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Optimal Influence Strategies in Social Networks

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Keywords influence strategy · monopoly · opinion formation · social networks.

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2 Introduction

The rapid growth of online social media (such as Facebook and Twitter) has raised the importance of social interactions for the analysis of consumer behavior. More specifically, consumers often communicate in online social networking platforms to exchange information about the quality of new goods introduced in the market. Indeed, the process of word-of-mouth communication has been identified by recent marketing research as an important informational channel affecting consumers' purchasing decisions [Iyengar et al.(2011)Iyengar, Van den Bulte, and Valente]. At the same time, the impact of traditional marketing techniques has been constantly decreasing [Schmitt et al.(2011)Schmitt, Skiera, and Van den Bulte]. In this context, a natural question is to study the various strategies developed by firms to manipulate the flow of information in social networks. A common strategy recently being followed by many firms involves exploiting certain agents' disproportionate influence on others to accelerate the diffusion of a new product [Hinz et al.(2014)Hinz, Schulze, and Takac]. For instance, Ford started a buzz marketing campaign in 2009 to spread information about a new brand (Ford Fiesta) in social media. The company selected a set of influential bloggers, gave a Fiesta to each of them and encouraged them to spread information about the new brand through the use of videos, tweets and blog posts. The campaign was successful, since it had a significantly positive impact both on brand awareness and on the volume of sales. Another prominent example is the online fashion store ASOS, which has recently launched a campaign on Twitter by using the "BestNightEver" hashtag and relying on the popularity of certain youth idols. This enabled the company to achieve a high level of consumer engagement and considerably increase its sales, thus registering its highest return on investment up to date. Such results can provide guidelines to firms for selecting influential agents in online social media and conducting buzz marketing campaigns to build brand loyalty and promote their profitability.

However, there is a body of literature which indicates that targeting the most influential agents might not always be the best practice for the diffusion of a product. For example, it has been argued that highly connected nodes receive a great amount of information because of their central position in the network. As a result, these influential agents normally need more time to decide which bits of information to transmit through the network [Dodds and Watts(2004)]. In this framework, Watts and Dodds [Watts and Dodds(2007)] use computer simulations to show that diffusion cascades are mainly driven by easily influenced individuals rather than by influential agents. The conditions under which a monopolist should target the more or less influential nodes of a network have been studied by Galeotti and Goyal [Galeotti and Goyal(2009)] in a theoretical model which studies the optimal advertising strategy of a firm intending to raise consumers' awareness or perceived quality of its brand. Their analysis shows that the monopolist's profit-maximizing strategy depends crucially on the content of social interaction. In particular, the firm should seed highly influential agents in the case of proportional adoption externalities but should target marginalized consumers when information about the product disseminates through a word-of-mouth communication process. Our paper also deals with the case of word-of-mouth communication to identify the structural conditions under which the monopolist should target the more or less influential consumers. However, we suggest an entirely different modeling framework - as explained below - and treat the firm as a node participating in the underlying network rather than as an external agent.

Influential agents have been generally defined as "individuals who are likely to influence other persons in their immediate environment" [Katz and Lazarsfeld(1955)]. The main properties of such an "opinion leader" include her competence, her strategic social location and the personification of certain values. Similarly, the concept of a maven refers to someone who has disproportionate influence on others due to her recognized expertise in a particular field [Gladwell(2006)]. A recent branch of the literature suggests various procedures to identify key players in a social network [Borgatti(2006)]. The alternative measures representing the centrality of individual agents are thoroughly described in [Jackson et al.(2008)]. In this context, the degree centrality of a node - i.e. the number of her direct neighbors - has often been suggested both as an empirical measure of influence [Hinz et al.(2011)Hinz, Skiera, Barrot, and Becker] and as an appropriate theoretical tool to define influential agents [Galeotti and Goyal(2009)]. However, the degree centrality is a simple measure of popularity which does not provide enough information about the strategic location of a node in the network. In order to capture this aspect, the notion of betweenness centrality has been used to characterize influential agents as those who act like bridges and facilitate the flow of information between different groups of nodes [Hinz and Spann(2008)]. Alternatively, the influence and the strategic position of a node in the network can be measured by her eigenvector centrality, which depends on how much central or important her neighbors are. According to this interpretation, an agent is influential to the extent that she has influential friends [Richardson and Domingos(2002)]. This measure of influence has been used by [Golub and Jackson(2010)] in the context of a social learning model. A closely related concept is that of Katz-Bonacich centrality applied by [Ballester et al.(2006)Ballester, Calvó-Armengol, and Zenou] to identify the key player in a noncooperative network game with local payoff complementarities. Our paper follows [Golub and Jackson(2010)] by identifying the most influential agent of the network in terms of eigenvector centrality, which corresponds better to iterated influence relationships involved in the process of opinion formation.

In general, there are two main approaches to modeling the formation of agents' beliefs in a social network. The first approach considers a set of fully rational agents using Bayes' rule to update their beliefs over time [Acemoglu et al.(2011)Acemoglu, Dahleh, Lobel, and Ozdaglar]. The second approach assumes that agents are boundedly rational and the process of updating in each period occurs according to an exogenously specified rule of thumb - e.g. by taking weighted averages of neighbors' beliefs from the previous period [DeGroot(1974), Golub and Jackson(2010)]. This article adopts the relatively more realistic perspective of bounded rationality to model a network of agents who form their beliefs about product quality according to the rule specified in the DeGroot model.

We consider a weighted and strongly connected network which includes a monopolistic firm and two consumers (representing two groups or types of customers) buying the good produced by the monopolist. The firm can spend an amount of resources to increase its influence on consumers' beliefs about the brand quality. This activity can be understood as a costly investment that raises the valuation of the good, thus also increasing the associated market demand. Agents repeatedly communicate in the network to form their beliefs about quality and consumers ultimately make their purchasing decisions given the monopolistic price set by the firm. The proposed formulation results in a linearly constrained, nonlinear (rational objective function) optimization problem which cannot be analytically tracked unless particular assumptions are made on the exogenous model parameters. Therefore, optimization has been computationally conducted through the utilization of the Sequential Quadratic Programming approach which constitutes a state of the art method for nonlinear optimization problems.

For the benchmark case where all consumers have the same initial beliefs, we analytically and numerically show that the monopolist always invests relatively more in affecting the most influential consumer's opinion. We also characterize the optimal overall amount of resources that should be allocated by the firm to the activity of manipulating the network. When consumers have different initial beliefs, we rely on the appropriate numerical methods to show that the firm might optimally target the least influential consumer if the latter's initial valuation of the good is low enough (relative to the other's valuation). In this context, we conduct a series of experiments showing that the monopolist's incentives to target the agent with the relatively lower influence increase both with the distance between consumers' initial beliefs and with the degree of trust attributed on the set of consumers by the firm. In both cases of uniform and non-uniform initial beliefs, the equilibrium valuation of the good and the monopolist's profit are minimized when consumers' influences become equal, implying that the firm benefits from the presence of consumers with divergent strategic locations in the network.

The above results hold either when the firm faces a binding constraint on total investment or not. In the latter case, however, the underlying forces driving the U-shaped behavior of the equilibrium profit curve are qualitatively different as shown by a decomposition of profit into its two main components (net revenue and investment cost). The study of comparative statics implies that an increase in consumers' initial beliefs weakens the monopolist's incentives to manipulate the network and a higher level of trust placed on consumers by the firm has an ambiguous (initially positive but eventually negative) impact on the latter's equilibrium investment strategy.

The rest of this article is organized as follows. Section 3 introduces the basic modeling framework and Section 4 derives the equilibrium outcome both analytically and computationally. Section 5 studies the implications of equilibrium when the firm faces a binding constraint on total investment and consumers have either uniform or different initial beliefs about brand quality. Section 6 deals with the case of a non-binding constraint on total investment. Finally, section 7 concludes and provides directions for future research.

3 Modeling Framework

Our analysis focuses on the case of a monopolistic firm (which is referred to as agent 0) but can be readily extended to account for an oligopolistic product market. The set of consumers buying the product of the firm is $N = \{1, ..., n\}$. The firm and consumers repeatedly communicate with each other in the context of a social network to form their beliefs about product quality. Therefore, the full set of nodes participating in the underlying network can be denoted by $M = \{0, 1, ..., n\}$. The pattern of interactions is captured by an $(n+1) \times (n+1)$ nonnegative matrix $\mathbf{T}$. This matrix is row stochastic, i.e. its entries across each row sum to one. Each entry $T_{ij} \in [0, 1]$ of the matrix represents the weight or trust placed by agent i on the current belief of agent j in forming i's beliefs for the next period (where $i, j \in M$). Let $p_i(t) \in [0, 1]$ be agent i's valuation of the good in period t. The vector of agents' beliefs in period t is $p(t) = [p_0(t), p_1(t), p_2(t), \cdots, p_n(t)]^{-1}$and the vector of initial beliefs is $p(0)$. In what follows, we make the reasonable assumption that the firm holds the best possible opinion about its product at the beginning of the communication process ($p(0) = 1$). Of course, this initial opinion is subject to change during the course of information exchange with the set of consumers. The possible influence of consumers on the firm's valuation of its own good is captured by letting T_{0j} be generally different than zero. Based on these assumptions, the network under study can be described by the following updating matrix:

$$\mathbf{T} = \begin{bmatrix} T_{00} & T_{01} & T_{02} & \dots & T_{0n} \\ T_{10} & T_{11} & T_{12} & \dots & T_{1n} \\ T_{20} & T_{21} & T_{22} & \dots & T_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ T_{n0} & T_{n1} & T_{n2} & \dots & T_{nn} \end{bmatrix} \tag{1}$$

A concrete example refers to a network with two consumers (1 and 2). In this case, the interaction matrix becomes:

$$\mathbf{T} = \begin{bmatrix} T_{00} & T_{01} & T_{02} \\ T_{10} & T_{11} & T_{12} \\ T_{20} & T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} 1-\lambda_1-\lambda_2 & \lambda_1 & \lambda_2 \\ T_1 & 1-T_1-\theta_2 & \theta_2 \\ T_2 & \theta_1 & 1-T_2-\theta_1 \end{bmatrix} \tag{2}$$

where $\lambda_j \geq 0$ (with $\sum_{j\in N} \lambda_j \leq 1$) shows how much consumer j directly affects the firm's valuation of the good, $\theta_j \in [0,1]$ shows j's direct influence on the other consumer's opinion and $T_j \geq 0$ shows the firm's direct influence on j's beliefs. The row stochasticity of the interaction matrix implies that the following set of constraints must necessarily hold:

$$\begin{aligned} 0 \leq T_1 \leq 1-\theta_2 \\ 0 \leq T_2 \leq 1-\theta_1 \end{aligned} \tag{3}$$

Note that each consumer might actually be perceived as a group or a representative agent of a particular type, where each group or type of customers is defined in terms of some characteristic (age, gender, education level etc.) that can influence agents' propensity to link with each other. It is generally admitted that homophily - i.e. the tendency of individuals to associate with others of the same type - is one of the most pervasive properties governing the structure of social networks [McPherson et al.(2001)McPherson, Smith-Lovin, and Cook]. In the presence of nontrivial homophily, convergence of beliefs is generally guaranteed and achieved at a relatively fast rate within each group golub2012homophily. Therefore, we can focus our attention on the question of reaching a consensus among different groups. In the context of such a group-level analysis, the parameter θ_j represents group j's direct influence on the other group and can be defined as an aggregate measure of weights placed on type-j agents by individuals of the other type [Golub and Jackson(2010)].

The network corresponding to the interaction matrix $\mathbf{T}$ is graphically depicted in Fig. 1.

We follow [DeGroot(1974)] and [Golub and Jackson(2010)] by assuming that beliefs are updated in each period according to the following rule:

$$p_i(t) = \sum_{j\in M} T_{ij} \cdot p_j(t-1), \ \forall i \in M \tag{4}$$

In words, each agent updates her belief in each period by taking a weighted average of her neighbors' beliefs in the previous period. Let $x_i = p_i(\infty)$ be the

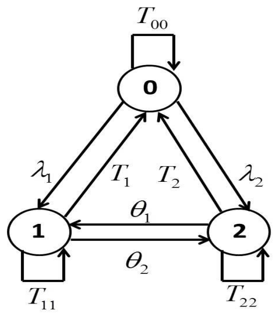

Fig. 1 The network of interactions for the case of two consumers

limiting belief reached by agent $i \in M$ in equilibrium. The quantity of output produced by the firm is q and the production cost function is $c(q) = c \cdot q$, where $c > 0$ is the marginal cost of production. The firm can invest an amount of resources to increase its influence (T_j) on consumer j's beliefs. The cost of this investment activity is given by $g(T_j) = \gamma \cdot T_j$, where $\gamma > 0$ is the marginal cost of investment. If we denote the price of the good by p, we can write the firm's overall profit as:

$$\Pi = (p - c) \cdot q - \sum_{j \in N} g(T_j) = (p - c) \cdot q - \gamma \cdot \sum_{j \in N} T_j \tag{5}$$

We also assume that each consumer j's demand (q_j) for the good increases with her valuation of the good (x_j) and decreases with the market price (p). Let $q_j(x_j, p) = x_j \cdot (a - p)$, where $a > c$ represents the gross market size. The difference $d = a - c$ measures the size of the market demand relative to the production cost.

