Conjugacy of One Dimensional Map

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2009 NCTS Workshop on Dynamical Systems

- A way to say that two systems are equivalent is by obtaining a conjugacy map between them.

$f : I \longrightarrow I$ and $g : J \longrightarrow J$ are continuous maps.

1. A continuous function $h : I \longrightarrow J$ is said to be a topological semi-conjugacy of f to g if

$$h \circ f = g \circ h.$$

$$\begin{tikzcd} I \arrow[r, "h"] \arrow[d, "f"'] & J \arrow[d, "g"] \\ I \arrow[r, "h"] & J \end{tikzcd}$$

2. In addition, if h is bijective, then h is said to be a topological conjugacy of f and g.

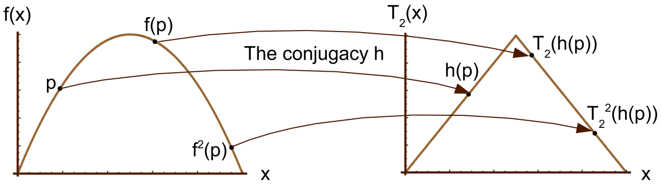
Conjugacy

- Conjugacy maps preserves the characteristic between two dynamical system.

Consider $([0,1], f)$ and $([0,1], T_2)$, if p is a periodic point of f with period 3. Then,

$$T_2^3 \circ h(p) = h \circ f^3(p) = h(p).$$

$h(p)$ is a periodic point of T_2 with period 3.



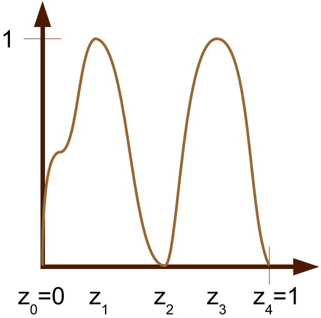
Conjugacy

- Milnor had proved the existance of the semi-conjugacy from a piecewise monotone map to a piecewise linear map.
  - Milnor & Thurston, *Dynamical systems : proceedings of the special year held at the University of Maryland, College Park, 1986-87*, Springer, 1988, On iterated maps of the interval, pp. 465-563.
- Banks had given a construction of the semi-conjugacy map from a 2-modal(unimodal) map to the tent map, which can computed numerically.
  - Banks & Dragan, *Chaos: A mathematical introduction*. Cambridge university press, 2003.

n-modal maps

A (strictly) *n*-modal map f is a continuous map on the interval $[0,1]$ with a partition $0 = z_0 < z_1 < \ldots < z_n = 1$ such that

1. $f(z_{2i}) = 0$ and $f(z_{2i+1}) = 1$, and
2. f is (strictly) increasing on $[z_{2i}, z_{2i+1}]$ and (strictly) decreasing on $[z_{2i+1}, z_{2i+2}]$.

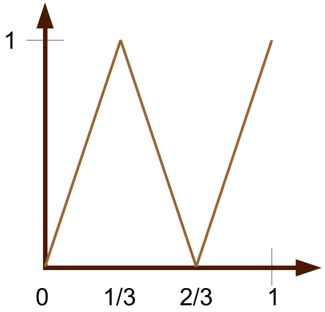


An example of a 4-modal map.

The generalized tent maps

A generalized tent map $T_n : [0,1] \to [0,1]$ is defined as

$$T_n(x) = \begin{cases} n(x - \frac{2i}{n}) & \text{if } x \in [\frac{2i}{n}, \frac{2i+1}{n}]; \\ 1 - n(x - \frac{2i+1}{n}) & \text{if } x \in [\frac{2i+1}{n}, \frac{2i+2}{n}]. \end{cases}$$



An example of T_3.

