Nonexistence of Wandering domain for strongly dissipative infinite renormalizable Hénon-like maps

Dyi-Shing Ou

Stony Brook University

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Outline

- Motivation

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- Introduction to infinite renormalizable Hénon-like map
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  - Why the method break down in dimension-two (Hénon)
  - Higher dimension solution

Wandering Interval

Definition (Wandering interval)

Let $I \subset \mathbb{R}$ (or $I = \mathbb{S}^1$) and $f : I \to I$ be continuous. A nonempty connected open subinterval $J \subset I$ is a **wandering interval** of f if

1. $f^m(J) \cap f^n(J) = \phi$ for all $m \neq n$
2. the orbit of J does not tend to a periodic orbit

Characteristic:

- Does not tend to go back to itself
- Have the same dynamical behavior

Real Dimension One

Solves the classification problem

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- Circle diffeomorphism

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$$\overbrace{\begin{array}{c} C^{1+bv} \text{ (1932 Denjoy)} \\ \text{analytic homeomorphism (1984 Yoccoz)} \\ C^1\text{+Zygmund (1995 Hu,Sullivan)} \end{array}}^{\text{regularity}}$$

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$$\underbrace{\text{regularity}}_{\substack{C^{1+bv} \text{ (1932 Denjoy)} \\ \text{analytic homeomorphism (1984 Yoccoz)} \\ C^1 + \text{Zygmund (1995 Hu,Sullivan)}}} \Longrightarrow \text{no wandering interval}$$

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- Unimodal map

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- Unimodal map

$$\overbrace{\begin{array}{c} C^3\text{+negative S (1979 Guckenheimer)} \\ C^3\text{+non-flat critical pt (1988,1989 de Melo, van Strien)} \\ C^{1+bv} \text{ (1992 Martens, de Melo, van Strien)} \end{array}}^{\text{regularity}}$$

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Complex Dimension One

$f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a **rational map**:

$$f(z) = \frac{p(z)}{q(z)}$$

Definition

The **Fatou set** of f (roughly speaking) consists of points with the property that **all nearby values behave similarly** under repeated iteration of the function (from Wikipedia)

Definition

A Fatou component is **wandering** if it is **not eventually periodic**.

Complex Dimension One

Theorem (1985 Sullivan)

A rational map of deg ≥ 2 ***does not have wandering Fatou component.***

Classification of periodic Fatou components U

1. U contains an attracting periodic point
2. U is parabolic
3. U is a Siegel disc
4. U is a Herman ring

Complex Dimension One

- Transcendental entire map
  - Non-existence:
    - 1986 Goldberg, Keen
    - 1992 Eremenko, Lyubich
    - 2013 Mihaljević-Brandt, Rempe-Gillen
  - Existence
    - 1976,1984 Baker
    - 1984 Herman
    - 1985 Sullivan
    - 1987 Eremenko, Lyubich
    - 2015 Bishop
    - 2015 Fagella, Godillon, Jarque

Real Higher Dimension Results

- Classification problem fails
  - 1975, 1979 Harrison (fail in any two level of differentiablity)
- Existence by non-hyperbolic phenomenon
  - 2011 Colli, Vargas (C^∞ diffeomorphisms on surface)
  - 2017 Kiriki, Soma (C^r-close to Hénon maps, $r < \infty$)
  - 2016 Kiriki, Nakano, Soma (C^1-close to diffeomorphism on 3-manifold)

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- Generalization of Denjoy's theory
  - Disc (1994 Bonatti, Gambaudo, Lion, Tresser)
  - Torus (1996 Norton, Sullivan) (2017 Navas) (1993, 1995 Mc Swiggen)

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- Generalization of Denjoy's theory
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  - Torus (1996 Norton, Sullivan) (2017 Navas) (1993, 1995 Mc Swiggen)
- Conditions between wandering domain and entropy
  - 2010 Kwakkel, Markovic (on surface, wandering domain of generic shape$\Longrightarrow$0 entropy)
- No wandering domain
  - 1991 Norton (compact 2-manifold)
  - 1994 Bonatti, Gambaudo, Lion, Tresser (disc)

Complex Higher Dimension

- Transendental biholomorphic
  - 1998 Fornæss, Sibony

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- Polynomial skew-product $F(t, z) = (f(t), g(t, z))$
  - 2014 Lilov (super-attracting)
  - 2017 Peters, Smit (subhyperbolic attracting)
  - 2016 Astorg, Buff, Dujardin, Peters, Raissy (parabolic)
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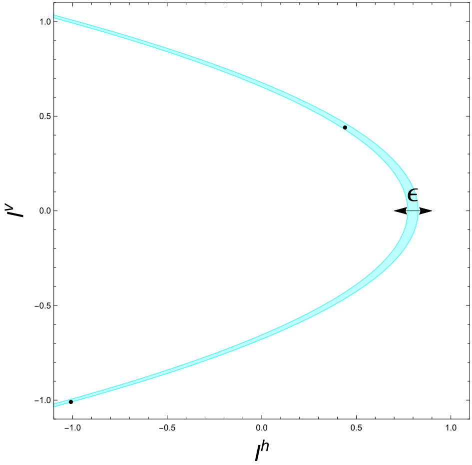
  - 2016 Raissy
- Problem on Complex Hénon map is still widely open
  - 2015 Bedford (Survey)

Hénon-like map = Unimodal map + perturbation

A **Hénon-like map** is a continuous map $F : I^h \times I^v \to \mathbb{R}^2$ with the form

$$F(x, y) \quad = \quad (\underbrace{f(x)}_{\text{unimodal}} \quad - \quad \underbrace{\epsilon(x, y)}_{\text{pertubation}} \ , x)$$

In this talk, F satisfies some regularity conditions that allows us to define renormalization (F: real analytic,...)



Jacobian

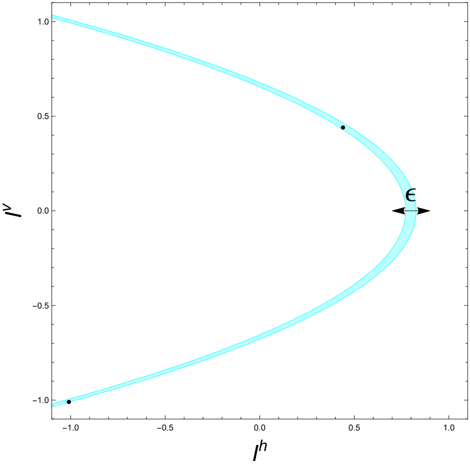
Hénon-like map = Unimodal map + perturbation

$$F(x,y) = (\underbrace{f(x)}_{\text{unimodal}} - \underbrace{\epsilon(x,y)}_{\text{pertubation}}, x)$$

- The **Jacobian**

$$J = \det(DF) = \frac{\partial \epsilon}{\partial y}$$

- Size of $J \approx$ Size of ϵ



Examples

- A Hénon-like map

$$F(x, y) = (\underbrace{f(x)}_{\text{unimodal}} - \underbrace{\epsilon(x, y)}_{\text{pertubation}}, x)$$

- **Degenerate (unimodal) case**: $\epsilon = 0$
  Reduces to the one-dimension unimodal map.
  The behavior is fully described by the dynamics of the unimodal map.

Examples

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- **Degenerate (unimodal) case**: $\epsilon = 0$
  Reduces to the one-dimension unimodal map.
  The behavior is fully described by the dynamics of the unimodal map.
- **Classical Hénon family**:
  $f(x)$: quadratic polynomial
  $\epsilon(x,y) = by$, b is a constant

Dynamics on the Partition

- A Hénon-like map $F : I^h \times I^v \to \mathbb{R}^2$

$$F(x,y) = (\underbrace{f(x)}_{\text{unimodal}} - \underbrace{\epsilon(x,y)}_{\text{pertubation}}, x)$$

- Study the dynamics: Need a domain D that makes F a self-map

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- Renormalization: First returning map on a subset $C \subset D$

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- Study the dynamics: Need a domain D that makes F a self-map
- Renormalization: First returning map on a subset $C \subset D$
- Define a partition on a subset of $I^h \times I^v$ by using the local stable manifolds of its two saddle fixed points

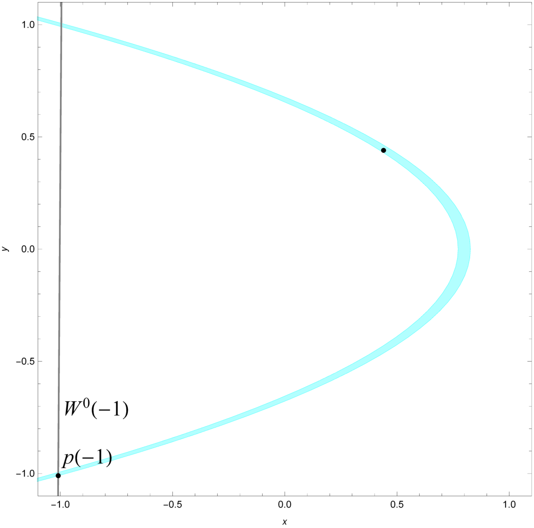
Partition for Hénon-like maps

Assume that ϵ is small

$F(x,y) = (f(x) - \epsilon(x,y), x)$

1. **Local Stable Manifold** of the fixed point with **Positive** Multiplier

[2011, M. Lyubich and M. Martens]



