SIMULATION OF INVERTED PENDULUM WITH ACTIVE SUSPENSION

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2.152 Nonlinear Controls Final Project

INTRODUCTION

This project simulates the control of an inverted pendulum with an active suspension (drawing inspiration from the Segway and Bose automotive suspension). As Figure 1 shows, the pendulum chassis is modeled to consist of a wheel of radius R with mass m_3 and a platform with mass m_2, with a link of negligible mass between the two. A load m_1 is attached to the platform via an active suspension. For illustrative purposes, the suspension is drawn as a spring and damper in Figure 1; however, it is modeled in the equations as a linear motor that generates a force F_m between the load m_1 and the platform m_2. In addition, the wheel is modeled as producing another force F_c that is used to propel the cart forward and backward. The value h_2 is a constant that represents the distance from the bottom of the wheel to the platform. The value $h_1(t)$ is the time-varying distance from the platform to the load m_1, and the value $d(t)$ is a time-varying displacement at the wheel that models the effect of going over potholes, etc. Finally, the pendulum can be tilted at an angle ϑ from the vertical, and its displacement is measured by variables x (measured at the wheel) and y (measured as the ride height of the load m_1).

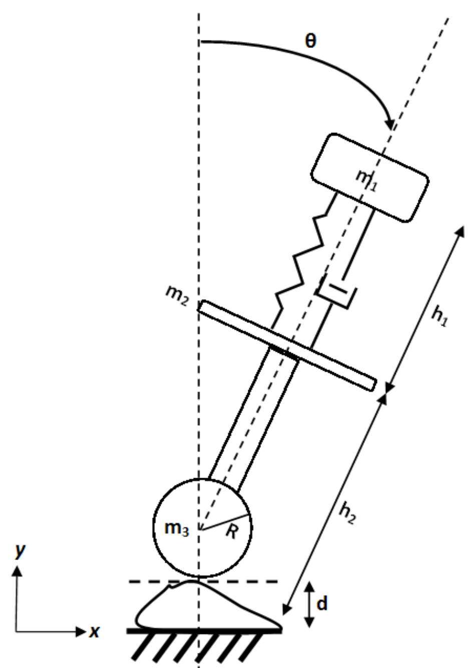

FIGURE 1: DIAGRAM OF SYSTEM MODEL.

EQUATIONS OF MOTION

The equations of motion for the system can be derived by using the Lagrangian. First, an intermediate variable *l* is defined as shown below to simplify the derivation. This value represents the center of mass of m_1 and m_2, and is the analogue of the length in a simple inverted pendulum. The difference here is that *l* varies with time, and thus its derivative is also a function of time.

$$l = \frac{m_2 h_2 + m_1(h_1 + h_2)}{m_1 + m_2}$$

$$\dot{l} = \frac{m_1}{m_1 + m_2}\dot{h}_1$$

Using this, the Lagrangian can be determined using the kinetic co-energy and the potential energy:

$$L = \frac{1}{2}m_3(\dot{x}^2 + \dot{d}^2) + \frac{1}{2}(m_1 + m_2)\left[\dot{x}^2 + \dot{d}^2 + \dot{l}^2 + l^2\dot{\theta}^2 + 2(\dot{x}\dot{l} - l\dot{d}\dot{\theta})\sin\theta + 2(l\dot{x}\dot{\theta} + \dot{d}\dot{l})\cos\theta\right] - m_3 g d - (m_1 + m_2)g(d + l\cos\theta)$$

The state of the system is chosen as the following:

$$state\colon \left[x\ \theta\ l\ \dot{x}\ \dot{\theta}\ \dot{l}\right]^T$$

Continuing with Lagrange's equation, the accelerations for *x*, *ϑ*, and *l* can be computed:

$$\ddot{x} = \frac{1}{m_1 + m_2 + m_3}\left\{F_c - (m_1 + m_2)\left[(\ddot{l} - l\dot{\theta}^2)\sin\theta + (2\dot{l}\dot{\theta} + l\ddot{\theta})\cos\theta\right]\right\}$$

$$\ddot{\theta} = \frac{1}{l}\left[(\ddot{d} + g)\sin\theta - \ddot{x}\cos\theta - 2\dot{l}\dot{\theta}\right]$$

$$\ddot{l} = \frac{F_m}{m_1 + m_2} - \ddot{x}\sin\theta - (\ddot{d} + g)\cos\theta + l\dot{\theta}^2$$