The main result

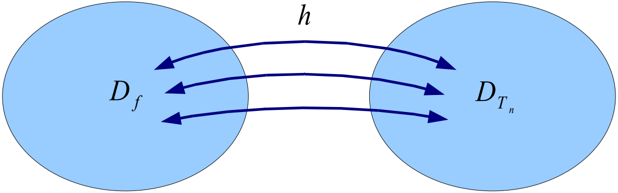
Theorem

Assume that f is a n-modal map on $[0,1]$ and T_n is the tent map. Then there exists an unique semi-conjugacy h of f to T_n such that $h(0)=0$, $h(1)=1$ and h is increasing. That is, $h \circ f = T_n \circ h$.

The approximated semi-conjugacy is constructed by matching the orbits.
We match "the inverse orbits" of 0 and 1.
Let

$$D_g = \{x \in [0,1] : \exists k \text{ such that } g^k(x) = 0 \text{ or } 1\}.$$

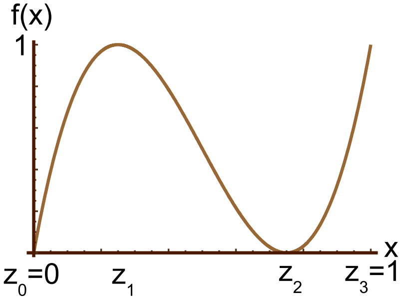
Then,



Example of constructing a conjugacy

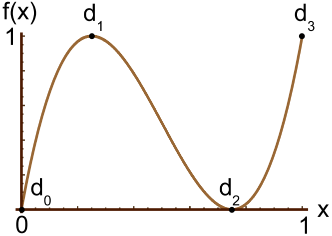
A 3-modal map $f(x) = x(3-4x)^2$.
A partition is given as:

$$z_0 = 0, z_1 = \frac{1}{4}, z_2 = \frac{3}{4}, z_3 = 1.$$

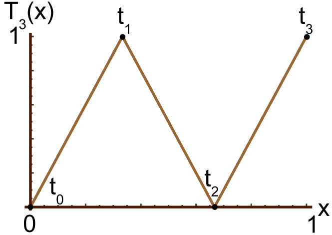


Example of constructing a conjugacy

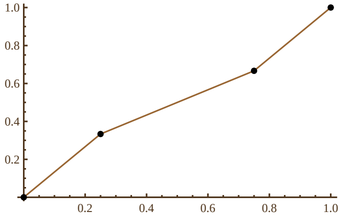
Step 1: $m = 1$
$f(x) = 0$ for $x = z_0, z_2$. $T_3(x) = 0$ for $x = 0, \frac{2}{3}$.
$f(x) = 1$ for $x = z_1, z_3$. $T_3(x) = 1$ for $x = \frac{1}{3}, 1$.



$d \xrightarrow{h_1} t$



h_1

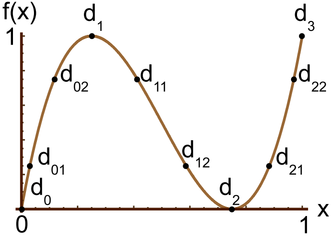


Example of constructing a conjugacy

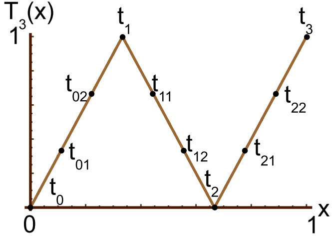
Step 2: $m = 2$

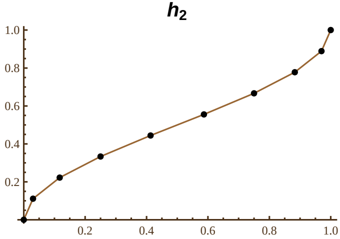
$f(x) = d_1$ for $x = d_{01}, d_{12}, d_{21}$. $T_3(x) = t_1$ for $x = t_{01}, t_{12}, t_{21}$.

$f(x) = d_2$ for $x = d_{02}, d_{11}, d_{22}$. $T_3(x) = t_1$ for $x = t_{02}, t_{11}, t_{22}$.