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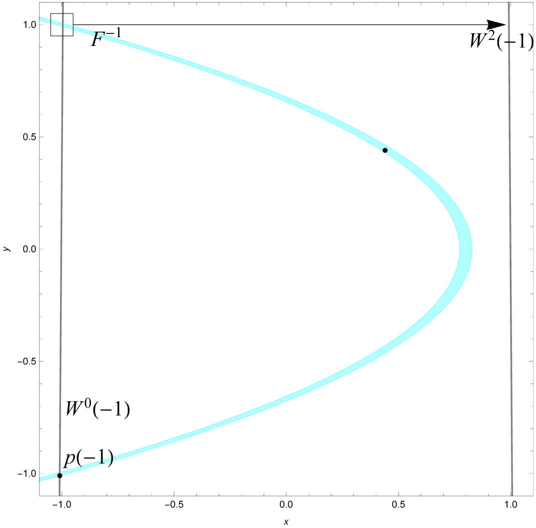

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3. Set $\boldsymbol{D}$ be the set in **between**

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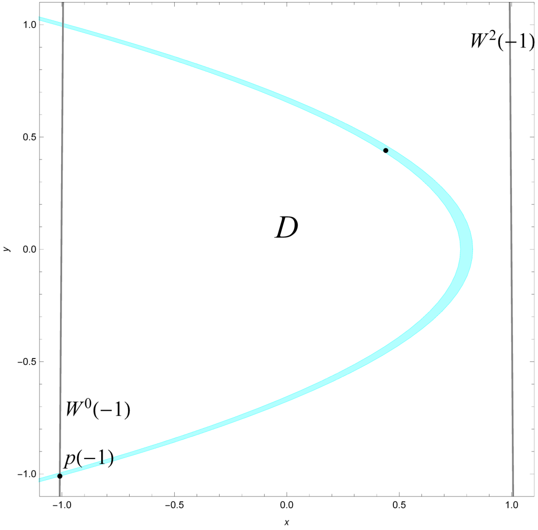
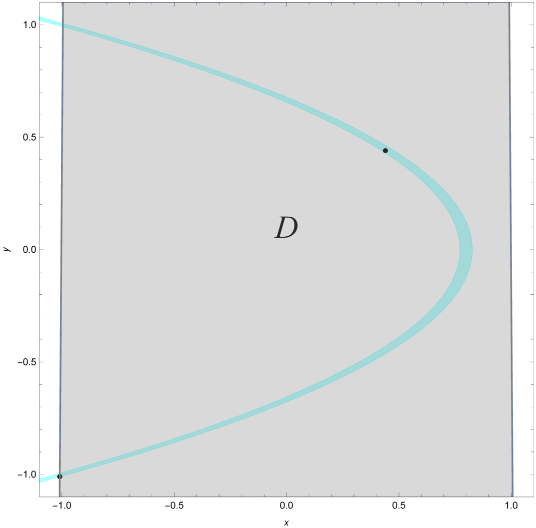


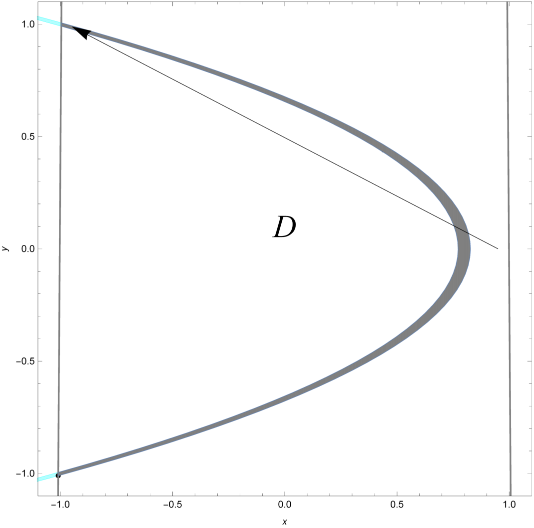
Dynamics on the Partition

D



Dynamics on the Partition

$\boldsymbol{D} \to \boldsymbol{D}$
F is a **self map** on D



Partition for Hénon-like maps

1. **Local Stable Manifold** of the fixed point with **Negative** Multiplier

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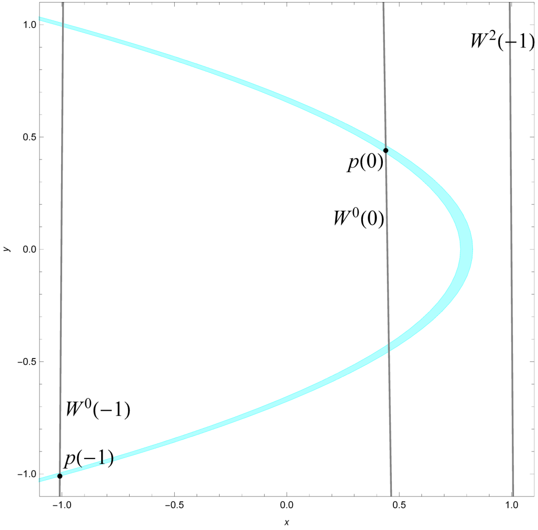

Partition for Hénon-like maps

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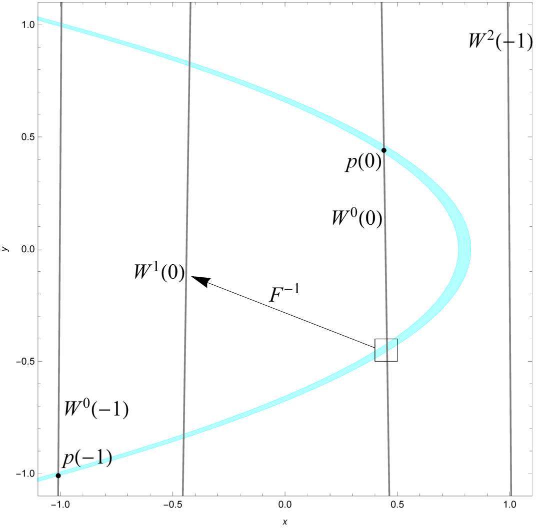
[2011, M. Lyubich and M. Martens]



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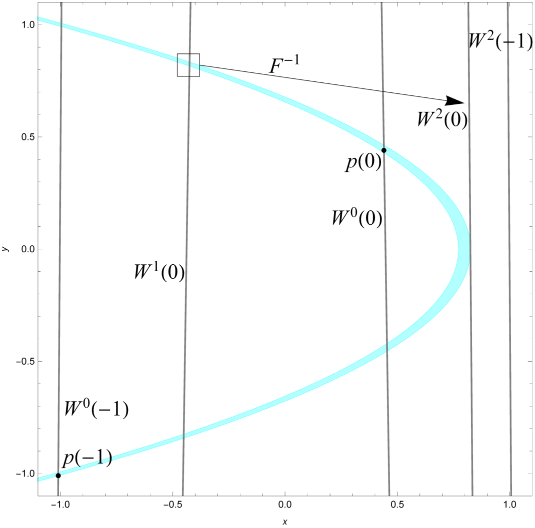
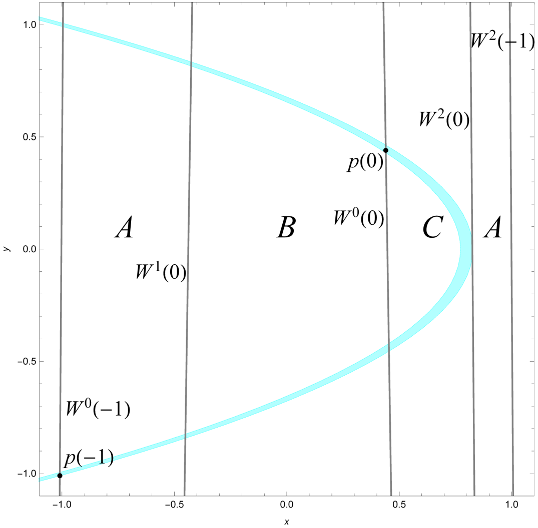
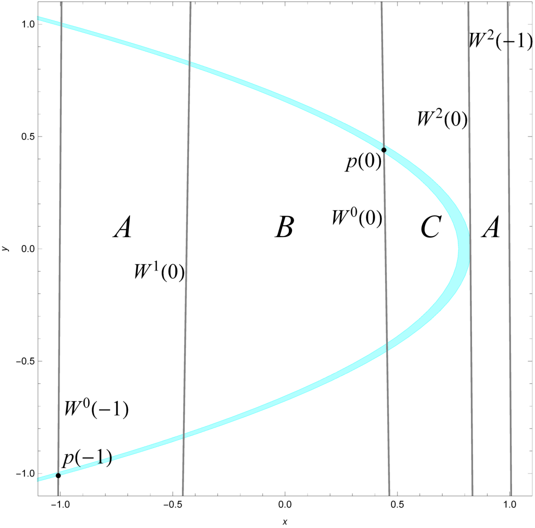


1. **Local Stable Manifold** of the fixed point with **Negative** Multiplier
2. Take the **preimage**
3. Take the **preimage**
4. Set ***A***, ***B***, ***C***, ***A*** be the set in **between**

[2011, M. Lyubich and M. Martens]

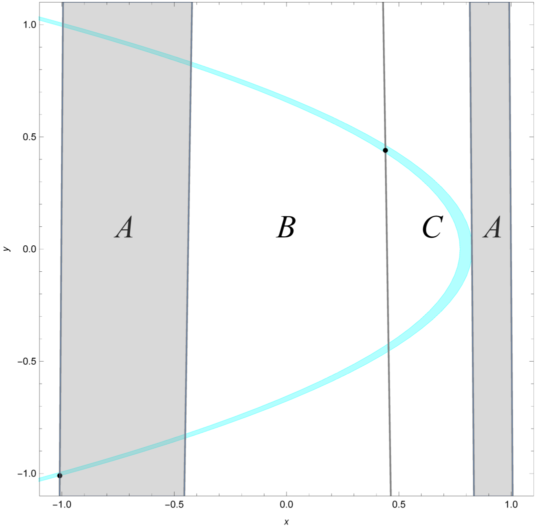


- **unimodal** (degenerate) case: $\epsilon = 0$
  - $F(x, y) = (f(x), x)$
  - preimages of the fixed points
  - local stable manifolds are vertical lines determined by the preimages



