \sin\theta = u$$

$$m_p l \ddot{x}\cos\theta + m_p l^2 \ddot{\theta} + m_p g l \sin\theta = 0$$

The cart-pole is an underactuated system by inspection: there is a force applied on the cart, but no actuator to directly apply torque to the pendulum.

Acrobot

The acrobot is a double pendulum that can apply torque at the elbow but not at the shoulder. It is so named because it resembles an acrobat, who can apply torque at his or her hips, but not at where he or she grips the bar. As with the cartpole, the acrobot is underactuated, and the standard control problem is to swing up to the unstable equilibrium point at the top. Taking q_1 to be the angle of the first link from the vertical, and q_2 to be the angle of the second link from the first link, the position and state vectors q and x are defined to be the following:

$$\underline{q} = [q_1, q_2]^T$$

$$\underline{x} = [q_1, q_2, \dot{q}_1, \dot{q}_2]^T$$

With $\underline{q} = 0$ being defined as the downwards position (i.e. the acrobot's stable equilibrium point), the Lagrangian L is given by the following equation:

$$L = \frac{1}{2}\left(I_1 + m_2{l_1}^2 + I_2 + 2m_2 l_1 l_{c2}\cos q_2\right){\dot{q}_1}^2 + \frac{1}{2}I_2{\dot{q}_2}^2 + (I_2 + m_2 l_1 l_{c2}\cos q_2)\dot{q}_1\dot{q}_2 \\ + m_1 g l_{c1}\cos q_1 + m_2 g(l_1\cos q_1 + l_{c2}\cos(q_1 + q_2))$$

Here, m_1 and m_2 are the masses of the first and second links, I_1 and I_2 are the rotational inertias of the two links, l_1 and l_2 are the lengths, and l_{c1} and l_{c2} are the lengths from the respective joints to the centers of mass for the two links. The equations of motion can be expressed using the standard manipulator equations:

$$H\left(\underline{q}\right)\underline{\ddot{q}} + C\left(\underline{q},\underline{\dot{q}}\right)\underline{\dot{q}} + G\left(\underline{q}\right) = B\underline{u}$$

In this case, the inertia matrix H, Coriolis matrix C, gravitational matrix G, and input shaping matrix B are the following:

$$H\left(\underline{q}\right) = \begin{bmatrix} I_1 + I_2 + m_2{l_1}^2 + 2m_2 l_1 l_{c2}\cos q_2 & I_2 + m_2 l_1 l_{c2}\cos q_2 \\ I_2 + m_2 l_1 l_{c2}\cos q_2 & I_2 \end{bmatrix}$$

$$C\left(\underline{q},\underline{\dot{q}}\right) = \begin{bmatrix} -2m_2 l_1 l_{c2}\dot{q}_2 & -m_2 l_1 l_{c2}\dot{q}_2\sin q_2 \\ m_2 l_1 l_{c2}\dot{q}_1\sin q_2 & 0 \end{bmatrix}$$

$$G\left(\underline{q}\right) = \begin{bmatrix} m_1 g l_{c1}\sin q_1 + m_2 g(l_1\sin q_1 + l_{c2}\sin(q_1 + q_2)) \\ m_2 g l_{c2}\sin(q_1 + q_2) \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Discrete Mechanics and Optimal Control (DMOC)

Discrete Mechanics and Optimal Control (DMOC) is a method to solve optimal control problems for mechanical systems. It is based on directly discretizing the Lagrange-d'Alembert principle for the system (rather than, for example, the equations of motion as with direct collocation), and then using the resulting discrete Euler-Lagrange equations as the constraints for an optimization solver. The DMOC approach is described below.