The model has the following time structure. First, the firm chooses the levels of investment (T_j) associated with each consumer j. Second, the firm sets the price of the good in the monopolistic product market. Third, all agents in the network repeatedly communicate until reaching a limiting belief vector $x = [x_0, x_1, ..., x_n]^{-1}$ by following the updating rule specified in Eq. 4. Finally, consumers make their purchasing decisions according to the demand functions described above.

4 Derivation of Equilibrium

4.1 Analytical Solution

We now proceed to characterize the equilibrium outcome of the model. Since the parameters λ_j and θ_j are generally non-zero and the firm can always make positive investment choices, the network is strongly connected and the interaction matrix is aperiodic. These two properties imply that all agents' beliefs converge and reach a consensus in the long-run, i.e. $x_i = x_j = x$ for all $i, j \in M$ [Golub and Jackson(2010)]. For two (groups of) consumers, the limiting consensus belief can be identified by first calculating the social influence vector $\mathbf{s} = (s_0, s_1, s_2)$ whose elements sum to one. This vector shows each agent j's final influence (s_j) on the consensus belief and is given by the left-hand eigenvector associated with the unit eigenvalue of the matrix $\mathbf{T}$, i.e. $\mathbf{sT} = \mathbf{s}$. By imposing the normalization $\sum_{j \in M} s_j = 1$, we get the following solution:

$$s_0 = \frac{T_1T_2 + T_1\theta_1 + T_2\theta_2}{T_1T_2 + (\lambda_2 + \theta_1)T_1 + (\lambda_1 + \theta_2)T_2 + (\lambda_1 + \lambda_2)(\theta_1 + \theta_2)} \tag{6}$$

$$s_1 = \frac{\lambda_1 T_2 + (\lambda_1 + \lambda_2)\theta_1}{T_1T_2 + (\lambda_2 + \theta_1)T_1 + (\lambda_1 + \theta_2)T_2 + (\lambda_1 + \lambda_2)(\theta_1 + \theta_2)} \tag{7}$$

$$s_2 = \frac{\lambda_2 T_1 + (\lambda_1 + \lambda_2)\theta_2}{T_1T_2 + (\lambda_2 + \theta_1)T_1 + (\lambda_1 + \theta_2)T_2 + (\lambda_1 + \lambda_2)(\theta_1 + \theta_2)} \tag{8}$$

Before going on, we note two extreme cases. First, if the firm does not invest at all in influence ($T_1 = T_2 = 0$) then its limiting influence will be zero ($s_0 = 0$) and the limiting influence of each consumer will be proportional to her direct influence on the other ($s_1 = \frac{\theta_1}{\theta_1+\theta_2}$, $s_2 = \frac{\theta_2}{\theta_1+\theta_2}$). Second, if no consumer has any direct influence on the firm ($\lambda_1 = \lambda_2 = 0$) the latter's initial opinion fully determines the consensus belief ($s_0 = 1$, $s_1 = s_2 = 0$). In general, the consensus belief will be the weighted average of initial opinions, where each agent j's weight will be given by the j-th element of the influence vector $\mathbf{s}$:

$$x = \sum_{j \in M} s_j p_j(0) \tag{9}$$

The monopolist sets the price of the good so as to maximize its profit $\Pi = (p - c) \cdot (q_1(x, p) + q_2(x, p)) - \gamma \cdot \sum_{j \in N} T_j$. This problem can be solved to yield $p = \frac{a+c}{2}$, implying $\Pi = \frac{(a-c)^2 \cdot x}{2} - \gamma \cdot (T_1 + T_2)$. The firm chooses the profit-maximizing level of investment associated with each consumer $j \in N$. Suppose that the firm's investment cannot exceed a fixed overall amount of resources available to manipulate consumers' beliefs. This constraint can be captured by the condition $\sum_{j \in N} T_j \leq k$.

Formulating the constrained optimization problem of determining the profit-maximizing distribution of influence-related investments requires the definition of the following quantities:

$$Q(T_1, T_2) = T_1T_2 + (\lambda_2 + \theta_1)T_1 + (\lambda_1 + \theta_2)T_2 + (\theta_1 + \theta_2)(\lambda_1 + \lambda_2) \tag{10}$$

and

$$\begin{aligned} F_0(T_1, T_2) &= T_1T_2 + \theta_1 T_1 + \theta_2 T_2 \\ F_1(T_1, T_2) &= \theta_1 P_1(\lambda_1 + \lambda_2) + \lambda_1 P_1 T_2 \\ F_2(T_1, T_2) &= \theta_2 P_2(\lambda_1 + \lambda_2) + \lambda_2 P_2 T_1 \end{aligned} \tag{11}$$

where $P_1 = p_1(0)$ and $P_2 = p_2(0)$. By combining the quantities defined in Eqs. 11 such that $F(T_1, T_2) = F_0(T_1, T_2) + F_1(T_1, T_2) + F_2(T_1, T_2)$, we may write that:

$$F(T_1, T_2) = T_1T_2 + (\lambda_2 P_2 + \theta_1)T_1 + (\lambda_1 P_1 + \theta_2)T_2 + (\lambda_1 + \lambda_2)(\theta_1 P_1 + \theta_2 P_2) \quad (12)$$

Therefore, the overall objective function can be formulated as follows:

$$\hat{F}(T_1, T_2) = \gamma(T_1 + T_2) - \frac{1}{2}(a-c)^2 \frac{F(T_1, T_2)}{Q(T_1, T_2)} = -\Pi(T_1, T_2) \quad (13)$$

within the framework of a minimization problem which is given by:

$$\min_{(T_1,T_2)} \quad \hat{F}(T_1, T_2) \quad (14a)$$

$$\text{s.t} \quad 0 \leq T_1 \leq 1 - \theta_2 \quad (14b)$$

$$0 \leq T_2 \leq 1 - \theta_1 \quad (14c)$$

$$T_1 + T_2 \leq k \quad (14d)$$

so that:

$$0 \leq \theta_1 \leq 1 \quad (15a)$$

$$0 \leq \theta_2 \leq 1 \quad (15b)$$

$$0 \leq \lambda_1 + \lambda_2 \leq 1 \quad (15c)$$

The inequality constraints of the optimization problem defined in Eqs. 14 can be re-expressed through the utilization of the following auxiliary functions:

$$g_1(T_1, T_2) = -T_1 \quad (16)$$

$$g_2(T_1, T_2) = T_1 + \theta_2 - 1 \quad (17)$$

$$g_3(T_1, T_2) = -T_2 \quad (18)$$

$$g_4(T_1, T_2) = T_2 + \theta_1 - 1 \quad (19)$$

$$g_5(T_1, T_2) = T_1 + T_2 - k \quad (20)$$

Thus, the overall minimization problem may be solved in the framework of Lagrangian optimization by defining the corresponding Lagrangian function as:

$$L(T_1, T_2) = \hat{F}(T_1, T_2) + \sum_{j \in [5]} \mu_j g_j(T_1, T2) \quad (21)$$

where $\mu_j \geq 0, \ \forall j \in [5]$ are the associated Lagrangian multipliers. Specifically, the Karush-Kuhn-Tucker (KKT) necessary and sufficient conditions for a normal point (T_1^*, T_2^*) to be a minimum is the existence of a unique quintuple $(\mu_1^*, \mu_2^*, \mu_3^*, \mu_5^*, \mu_5^*)$ of Lagrangian multipliers such that:

$$\frac{\partial L(T_1^*, T_2^*)}{\partial T_i} = 0, \ \forall i \in [2] \quad (22a)$$

$$\mu_j^* \cdot g_j(T_1^*, T_2^*) = 0, \ \forall j \in [5] \quad (22b)$$

$$g_j(T_1^*, T_2^*) \leq 0, \ \forall j \in [5] \quad (22c)$$

$$\mu_j^* \geq 0, \ \forall j \in [5] \quad (22d)$$

Letting $\bar{T} = (T_1, T_2)$, one should also impose positive definite constraints on $\nabla_{\bar{T}\bar{T}} L(T_1^*, T_2^*)$ in order to ensure that the normal point defined by the previous conditions corresponds to a minimum of the Lagrangian function.

The First Order Conditions (FOCs) defined in Eqs. 22a imply that

$$\frac{\partial \hat{F}(T_1^*, T_2^*)}{\partial T_1} - \mu_1^* + \mu_2^* + \mu_5^* = 0 \tag{23a}$$

$$\frac{\partial \hat{F}(T_1^*, T_2^*)}{\partial T_2} - \mu_3^* + \mu_4^* + \mu_5^* = 0 \tag{23b}$$

where

$$\frac{\partial \hat{F}(T_1, T_2)}{\partial T_1} = \gamma - \frac{1}{2}(a-c)^2 \frac{G_1(T_1, T_2)}{Q^2(T_1, T_2)} \tag{24a}$$

$$\frac{\partial \hat{F}(T_1, T_2)}{\partial T_2} = \gamma - \frac{1}{2}(a-c)^2 \frac{G_2(T_1, T_2)}{Q^2(T_1, T_2)} \tag{24b}$$

and

$$G_i(T_1, T_2) = \frac{\partial F(T_1, T_2)}{\partial T_i} Q(T_1, T_2) - F(T_1, T_2) \frac{\partial Q(T_1, T_2)}{\partial T_i}, \ \forall i \in [2] \tag{25}$$

It can be easily deduced that the quantity $G_1(T_1, T_2)$ is in fact a second degree polynomial with respect to T_2 so that it can be expressed in the following form:

$$G_1(T_1, T_2) = G_1(T_2) = A_2 \cdot T_2^2 + A_1 \cdot T_2 + A_0 \tag{26}$$

where

$$\begin{aligned} A_2 &= \lambda_1(1 - P_1) \\ A_1 &= \lambda_1[\theta_1(1-P_1) + \theta_2(1-P_2) + \lambda_2(P_2 - P_1)] - \theta_1(\lambda_1+\lambda_2)(P_1 - 1) \\ A_0 &= \theta_1(\lambda_1+\lambda_2)[\theta_1(1-P_1) + \theta_2(1-P_2) + \lambda_2(P_2 - P_1)] \end{aligned} \tag{27}$$

Likewise, it is straightforward to show that the quantity $G_2(T_1, T_2)$ is in fact a second degree polynomial with respect to T_1 such that:

$$G_2(T_1, T_2) = G_2(T_1) = B_2 \cdot T_1^2 + B_1 \cdot T_1 + B_0 \tag{28}$$

where

$$\begin{aligned} B_2 &= \lambda_2(1 - P_2) \\ B_1 &= \lambda_2[\theta_1(1-P_1) + \theta_2(1-P_2) + \lambda_1(P_1 - P_2)] - \theta_2(\lambda_1+\lambda_2)(P_2 - 1) \\ B_0 &= \theta_2(\lambda_1+\lambda_2)[\theta_1(1-P_1) + \theta_2(1-P_2) + \lambda_1(P_1 - P_2)] \end{aligned} \tag{29}$$

Having in mind Eqs. 23a, 23b, 24a and 23b, the FOCs initially defined by Eqs. 22a may be reformulated according to:

$$\gamma - \frac{1}{2}(a-c)^2 \frac{G_1(T_2^*)}{Q^2(T_1^*, T_2^*)} - \mu_1^* + \mu_2^* + \mu_5^* = 0 \tag{30a}$$

$$\gamma - \frac{1}{2}(a-c)^2 \frac{G_2(T_1^*)}{Q^2(T_1^*, T_2^*)} - \mu_3^* + \mu_4^* + \mu_5^* = 0 \tag{30b}$$

The KKT conditions defined in Eqs. 22b, 22c and 22d may be combined as:

$$\forall j \in [5] : \mu_j^* > 0 \Rightarrow g_j(T_1^*, T_2^*) = 0 \tag{31}$$

so that the following clarifications can be made:

1. (μ_1^*, μ_2^*) are the optimal Lagrangian multipliers controlling the existence of interior or boundary solutions for T_1^*. Interior solutions for T_1^* are those lying in the interval $(0, 1-\theta_2)$, while boundary solutions for T_1^* are those taking the extreme values $T_1^* = 0$ or $T_2^* = 1-\theta_2$. In particular, there exist 4 possible combinations for the values of (μ_1^*, μ_2^*) listed below:
   (a) ($\mu_1^* > 0$ and $\mu_2^* = 0$) which yields a boundary minimum solution ($T_1^* = 0$) since $g_1(T_1^*, T_2^*) = 0$, given that $\theta_2 < 1$.
   (b) ($\mu_1^* = 0$ and $\mu_2^* > 0$) which yields a boundary maximum solution ($T_1^* = 1-\theta_2$) since $g_2(T_1^*, T_2^*) = 0$, given that $\theta_2 < 1$.
   (c) ($\mu_1^* = 0$ and $\mu_2^* = 0$) which yields an interior solution for T_1^* such that $0 < T_1^* < 1-\theta_2$, given that $\theta_2 < 1$.
   (d) ($\mu_1^* > 0$ and $\mu_2^* > 0$) which gives rise to a pathological boundary solution ($T_1^* = 0 = 1-\theta_2$) since $g_1(T_1^*, T_2^*) = 0$ and $g_2(T_1^*, T_2^*) = 0$, given that $\theta_2 = 1$.
2. (μ_3^*, μ_4^*) are the optimal Lagrangian multipliers controlling the existence of interior or boundary solutions for T_2^*. Interior solutions for T_2^* are those lying in the interval $(0, 1-\theta_1)$, while boundary solutions for T_2^* are those taking the extreme values $T_2^* = 0$ or $T_2^* = \theta_1$. In particular, there exist 4 possible combinations for the values of (μ_3^*, μ_4^*) listed below:
   (a) ($\mu_3^* > 0$ and $\mu_4^* = 0$) which yields a boundary minimum solution ($T_2^* = 0$) since $g_3(T_1^*, T_2^*) = 0$, given that $\theta_1 < 1$.
   (b) ($\mu_3^* = 0$ and $\mu_4^* > 0$) which yields a boundary maximum solution ($T_2^* = 1-\theta_1$) since $g_4(T_1^*, T_2^*) = 0$, given that $\theta_1 < 1$.
   (c) ($\mu_3^* = 0$ and $\mu_4^* = 0$) which yields an interior solution for T_2^* such that $0 < T_2^* < 1-\theta_1$, given that $\theta_1 < 1$.
   (d) ($\mu_3^* > 0$ and $\mu_4^* > 0$) which gives rise to a pathological boundary solution ($T_2^* = 0 = 1-\theta_1$) since $g_3(T_1^*, T_2^*) = 0$ and $g_4(T_1^*, T_2^*) = 0$, given that $\theta_2 = 1$.
3. μ_5^* is the optimal Lagrangian multiplier controlling the existence of interior or boundary solutions for $T_1^* + T_2^*$. Interior solutions for $T_1^* + T_2^*$ are those for which $T_1^* + T_2^* < k$ when $\mu_5^* = 0$, while the boundary ones are those for which $T_1^* + T_2^* = k$ when $\mu_5^* > 0$.

Ideally, one would be particularly interested in finding an exact analytical solution to the nonlinear optimization problem defined in Eqs. 14. Obtaining a closed form analytical solution would in turn allow us to express the optimal influence-related investment levels as a function of the exogenous parameters of the model as follows:

$$\bar{T}^* = (T_1^*, T_2^*) = \mathcal{F}(P_0(0), P_1(1), P_2(0), \theta_1, \theta_2, \lambda_1, \lambda_2, \gamma, \delta, k) \tag{32}$$

where $\delta = (a-c)$. Such a functional relationship would be of particular importance towards analyzing the behavior of the corresponding interior optimal solutions, satisfying $((T_1^*, T_2^*) \in (0, 1-\theta_2) \times (0, 1-\theta_1))$. According to the previous discussion, interior solutions are associated with zero-valued Lagrangian multipliers such that $(\mu_1^* = \mu_2^* = \mu_3^* = \mu_4^* = 0)$. Therefore, the major distinction that can be made on the nature of the optimal solutions concerns the activation or deactivation of the binding constraint on the total amount of the optimal investment intensities $(T_1^* + T_2^*)$, controlled by μ_5^*.

In this context, the assumption of strictly interior optimal solutions transforms Eqs. 30a and 30b into the following form:

$$\gamma - \frac{1}{2}(a-c)^2 \frac{G_1(T_2^*)}{Q^2(T_1^*, T_2^*)} + \mu_5^* = 0 \tag{33a}$$

$$\gamma - \frac{1}{2}(a-c)^2 \frac{G_2(T_1^*)}{Q^2(T_1^*, T_2^*)} + \mu_5^* = 0 \tag{33b}$$

By performing pairwise substraction on the set of Eqs. 33a and 33b, it is easy to deduce that the optimal solutions (T_1^*, T_2^*) will be provided by finding the roots of the polynomial given below:

$$G(T_1^*, T_2^*) = G_1(T_2^*) - G_2(T_1^*) \tag{34}$$

Determining the roots of the polynomial defined in Eq. 34 is not a trivial task, unless one assumes that all consumers hold identical initial beliefs $(P_1 = P_2 = P)$ about the brand quality and the firm puts equal direct influence $(\lambda_1 = \lambda_2 = \lambda)$ on each consumer's opinion. Under these assumptions, the coefficients of the polynomials $G_1(T_2)$ and $G_2(T_1)$ will be given by:

$$\begin{aligned} A_2 &= \lambda(1-P) \\ A_1 &= \lambda[(1-P)(\theta_1+\theta_2)] + 2\lambda\theta_1(1-P) \\ A_0 &= 2\lambda\theta_1[(1-P)(\theta_1+\theta_2)] \end{aligned} \tag{35}$$

and

$$\begin{aligned} B_2 &= \lambda(1-P) \\ B_1 &= \lambda[(1-P)(\theta_1+\theta_2)] + 2\lambda\theta_2(1-P) \\ B_0 &= 2\lambda\theta_1[(1-P)(\theta_1+\theta_2)] \end{aligned} \tag{36}$$

which yields the following parameterized solution:

$$\begin{aligned} T_1^* &= \theta_1 - \theta_2 + z \\ T_2^* &= z \end{aligned} \tag{37}$$

where z must lie in the interval $[\max(0, \theta_2 - \theta_1), \min(1-\theta_1, 1-\theta_2)]$. Thus, when the binding constraint is activated $(\mu_5^* > 0)$, the parameter z may be derived from the corresponding equality $(T_1^* + T_2^* = k)$ which entails:

$$\begin{aligned} T_1^* &= \frac{\theta_1 - \theta_2 + k}{2} \\ T_2^* &= \frac{\theta_2 - \theta_1 + k}{2} \end{aligned} \tag{38}$$

This equilibrium solutions are indeed interior when $\max(\theta_1 - \theta_2, \theta_2 - \theta_1) \leq k \leq 2 - (\theta_1 + \theta_2)$. On the contrary, when the binding constraint is deactivated ($\mu_5^* = 0$), parameter z ,may be obtained from $G(T_1^*(z), T_2^*(z)) = 0$ as the root of a fourth degree polynomial with respect to z.

4.2 Computational Solution

It is clear from the analysis in the previous subsection that apart from exceptionally special cases the constrained nonlinear optimization problem defined in Eq. 14 cannot be analytically solved. Therefore, the optimal influence investment levels (and the associated Lagrangian multipliers) cannot be obtained in order to derive an exact functional relationship connecting these quantities with the exogenous parameters of the model. These limitations imply that a numerical optimization algorithm must be incorporated within a multidimensional grid traversing procedure in order to obtain a computational approximation of the functional form defined in Eq. 32. The graphical representation of this computationally approximated function, may be subsequently utilized to assess the influence of each exogenous parameter on the optimal influence investment levels. In particular, each parameter is assigned with a discrete range of values to be traversed, designated by the corresponding minimum, maximum and step-size values as:

1. $P_0(0) : \{P_0^{min}(0), P_0^{max}(0), \Delta P_0(0)\}$
2. $P_1(0) : \{P_1^{min}(0), P_1^{max}(0), \Delta P_1(0)\}$
3. $P_2(0) : \{P_2^{min}(0), P_2^{max}(0), \Delta P_2(0)\}$
4. $\theta_1 : \{\theta_1^{min}, \theta_1^{max}, \Delta\theta_1\}$
5. $\theta_2 : \{\theta_2^{min}, \theta_2^{max}, \Delta\theta_2\}$
6. $\lambda_1 : \{\lambda_1^{min}, \lambda_1^{max}, \Delta\lambda_1\}$
7. $\lambda_2 : \{\lambda_2^{min}, \lambda_2^{max}, \Delta\lambda_2\}$
8. $\gamma : \{\gamma_{min}, \gamma_{max}, \Delta\gamma\}$
9. $\delta : \{\delta_{min}, \delta_{max}, \Delta\delta\}$
10. k: $\{k_{min}, k_{max}, \Delta k\}$

Having defined the overall set of parameters as:

$$\begin{aligned}\mathcal{P} = \{&P_0^{min}(0), P_0^{max}(0), \Delta P_0(0),\\ &P_1^{min}(0), P_1^{max}(0), \Delta P_1(0),\\ &P_2^{min}(0), P_2^{max}(0), \Delta P_2(0),\\ &\theta_1^{min}, \theta_1^{max}, \Delta\theta_1,\\ &\theta_2^{min}, \theta_2^{max}, \Delta\theta_2,\\ &\lambda_1^{min}, \lambda_1^{max}, \Delta\lambda_1,\\ &\lambda_2^{min}, \lambda_2^{max}, \Delta\lambda_2,\\ &\gamma_{min}, \gamma_{max}, \Delta\gamma,\\ &\delta_{min}, \delta_{max}, \Delta\delta,\\ &k_{min}, k_{max}, \Delta k\}\end{aligned} \tag{39}$$

the multidimensional grid traversal procedure may then be given by Algorithm 1 below. It is easy to see that by setting the minimum value of a particular parameter range equal to the corresponding maximum value, this parameter behaves as a constant.

Algorithm 1 $\hat{T}^* =$ OPTIMAL-INFLUENCES($\mathcal{P}$)

```
INPUT: 𝒫;
OUTPUT: T̂* = ℱ(𝒫);
T̂* ← ∅;
for all P_0(0) ∈ [P_0^min(0) : ΔP_0(0) : P_0^max(0)] do
  for all P_1(0) ∈ [P_1^min(0) : ΔP_1(0) : P_1^max(0)] do
    for all P_2(0) ∈ [P_2^min(0) : ΔP_2(0) : P_2^max(0)] do
      for all θ_1 ∈ [θ_1^min : Δθ_1 : θ_1^max] do
        for all θ_2 ∈ [θ_2^min : Δθ_2 : θ_2^max] do
          for all λ_1 ∈ [λ_1^min : Δλ_1 : λ_1^max] do
            for all λ_2 ∈ [λ_2^min : Δλ_2 : λ_2^max] do
              for all γ ∈ [γ_min : Δγ : γ_max] do
                for all δ ∈ [δ_min : Δδ : δ_max] do
                  for all k ∈ [k_min : Δk : k_max] do
                    (T_1*, T_2*) ← SQP(𝒫)
                    T̂* ← T̂* ∪ (T_1*, T_2*);
                  end for
                end for
              end for
            end for
          end for
        end for
      end for
    end for
  end for
end for
```

In this paper, optimization has been algorithmically conducted within the framework of Sequential Quadratic Programming (SQP), as described by the works of [Biggs(1973)], [Han(1977)] and [Powell(1978a), Powell(1978b)]. SQP methods after their popularization in the late 1970s, have arguably become the state of the art in nonlinear programming. As it is the case with the majority of optimization methods, SQP is not merely a single algorithm, but rather a conceptual framework from which numerous specific algorithms have evolved. [Schittkowski(1986)], for example, formulated and implemented a variation of SQP that outperformed every other tested method in terms of accuracy, and percentage of successful solutions, over a wide range of experimentation scenarios.

The basic idea of SQP, as it may be found in [Fletcher(2013)], [Gill et al.(1981)Gill, Murray, and Wright], [Powell(1983)] and [Schittkowski(1986)], consists in modeling the original Non Linear Program (NLP) at a given approximate solution, $\bar{T}_k$, as a quadratic programming subproblem, and then using the solution obtained by this subproblem so as to construct a better approximation $\bar{T}_{k+1}$. Specifically, the utilization of an appropriate choice for the quadratic subproblem allows the SQP methods to handle constrained optimization problems in a manner that may viewed as a natural extension to the traditional Newton-like approaches. Therefore, SQP methods exhibit the property of rapid convergence when the iterates are close to the solution, which is the case for the optimization problem at hand since $\hat{T}$ is by definition constrained within the $[0,1]^2$ interval. Moreover, SQP is not a feasible-point method in the sense that neither the initial point $\bar{T}_0$ nor any of the subsequent iterates $\bar{T}_k$ need to satisfy all of the constraints of the original NLP. This feature constitutes a major advantage since identifying a feasible point when there are nonlinear constraints may be nearly as hard as solving the NLP itself. More importantly, SQP methods can be easily modified so that linear constraints, including simple bounds, are always satisfied which is also the case for the problem at hand appearing in Eqs. 14. In this work we utilized the programming environment of Matlab and its built-in procedures for the implementation of the SQP method along with the multidimensional grid traversal procedure given by Algorithm 1.