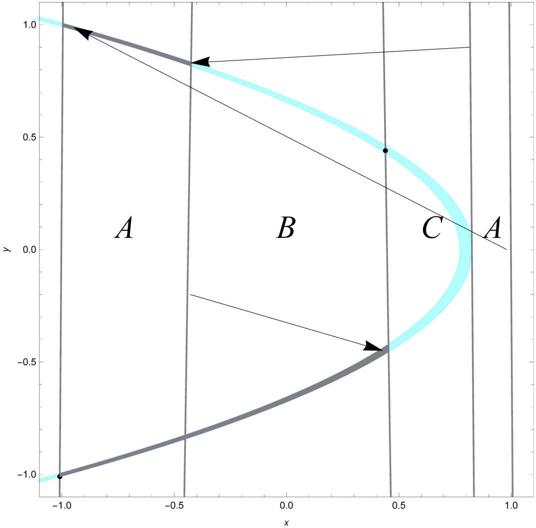
Dynamics on the Partition

A



Dynamics on the Partition

$$\begin{tikzcd} \boldsymbol{A} \arrow[r] \arrow[loop below] & \boldsymbol{B} \end{tikzcd}$$



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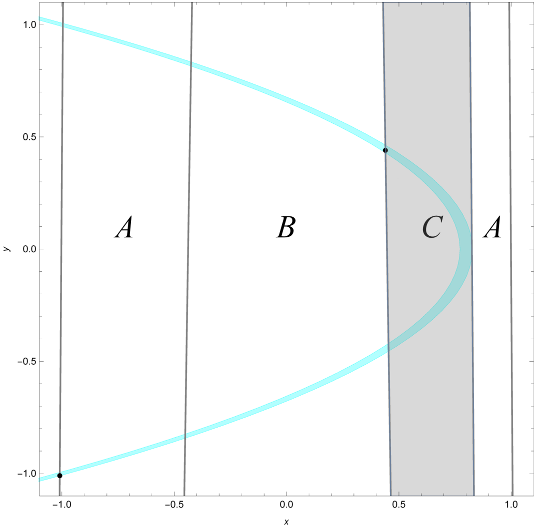
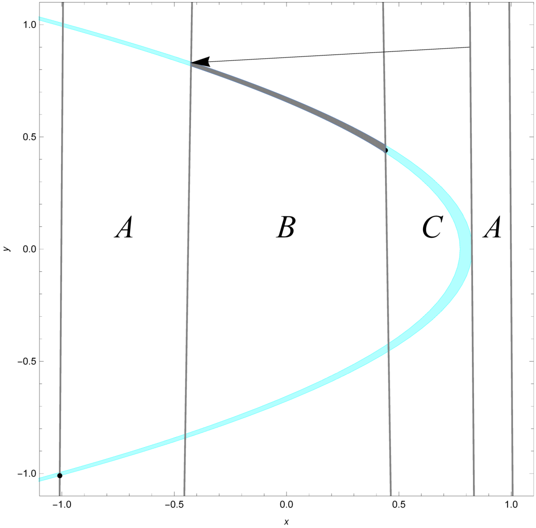


Dynamics on the Partition

$$\begin{tikzcd} A \arrow[r] \arrow[loop below] & B & C \arrow[l] \end{tikzcd}$$



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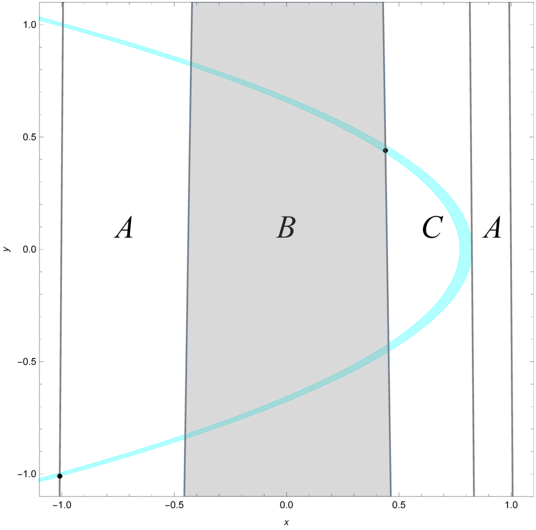
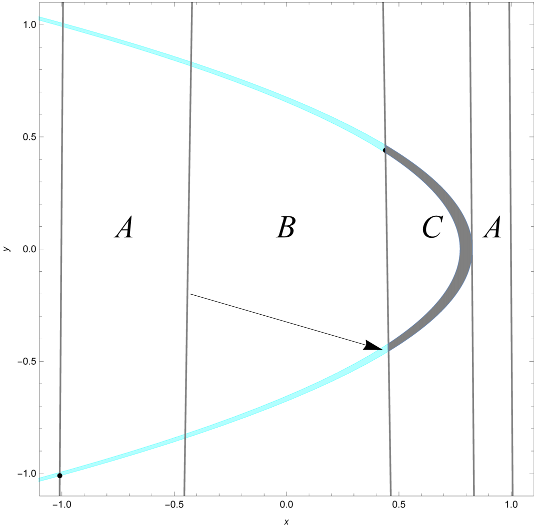


Dynamics on the Partition

$$\begin{tikzcd} A \arrow[r] \arrow[loop below] & \boldsymbol{B} \arrow[r, leftrightarrow] & \boldsymbol{C} \end{tikzcd}$$

A Hénon-like map F is (period-doubling) ***renormalizable*** if

$$F(B) \subset C.$$



Nonlinear rescaling and the Renormalization operator

- For a Hénon-like map

$$F(x,y) = (h_y(x), x) = (f(x) - \epsilon(x,y), x)$$

- Unfortunately, the second iterate

$$F^2(x,y) = (h_x(h_y(x)), \underbrace{h_y(x)}_{\text{not } x})$$

  is not a Hénon-like map

Nonlinear rescaling and the Renormalization operator

[2005, A. de Carvalho, M. Lyubich, and M. Martens]

1. For small ϵ, CLM introduced a **nonlinear rescaling** ϕ that turns the first return map on C into a Hénon-like map

   x-direction preserves orientation

   y-direction reverses orientation & affine

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2. The **renormalization** $\hat{F} = RF = \phi \circ F^2 \circ \phi^{-1}$ is a Hénon-like map

$$\hat{F}(x, y) = (\underbrace{\hat{f}(x)}_{\text{unimodal}} - \underbrace{\hat{\epsilon}(x, y)}_{\text{perturbation}}, x)$$

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   1. Unimodal: ϵ-close to the renormalized unimodal map

$$\left\|\hat{f} - R_c f\right\| \leq c \left\|\epsilon\right\|$$

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   1. Unimodal: ϵ-close to the renormalized unimodal map

$$\left\| \hat{f} - R_c f \right\| \leq c \left\| \epsilon \right\|$$

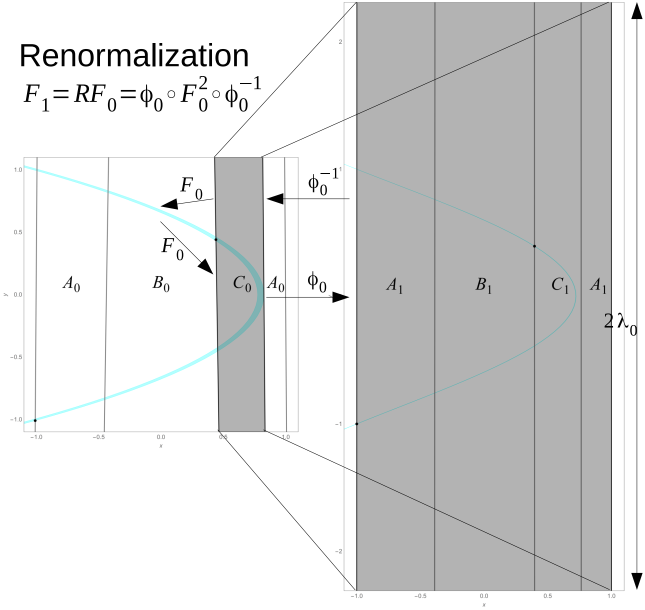
   2. Perturbation: has order ϵ^2

$$\left\| \hat{\epsilon} \right\| \leq c \left\| \epsilon \right\|^2$$

Nonlinear rescaling and the Renormalization operator

Renormalization

$$\begin{tikzcd} D_1 \arrow[r, "F_1"] & D_1 \\ C_0 \arrow[u, "\phi_0"] \arrow[r, "F_0^2"] & C_0 \arrow[u, "\phi_0"'] \end{tikzcd}$$



Infinite Renormalizable Hénon-like maps

- $F \overset{\text{renormalizable}}{\longrightarrow} RF \overset{\text{renormalizable}}{\longrightarrow} R^2F \to \cdots$
- If F can be renormalized infinite many times, we say that F is **infinite renormalizable**

[2005, A. de Carvalho, M. Lyubich, and M. Martens]
Summary for infinite renormalizable Hénon-like maps with **small** ϵ