$d \xrightarrow{h_2} t$

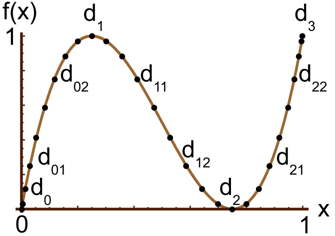




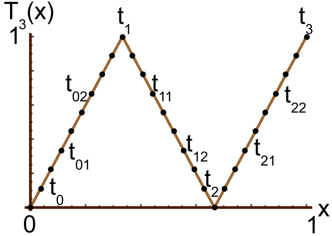
Example of constructing a conjugacy

Step 3: $m = 3$

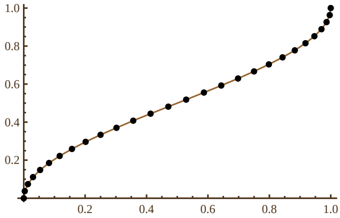
We map the solutions of $f(x) = d_{01}, d_{02}, d_{11}, d_{12}, d_{21}, d_{22}$ to the solutions of $T_3(x) = t_{01}, t_{02}, t_{11}, t_{12}, t_{21}, t_{22}$ as the similarly way.



$d \xrightarrow{h_3} t$



$\boldsymbol{h_3}$



Definition of the approximated semi-conjugacy

$t_a = \sum_{i=1}^{l(a)} a_i \frac{1}{n^i}$.
d_a is defined by solving $f(d_a) = d_{\tilde{a}}$.

The approximated semi-conjugacy $h_m : [0,1] \to [0,1]$ is defined to be piecewise linear with turning points $h_m(d_a) = t_a$.

Fact

1. *h_m is increasing.*
2. *The limit $h = \lim_{m\to\infty} h_m$ exists and h is the unique semi-conjugacy.*

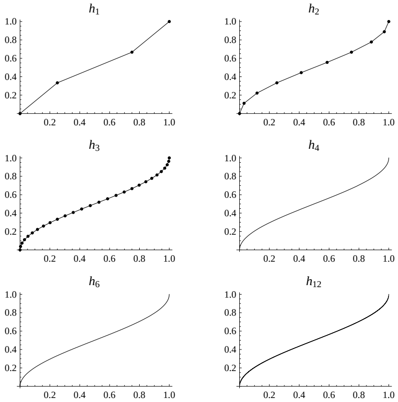
Algorithm for numerical implement

1. Give a function f with its turning points $z_1, ..., z_{n-1}$ and the number of iterations k.
2. Set $m = 1$, $z_0 = 0$, $z_n = 1$ and $d_0 = 0, d_1 = z_1, d_2 = z_2, ..., d_{n-1} = z_{n-1}$.
3. Set $\tilde{d}_{in} = d_i$ for $i = 0, ..., n^m - 1$.
4. For each $i = 0, ..., n^m - 1$, if i does not divide n, then
   1. set $\tilde{d}_{jn^{m-1}+i}$ be the solution of $f(x) = d_i$ on $[z_j, z_{j+1})$ for $j = 0, 2, ... < n$, and
   2. set $\tilde{d}_{jn^{m-1}+j-i}$ be the solution of $f(x) = d_i$ on $[z_j, z_{j+1})$ for $j = 1, 3, ... < n$.
5. Set $d = \tilde{d}$ and $m = m + 1$.
6. Redo steps 3 to 5 until $m = k$.
7. For $i = 0, ..., n^k$, set $t_i = \frac{i}{n^k}$.
8. Output the data $\{(d_i, t_i)\}_{i=0}^{n^k}$. The points constructs the piecewise linear approximation of the semi-conjugacy map.