5 Implications with a Binding Constraint on Total Investment

As it was thoroughly described in Section 4, the binding constraint on the total amount of optimal investment intensity becomes activated when the corresponding Lagrange multiplier takes on a nonzero positive value ($\mu_5^* > 0$). In this case, according to Eq. 33a (or equivalently Eq. 33b, since $G_1(T_2^* = G_2(T_1^*))$ the optimal value for μ_5^* will be given by the following equation:

$$\mu_5^* = \frac{1}{2}(a-c)^2 \frac{G_1(T_2^*)}{Q^2(T_1^*, T_2^*)} - \gamma \tag{40}$$

By defining the quantity $\tilde{\gamma}$ as:

$$\tilde{\gamma} = \frac{1}{2}(a-c)^2 \frac{G_1(T_2^*)}{Q^2(T_1^*, T_2^*)} \tag{41}$$

it is obvious that the binding constraint becomes activated when $\gamma < \tilde{\gamma}$.

5.1 Benchmark Outcome: Uniform Initial Beliefs

We first consider the benchmark case where all consumers hold identical initial beliefs about the brand quality ($p_1(0) = p_2(0) = p$) and the firm puts equal trust ($\lambda_1 = \lambda_2 = \lambda$) on each consumer's opinion. Under these assumptions, the optimal solutions for T_1^* and T_2^* will be given by Eqs. 38 so that the actual value of $\tilde{\gamma}$ becomes:

$$\tilde{\gamma} = -\frac{2\,\lambda\,(a-p)^2\,(p-1)\,\left(k^2 + 4\,k\,\theta_1 + 4\,k\,\theta_2 + 3\,{\theta_1}^2 + 10\,\theta_1\,\theta_2 + 3\,{\theta_2}^2\right)}{\left(k^2 + 2\,k\,\theta_1 + 2\,k\,\theta_2 + 4\,\lambda\,k - 3\,{\theta_1}^2 + 6\,\theta_1\,\theta_2 + 8\,\lambda\,\theta_1 - 3\,{\theta_2}^2 + 8\,\lambda\,\theta_2\right)^2} \tag{42}$$

In this case, the investment level optimally chosen by the monopolist to affect consumer j's belief equals half of the total budget ($\frac{k}{2}$) plus an amount which depends positively on j's net influence ($\theta_j - \theta_i$) on consumer i. We can also compare $T_1^*$and T_2^* in terms of θ_1 and θ_2 to find: $T_2^* \geq (\leq) T_1^* \Leftrightarrow \theta_2 \geq (\leq) \theta_1 = \hat{\theta}$.

The solution in Eq. 38 can be used to calculate the limiting influence vector $s^* = (s_0^*, s_1^*, s_2^*)$, the equilibrium valuation of the good and the optimal level of profit:

$$s_0^* = \frac{\theta_1^2 + 2\theta_1(k-\theta_2) + (k+\theta_2)^2}{\theta_1^2 + 2\theta_1(k-\theta_2+4\lambda) + 4\lambda(k+2\theta_2) + (k+\theta_2)^2} \tag{43}$$

$$s_1^* = \frac{2\lambda(k+3\theta_1+\theta_2)}{\theta_1^2 + 2\theta_1(k-\theta_2+4\lambda) + 4\lambda(k+2\theta_2) + (k+\theta_2)^2} \tag{44}$$

$$s_2^* = \frac{2\lambda(k+3\theta_2+\theta_1)}{\theta_1^2 + 2\theta_1(k-\theta_2+4\lambda) + 4\lambda(k+2\theta_2) + (k+\theta_2)^2} \tag{45}$$

$$x^* = \frac{\theta_1^2 + 2\theta_1(k-\theta_2+4\lambda p) + (k+\theta_2)^2 + 4\lambda p(k+2\theta_2)}{\theta_1^2 + 2\theta_1(k-\theta_2+4\lambda) + 4\lambda(k+2\theta_2) + (k+\theta_2)^2} \tag{46}$$

$$\Pi^* = \frac{(a-c)^2 x^*}{2} - \gamma k \tag{47}$$

From Eqs. 44 and 45 we can easily get: $s_2^* \geq (\leq) s_1^* \Leftrightarrow \theta_2 \geq (\leq) \theta_1 = \theta^*$. Since $\hat{\theta} = \theta^*$ we deduce that the firm always invests more in affecting the consumer with the highest limiting influence, implying Proposition 1 below.

Proposition 1 *When all consumers have the same initial belief, the monopolist always targets the most influential agent in equilibrium:* $T_2^* \geq (\leq) T_1^* \Leftrightarrow s_2^* \geq (\leq) s_1^*$.

The result stated in Proposition 1 is graphically depicted in Figs. 2 and 3, where we have drawn the firm's optimal influence strategy and each agent's limiting influence in terms of θ_2 for a specific set of parameter values ($\lambda = 0.25, p = 0.2, \theta_1 = 0.4, \delta = \alpha - c = 2.1, \gamma = 0.59, k = 1$). In Fig. 2 the red line depicts T_1^* and the green line depicts T_2^*, while in Fig. 3 the blue curve depicts s_0^*, the red curve depicts s_1^* and the green curve depicts s_2^*.

Note that the optimal allocation of the firm's total investment and the social influence vector do not vary with the marginal cost of investment (γ), with the size of the market (δ) or with the level of consumers' initial beliefs (P). Furthermore, an increase in consumers' direct influence (λ) on firm's beliefs does not affect investment choices but decreases the monopolist's limiting influence and increases both consumers' limiting influences. The equilibrium valuation is invariant to the marginal investment cost and to the market size. However, the optimal level of profit naturally decreases with γ and increases with δ. It is also straightforward to show that both the consensus belief and the optimal level of profit increase with consumers' initial beliefs but decrease with consumers' direct influence on the firm.

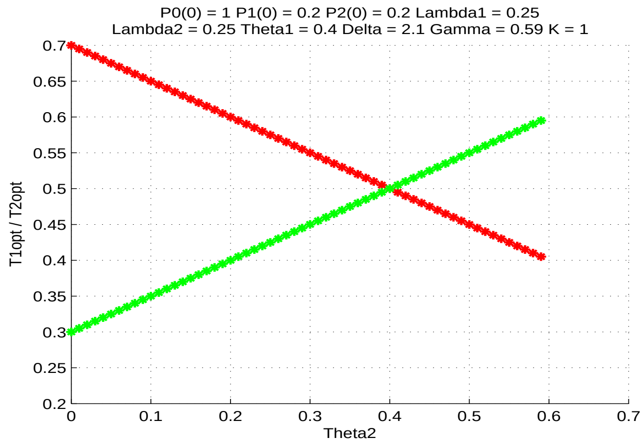

Fig. 2 The firm's optimal influence strategy.

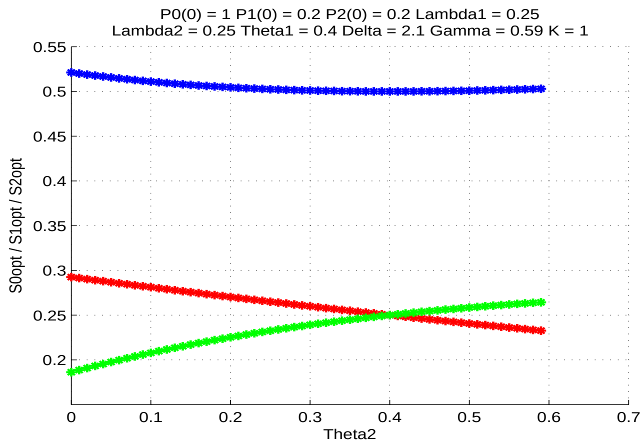

Fig. 3 Limiting influences.

It can be easily seen that the optimal levels of investment, the firm's limiting influence and the consensus belief increase with the budget (k) available to the firm. Direct calculations imply that the sign of the derivatives $\frac{\partial s_1^*}{\partial \theta_j}$ and $\frac{\partial s_2^*}{\partial \theta_j}$ is ambiguous. Therefore, an increase in one consumer's direct influence on the other's beliefs can have either a positive or a negative impact on her own as well as on the other's limiting influence. More interestingly, we study how the values of θ_1 and θ_2 affect the monopolist's limiting influence and profit as well as the society's consensus belief:

$$\frac{\partial s_0^*}{\partial \theta_j} = \frac{8\lambda(\theta_j - \theta_i)(\theta_1 + 3\theta_2 + k)}{\phi}$$
$$\frac{\partial x^*}{\partial \theta_j} = (1-p)\cdot\frac{\partial s_0^*}{\partial \theta_j}$$
$$\frac{\partial \Pi^*}{\partial \theta_j} = (a-c)^2\cdot\frac{1}{2}\cdot\frac{\partial x^*}{\partial \theta_j} \tag{48}$$

where $\phi = (\theta_1^2 + 2\theta_1(k - \theta_2 + 4\lambda) + 4\lambda(k + 2\theta_2) + (k + \theta_2)^2)^2$.

Proposition 2 *The monopolist's limiting influence and profit as well as the equilibrium valuation of the good are minimized when $s_1^* = s_2^*$ (or, equivalently, $\theta_2 = \theta^*$), i.e. when consumers' limiting influences become equal.*

The behavior of s_0^* as a function of θ_2 is graphically depicted in Fig.3 above. The results concerning x^* and Π^* are shown in Figs. 4 and 5, respectively. Intuitively, if consumers have relatively similar limiting influences the firm has to make a significantly positive investment in both of them and the impact on profitability is negative. As the divergence of consumers' limiting influences increases, the firm only needs to invest a considerable amount of resources in the most influential consumer and thus can reap a higher level of profit.

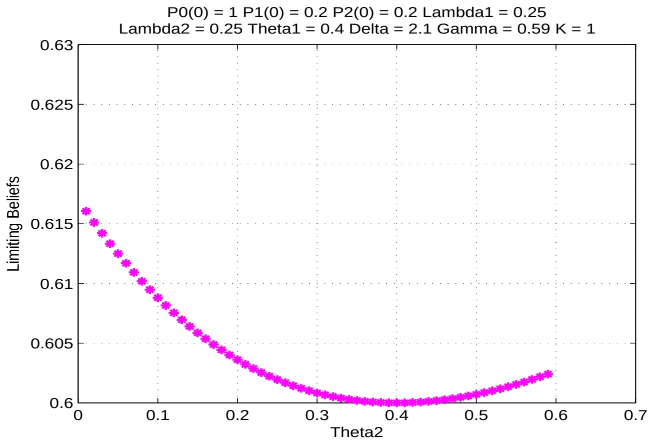

Fig. 4 The limiting consensus belief.

Finally, we can calculate the minimum consensus belief by setting $\theta_1 = \theta_2 = \theta$ in Eq. 46 to get:

$$x^*_{\min} = \frac{4\lambda p + k}{4\lambda + k} \tag{49}$$

The minimum belief is independent of θ, whereas it increases with consumers' initial beliefs p and the firm's budget k but decreases with the weight λ placed

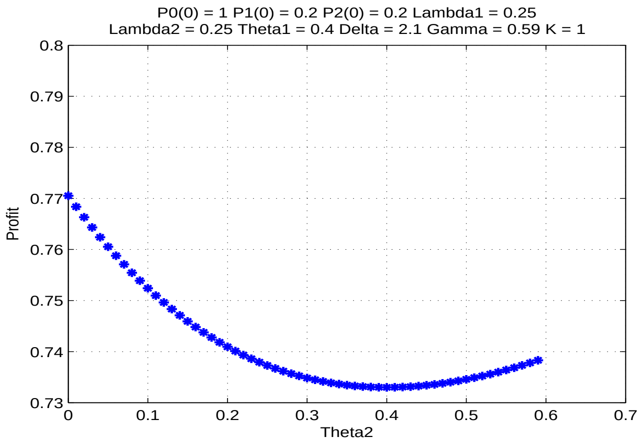

Fig. 5 The firm's equilibrium profit as a function of θ_2

by the firm on consumers' opinion. The corresponding level of profit will be $\Pi^*_{\min} = \frac{(a-c)^2(4\lambda p+k)}{2(4\lambda+k)} - \gamma k$. This expression immediately implies the following result.