- $F_n = R^n F =$ unimodal part f_n + perturbation part ϵ_n

Properties for Infinite renormalizable Hénon-like maps

[2005, A. de Carvalho, M. Lyubich, and M. Martens]
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- $F_n = R^n F =$ unimodal part f_n + perturbation part ϵ_n

unimodal $f_n \to g$ **geometrically**

$$\|f_n - g\| < c\rho^n \|F - G\|$$

- g: unimodal map, $R_c g = g$
- $G(x, y) = (g(x), x)$

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pertubation $\epsilon_n \to 0$ **super exponentially**

$$\|\epsilon_{n+t}\| < (c\|\epsilon_n\|)^{2^t}$$

Scope

In this talk, we focus on Hénon-like maps that are:

1. non-degenerate ($\epsilon \neq 0$, $\frac{\partial \epsilon}{\partial y} \neq 0$)
2. strongly dissipative (ϵ: small)
3. infinite (periodic doubling) renormalizable

Dynamics and Wandering Domain

Theorem (2011, M. Lyubich and M. Martens)

F: strongly dissipative infinite (period-doubling) renormalizable Hénon-like map. Then $\omega(x)$ is one of the following

1. *periodic orbit of period 2^n*
2. *the attracting Cantor set*

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Theorem (2011, M. Lyubich and M. Martens)

There exist an infinite renormalizable Hénon-like map such that the stable lamination is nowhere laminar

Dynamics and Wandering Domain

Definition

Assume that $F : D \to D$ is a non-degenerate Hénon-like map. A nonempty connected open set $J \subset D$ is a **wandering domain** of F if J does not intersect any of the stable manifolds for periodic points.

This means that a wandering domain is an open subset of the attracting Cantor set.

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Corollary

The union of the stable manifolds for the periodic points is dense.

Iteration and rescaling of wandering domain

- Assume that $F : D \to D$ is a Hénon-like map and J is a wandering domain.
- Properties:

  Iteration J: wandering domain $\Longrightarrow F(J)$: wandering domain

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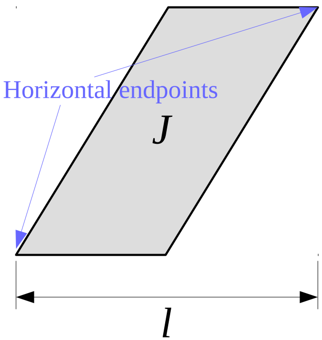
  Subset nonempty open subset of a wandering domain is also a wandering domain

Horizontal size

- **Horizontal size l of J**

$$l(J) = \sup\{|x_1 - x_2|\}$$

for $(x_1, y_1), (x_2, y_2) \in J$.



General Idea of the Proof

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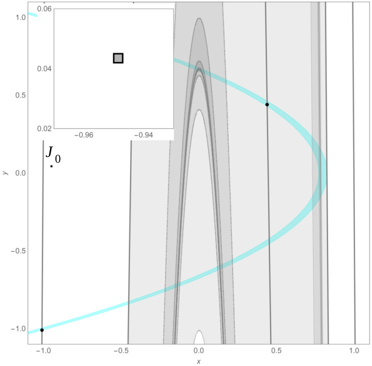
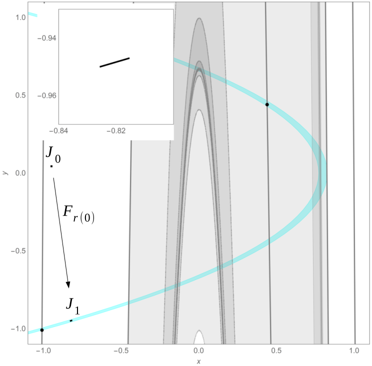
- **Goal**: Prove that $l_n \to \infty$

Prove by contradiction.
Assume that F has an wandering domain $J_0 \subset D$

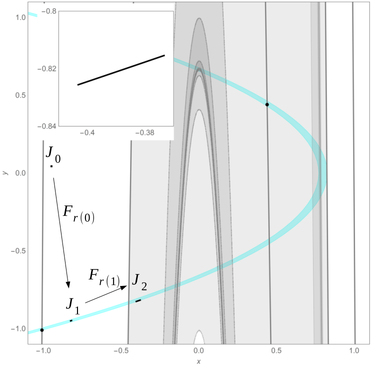


$J_0 \subset A_{r(0)}$: iterate $J_0 \xrightarrow{F_{r(0)}} J_1$



General Idea of the Proof

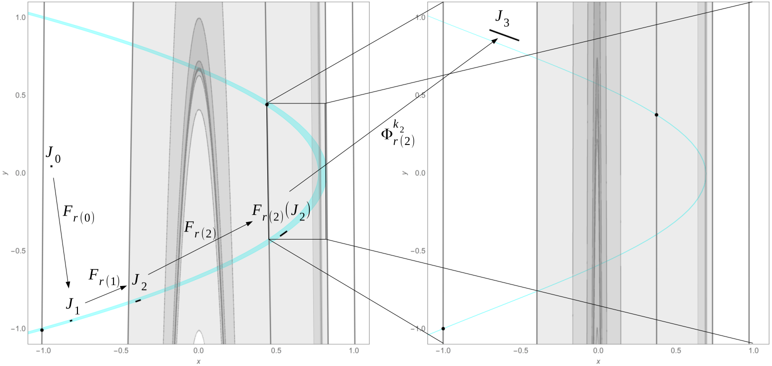
$J_1 \subset A_{r(1)}$: iterate $J_1 \overset{F_{r(1)}}{\Longrightarrow} J_2$



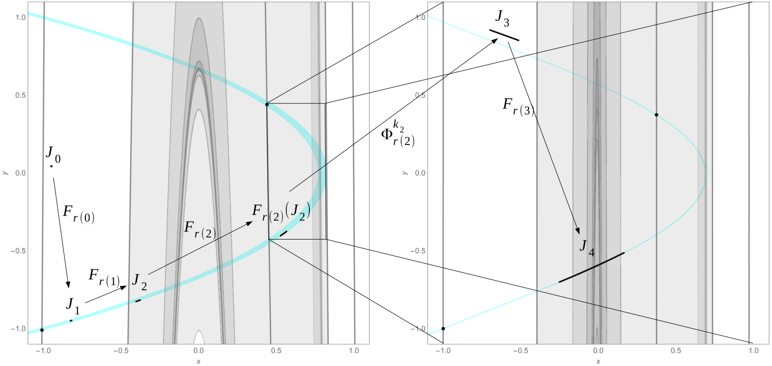
$J_2 \subset B_{r(2)}$: iterate then rescale as many times as possible

$$J_2 \xrightarrow{F_{r(2)}} F_{r(2)}(J_2) \xrightarrow{\Phi_{r(2)}^{k_2}} J_3$$

1 step=iteration+rescaling (if rescaling happens)



$J_3 \subset A_{r(3)}$: iterate $J_3 \overset{F_{r(3)}}{\Longrightarrow} J_4$
The horizontal size is too large that it intersects with some stable manifolds!
Contradiction!



General Idea of the Proof

- Goal: Want to show ($E > 1$)

$$l_{n+1} \geq E l_n \qquad \begin{tikzcd} A \arrow[loop below] \arrow[r] & B \arrow[r, leftrightarrow] & C \end{tikzcd}$$

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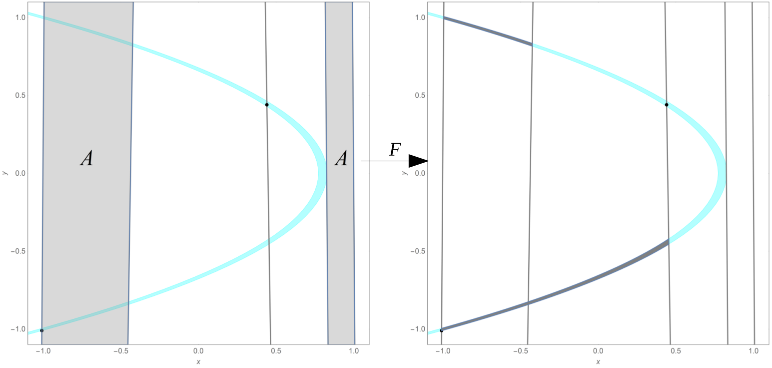

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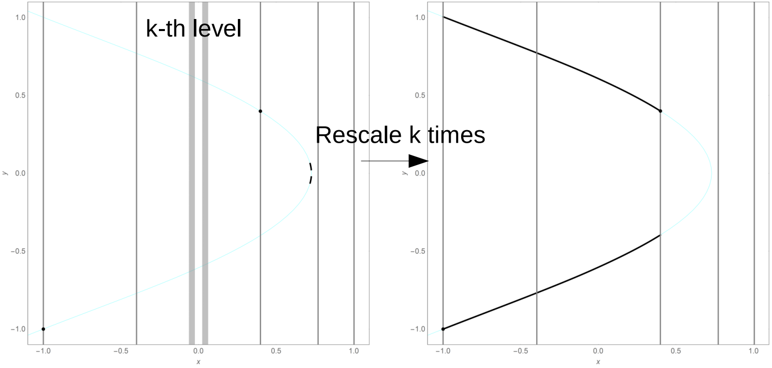
- $A \to A$ or B: $\checkmark$
- $B \overset{\text{contract}}{\to} C \overset{\text{rescale}}{\to} A$ or B: $\checkmark$ in the unimodal case



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- $A \to A$ or B: ✓
- $B \overset{\text{contract}}{\to} C \overset{\text{rescale}}{\to} A$ or B: ✓ in the unimodal case
- $B \overset{\text{contract}}{\to} C \overset{\text{rescale}}{\to} A$ or B: **False** in the Hénon case

Difficulty

- What makes it different between the **Hénon** and **unimodal** case?
  - Why does the expansion argument **fails** in the Hénon case?
  - How **small** will the horizontal size become when the expansion fails?

