鬼脚圖

1. Describe how this game plays.

(1) Everyone can walk along with the vertical line which he chooses and moving from the bottom to the top. He needs to turn to the other vertical line and keep going up when he meets the horizontal line. The rule will be repeated until he reaches the goal.

(2) In the ghost foot diagram, every horizontal line represents to a change with the position, and the horizontal line will only appear between two vertical lines. It won't be cross among three vertical lines at the same time.

(3) When we draw the ghost foot diagram, there are N initial positions which can be selected by N players.

2. Construct a mathematical model.

From the rules of the game, we can see that it is a kind of relation "$\to$" mapping from the top to the bottom. If *m* connects to *n* in the game, we can say $m \to n$. At first, the game starts from a relation $1 \to 1, 2 \to 2$, …, $N \to N$. The relation is actually a permutation $\begin{pmatrix} 1 & 2 & ... & N \\ 1 & 2 & ... & N \end{pmatrix}$ from the top to the bottom. As we start adding some bars in the game, the relation will change. If we add a bar between *k* and *k+1*, the path of *k* and *k+1* will exchange. As a result, we can define the line between *k* and *k+1* as $a_{k,k+1} = \begin{pmatrix} k & k+1 \\ k+1 & k \end{pmatrix} = (k, k+1)$. Hence, $a_{k,k+1}$ is also a permutation. As we finish adding lines, the relation r mapping from the top to the bottom can be expressed as

$r = a_{t_n,t_n+1} \circ a_{t_{n-1},t_{n-1}+1} \circ ... \circ a_{t_1,t_1+1}$, where a_{t_1,t_1+1} is highest line and a_{t_n,t_n+1} is the lowest line in the game (Figure 1). Thus, r is also a permutation. Let G_N be all kinds of relation in the game and $a_{1,2}, a_{2,3}, ..., a_{N-1,N}$ are the generators of G_N, we can get the theorem.

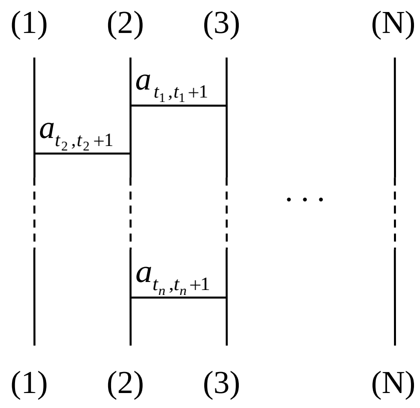

Figure 1

Theorem 1. G_N is a group of permutations with generators .

Since G_N is a group of permutations, we can see that the mapping from the top to the bottom is always one to one and onto.

There is also another kind of representation of the elements of G_N. Let $\phi_{k,k+1}(x)$ be the line of exchange after the relation N. Then, $r \in G_N$ can be expressed as $r = \phi_{t_n,t_n+1}(\phi_{t_{n-1},t_{n-1}+1}(\ldots\phi_{t_1,t_1+1}(e)\ldots))$. By Cayley's Theorem, G_N is isomorphic to a group of permutations of G_N. By letting $\phi_{k,k+1}(x) = a_{k,k+1}x$, we can see actually the two kinds of representation are the same thing. In convenience, we will use the first kind of representation in the rest of the test.

Example.

For $N = 3$, $G_3 = \{g_n\}_{n=1}^{6}$ where $g_1 = e$, $g_2 = (1,2)$, $g_3 = (2,3)$, $g_4 = (1,3)$, $g_5 = (1,2,3)$, $g_6 = (1,3,2)$. Then,

for x=1

$a_{1,2} \bullet 1 = 2$
$a_{2,3} \bullet 2 = 3$
$a_{1,2} \bullet 2 = 1$
$a_{3,4} \bullet 1 = 2$
$a_{2,3} \bullet 3 = 2$
$a_{3,4} \bullet 2 = 1$

$$\Rightarrow \quad \begin{array}{l} \phi_{1,2}(x) = a_{2,3} \bullet a_{1,2} \bullet 1 = 3 \\ \phi_{2,3}(x) = a_{3,4} \bullet a_{1,2} \bullet 2 = 2 \\ \phi_{1,2}(x) = a_{3,4} \bullet a_{1,2} \bullet 2 = 2 \\ a_{3,4} \bullet a_{1,2} \bullet 3 = 1 \end{array}$$

So, we can get a permutation $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

3. Can the generators generate S_N?

From the beginning, we had already shown that G_N is a subgroup of S_N. We want to see if the generators of G_N can spans S_N. We will start the discussing by observing the simplest case G_3. We have the generator (1,2), (2,3). How can we make a permutation (1,3) by the generators? It is simple. As figure 2 illustrates, we can see $1 \to 3$, $3 \to 1$, and $2 \to 2$. That is (1,3)=(1,2)(2,2)(1,2). As far as we can construct all of the transposition (1,2), (2,3), and (1,3) in S_3, by corollary 9.12 in the text book, we can construct all elements in S_3. Thus, all permutations can be constructed by the game; $G_3 = S_3$.