Proposition 3 *1. If* $\gamma < \frac{(a-c)^2(1-p)}{8\lambda}$*, then:* $\frac{\partial \Pi^*_{\min}}{\partial k} > (<) 0$ *for* $k < (>) \hat{k}$*, where:* $\hat{k} = (a-c)\sqrt{\frac{2\lambda(1-p)}{\gamma}} - 4\lambda$.

2. If $\gamma > \frac{(a-c)^2(1-p)}{8\lambda}$*, then* $\frac{\partial \Pi^*_{\min}}{\partial k} < 0, \ \forall k \geq 0$.

More generally, if the marginal cost of investment (γ) is not too high then the firm's equilibrium profit initially increases but eventually decreases with the budget available for influencing consumers' beliefs. Intuitively, an increase in k enables the firm to achieve a higher equilibrium evaluation of its product (thus increasing the aggregate level of demand) but also implies a higher total cost of investment. Therefore, the overall impact on profitability is ambiguous. The optimal amount of resources that should be allocated by the firm to the activity of manipulating the network is determined through the interplay of these two opposite effects. Naturally, the profit-maximizing level of budget is increasing in the net size of the market (δ) but decreasing in γ as well as in the initial belief held by the set of consumers. The ambiguous relationship between k and Π^* is graphically depicted in Fig. 6 below.

5.2 Non-Uniform Initial Beliefs

This section relies on numerical simulations to investigate how the introduction of non-uniform initial beliefs held by consumers with respect to product quality affects the firm's optimal influence strategy and the overall equilibrium outcome. In order to draw comparisons with the case of uniform initial beliefs,

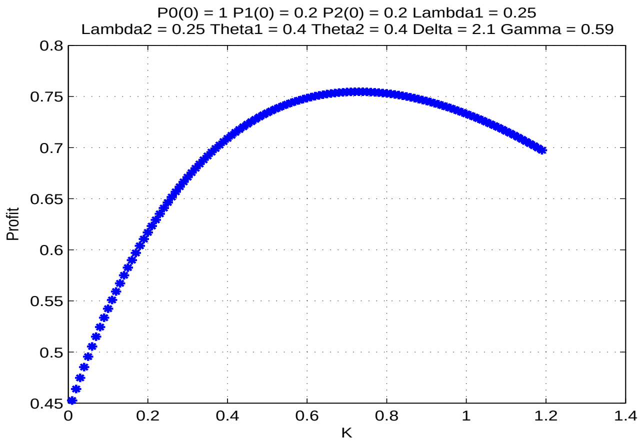

Fig. 6 The firm's equilibrium profit as a function of k

we use as a benchmark the set of parameter values considered above: $\lambda = 0.25$, $p_1(0) = p_2(0) = 0.2$, $\theta_1 = 0.4$, $\delta = 2.1$, $\gamma = 0.59$ and $k = 1$.

We first consider an example where consumer 1 has the lowest initial valuation of the good by setting $p_1(0) = 0.1 < p_2(0) = 0.2$. In this context, we can compare Figs. 2 and 7 to see that the reduction in consumer 1's valuation strengthens the firm's incentives to affect 1's beliefs and weakens its incentives to affect 2's beliefs (i.e. it increases T_1^* and decreases T_2^*). This kind of effect shifts the point $\hat{\theta}$ to the right. There exists a critical level of θ_2 (between 0.5 and 0.6) above which T_1^* gets its maximum value ($T_1^* = 1 - \theta_2$) and, therefore, T_2^* gets its minimum value ($T_2^* = \theta_2$). Similarly, we can compare Figs. 3 and 8 to see that the curve depicting s_1^* now shifts downwards and the curve depicting s_2^* shifts upwards.Therefore, the point θ^* shifts to the left and we can distinguish the following cases:

1. If $\theta_2 < \theta^*$ then $T_1^* > T_2^*$ and $s_1^* > s_2^*$.
2. If $\theta^* < \theta_2 < \hat{\theta}$ then $T_1^* > T_2^*$ and $s_1^* < s_2^*$.
3. If $\theta_2 > \hat{\theta}$ then $T_1^* < T_2^*$ and $s_1^* < s_2^*$.

The analysis of the previous section has shown that the firm always targets the most influential consumer when initial beliefs are the same. For the case of different initial beliefs, however, the monopolist also has an incentive to invest more in the consumer with the lowest initial valuation of the good. If this incentive is strong enough, the firm might optimally target the least influential consumer provided the latter's initial belief is low enough (relative to the belief of the other consumer). In the previous example (where consumer 1 has the lowest initial valuation), for $\theta_2 \in (\theta^*, \hat{\theta})$ the firm invests more in consumer 1 despite the fact that consumer 2 is relatively more influential in the limit. It should also be noted that the equilibrium valuation and the firm's profit are still minimized when consumers' limiting influences become equal (as in

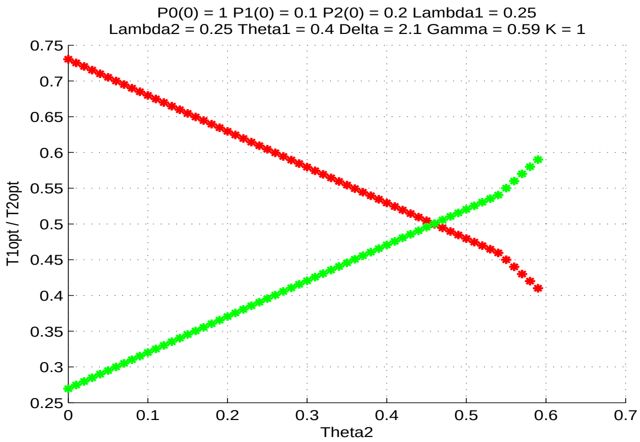

Fig. 7 The firm's optimal influence strategy with $p_1(0) < p_2(0)$.

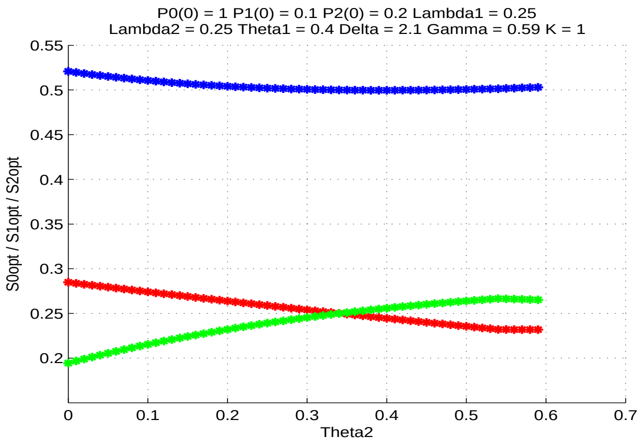

Fig. 8 Limiting influences with $p_1(0) < p_2(0)$.

the case of uniform initial beliefs). These results are shown in Figs. 9 and 10 below.

A similar pattern of results can be drawn when $p_2(0) < p_1(0)$. For example, consider the case where $p_1(0) = 0.2$, $p_2(0) = 0.1$ and the other parameter values remain the same as before. The decrease in $p_2(0)$ (relative to the benchmark case with $p_1(0) = p_2(0) = 0.2$) has a negative impact on $T_1^*$and a positive impact on T_2^*, thus shifting the point $\hat{\theta}$ to the left (Fig. 11). There also exists a critical level of θ_2 above which T_2^* gets its maximum value ($T_2^* = 1 - \theta_1 = 0.6$) and T_1^* gets its minimum value ($T_1^* = \theta_1 = 0.4$). The upward shift in the curve depicting s_1^* and the downward shift in the curve depicting s_2^* now move the

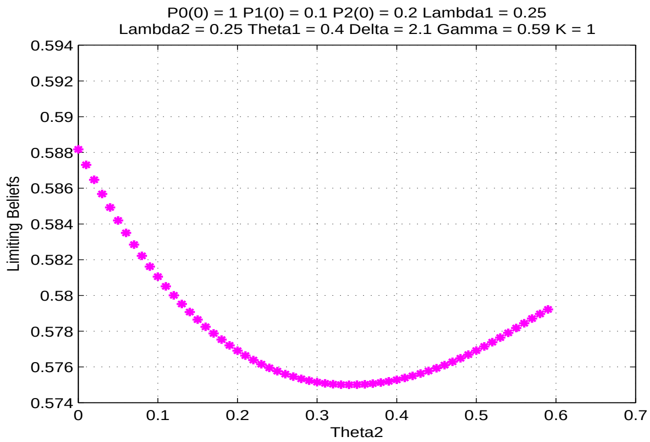

Fig. 9 The limiting consensus belief with $p_1(0) < p_2(0)$.

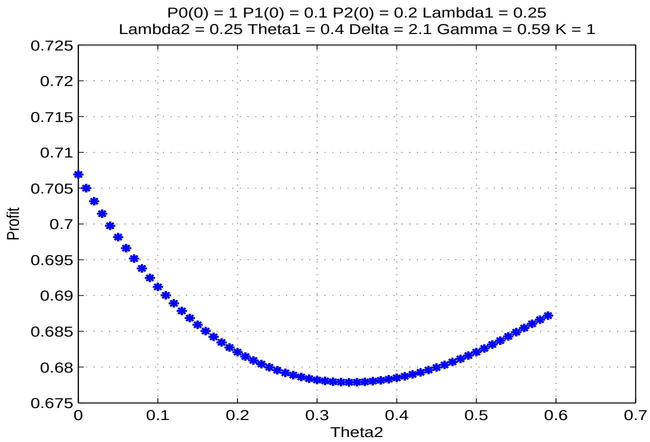

Fig. 10 The firm's equilibrium profit with $p_1(0) < p_2(0)$.

point θ^* to the right (Fig. 12). As a result, we can distinguish the following cases:

1. If $\theta_2 < \hat{\theta}$ then $T_1^* > T_2^*$ and $s_1^* > s_2^*$.
2. If $\hat{\theta} < \theta_2 < \theta^*$ then $T_1^* < T_2^*$ and $s_1^* > s_2^*$.
3. If $\theta_2 > \theta^*$ then $T_1^* < T_2^*$ and $s_1^* < s_2^*$.

Again, we can identify a parameter interval (now given by $\theta_2 \in (\hat{\theta}, \theta^*)$) where the firm invests more in the least influential consumer (in this example, consumer 2) due to the fact that the latter has a relatively lower valuation of the good. Therefore, we can state the following general result which is our main finding in this section.

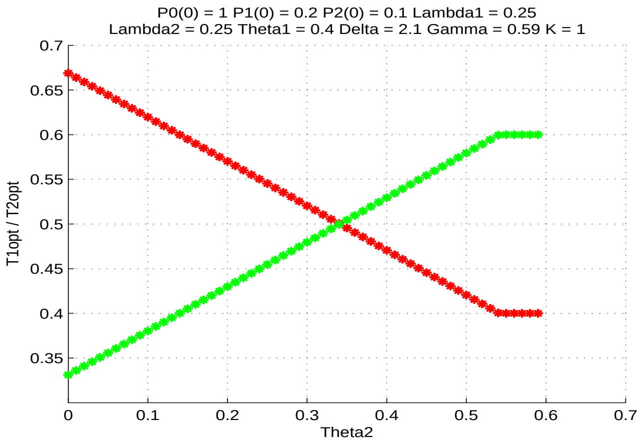

Fig. 11 The firm's optimal influence strategy with $p_2(0) < p_1(0)$.

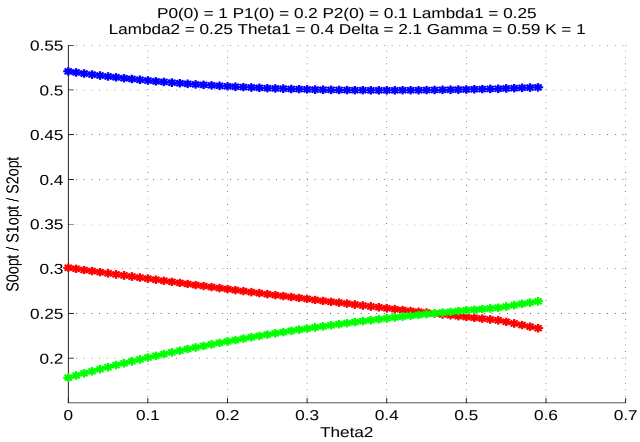

Fig. 12 Limiting influences with $p_2(0) < p_1(0)$.