Difficulty

- What makes it different between the **Hénon** and **unimodal** case?
  - Why does the expansion argument **fails** in the Hénon case?
  - How **small** will the horizontal size become when the expansion fails?
- Two features make Hénon-like maps different from the unimodal maps
  - **Good and Bad regions**
  - **Thickness**

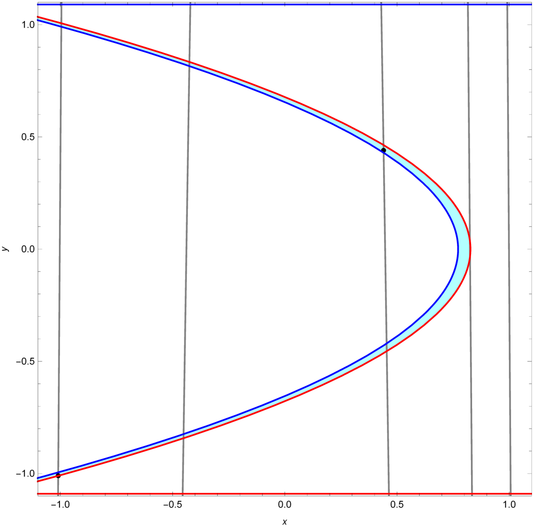
Bad News

- Assume the simple case

$$\epsilon(x, y) = ay$$

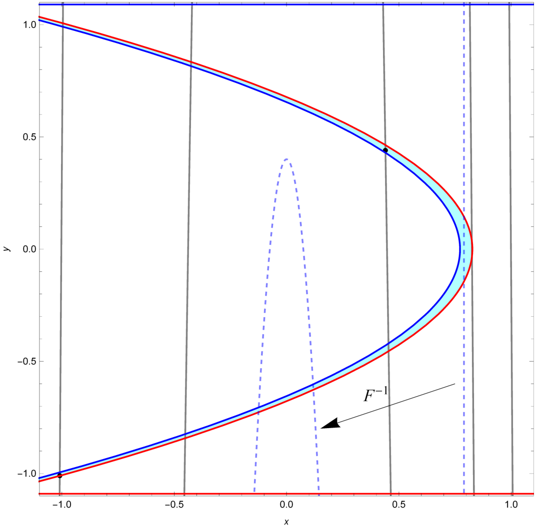
for $a > 0$. i.e.

$$F(x, y) = (f(x) - ay, x)$$



Bad News

- **Draw a vertical line** that intersects the image once
- Take the **preimage** of the vertical line



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- **Draw a vertical line** that intersects the image once
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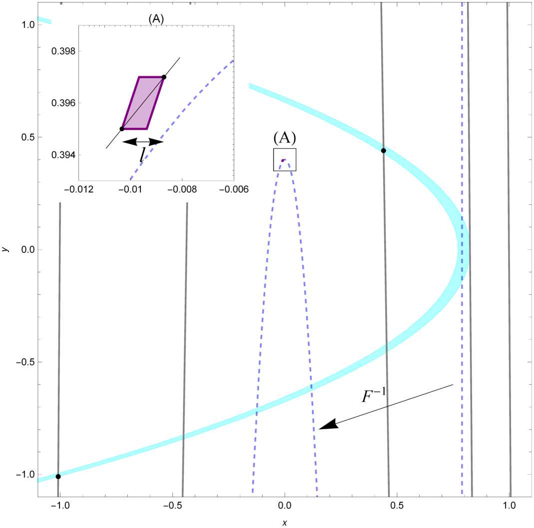
- If the horizontal endpoints of a wandering domain is parallel to the preimage, then the image of the horizontal endpoints ~ 0.



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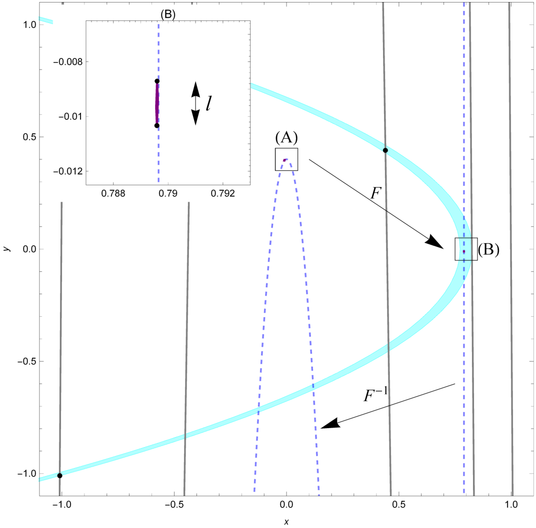
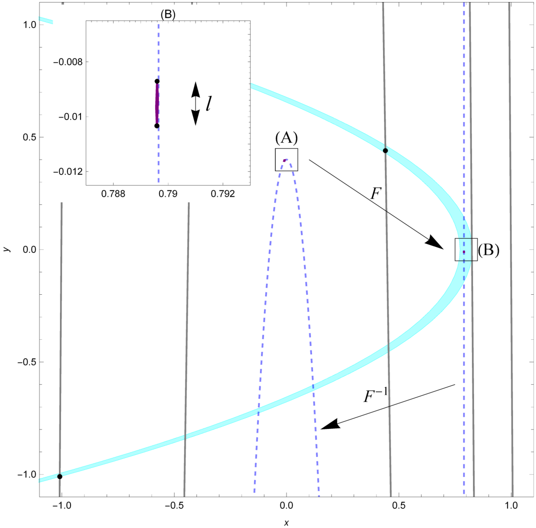


Bad News

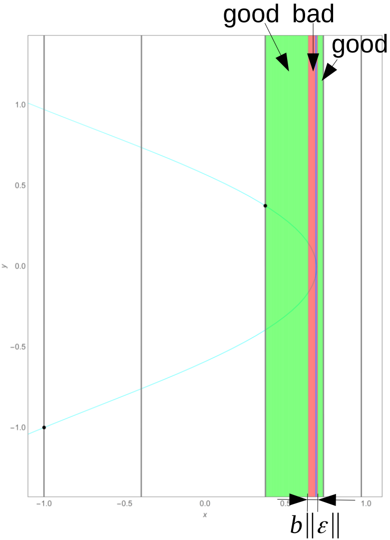
- Want to avoid "vertical line intersects the image once"
- Size of the image is $\|\epsilon_n\|$



Good and Bad regions

Let $b > 0$ be a large number and $F_n = R^n F$.

- **Bad region** in C_n:
  Vertical strip in C that covers the "intersection once" area
  Width: $\sim \|\epsilon_n\|$
  # of rescale: $\sim \|\epsilon_n\|^{-1}$



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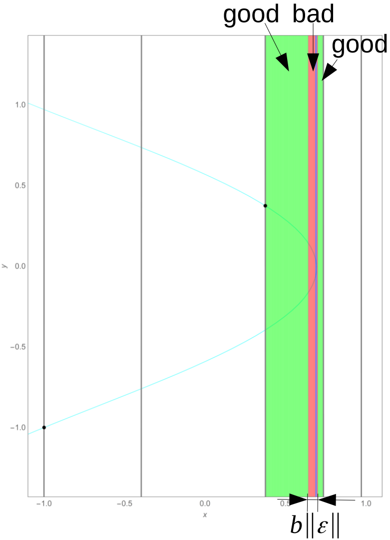

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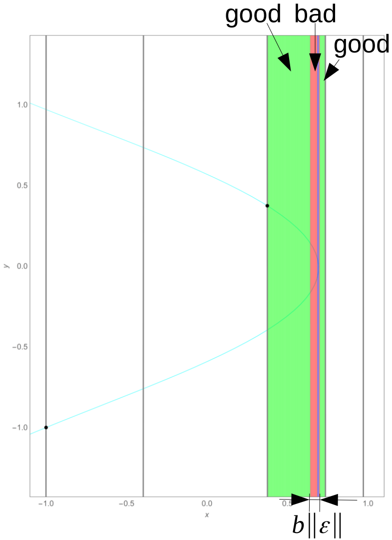
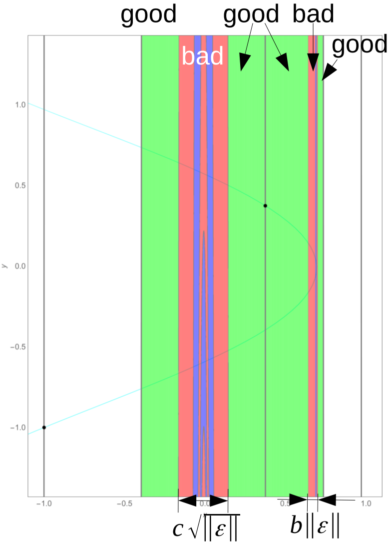
- **Good region** in C_n:
  Complement of the bad region in C_n
- "Good and Bad regions" is a dimension-two feature
  Degenerate Hénon-like maps does **NOT** have bad region



Good and Bad regions

- **Bad region** in B_n:
  Preimage of the bad region in C
  Width: $\sim \sqrt{\|\epsilon_n\|}$