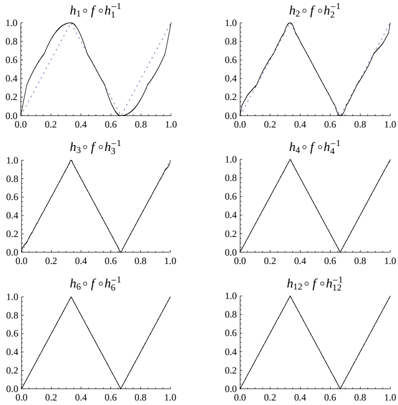
The running time for each m is of order $O(n^m)$.

Example of constructing a conjugacy

Numerical results for the 3-modal map $f(x) = x(3-4x)^2$.



The approximated semi-conjugacy h_m.



The comparison of $h_m \circ f \circ h_m^{-1}$ with T_3.

We know that $h^{-1}(x) = \frac{1}{2}(1 - \cos \pi x) = \sin^2(\frac{\pi}{2} x)$.

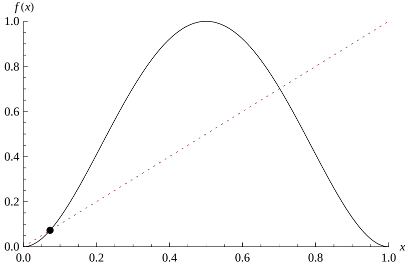
Example of constructing a conjugacy

m	CPU(s)	$-\log_3$	$h_m - h$	$-\log_3$	$h_m^{-1} - h^{-1}$	$-\log_3$	$h_m \circ f \circ h_m^{-1} - T_3$	$-\log_3$
1			7.75×10^{-2}	2.33	5.82×10^{-2}	2.59	2.32×10^{-1}	1.33
2	0.000014	10.15	2.75×10^{-2}	3.27	7.48×10^{-3}	4.46	7.43×10^{-2}	2.37
3	0.000034	9.35	9.19×10^{-3}	4.27	8.45×10^{-4}	6.44	2.47×10^{-2}	3.37
4	0.000115	8.25	1.94×10^{-3}	5.69	9.40×10^{-5}	8.44	8.23×10^{-3}	4.37
5	0.000343	7.26	3.29×10^{-4}	7.30	1.05×10^{-5}	10.44	2.74×10^{-3}	5.37
6	0.00101	6.28	6.72×10^{-5}	8.75	1.16×10^{-6}	12.44	9.15×10^{-4}	6.37
7	0.00302	5.28	8.41×10^{-6}	10.64	1.29×10^{-7}	14.44	3.05×10^{-4}	7.37
8	0.00767	4.43	2.10×10^{-6}	11.90	1.43×10^{-8}	16.44	1.02×10^{-4}	8.37
9	0.0225	3.45	4.73×10^{-7}	13.26	1.59×10^{-9}	18.44	3.39×10^{-5}	9.37
10	0.0681	2.45	4.23×10^{-8}	15.45	1.77×10^{-10}	20.44	1.13×10^{-5}	10.37
11	0.202	1.46	1.28×10^{-8}	16.54	1.97×10^{-11}	22.44	3.76×10^{-6}	11.37
12	0.584	0.49	3.34×10^{-9}	17.77	2.18×10^{-12}	24.44	1.25×10^{-6}	12.37
13	1.73	−0.50	6.02×10^{-10}	19.33	2.43×10^{-13}	26.44	4.18×10^{-7}	13.37
14	5.52	−1.56	9.36×10^{-11}	21.02	2.70×10^{-14}	28.44	1.39×10^{-7}	14.37
15	19.5	−2.71	1.52×10^{-11}	22.67	3.63×10^{-14}	28.17	4.65×10^{-8}	15.37
16	51.9	−3.60	4.15×10^{-12}	23.86	6.52×10^{-14}	27.64	1.39×10^{-8}	16.37
	$O(3^m)$		$O(3^{-m})$		$O(3^{-2m})$		$O(3^{-m})$	

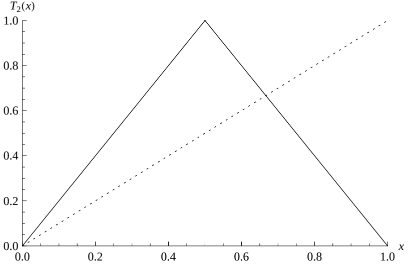
Table: The progressing time and error versus the number of iterations.