Proposition 4 *If consumers have different initial beliefs about the product quality, the firm might optimally target the least influential consumer in equilibrium if the latter's initial valuation of the good is low enough relative to the initial valuation of the other.*

We proceed with a series of experiments aiming to identify the key determinants of the interval where the monopolist invests more in the least influential consumer. Consider again the case $p_1(0) = 0.1 < p_2(0) = 0.2$, $\lambda = 0.25$, $\delta = 2.1$, $\gamma = 0.59$, $k = 1$ and suppose the value of θ_1 increases from 0.15 to 0.30 and then to 0.40. As shown in Fig. 13, the increase in θ_1 (i.e. the higher level of direct influence exerted by consumer 1 on 2's beliefs) has a positive impact on T_1^* and a negative impact on T_2^*, thus shifting the critical point $\hat{\theta}$

to the right. At the same time, Fig. 14 reveals that the point θ^* also moves to the right due to an upward shift of the s_1^*-curve and a downward shift of the s_2^*-curve. Moreover, it can be verified that the increase in the value of $\hat{\theta}$ has exactly the same magnitude as the increase in the value of θ^*, implying that the size of the interval $(\theta^*, \hat{\theta})$ remains unaffected. For the selected parameter values, the size of this region is calculated in Table 1 below. More generally, we can conclude that a variation in the direct influence (θ_j) of consumer j on the other consumer's beliefs does not affect the monopolist's incentives to target the least influential consumer in equilibrium.

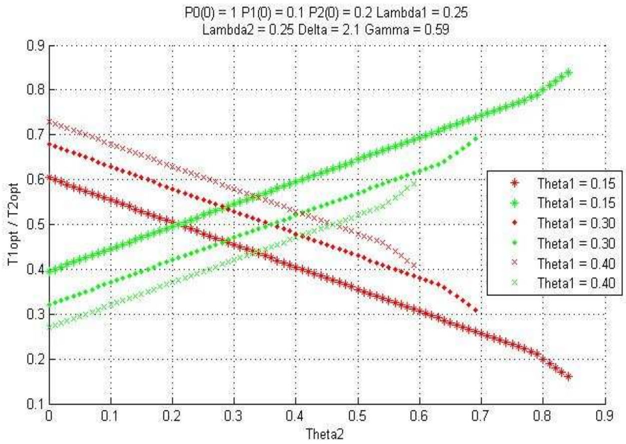

Fig. 13 The firm's optimal influence strategy with $p_1(0) < p_2(0)$ for $\theta_1 \in \{0.15, 0.30, 0.40\}$.

Table 1 The size of $\Delta\theta$ for different values of θ_1

	θ^*	$\hat{\theta}$	$\Delta\theta = \theta^* - \hat{\theta}$
$\theta_1 = 0.15$	0.091	0.209	0.118
$\theta_1 = 0.30$	0.241	0.359	0.118
$\theta_1 = 0.40$	0.341	0.459	0.118

We now investigate the impact of variations in the value of λ (which represents the level of trust initially attributed by the firm on consumers' beliefs). The first thing to note here is that a higher value of λ increases investment in the consumer with the lower initial opinion, contrasting the case of uniform beliefs where the firm's investment strategy remains unaffected by the value of λ. This is clearly shown in Fig. 15, where increasing λ (from 0.05 to 0.25 and then to 0.45) has a positive impact on T_1^* and a negative impact on T_2^* shifting the point $\hat{\theta}$ to the right. Since the curves depicting s_1^* and s_2^* in Fig. 16 move upwards in such a way that θ^* shifts to the left (reflecting the fact that

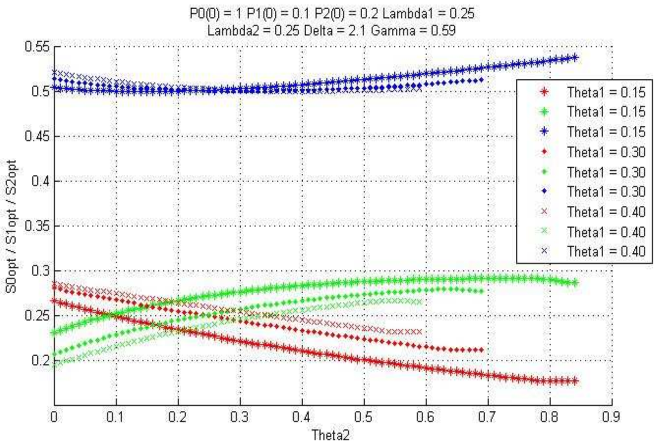

Fig. 14 Limiting influences with $p_1(0) < p_2(0)$ for $\theta_1 \in \{0.15, 0.30, 0.40\}$.

s_2^* increases relatively more than s_1^*), the interval $(\theta^*, \hat{\theta})$ becomes wider. This is clearly shown in the calculations of Table 2. Therefore, a higher degree of direct influence exerted by the set of consumers on the monopolist strengthens the latter's incentives to invest more in the least influential consumer.

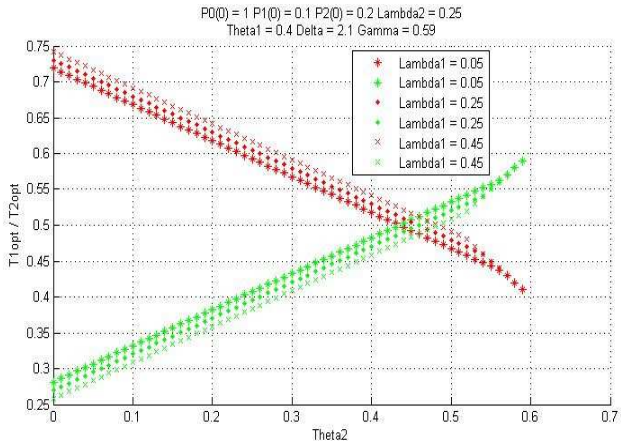

Fig. 15 The firm's optimal influence strategy with $p_1(0) < p_2(0)$ for $\lambda \in \{0.05, 0.25, 0.45\}$.

Finally, we study changes in the distance between consumers' initial beliefs. Specifically, we hold the value of $p_1(0)$ constant at 0.1 and increase the value of $p_2(0)$ from 0.18 to 0.26 and then to 0.34. In Fig. 17, this kind of variation enhances T_1^* and reduces T_2^* shifting $\hat{\theta}$ to the right again. However, the s_1^*-curve shifts downwards and the s_2^*-curve shifts upwards in Fig. 18, resulting in

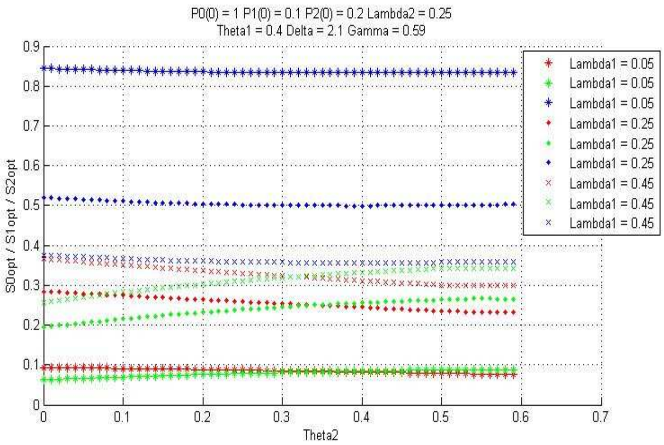

Fig. 16 Limiting influences with $p_1(0) < p_2(0)$ for $\lambda \in \{0.05, 0.25, 0.45\}$.

Table 2 The size of $\Delta\theta$ for different values of $\lambda = \lambda_1 = \lambda_2$

	θ^*	$\hat{\theta}$	$\Delta\theta = \theta^* - \hat{\theta}$
$\lambda = 0.05$	0.365	0.435	0.070
$\lambda = 0.25$	0.341	0.459	0.118
$\lambda = 0.45$	0.318	0.483	0.165

a significant downward movement of θ^* which leads to a much wider interval $(\theta^*, \hat{\theta})$ as calculated in Table 3.

Table 3 The size of $\Delta\theta$ for different values of $\Delta P = p_2(0) - p_1(0)$

	θ^*	$\hat{\theta}$	$\Delta\theta = \theta^* - \hat{\theta}$
$\Delta P = 0.08$	0.153	0.247	0.094
$\Delta P = 0.16$	0.102	0.299	0.197
$\Delta P = 0.32$	0.046	0.357	0.311

All the above results are summarized in Proposition 5 below.

Proposition 5 *The firm's incentives to target the least influential consumer:*

1. *do not vary with the direct influence (θ_j) exerted by consumer j on the other one.*
2. *increase with the trust (λ) placed by the firm on the set of consumers.*
3. *increase with the distance (ΔP) between consumers' initial beliefs.*

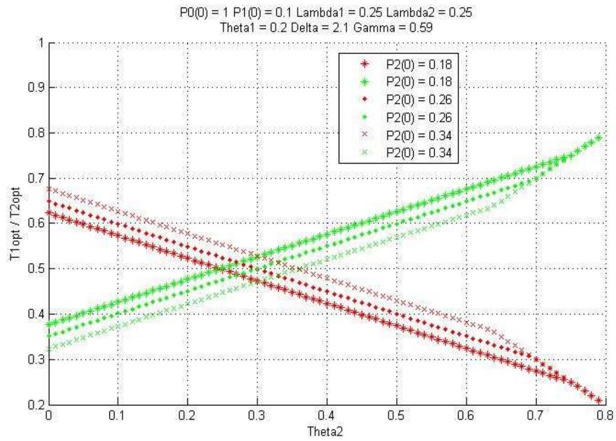

Fig. 17 The firm's optimal influence strategy with $p_1(0) = 0.1$ for $p_2(0) \in \{0.18, 0.26, 0.34\}$.

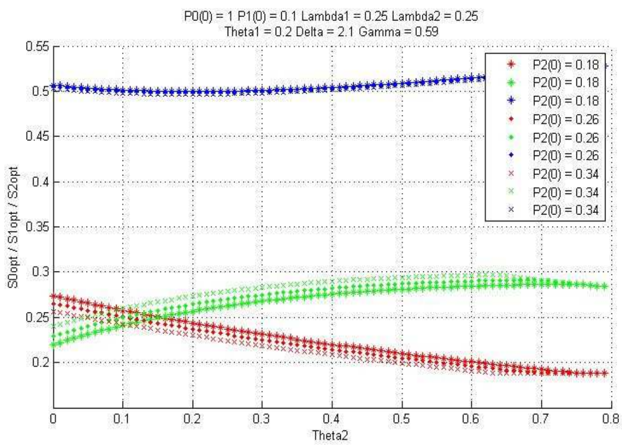

Fig. 18 Limiting influences with $p_1(0) = 0.1$ for $p_2(0) \in \{0.18, 0.26, 0.34\}$.

6 Implications with a Non-Binding Constraint (NBC) on Total Investment

We now turn to the case where the constraint on total investment ($T_1+T_2 \leq k$) holds with a strict inequality, reflecting the fact that the marginal cost of investment (γ) is sufficiently high. Throughout this section we assume that consumers 1 and 2 have the same initial beliefs, since the case of non-uniform beliefs yields qualitatively the same results as those summarized in Propositions 4 and 5. It can be seen that the critical points $\hat{\theta}$ and θ^* coincide again in Figs. 19 and 20, thus verifying that the monopolist always targets the most influential consumer under the assumption of identical initial opinions. Despite this similarity, there is a major difference with the case of the binding constraint ($T_1 + T_2 = k$) primarily associated with the behavior of the s_0^*-curve. As shown in Fig. 20, initial increases in θ_2 (up to θ^*) have a positive impact on the firm's limiting influence s_0^* directly translated into an increasing limiting belief x^* in Fig. 21. However, Fig. 22 reveals that profits are minimized precisely at the point θ^* (where the equilibrium belief about brand quality reaches its maximum).