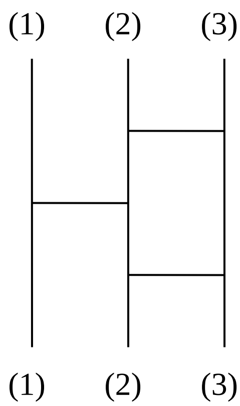

Figure 2

Next we can generalize the result.

Lemma 2. $(m,n) \in G_N$

Proof: We need to show that the generator of G_N can generates (m,n). We use the idea of figure 3 to prove this.

We can assume that m<n<N.

When n=m+1; (m,n) is a generator of G.

If n=k<N, $(m,n)\in G_N$

When n=k+1

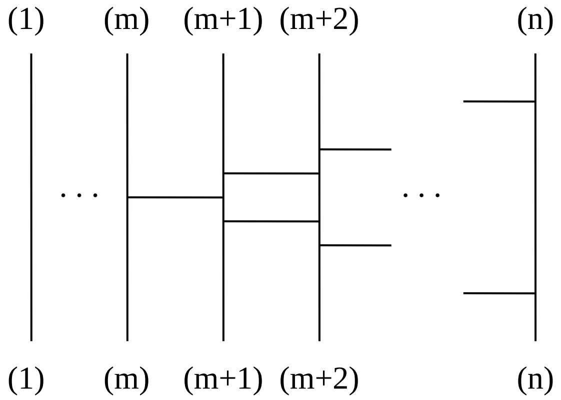

Figure 3

$\because k+1<N$, $\therefore k+1\le N \Rightarrow (k,k+1)$ is a generators of G_N.

$$
\begin{aligned}
(k,k+1)(m,k)(k,k+1) &= \begin{pmatrix} m & k & k+1 \\ m & k+1 & k \end{pmatrix}\begin{pmatrix} m & k & k+1 \\ k & m & k+1 \end{pmatrix}\begin{pmatrix} m & k & k+1 \\ m & k+1 & k \end{pmatrix} \\
&= \begin{pmatrix} m & k & k+1 \\ m & k+1 & k \end{pmatrix}\begin{pmatrix} m & k & k+1 \\ k & k+1 & m \end{pmatrix} \\
&= \begin{pmatrix} m & k & k+1 \\ k+1 & k & m \end{pmatrix} \\
&= (m,k+1) \\
&= (m,n)
\end{aligned}
$$

$\because (m,k),(k,k+1)$ can be generated by the generators of G_N,

$\therefore (m,n)\in G_N$

By principle of mathematical induction, $(m,n)\in G_N$ for $m,n\le N$. Q.E.D.

By corollary 9.12 in the text book, we can get

Theorem 3. $G_N = S_N$.

4. Number of methods to construct $a_{j,k}$.

By part 3 in the preceding text, we had shown that every kind of permutation can be constructed by the generators. Then we will want to ask, what is the minimum number of bars to construct a certain permutation? For a certain numbers of bars, how many kinds of methods are there to construct a certain permutation? First we will derive a recurrent formula for the function of number of bars and a certain permutation mapping to the number of method to construct the permutation.

Let $n_t(a)$ be the number of methods for constructing the permutation a when t bars had occurred in the game. Since the last bar is $a_{k,k+1}$, when we omitted the last bar, then the number of methods will be $n_{t-1}(a \circ a_{k,k+1}{}^{-1})$. The total number of method $n_t(a)$ will be the sum of number of methods omitted the all kinds of last lines. Thus, we can get a recurrent relation

(1) $$n_t(a) = n_{t-1}(a \circ a_{1,2}{}^{-1}) + n_{t-1}(a \circ a_{2,3}{}^{-1}) + \ldots + n_{t-1}(a \circ a_{N-1,N}{}^{-1})$$

for all $t > 1$. Then we need the initial conditions, which is when $t = 1$, for calculating the latter terms. The initial conditions are

(2) $n_1(a_{k,k+1}) = 1$ for all k, and $n_1(a) = 0$ for others.