Consider the following cost functional:

$$J(q,\dot{q},u) = \int_0^T C(q(t),\dot{q}(t),f(t))dt$$

Note that q, $\dot{q}$, and *f* may be vectors. The force *f* is a function of the states and the control input:

$$f(t) = (q(t),\dot{q}(t),u(t))$$

The continuous **Lagrange-d'Alembert principle** is given by the following equation:

$$\delta \int_0^T L(q,\dot{q})dt + \int_0^T f(t) \cdot \delta q(t)dt = 0$$

This is required for all variations δq with $\delta q(0) = \delta q(T) = 0$, and *L* is the Lagrangian of the system. The Lagrange-D'Alembert principle can be shown (via integration by parts) to be equivalent to the **forced Euler-Lagrange equations**:

$$\frac{\partial L}{\partial q}(q,\dot{q}) - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}(q,\dot{q})\right] + f_L(q,\dot{q},u) = 0$$

Here, we discretize the Lagrangian *L* with a discrete Lagrangian L_d:

$$L_d(q_k,q_{k+1}) \approx \int_{kh}^{(k+1)h} L(q(t),\dot{q}(t))dt$$

Likewise, the virtual work is discretized into the *left* and *right discrete forces* as follows:

$$f_k^- \cdot \delta q_k + f_k^+ \cdot \delta q_{k+1} \approx \int_{kh}^{(k+1)h} f(t) \cdot \delta(t)dt$$

The discrete Lagrange-d'Alembert principle, also called the *forced discrete Euler-Lagrange equations*, are given by the following:

$$D_2 L_d(q_{k-1},q_k) + D_1 L_d(q_k,q_{k+1}) + f_{k-1}^+ + f_k^- = 0$$

Here, the D_i notation is used to specify the derivative with respect to the *i-th* element, e.g. $D_2L_d(q_{k-1},q_k)$ indicates the derivative of the discrete Lagrangian with respect to q_k.

DMOC Optimization Problem Statement

The optimization problem can be solved using an optimization tool such as Matlab's *fmincon* function, or the SNOPT package. The goal is to minimize the following discrete cost function with respect to f_d:

$$J_d(q_d, f_d) = \sum_{k=0}^{N-1} C_d(q_k, q_{k+1}, f_k, f_{k+1})$$

The constraints are given by the following equations:

$$\underline{q}_0 = \underline{q}(t_i)$$

$$\underline{q}_N = \underline{q}(t_f)$$

$$D_2 L(q_0, \dot{q}_0) + D_1 L_d(q_0, q_1) + f_0^- = 0$$

$$D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) + f_{k-1}^+ + f_k^- = 0$$

$$-D_2 L(q_N, \dot{q}_N) + D_2 L_d(q_{N-1}, q_N) + f_{N-1}^+ = 0$$

The first two constraints specify that the initial and final positions from the optimization must match the specified initial and final states. The fourth constraint is the *forced discrete Euler-Lagrange equation*, as discussed earlier. The third and fifth constraints are the discrete boundary conditions for the problem.

In practice, the Midpoint Rule is used to approximate the integrals:

$$C_d(q_k, q_{k+1}, f_k, f_{k+1}) = hC(\frac{q_{k+1} + q_k}{2}, \frac{q_{k+1} - q_k}{h}, \frac{f_{k+1} + f_k}{2})$$

$$L_d(q_k, q_{k+1}) = hL(\frac{q_{k+1} + q_k}{2}, \frac{q_{k+1} - q_k}{h})$$

$$f_k^- = f_k^+ = \frac{h}{4}(f_{k+1} + f_k)$$

DMOC Equations for the Cart-Pole and Acrobot

The problem to be solved here is the swing-up problem for the cart-pole with minimum control effort. Therefore, the cost function is chosen to be the following:

$$J(q, u) = \int_0^T u(t)^2 dt$$

By plugging in the discrete approximations for the system states and velocities, the discrete Lagrangian L_d for the cart-pole is found to be the following:

$$L_d(q_k, q_{k+1}) = h\left[\frac{1}{2}(m_c + m_p)\left(\frac{x_{k+1} - x_k}{h}\right)^2 + m_p l\left(\frac{x_{k+1} - x_k}{h}\right)\left(\frac{\theta_{k+1} - \theta_k}{h}\right)\cos\left(\frac{\theta_{k+1} + \theta_k}{2}\right) + \frac{1}{2}m_p l^2 (\frac{\theta_{k+1} - \theta_k}{h})^2 + m_p g l \cos(\frac{\theta_{k+1} + \theta_k}{2})\right]$$