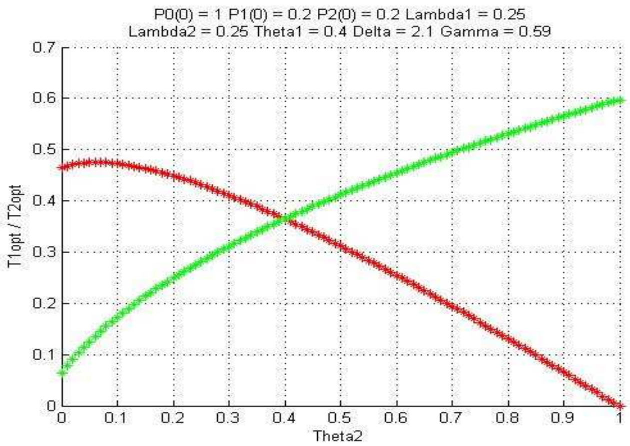

Fig. 19 The firm's optimal influence strategy. (NBC)

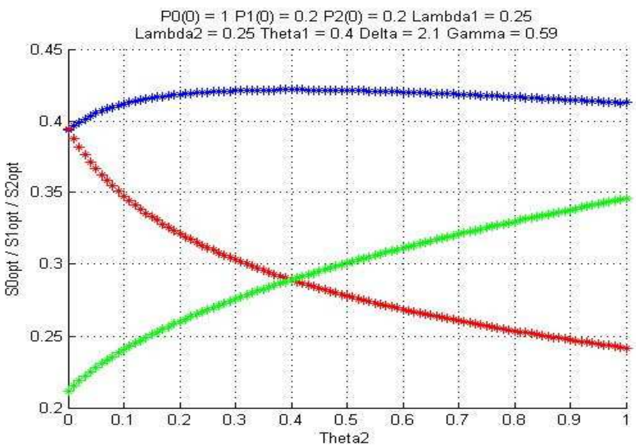

Fig. 20 Limiting influences with $p_1(0) = p_2(0)$ as a function of θ_2. (NBC)

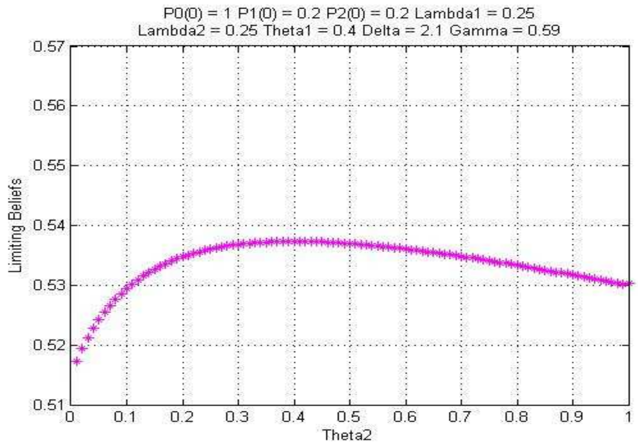

Fig. 21 The limiting consensus belief with $p_1(0) = p_2(0)$ as a function of θ_2. (NBC)

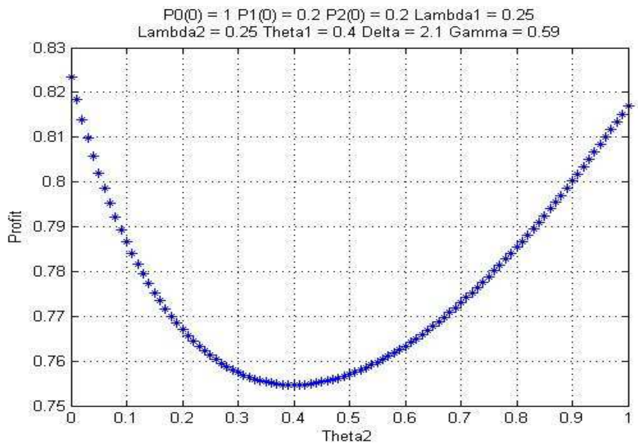

Fig. 22 The firm's equilibrium profit with $p_1(0) = p_2(0)$ as a function of θ_2. (NBC)

This apparently paradoxical result can be understood by decomposing the profit into its two main components, namely the revenue (net of production costs) and the total investment cost. Such a decomposition has been performed in Fig.23, where the revenue curve (on the top of the figure) closely follows the behavior of x^* - i.e. it increases up to θ^* and decreases afterwards. At the same time, initial increases in θ_2 tend to close the gap between consumers' limiting influences (s_1^* and s_2^*) and lead to an increase in the total investment intensity ($T_1^* + T_2^*$) as well as in the associated investment cost (represented by the curve at the bottom of Fig. 23) which also reaches a maximum at θ^*. In this framework, the negative effect associated with cost increases dominates the positive effect on profitability resulting from increases in revenue, thus generating the U-shaped profit curve in Fig. 22 (reproduced in the middle part of Fig. 23). It should be noted that the underlying forces driving the U-shaped behavior of profit with a binding constraint on total investment are qualitatively different. Indeed, in the latter case initial increases in θ_2 have a downward impact on s_0^* as well as on x^* and on the firm's revenue. Since the total investment cost (which equals $\gamma \cdot (T_1^* + T_2^*) = \gamma \cdot k$) remains constant in this case, the fall in revenue directly implies a fall in profit which reaches a minimum at θ^* (as shown, for example, in Fig. 5).

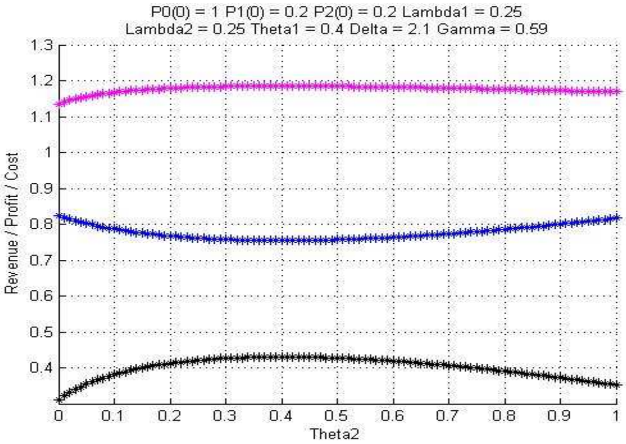

Fig. 23 The firm's equilibrium profit components as a function of θ_2. (NBC)

In the absence of a binding constraint on total investment, the key equilibrium variables are now more sensitive to changes in the various parameters of the model. The following experiments consider the case $\theta_1 = \theta_2 = 0.4$, thus implying $T_1^* = T_2^*$ and $s_1^* = s_2^*$. This is also the reason for which the red and green lines coincide in the following figures. Not surprisingly, an increase in the marginal cost of investment (γ) reduces the firm's investment choices (T_1^* and T_2^*) and has a negative impact on s_0^* but a positive impact on s_1^* and s_2^*. Consequently, both the consensus belief (x^*) and the equilibrium profit (Π^*) are decreasing in γ. These results are depicted in Figs. 24 to 27 and are qualitatively similar to those associated with a decrease in the market size (δ).

Furthermore, an improvement in consumers' (uniform) initial beliefs (i.e. an increase in $p_j(0)$, where $j \in N$) naturally weakens the firm's incentives to manipulate the network, leading to a decrease in T_1^* and T_2^*. A higher level of $p_j(0)$ reduces s_0^*, increases s_1^* and s_2^* and has a positive impact on both the limiting belief x^* and on equilibrium profits (Π^*). These results are depicted in Figs. 28 to 31.

Perhaps more interestingly, an increase in consumers' direct influence (λ) on firm's beliefs has an ambiguous (initially positive but eventually negative) effect on the latter's investment choices (T_1^* and T_2^*) as shown in Fig. 32.

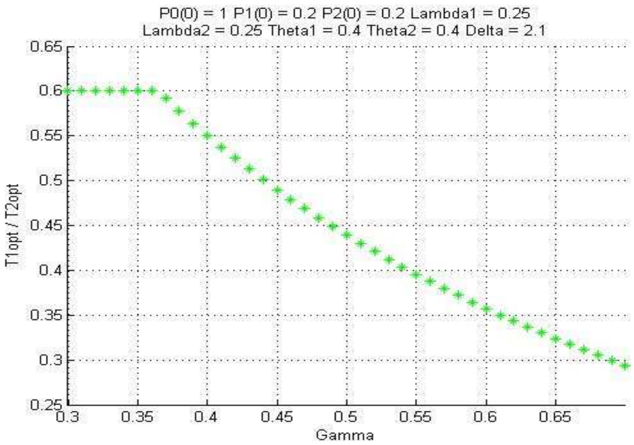

Fig. 24 The firm's optimal influence strategy as a function of γ

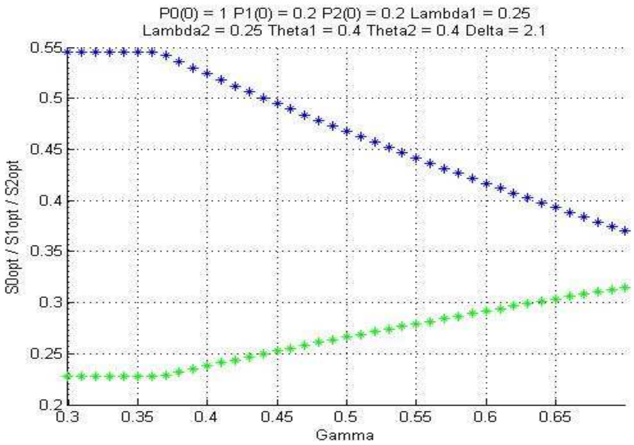

Fig. 25 Limiting influences as a function of γ

Finally, the negative impact of λ on s_0^*, x^* and Π^* as well as its positive impact on s_1^* and s_2^* are depicted in Figs. 33 to 35.

7 Conclusions & Future Work

This paper has developed a modeling framework to study the optimal strategy followed by a monopolistic firm to manipulate the flow of information in a social network. The network includes the monopolist and a set of consumers who repeatedly communicate with each other to form their beliefs about the underlying product quality. We assume that the firm has an amount of resources

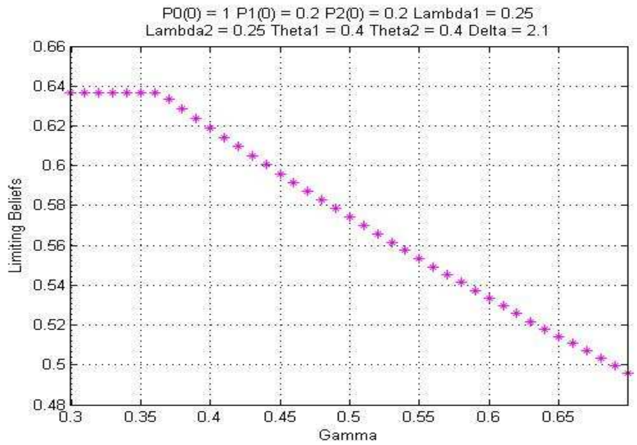

Fig. 26 The limiting consensus belief as a function of γ

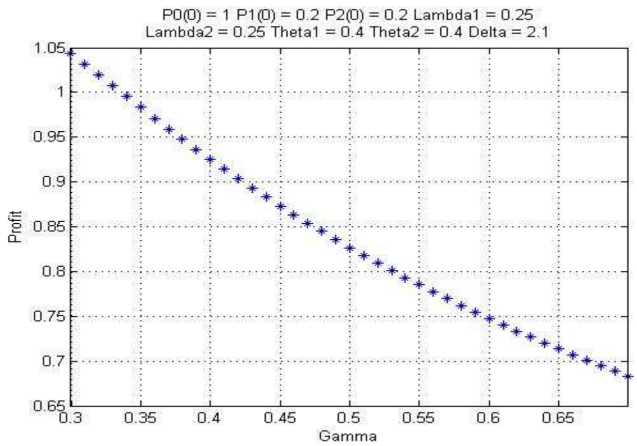

Fig. 27 The firm's equilibrium profit as a function of γ

available to invest in influencing consumers' valuation of the good. For the benchmark case where all consumers have the same initial beliefs and the firm faces a binding constraint on total investment, we have analytically and numerically shown that the monopolist always invests more in the most influential consumer. An increase in the budget available to the firm for manipulating the network has an ambiguous impact on profitability, since the benefits associated with the enlarged possibility to improve the equilibrium valuation of the good might be outweighed by the increase in the overall investment cost. The equilibrium profit is minimized when consumers' limiting influences become equal. In other words, the monopolist benefits from the presence of consumers who have divergent strategic locations in the network. When consumers have

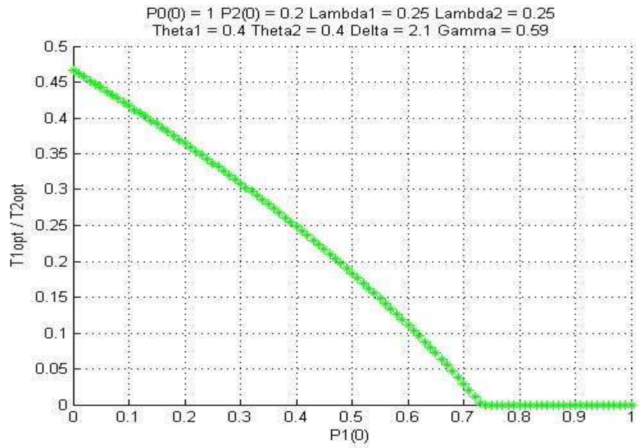

Fig. 28 The firm's optimal influence strategy as a function of $p_1(0)$

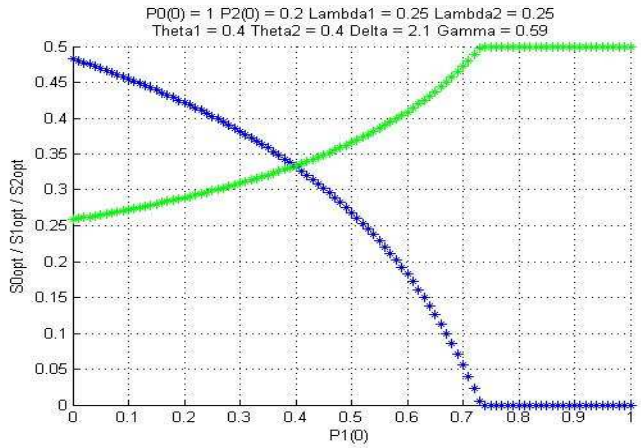

Fig. 29 Limiting influence levels as a function of $p_1(0)$

different initial beliefs about the brand quality, the firm's profit-maximizing strategy might be to invest more in the least influential consumer if the latter's initial belief is low enough (relative to the belief of the most influential consumer). The probability of targeting the agent with the lower influence increases both with the distance between consumers' initial beliefs and with the trust attributed by the monopolist on consumers. If the firm does not face a binding constraint on total investment, an improvement in consumers' initial beliefs makes the firm less willing to manipulate the network and a higher degree of trust placed by the firm on consumers can either enhance or reduce the total intensity of investment in equilibrium.