By inspection, the above set of governing equations can be reduced to those for a simple inverted pendulum when *l* is considered to be a constant, and there is no ground disturbance. Using algebra and rearranging terms to show the inputs F_c and F_m explicitly provides the following equations:

$$\ddot{x} = \frac{1}{m_3}F_c - \frac{\sin\theta}{m_3}F_m$$

$$\ddot{\theta} = \frac{(\ddot{d} + g)\sin\theta - 2\dot{l}\dot{\theta}}{l} - \frac{1}{m_3 l}F_c + \frac{\sin\theta}{m_3 l}F_m$$

$$\ddot{l} = l\dot{\theta}^2 - (\ddot{d} + g)\cos\theta - \frac{\sin\theta}{m_3}F_c + \frac{m_3 + (m_1 + m_2)(\sin\theta)^2}{m_3(m_1 + m_2)}F_m$$

Finally, it is noted that *y* and not *l* is the variable of interest, and thus a state transformation is used to obtain the second derivative of *y* as shown below:

$$\ddot{y} = \ddot{d} - \left[\frac{2\dot{\theta}^2 (\sin\theta)^2 (y-d)}{(\cos\theta)^2} + \frac{2\dot{\theta}(\dot{y}-\dot{d})\sin\theta}{\cos\theta}\right] + \frac{(\ddot{d}+g)\sin\theta - 2\dot{l}\dot{\theta}}{l}(h_1+h_2)\sin\theta$$
$$+ \left\{\left(\frac{m_1+m_2}{m_1}\right)\left[l\dot{\theta}^2 - (\ddot{d}+g)\cos\theta\right] - (h_1+h_2)\dot{\theta}^2\cos\theta\right\}\cos\theta$$
$$- \left[\frac{(m_1+m_2)\sin\theta\cos\theta}{m_1 m_3} + \frac{(h_1+h_2)\sin\theta}{m_3 l}\right]F_c$$
$$+ \left[\frac{m_3 + (m_1+m_2)(\sin\theta)^2\cos\theta}{m_1 m_3} + \frac{(h_1+h_2)(\sin\theta)^2}{m_3 l}\right]F_m$$

This allows the state of the system in simulation to be:

$$state: \left[x\ \theta\ y\ \dot{x}\ \dot{\theta}\ \dot{y}\right]^T$$

Controller Design

First, it is apparent from intuition that using only the control inputs provided, it is not possible to generate any arbitrary trajectory of the system state. For example, the inverted pendulum cannot be held motionless at a diagonal angle; it will fall over, or it must move horizontally to avoid tipping over. To address this issue, the controller is broken into an "outer loop" that controls horizontal velocity, and an "inner loop" that controls pendulum angle and ride height.

Inner Loop Control for Pendulum Angle and Ride Height

In order to control pendulum angle *ϑ* and ride height *y*, the following intermediate variables are defined:

$$q = [\theta\ y]^T, \qquad \dot{q} = \left[\dot{\theta}\ \dot{y}\right]^T, \qquad \underline{\tilde{q}} = \underline{q} - \underline{q}_d, \qquad s = \underline{\dot{\tilde{q}}} + \lambda\underline{\tilde{q}}$$

From here, it is noted that the state equations, while appearing complicated, are of the following form:

$$\underline{\dot{q}} = f\left(\underline{q}, d, \dot{d}, \ddot{d}\right) + B(\underline{q})\underline{u}$$

In other words, the state derivatives consist of a function *f* of the state, along with the time-varying unknown ground disturbance *d* and its derivatives, plus a function *B* of the state times the control inputs. Based on the system's properties, a sliding mode controller is chosen to provide trajectory tracking functionality for the system. A control law that is the best estimate to make $\dot{s}$ zero is the following:

$$\underline{\hat{u}} = B^{-1}\left[\underline{\ddot{q}}_d - \hat{f} - \lambda\underline{\dot{\tilde{q}}}\right]$$

Then, the control law is modified with an additional term to satisfy the sliding condition with a boundary layer:

$$\underline{u} = \underline{\hat{u}} + (F+\eta)sat\left(\frac{s}{\Phi}\right)$$

In the above control equations, $\hat{f}$ is the estimate of the dynamics *f*, and F is based on the assumed bounds on the ground disturbance. The value η is a strictly positive constant from the sliding condition, λ is a control parameter chosen based on unmodeled dynamics, and Φ is the boundary layer of the sliding surface.

Outer Loop Control for Horizontal Velocity

The outer loop control for velocity is simply based on the idea that the inverted pendulum's velocity can be controlled by specifying the pendulum angle. To this end, a simple proportional-integral controller is implemented:

$$\theta_d = -K_P(\dot{x} - \dot{x}_d) - K_I \int_0^t (\dot{x} - \dot{x}_d)dt$$

The output ϑ_d of this control law is capped to prevent the controller from specifying desired angles that would make the inverted pendulum tip over.

Results

In simulation, the sliding mode controller for pendulum angle and ride height, along with the PI controller for velocity, was able to stabilize the system. Shown in Figure 2 is a screenshot of the animated simulation in which the pendulum moves along an otherwise flat road with a sinusoidal ground disturbance. Shown in Figure 3 are the ground disturbance and ride height error. As the plots indicate, the controller effectively isolates the load from the bumps in the ground.