For simplification in the latter discussion, we can also get

(3) $$n_t(a) = \sum_{g \in G_N} n_{t-1}(a \circ g^{-1}) \times n_1(g) .$$

Let $k = a \circ g^{-1}$, it follows that $g = k^{-1} \circ a$. Hence,

(4)
$$\begin{aligned} n_t(a) &= \sum_{k^{-1}\circ a\in G_N} n_{t-1}(k)\times n_1(k^{-1}\circ a) \\ &= \sum_{k\in G_N} n_{t-1}(k)\times n_1(k^{-1}\circ a). \end{aligned}$$

Lemma 4. $n_t(a) = \sum_{k\in G_N} n_{t-1}(k)\times n_1(k^{-1}\circ a).$

Let $g_1, g_2, ..., g_{N!}$ be the elements of G_N. Then by lemma 4, we will get the equations

$$\begin{cases} n_t(g_1) = n_1(g_1^{-1}\circ g_1)\times n_{t-1}(g_1) + n_1(g_2^{-1}\circ g_1)\times n_{t-1}(g_2) + ... + n_1(g_{N!}^{-1}\circ g_1)\times n_{t-1}(g_{N!}) \\ n_t(g_2) = n_1(g_1^{-1}\circ g_2)\times n_{t-1}(g_1) + n_1(g_2^{-1}\circ g_2)\times n_{t-1}(g_2) + ... + n_1(g_{N!}^{-1}\circ g_2)\times n_{t-1}(g_{N!}) \\ \qquad\vdots \\ n_t(g_{N!}) = n_1(g_1^{-1}\circ g_{N!})\times n_{t-1}(g_1) + n_1(g_2^{-1}\circ g_{N!})\times n_{t-1}(g_2) + ... + n_1(g_{N!}^{-1}\circ g_{N!})\times n_{t-1}(g_{N!}) \end{cases}$$

Let $M = \begin{pmatrix} n_1(g_1^{-1}\circ g_1) & n_1(g_2^{-1}\circ g_1) & ... & n_1(g_{N!}^{-1}\circ g_1) \\ n_1(g_1^{-1}\circ g_2) & n_1(g_2^{-1}\circ g_2) & ... & n_1(g_{N!}^{-1}\circ g_2) \\ \vdots & \vdots & \vdots & \vdots \\ n_1(g_1^{-1}\circ g_{N!}) & n_1(g_1^{-1}\circ g_{N!}) & ... & n_1(g_1^{-1}\circ g_{N!}) \end{pmatrix}$ and $m_t = \begin{pmatrix} n_t(g_1) \\ n_t(g_2) \\ \vdots \\ n_t(g_{N!}) \end{pmatrix}$,

then we can express the equations as matrix. That is, $m_t = Mm_{t-1}$. Thus, we can get

Theorem 5. $m_t = M^{t-1}m_1$, where $m_1 = \left(n_1(g_1), n_1(g_2), \ldots, n_1(g_{N!})\right)^t$.

If the matrix M is Hermitian (in our case, M is usually Hermitian because the diagram is symmetric), we can find a diagonal matrix D with the entries $d_1, d_2, \ldots, d_{N!}$, by using the eigenvalue method, such that $M = ADA^{-1}$ for some invertible matrix A, then it will be easier to calculate m_t. Hence,

(5)
$$\begin{aligned} m_t &= M^{t-1}m_1 = (ADA^{-1})^{t-1}m_1 = AD^{t-1}Am_1 \\ &= A\begin{pmatrix} d_1^{\,t-1} & 0 & \cdots & 0 \\ 0 & d_2^{\,t-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{N!}^{\,t-1} \end{pmatrix} A^{-1}m_1. \end{aligned}$$

Therefore, we can find the function $n_t(g)$ for all $g \in G_{N!}$.

Example:

For $N = 3$, $G_3 = \{g_n\}_{n=1}^{6}$ where $g_1 = e$, $g_2 = (1,2)$, $g_3 = (2,3)$, $g_4 = (1,3)$, $g_5 = (1,2,3)$, $g_6 = (1,3,2)$. Then

$$
\begin{aligned}
m_1 &= \left(n_1(g_1), n_1(g_2), n_1(g_3), n_1(g_4), n_1(g_5), n_1(g_6)\right)^t \\
&= \left(n_1(e), n_1((1,2)), n_1((2,3)), n_1((1,3)), n_1((1,2,3)), n_1((1,3,2))\right)^t \\
&= (0,1,1,0,0,0)
\end{aligned}
$$