The following derivatives of the Lagrangian are then taken:

$$D_2 L\left(\underline{q}, \underline{\dot{q}}\right) = \frac{\partial L}{\partial \underline{\dot{q}}} = [\frac{\partial L}{\partial \dot{x}}, \frac{\partial L}{\partial \dot{\theta}}]^T$$

$$D_1 L_d\left(\underline{q}_k, \underline{q}_{k+1}\right) = [\frac{\partial L_d\left(\underline{q}_k, \underline{q}_{k+1}\right)}{\partial x_k}, \frac{\partial L_d\left(\underline{q}_k, \underline{q}_{k+1}\right)}{\partial \theta_k}]^T$$

$$D_2 L_d\left(\underline{q}_{k-1}, \underline{q}_k\right) = [\frac{\partial L_d\left(\underline{q}_{k-1}, \underline{q}_k\right)}{\partial x_k}, \frac{\partial L_d\left(\underline{q}_{k-1}, \underline{q}_k\right)}{\partial \theta_k}]^T$$

The procedure of discretizing the Lagrangian and taking its derivatives are followed similarly for the acrobot.

Results: Cart-Pole

Matlab's *fmincon* was used the optimization solver for this problem, specifically using the *sequential quadratic programming* (SQP) method. For comparison, the problem was solved using the direct collocation ("dircol") technique in addition to DMOC. The dircol method uses the equations of motion as constraints in the optimization problem. Simulations were run for a cart-pole system with parameters $m_c = 1$, $m_p = 1$, $l = 1$, and $g = 9.8$.

First, a series of straight lines was used for the initial guess, i.e. the position x was set to zero for all time, the angle ϑ was set to a line from 0 to π, and the control effort u was set to zero for all time. This initial guess obviously does not obey the system dynamics—it is left to the optimization solver to find one that does. The results are shown in Figure 1, and as the plots show, both DMOC and direct collocation found successful swing-up solutions. For the specified parameters, DMOC returned a solution with a cost of 64.9, and direct collocation returned a solution with a cost of 42.9. This tends to suggest that the direct collocation solution is preferable; however, closer inspection of the control effort plot also shows the dircol solution has some chattering of the control input at the beginning. In addition, DMOC provides a control trajectory that is smoother, and bounded within tighter force limits than the dircol trajectory. This indicates that the DMOC solution may be better-suited for an actual mechanical system.