- **Bad region** in B_n:
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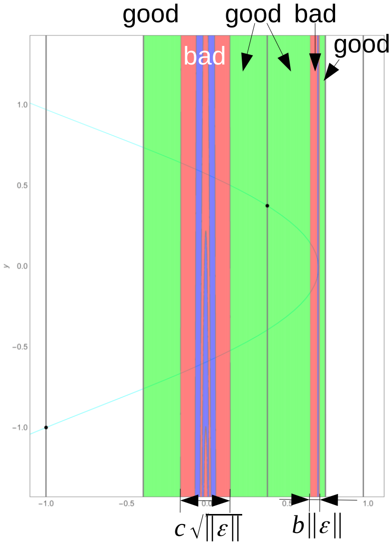
  Width: $\sim \sqrt{\|\epsilon_n\|}$

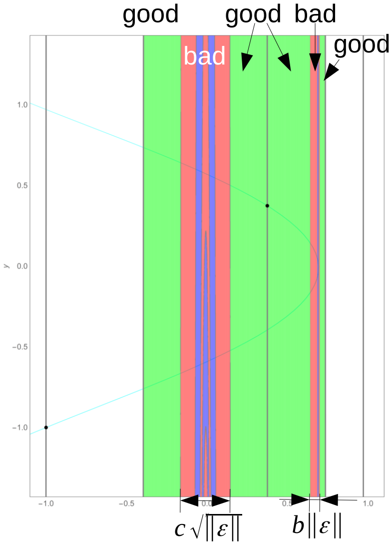
- **Good region** in B_n:
  Preimage of the good region in C



Properties for the Good and Bad regions

- **Good region** is GOOD (**like** unimodal)
  - Dynamics: Horizontal size expands

$$l_{n+1} \geq E l_n$$



- **Good region** is GOOD (**like** unimodal)
  - Dynamics: Horizontal size expands

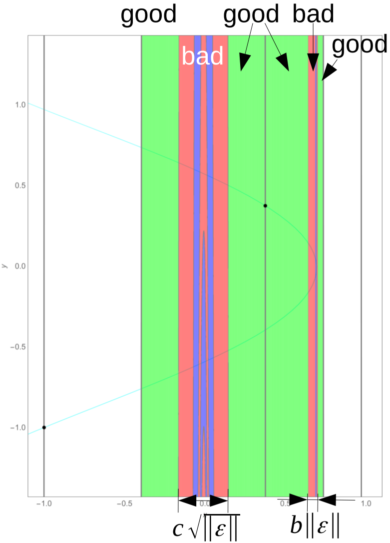
$$I_{n+1} \geq E I_n$$

- **Bad region** is BAD (**unlike** unimodal)
  - Dynamics: Horizontal size may NOT expand

$$I_{n+1} \not\geq E I_n$$

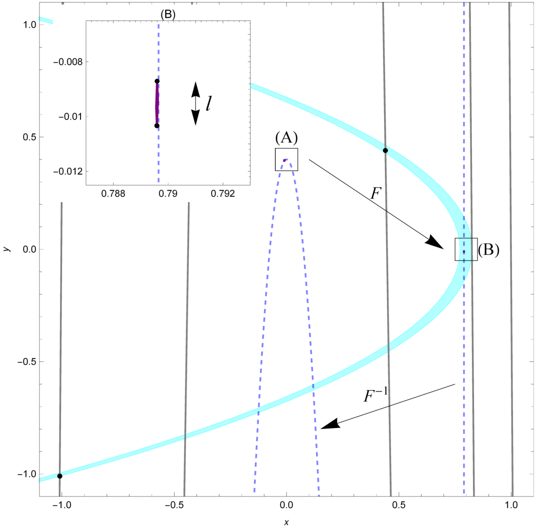
The contraction can be as strong as possible!

$$\frac{I_{n+1}}{I_n} \searrow 0$$



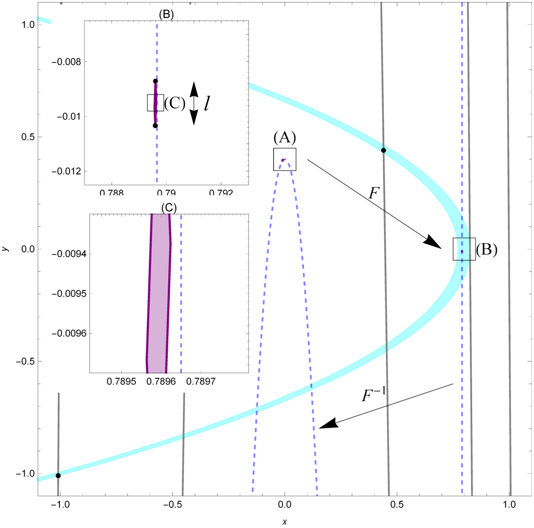
Bad News

- How **small** can the horizontal size be after the wandering domain enters the bad region?



Bad News

- How **small** can the horizontal size be after the wandering domain enters the bad region?
- area/cross-section determines the horizontal size

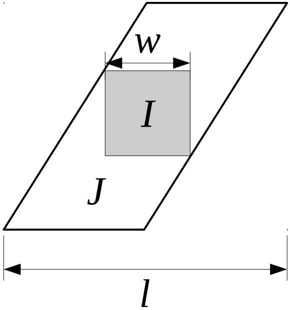


Thickness

square a closed set $I \subset \mathbb{R}^2$ with horizontal and vertical sides of the same length

thickness $w(J)$ the side of the "largest" square subset of J

- Thickness measures the size of horizontal cross-section



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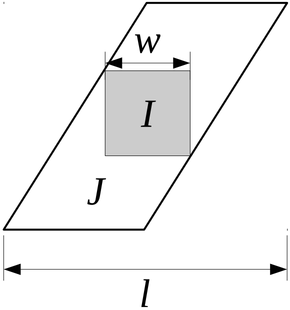

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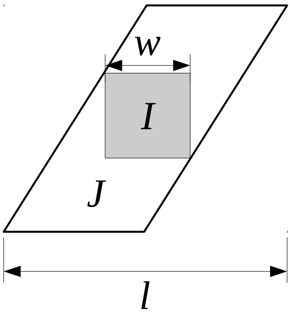
- Thickness is also a dimension-two feature (characterized by the area)



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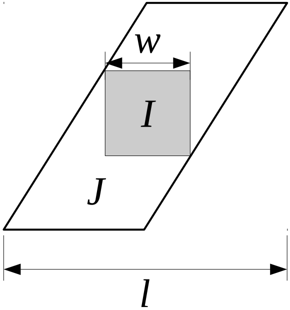
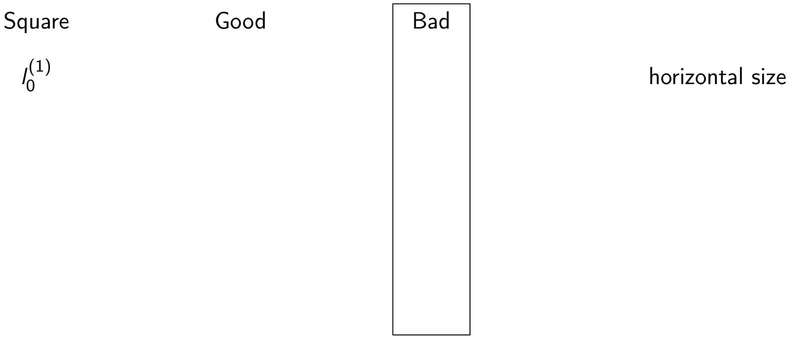
- Thickness is also a dimension-two feature (characterized by the area)
- Contraction $\cong$ Jacobian of $F \cong \|\epsilon\|$



Relation Between Horizontal Size with Thickness



- Start from a Square

Relation Between Horizontal Size with Thickness

Square	Good	Bad	
$I_0^{(1)}$ $\longrightarrow$	expands $E^{n^{(1)}}$ $\longrightarrow$	$I_{n^{(1)}}^{(1)}$	horizontal size

- Start from a Square
- Horizontal size **Expands** uniformly

Relation Between Horizontal Size with Thickness

Square | Good | Bad

$I_0^{(1)} \longrightarrow$ expands $E^{n^{(1)}} \longrightarrow I_{n^{(1)}}^{(1)} \xrightarrow{\times} I_{n^{(1)}+1}^{(1)}$ horizontal size

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad w_{n^{(1)}+1}^{(1)}$ thickness

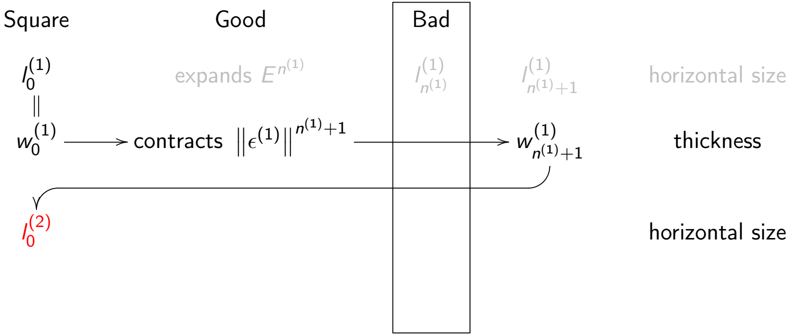
- Horizontal size false to expands
- **Thickness** determines the Horizontal size

Relation Between Horizontal Size with Thickness

Square	Good	Bad		
$I_0^{(1)}$	expands $E^{n^{(1)}}$	$I_{n^{(1)}}^{(1)}$	$I_{n^{(1)}+1}^{(1)}$	horizontal size
$\Vert$			$\Vert$	
$w_0^{(1)}$ $\longrightarrow$	contracts $\left\|\epsilon^{(1)}\right\|^{n^{(1)}+1}$	$\longrightarrow$	$w_{n^{(1)}+1}^{(1)}$	thickness