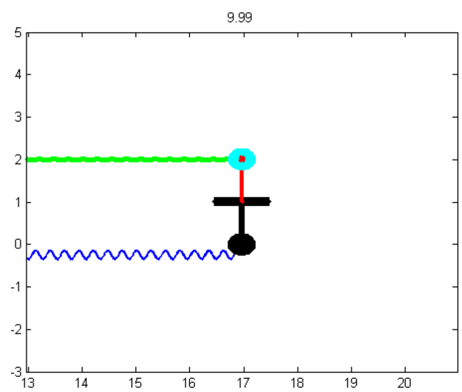

FIGURE 2: SCREENSHOT OF ANIMATED SIMULATION.
The black shapes are the rigid parts of the inverted pendulum chassis, the red represents the active suspension, and the cyan circle represents the load. The blue and green curves mark the paths of the wheel and the load, respectively.

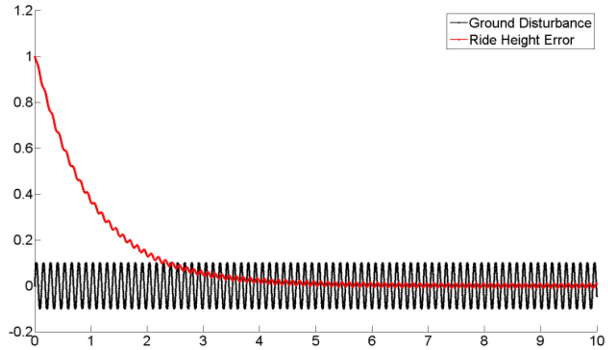

FIGURE 3: GROUND DISTURBANCE AND RIDE HEIGHT ERROR.

The ground disturbance has an amplitude of 0.1, and the ride height has an amplitude of 0.01. This 90% reduction in vibration can be improved with a tighter boundary layer in the sliding controller, at the cost of additional chattering in the control input.

Shown in Figure 4 and Figure 5 are the effects of the outer loop control for velocity. As the plots indicate, the controller successfully keeps the inverted pendulum moving at a constant velocity while also maintaining stability and ride height.

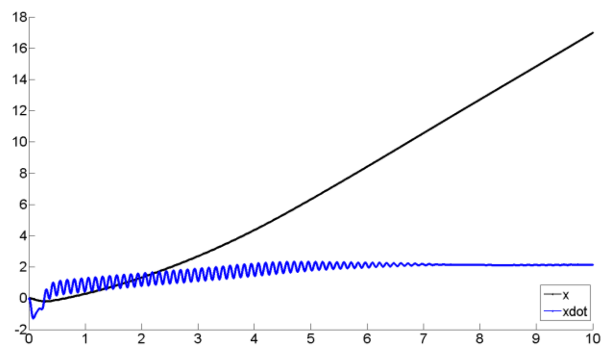

FIGURE 4: POSITION AND VELOCITY OF INVERTED PENDULUM.

As shown, the velocity "outer loop" proportional-integral control allows the vehicle to keep a constant speed over bumpy terrain.

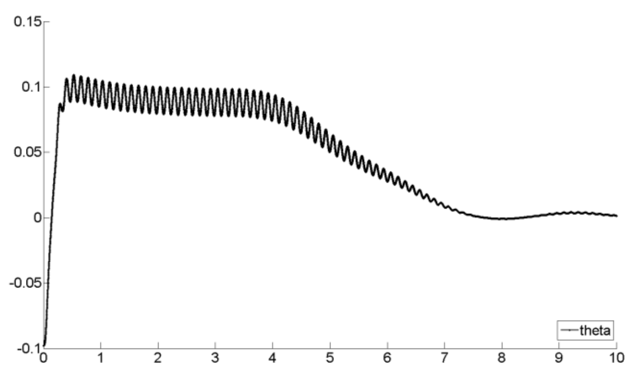

FIGURE 5: INVERTED PENDULUM ANGLE.

As shown, the velocity "outer loop" control pushes the pendulum forwards for some time in order to gain velocity, and then settles to an upright position once the target speed has been reached.

CONCLUSION

The equations of motion for an inverted pendulum with an active suspension were derived. The suspension was modeled as a motor producing linear force along the length of the pendulum, and bumps in the road were modeled as a time-varying disturbance at the wheel. It can be shown that these equations reduce to those for a simple inverted pendulum when the length is considered to be constant, and the ground disturbance is zero. Then, a sliding controller was successfully implemented in simulation to control pendulum angle and ride height, with an outer loop PI controller for horizontal velocity.

Future directions for this topic could include: modeling the disturbance as a function of x rather than t, implementing observers to estimate state variables that may not be reliably measured, and using adaptation to deal with unknown loads.