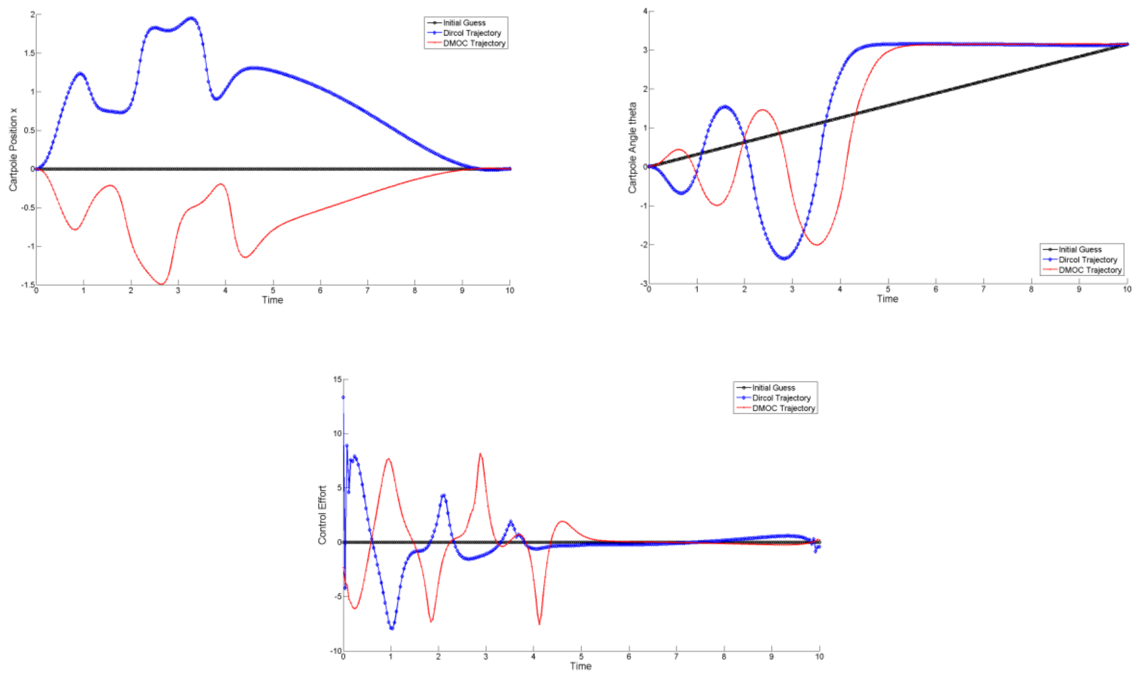
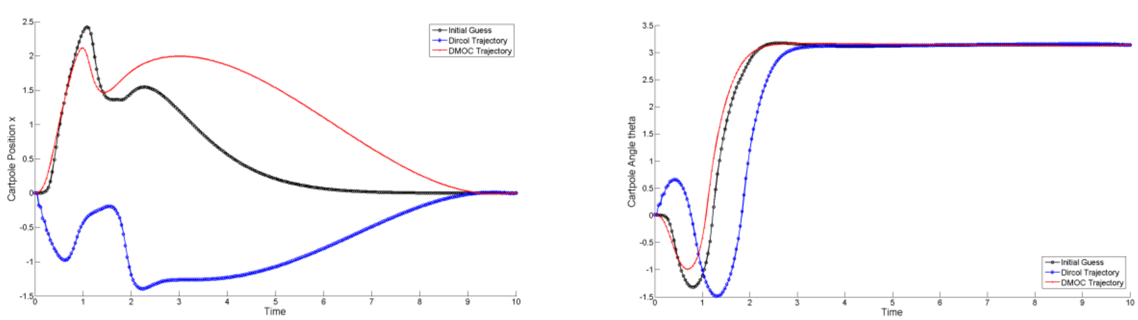


Figure 1: Cartpole position, angle, and control effort with straight lines for initial guess.

Next, a trajectory that does obey the system dynamics was used for the initial guess in the optimization solver. To obtain this initial trajectory, an energy shaping controller with partial feedback linearization (PFL) was used for swing-up, with a linear quadratic regulator (LQR) for stability at the goal. It is worth noting that the initial control trajectory is not smooth at the point where the controller switches from energy shaping to LQR; however, this did not prevent the optimization solver from finding a solution. The results for both DMOC and direct collocation are shown in Figure 2. For the specified parameters, the initial guess had a cost of 308.2, with DMOC returning a solution of cost 164.4, and dircol returning a solution of cost 47.3. Again, we see that the solution provided by DMOC has a higher cost than the one provided by direct collocation; however, the DMOC control trajectory is much smoother and more tightly bounded.



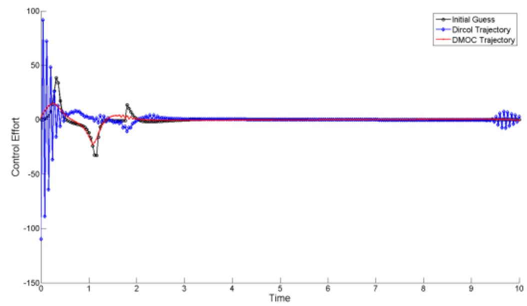


Figure 2: Cartpole position, angle, and control effort with energy shaping for initial guess.