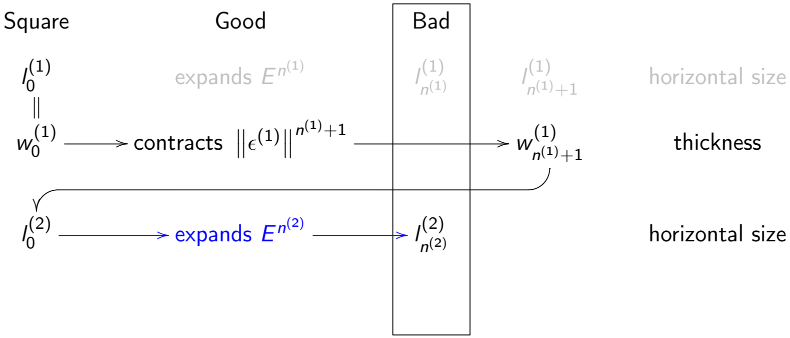
- Thickness **Contracts** by $\left\|\epsilon^{(1)}\right\|$

Relation Between Horizontal Size with Thickness



- Thickness **Contracts** by $\left\|\epsilon^{(1)}\right\|$
- Select a Largest Square and start a new row

Relation Between Horizontal Size with Thickness



- Horizontal size expands uniformly

Relation Between Horizontal Size with Thickness

Square	Good	Bad		
$I_0^{(1)}$	expands $E^{n^{(1)}}$	$I_{n^{(1)}}^{(1)}$	$I_{n^{(1)}+1}^{(1)}$	horizontal size
$\Vert$				
$w_0^{(1)} \longrightarrow$	contracts $\left\Vert\epsilon^{(1)}\right\Vert^{n^{(1)}+1}$	$\longrightarrow$	$w_{n^{(1)}+1}^{(1)}$	thickness
$I_0^{(2)}$	expands $E^{n^{(2)}}$	$I_{n^{(2)}}^{(2)}$	$\xrightarrow{\times} I_{n^{(2)}+1}^{(2)}$	horizontal size
$\Vert$			$\Vert$	
$w_0^{(2)} \longrightarrow$	contracts $\left\Vert\epsilon^{(2)}\right\Vert^{n^{(2)}+1}$	$\longrightarrow$	$w_{n^{(2)}+1}^{(2)}$	thickness

- Horizontal size is determined by the Thickness

Square	Good	Bad		
$I_0^{(1)}$	expands $E^{n^{(1)}}$	$I^{(1)}_{n^{(1)}}$	$I^{(1)}_{n^{(1)}+1}$	horizontal size
$\Vert$				
$w_0^{(1)} \longrightarrow$	contracts $\left\Vert \epsilon^{(1)} \right\Vert^{n^{(1)}+1} \longrightarrow$		$w^{(1)}_{n^{(1)}+1}$	thickness
$I_0^{(2)}$	expands $E^{n^{(2)}}$	$I^{(2)}_{n^{(2)}}$	$I^{(2)}_{n^{(2)}+1}$	horizontal size
$\Vert$				
$w_0^{(2)} \longrightarrow$	contracts $\left\Vert \epsilon^{(2)} \right\Vert^{n^{(2)}+1} \longrightarrow$		$w^{(2)}_{n^{(2)}+1}$	thickness
$I_0^{(3)} \longrightarrow \cdots$				horizontal size

- Continue to add new rows until the sequence of wandering domain stops to enter the bad region

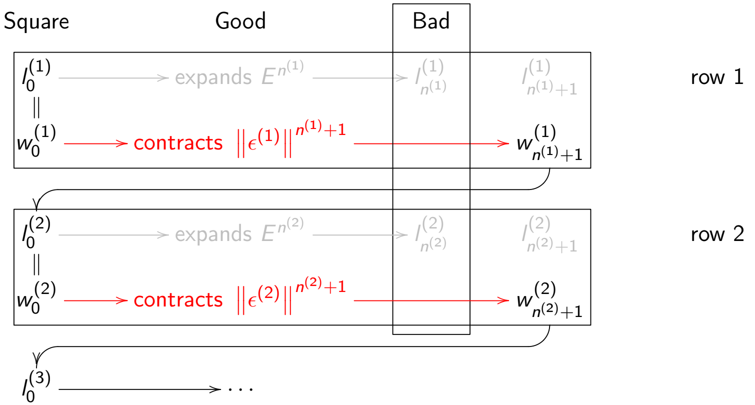
Relation Between Horizontal Size with Thickness

$$
\begin{tikzcd}[column sep=small]
\text{Square} & & \text{Good} & & \text{Bad} & & & \\
I_0^{(1)} \arrow[rr] \arrow[d, equal] & & \text{expands } E^{n^{(1)}} \arrow[rr] & & I_{n^{(1)}}^{(1)} \arrow[rr, "\times"] & & I_{n^{(1)}+1}^{(1)} \arrow[d, equal] & \text{row } 1 \\
w_0^{(1)} \arrow[r] & \text{contracts } \left\|\epsilon^{(1)}\right\|^{n^{(1)}+1} \arrow[rrrrr] & & & & & w_{n^{(1)}+1}^{(1)} \arrow[dllllll, bend left] & \\
I_0^{(2)} \arrow[rr] \arrow[d, equal] & & \text{expands } E^{n^{(2)}} \arrow[rr] & & I_{n^{(2)}}^{(2)} \arrow[rr, "\times"] & & I_{n^{(2)}+1}^{(2)} \arrow[d, equal] & \text{row } 2 \\
w_0^{(2)} \arrow[r] & \text{contracts } \left\|\epsilon^{(2)}\right\|^{n^{(2)}+1} \arrow[rrrrr] & & & & & w_{n^{(2)}+1}^{(2)} \arrow[dllllll, bend left] & \\
I_0^{(3)} \arrow[rr] & & \cdots & & & & &
\end{tikzcd}
$$

Conclusion:

- Row = Enter the Bad Region once

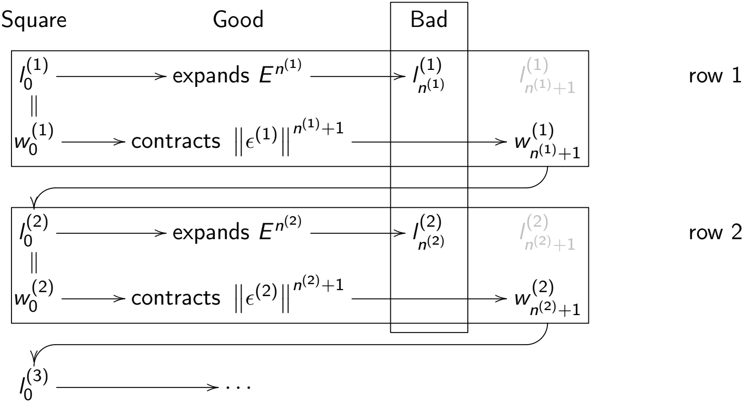
Relation Between Horizontal Size with Thickness



Conclusion:

- Row = Enter the Bad Region once
- Enter the Bad Region = strong Contraction on Horizontal Size

Relation Between Horizontal Size with Thickness



Conclusion:

- Row = Enter the Bad Region once
- Enter the Bad Region = strong Contraction on Horizontal Size

Key: can only enter the bad region at most **Finitely** many times!

Relation Between Horizontal Size with Thickness

Square | Good | Bad

row 1:

$I_0^{(1)} \longrightarrow$ expands $E^{n^{(1)}} \longrightarrow \boldsymbol{I_{n^{(1)}}^{(1)}}$ $\quad I_{n^{(1)}+1}^{(1)}$

$\|$

$w_0^{(1)} \longrightarrow$ contracts $\left\|\epsilon^{(1)}\right\|^{n^{(1)}+1} \longrightarrow w_{n^{(1)}+1}^{(1)}$

row 2:

$I_0^{(2)} \longrightarrow$ expands $E^{n^{(2)}} \longrightarrow \boldsymbol{I_{n^{(2)}}^{(2)}}$ $\quad I_{n^{(2)}+1}^{(2)}$

$\|$

$w_0^{(2)} \longrightarrow$ contracts $\left\|\epsilon^{(2)}\right\|^{n^{(2)}+1} \longrightarrow w_{n^{(2)}+1}^{(2)}$

$I_0^{(3)} \longrightarrow \cdots$

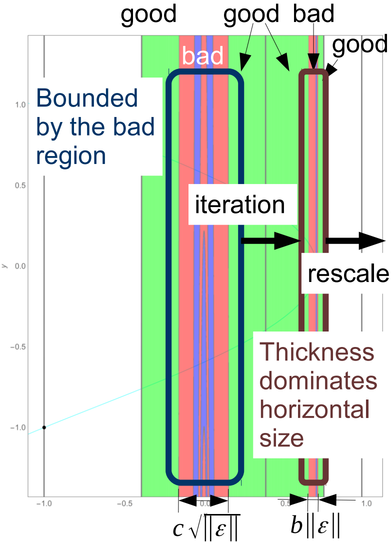
Conclusion:

- Row = Enter the Bad Region once = **Bounded** by the Bad Region
- Enter the Bad Region = strong Contraction on Horizontal Size

Key: can only enter the bad region at most **Finitely** many times!