Example of constructing a semi-conjugacy

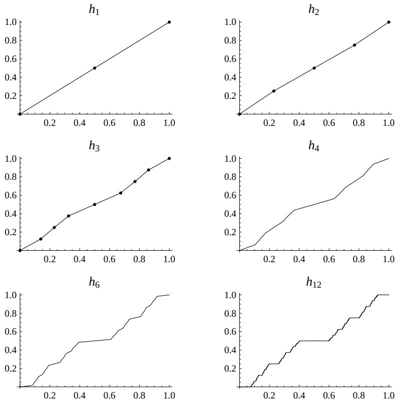
A 2-modal map $f(x) = 16x^2(1-x)^2$.



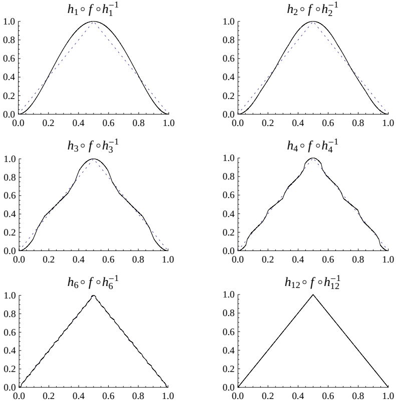
The tent map T_2



Example of constructing a semi-conjugacy



The approximated semi-conjugacy h_m.



The comparison of $h_m \circ f \circ h_m^{-1}$ with T_2.

Example of constructing a semi-conjugacy

m	CPU(s)	$-\log_2$	$h_m - h_{m-1}$	$-\log_2$	$h_m^{-1} - h_{m-1}^{-1}$	$-\log_3$	$h_m \circ f \circ h_m^{-1} - T_2$	$-\log_3$
1							1.30×10^{-1}	2.95
2	0.0000106	16.52	2.05×10^{-2}	5.61	2.04×10^{-2}	5.61	1.00×10^{-1}	3.32
3	0.0000132	16.21	3.62×10^{-2}	4.79	3.93×10^{-2}	4.67	7.27×10^{-2}	3.78
4	0.0000207	15.56	3.35×10^{-2}	4.90	4.68×10^{-2}	4.62	4.60×10^{-2}	4.44
5	0.0000365	14.74	2.25×10^{-2}	5.48	4.81×10^{-2}	4.38	2.64×10^{-2}	5.25
6	0.0000657	13.89	1.31×10^{-2}	6.25	4.83×10^{-2}	4.37	1.42×10^{-2}	6.13
7	0.0001341	12.86	7.11×10^{-3}	7.14	4.81×10^{-2}	4.38	7.43×10^{-3}	7.07
8	0.0002510	11.96	3.71×10^{-3}	8.07	4.81×10^{-2}	4.38	3.80×10^{-3}	8.04
9	0.0004672	11.06	1.90×10^{-3}	9.04	4.81×10^{-2}	4.38	1.92×10^{-3}	9.02
10	0.0009223	10.08	9.62×10^{-4}	10.02	4.81×10^{-2}	4.38	9.68×10^{-4}	10.01
11	0.0017604	9.15	4.84×10^{-4}	11.01	4.81×10^{-2}	4.38	4.86×10^{-4}	11.01
12	0.0032727	8.26	2.43×10^{-4}	12.01	4.81×10^{-2}	4.38	2.43×10^{-4}	12.00
13	0.0064051	7.29	1.22×10^{-4}	13.00	4.81×10^{-2}	4.38	1.22×10^{-4}	13.00
14	0.0122889	6.35	6.10×10^{-5}	14.00	4.81×10^{-2}	4.38	6.10×10^{-5}	14.00
15	0.0238768	5.39	3.05×10^{-5}	15.00	4.81×10^{-2}	4.38	3.05×10^{-5}	15.00
16	0.0518212	4.27	1.53×10^{-5}	16.00	4.81×10^{-2}	4.38	1.52×10^{-5}	16.00
	$O(2^m)$		$O(2^{-m})$		x		$O(2^{-m})$	

Table: The progressing time and error versus the number of iterations.