One interesting point to note is that DMOC seems to have provided a better solution for the initial guess of straight lines than for the initial guess obeying system dynamics. In the first case, DMOC uses five swings to get to the top, which uses less energy than the second case, in which only two swings were used. This may be an effect of local minima, as the optimization solution is not necessarily globally optimal.

Results: Acrobot

The DMOC and direct collocation procedures were repeated for the acrobot, with similar results. Shown in Figure 3 and Figure 4 are plots for the acrobot, once again with straight lines and energy shaping, respectively, for the initial guess. As before, we see that both methods successfully complete the swing-up problem; however, the DMOC trajectory is smoother than the one provided by direct collocation.

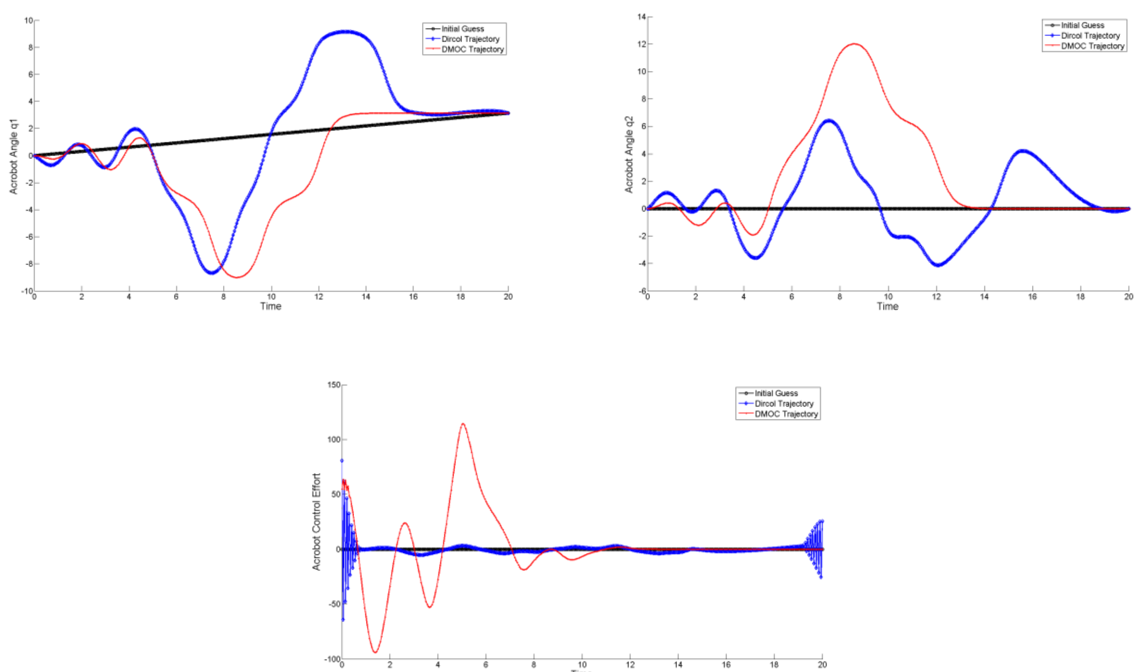


Figure 3: Acrobot angles and control effort with straight lines for initial guess.