Good and Bad regions

- **Bad region** in B_n:
  Preimage of the bad region in C
  Width: $\sim \sqrt{\|\epsilon_n\|}$
- Horizontal size is bounded by $\sqrt{\|\epsilon_n\|}$

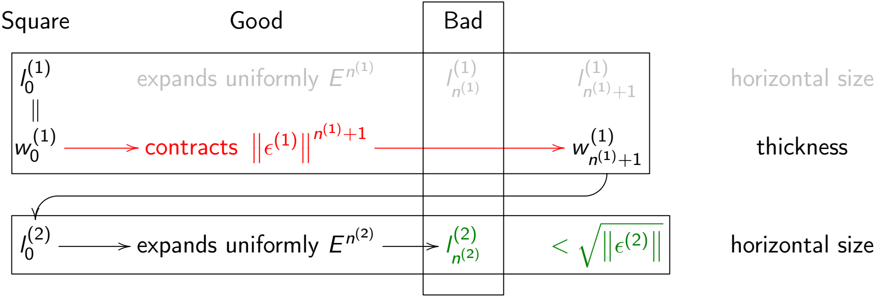


Two Row Lemma

Square	Good	Bad		
$I_0^{(1)}$	expands uniformly $E^{n^{(1)}}$	$I_{n^{(1)}}^{(1)}$	$I_{n^{(1)}+1}^{(1)}$	horizontal size
$\Vert$				
$w_0^{(1)}$	$\longrightarrow$ contracts $\left\|\epsilon^{(1)}\right\|^{n^{(1)}+1} \longrightarrow$		$w_{n^{(1)}+1}^{(1)}$	thickness
$I_0^{(2)}$	$\longrightarrow$ expands uniformly $E^{n^{(2)}} \longrightarrow$	$I_{n^{(2)}}^{(2)}$	$< \sqrt{\left\|\epsilon^{(2)}\right\|}$	horizontal size

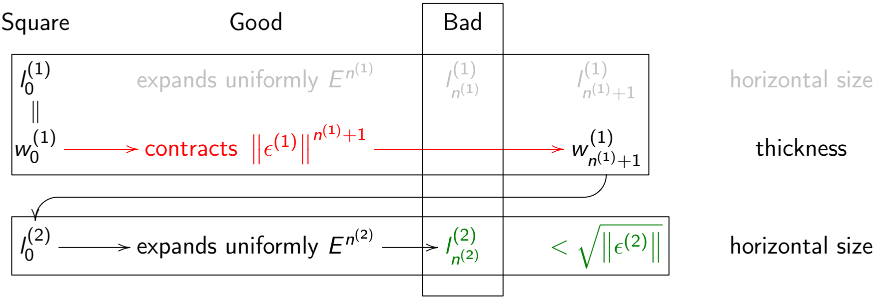
- Contraction of thickness: $\left\|\epsilon^{(1)}\right\|$

Two Row Lemma



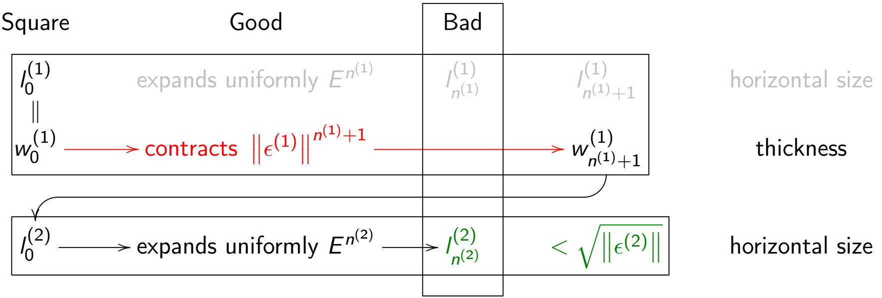
- Contraction of thickness: $\left\|\epsilon^{(1)}\right\|$
- Size of Bad Region: $\sqrt{\left\|\epsilon^{(2)}\right\|} = \left\|\epsilon^{(1)}\right\|^{2^{\#\text{ of rescaling}}}$

Two Row Lemma



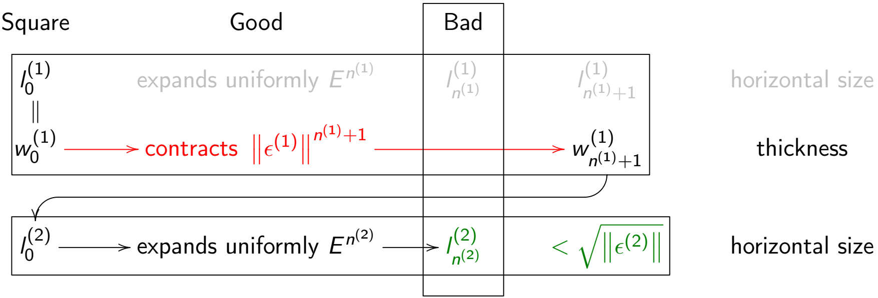
- Contraction of thickness: $\|\epsilon^{(1)}\|$
- Size of Bad Region: $\sqrt{\|\epsilon^{(2)}\|} = \|\epsilon^{(1)}\|^{\|\epsilon^{(1)}\|^{-1}}$

Two Row Lemma



- Contraction of thickness: $\left\|\epsilon^{(1)}\right\|$
- Contraction of the bad region: $\dfrac{\sqrt{\left\|\epsilon^{(2)}\right\|}}{\sqrt{\left\|\epsilon^{(1)}\right\|}} = \left\|\epsilon^{(1)}\right\|^{\left\|\epsilon^{(1)}\right\|^{-1} - \frac{1}{2}}$

Two Row Lemma



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- Roughly speaking: Bad Region contracts **Faster** than the Thickness

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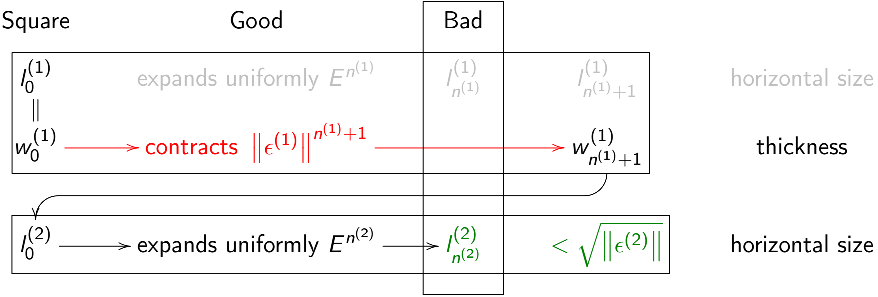

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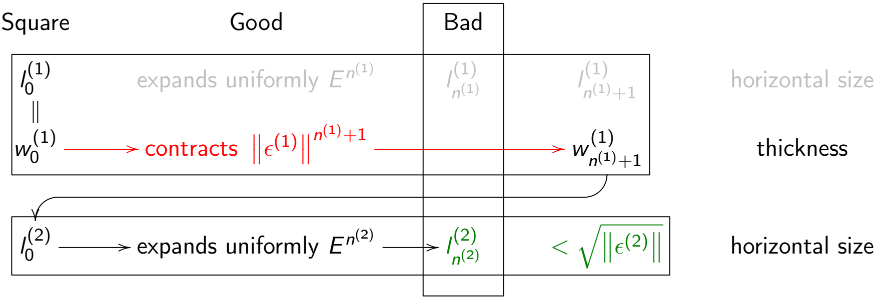
- Roughly speaking: Bad Region contracts **Faster** than the Thickness
- Need to introduce $n^{(1)}$: time span in the good region

Two Row Lemma



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- $n^{(1)} \geq \infty$ if enters the bad region ∞ times (contradiction)

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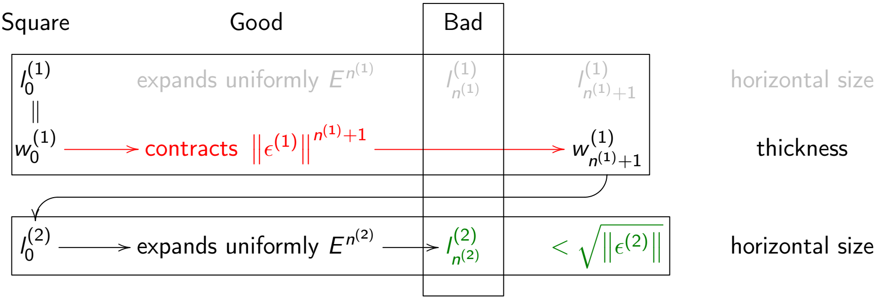

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- Roughly speaking: Bad Region contracts **Faster** than the Thickness
- $n^{(1)} \geq \infty$ if enters the bad region ∞ times (contradiction)
- This shows: enter the bad region at most **Finitely** many times!

Nonexistence of wandering domain

Summary

- Prove by contradiction: assume a wandering domain J

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Nonexistence of wandering domain

Summary

- Prove by contradiction: assume a wandering domain J
- Generate a sequence of wandering domain J_n
- Study the change of horizontal size l_n
  - Good region: l_n expands (∞ times)
  - Bad region: l_n contracts (finite times)

Nonexistence of wandering domain

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- Prove by contradiction: assume a wandering domain J
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- Study the change of horizontal size l_n
    - Good region: l_n expands (∞ times)
    - Bad region: l_n contracts (finite times)
- $l_n \to \infty$ contradiction

Nonexistence of wandering domain

Summary

- Prove by contradiction: assume a wandering domain J
- Generate a sequence of wandering domain J_n
- Study the change of horizontal size I_n
  - Good region: I_n expands (∞ times)
  - Bad region: I_n contracts (finite times)
- $I_n \to \infty$ contradiction

Theorem (Main Theorem, arXiv:1705.05036)

A strongly dissipative infinite (period-doubling) renormalizable Hénon-like map does not have wandering domains