The error analysis

	conditions	convergence rate
Convergence of h_m	$h \in C$	$O(n^{-m})$
	$h \in C^2, h'(x) \neq 0$	$O(n^{-2m})$
Convergence of h'_m	$h \in C^2, h'(x) \neq 0$	$O(n^{-m})$
Convergence of h_m^{-1}	$h^{-1} \in C^2$	$O(n^{-2m})$
Convergence of $h_m^{-1\prime}$	$h^{-1} \in C^2$	$O(n^{-m})$
Convergence of $h_m \circ f \circ h_m^{-1}$	$h \in C$	$O(n^{-m})$

$$h'_m \text{ converges in } O(n^{-m}) \Longrightarrow h_m \text{ converges in } O(n^{-2m})$$
$$h_m^{-1\prime} \text{ converges in } O(n^{-m}) \Longrightarrow h_m^{-1} \text{ converges in } O(n^{-2m})$$

Equivalent conditions for the existence of conjugacy

A continuous map $f : I \longrightarrow I$ is said to be topologically transitive if for any pair of open subintervals (U, V) of I, there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$.

Theorem

Assume that f is a n-modal map and h is the semi-conjugacy of f to T_n. Then the following conditions are equivalent:

1. *h is the conjugacy of f to T_n,*
2. *$\{$the sequence d that we had constructed$\}$ is dense in $[0, 1]$,*
3. *f is topological transitive.*

- Coincide with Parry's theorem.(1966)
- Gives a connection with the chaotic behavior.

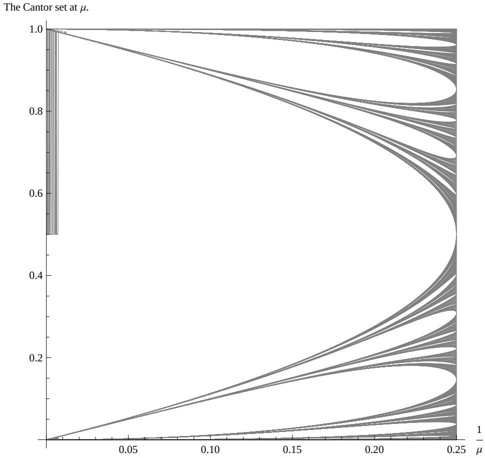
- A set Λ is said to be invariant under the map f if $f(\Lambda) \subset \Lambda$.
- The logistic map is defined as $l_\mu(x) = \mu x(1-x)$ for a parameter $\mu \in [0, \infty)$.
- The bounded orbits Λ_μ of l_μ forms an invariant cantor set.

Applications

The trajectory of the invatiant Cantor set for the logistic map

The result is similar as Chen's result.

- Y.-C. Chen, "Family of invariant cantor sets as orbits of differential equations," International Journal of Bifurcation and Chaos, vol. 18, no. 7, pp. 1825–1843, 2008.



Assume that (X, Σ, μ) is a measure space. We say that $T : X \to X$ is called a measure preserving transformation (mpt) if $T^{-1}(\Sigma) \subset \Sigma$ and

$$\mu(T^{-1}E) = \mu(E)$$

for every $E \in \Sigma$.

Theorem

Assume that f is a n-modal map with semi-conjugacy h to T_n and ν is the Borel measure generated by the cumulative distribution h.
(i.e. $\nu(a,b] = h(b) - h(a)$.)
Then, f preserves the measure $([0,1], \mathscr{B}, \nu)$.