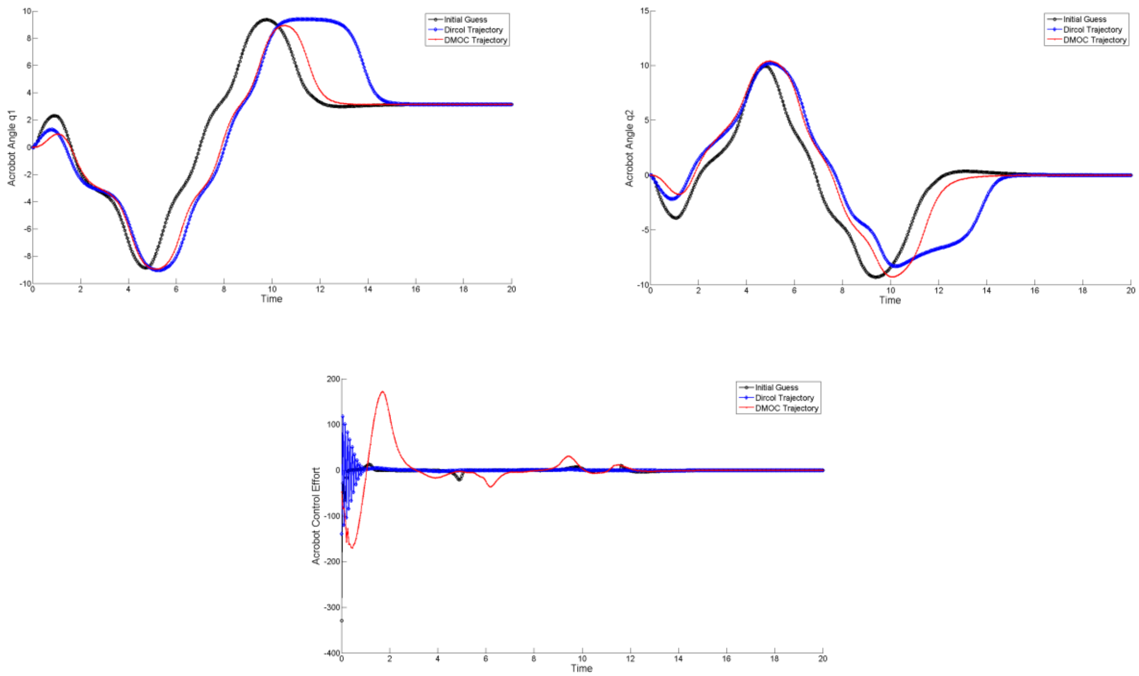


Figure 4: Acrobot angles and control effort with energy shaping for initial guess.

Conclusion

In this project, the discrete mechanics and optimal control (DMOC) method was explored and used to solve the swing-up problem for the cart-pole and acrobot. Results showed that DMOC could find a solution even with an unreasonable trajectory for the initial guess; however, the results were still subject to local minima, as sequential quadratic programming (SQP) was used for the optimization solver.

In practice, DMOC would be favored over direct collocation for systems that are nearly conservative. For example, a low-thrust spacecraft operating over long periods of time would be a good system for DMOC to be applied. This is because DMOC discretizes at the level of the Lagrangian, which may better respect energy budgets for mechanical systems.

Bibliography

J. Timmermann, S. K.-B. (2011). Discrete Mechanics and Optimal Control and Its Application to a Double Pendulum on a Cart. *Preprints of the 18th IFAC World Congress.* Milano, Italy.

Martin, J. (n.d.). *Discrete Mechanics and Optimal Control.*

Moore, A. (2008). *Optimization of Space Trajectories: Invariant Manifolds and DMOC.*

Moore, A. (n.d.). *Two Approaches Utilizing Invariant Manifolds to Design Trajectories for DMOC Optimization.*

Oliver Junge, J. E.-B. (n.d.). *Discrete Mechanics and Optimal Control.*

Sina Ober-Blobaum, O. J. (2010). *Discrete Mechanics and Optimal Control: An Analysis.* ESAIM: Control, Optimisation and Calculus of Variations.

Spong, M. W. (1995). *The Swing Up Control Problem for the Acrobot.* IEEE Control Systems.

Spong, M. W. (n.d.). *Energy Based Control of a Class of Underactuated Mechanical Systems.*

Stern, A. (2006). *Discrete Geometric Mechanics and Variational Integrators.* Siggraph 2006.

Tedrake, R. (n.d.). *Course Notes for MIT 6.832 Underactuated Robotics.*