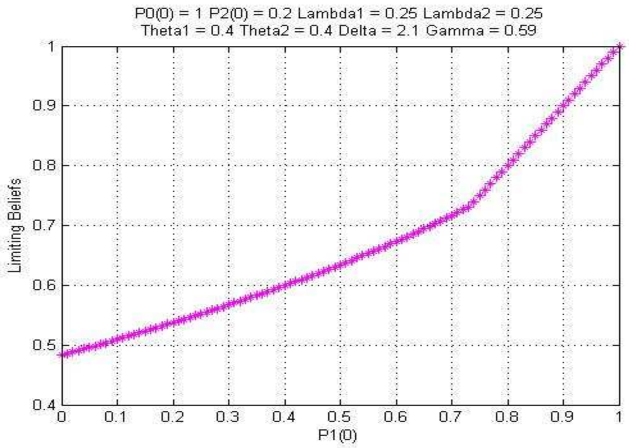

Fig. 30 The limiting consensus belief as a function of $p_1(0)$

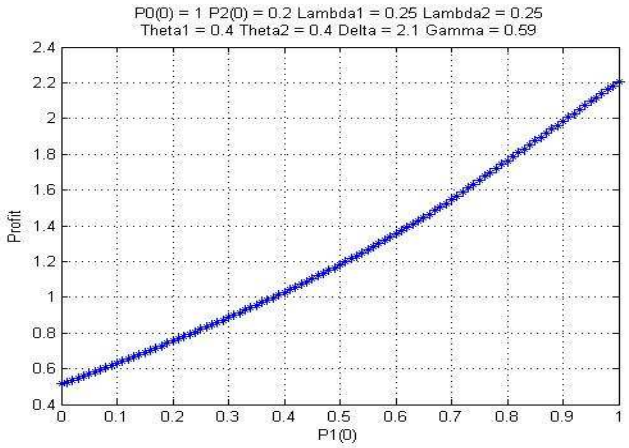

Fig. 31 The firm's equilibrium profit as a function of $p_1(0)$

The model developed above can be extended in multiple directions. First of all, the analysis could be generalized to account for the presence of more than two groups or types of consumers in the network. Apart from examining the pattern of optimal influence strategies in this framework, the speed of reaching a consensus among different groups of customers as a function of the homophily characterizing the network might be another question deserving closer scrutiny. The assumption of a monopolistic product market might also be relaxed to consider the possibility of oligopolistic competition between multiple firms producing either a homogeneous or a differentiated good. In contrast to the one-shot choices of price and investment intensities considered here, we might study the problem of choosing entire time paths for these variables - which

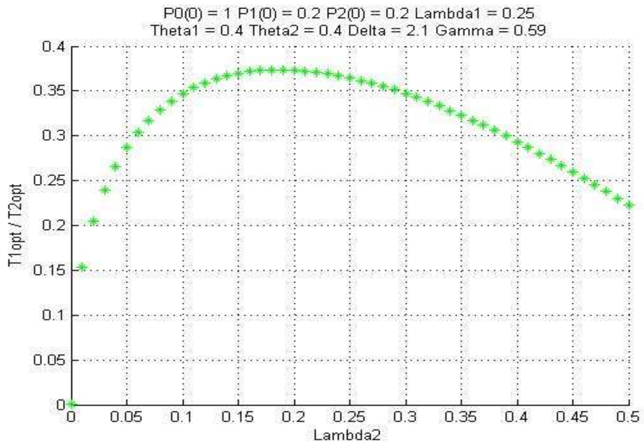

Fig. 32 The firm's optimal influence strategy as a function of λ_2

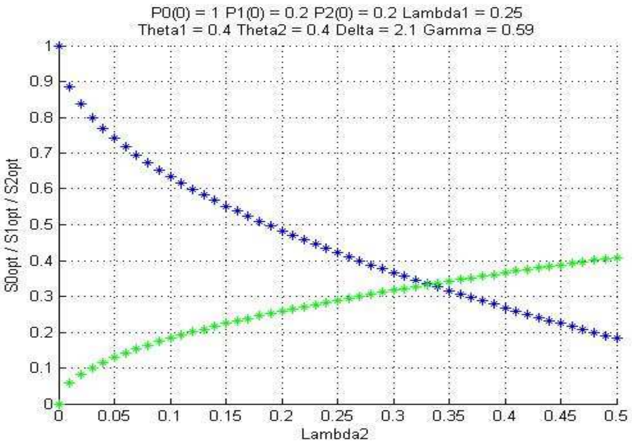

Fig. 33 Limiting influence levels as a function of λ_2

would be treated as control variables in such an enriched setting - by using the appropriate techniques of dynamic optimization. Finally, the formulation of a weighted network adopted above might be replaced by a standard buyer-seller network to investigate the structure of links strategically formed between consumers and firms as well as to evaluate the equilibrium network structure from a welfare point of view. These extensions are left for future research.

References

[Acemoglu et al.(2011)Acemoglu, Dahleh, Lobel, and Ozdaglar] D. Acemoglu, M. A. Dahleh, I. Lobel, and A. Ozdaglar. Bayesian learning in social networks. *The Review*

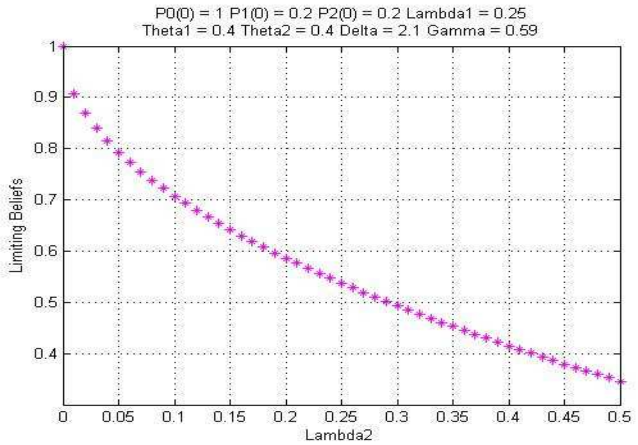

Fig. 34 The limiting consensus belief as a function of λ_2

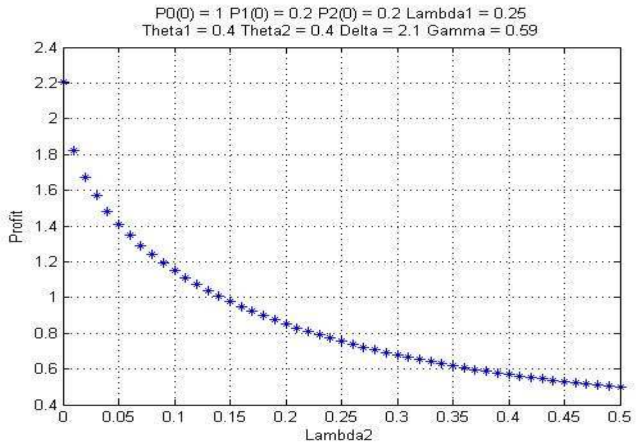

Fig. 35 The firm's equilibrium profit as a function of λ_2

of Economic Studies, 78(4):1201–1236, 2011.

[Ballester et al.(2006)Ballester, Calvó-Armengol, and Zenou] C. Ballester, A. Calvó-Armengol, and Y. Zenou. Who's who in networks. wanted: the key player. *Econometrica*, 74(5):1403–1417, 2006.

[Biggs(1973)] M. Biggs. *Constrained minimisation using recursive quadratic programming: some alternative subproblem formulations.* Hatfield Polytechnic. Numerical Optimisation Centre, 1973.

[Borgatti(2006)] S. P. Borgatti. Identifying sets of key players in a social network. *Computational & Mathematical Organization Theory*, 12(1):21–34, 2006.

[DeGroot(1974)] M. H. DeGroot. Reaching a consensus. *Journal of the American Statistical Association*, 69(345):118–121, 1974.

[Dodds and Watts(2004)] P. S. Dodds and D. J. Watts. Universal behavior in a generalized model of contagion. *Physical review letters*, 92(21):218701, 2004.

[Fletcher(2013)] R. Fletcher. *Practical methods of optimization.* John Wiley & Sons, 2013.

[Galeotti and Goyal(2009)] A. Galeotti and S. Goyal. Influencing the influencers: a theory of strategic diffusion. *The RAND Journal of Economics*, 40(3):509–532, 2009.

[Gill et al.(1981)Gill, Murray, and Wright] P. E. Gill, W. Murray, and M. H. Wright. Practical optimization. 1981.

[Gladwell(2006)] M. Gladwell. *The tipping point: How little things can make a big difference*. Little, Brown, 2006.

[Golub and Jackson(2010)] B. Golub and M. O. Jackson. Naive learning in social networks and the wisdom of crowds. *American Economic Journal: Microeconomics*, pages 112–149, 2010.

[Han(1977)] S.-P. Han. A globally convergent method for nonlinear programming. *Journal of optimization theory and applications*, 22(3):297–309, 1977.

[Hinz and Spann(2008)] O. Hinz and M. Spann. The impact of information diffusion on bidding behavior in secret reserve price auctions. *Information Systems Research*, 19(3): 351–368, 2008.

[Hinz et al.(2011)Hinz, Skiera, Barrot, and Becker] O. Hinz, B. Skiera, C. Barrot, and J. U. Becker. Seeding strategies for viral marketing: An empirical comparison. *Journal of Marketing*, 75(6):55–71, 2011.

[Hinz et al.(2014)Hinz, Schulze, and Takac] O. Hinz, C. Schulze, and C. Takac. New product adoption in social networks: Why direction matters. *Journal of Business Research*, 67(1):2836–2844, 2014.

[Iyengar et al.(2011)Iyengar, Van den Bulte, and Valente] R. Iyengar, C. Van den Bulte, and T. W. Valente. Opinion leadership and social contagion in new product diffusion. *Marketing Science*, 30(2):195–212, 2011.

[Jackson et al.(2008)] M. O. Jackson et al. *Social and economic networks*, volume 3. Princeton University Press Princeton, 2008.

[Katz and Lazarsfeld(1955)] E. Katz and P. F. Lazarsfeld. *Personal Influence, The part played by people in the flow of mass communications.* Transaction Publishers, 1955.

[McPherson et al.(2001)McPherson, Smith-Lovin, and Cook] M. McPherson, L. Smith-Lovin, and J. M. Cook. Birds of a feather: Homophily in social networks. *Annual review of sociology*, pages 415–444, 2001.

[Powell(1983)] M. Powell. Variable metric methods for constrained optimization. In *Mathematical Programming The State of the Art*, pages 288–311. Springer, 1983.

[Powell(1978a)] M. J. Powell. The convergence of variable metric methods for non-linearly constrained optimization calculations. *Nonlinear programming*, 3, 1978a.

[Powell(1978b)] M. J. Powell. A fast algorithm for nonlinearly constrained optimization calculations. In *Numerical analysis*, pages 144–157. Springer, 1978b.

[Richardson and Domingos(2002)] M. Richardson and P. Domingos. Mining knowledge-sharing sites for viral marketing. In *Proceedings of the eighth ACM SIGKDD international conference on Knowledge discovery and data mining*, pages 61–70. ACM, 2002.

[Schittkowski(1986)] K. Schittkowski. Nlpql: A fortran subroutine solving constrained nonlinear programming problems. *Annals of operations research*, 5(2):485–500, 1986.

[Schmitt et al.(2011)Schmitt, Skiera, and Van den Bulte] P. Schmitt, B. Skiera, and C. Van den Bulte. Referral programs and customer value. *Journal of Marketing*, 75(1): 46–59, 2011.

[Watts and Dodds(2007)] D. J. Watts and P. S. Dodds. Influentials, networks, and public opinion formation. *Journal of consumer research*, 34(4):441–458, 2007.