$$
\begin{aligned}
M &= \begin{pmatrix}
n_1(g_1^{-1}\circ g_1) & n_1(g_2^{-1}\circ g_1) & n_1(g_3^{-1}\circ g_1) & n_1(g_4^{-1}\circ g_1) & n_1(g_5^{-1}\circ g_1) & n_1(g_6^{-1}\circ g_1) \\
n_1(g_1^{-1}\circ g_2) & n_1(g_2^{-1}\circ g_2) & n_1(g_3^{-1}\circ g_2) & n_1(g_4^{-1}\circ g_2) & n_1(g_5^{-1}\circ g_2) & n_1(g_6^{-1}\circ g_2) \\
n_1(g_1^{-1}\circ g_3) & n_1(g_2^{-1}\circ g_3) & n_1(g_3^{-1}\circ g_3) & n_1(g_4^{-1}\circ g_3) & n_1(g_5^{-1}\circ g_3) & n_1(g_6^{-1}\circ g_3) \\
n_1(g_1^{-1}\circ g_4) & n_1(g_2^{-1}\circ g_4) & n_1(g_3^{-1}\circ g_4) & n_1(g_4^{-1}\circ g_4) & n_1(g_5^{-1}\circ g_4) & n_1(g_6^{-1}\circ g_4) \\
n_1(g_1^{-1}\circ g_5) & n_1(g_2^{-1}\circ g_5) & n_1(g_3^{-1}\circ g_5) & n_1(g_4^{-1}\circ g_5) & n_1(g_5^{-1}\circ g_5) & n_1(g_6^{-1}\circ g_5) \\
n_1(g_1^{-1}\circ g_6) & n_1(g_2^{-1}\circ g_6) & n_1(g_3^{-1}\circ g_6) & n_1(g_4^{-1}\circ g_6) & n_1(g_5^{-1}\circ g_6) & n_1(g_6^{-1}\circ g_6)
\end{pmatrix} \\
&= \begin{pmatrix}
n_1(e) & n_1((1,2)) & n_1((2,3)) & n_1((1,3)) & n_1((1,3,2)) & n_1((1,2,3)) \\
n_1((1,2)) & n_1(e) & n_1((1,3,2)) & n_1((1,2,3)) & n_1((2,3)) & n_1((1,3)) \\
n_1((2,3)) & n_1((1,2,3)) & n_1(e) & n_1((1,3,2)) & n_1((1,3)) & n_1((1,2)) \\
n_1((1,3)) & n_1((1,3,2)) & n_1((1,2,3)) & n_1(e) & n_1((1,2)) & n_1((2,3)) \\
n_1((1,2,3)) & n_1((2,3)) & n_1((1,3)) & n_1((1,2)) & n_1(e) & n_1((1,3,2)) \\
n_1((1,3,2)) & n_1((1,3)) & n_1((1,2)) & n_1((2,3)) & n_1((1,2,3)) & n_1(e)
\end{pmatrix} \\
&= \begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0
\end{pmatrix}
\end{aligned}
$$

Since -2, -1, -1, 1, 1, 2 are the eigenvalues of the matrix M, and the corresponding eigenvectors are $(1,-1,-1,-1,1,1)^t$, $(-1,1,0,-1,0,1)^t$, $(-1,0,1,-1,1,0)$, $(-1,-1,0,1,0,1)^t$, $(-1,0,-1,1,1,0)^t$, $(1,1,1,1,1,1)^t$. We can let

$$A=\begin{pmatrix}1&-1&-1&-1&-1&1\\-1&1&0&-1&0&1\\-1&0&1&0&-1&1\\-1&-1&-1&1&1&1\\1&0&1&0&1&1\\1&1&0&1&0&1\end{pmatrix}.$$

Because M is a Hermitian matrix, the eigenvectors form a base of V. So A is an invertible matrix. Hence, we can write M as

$$M=ADA^{-1}$$

$$=\begin{pmatrix}1&-1&-1&-1&-1&1\\-1&1&0&-1&0&1\\-1&0&1&0&-1&1\\-1&-1&-1&1&1&1\\1&0&1&0&1&1\\1&1&0&1&0&1\end{pmatrix}\begin{pmatrix}-2&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&2\end{pmatrix}\begin{pmatrix}1&-1&-1&-1&-1&1\\-1&1&0&-1&0&1\\-1&0&1&0&-1&1\\-1&-1&-1&1&1&1\\1&0&1&0&1&1\\1&1&0&1&0&1\end{pmatrix}^{-1}$$

Therefore,

$$m_t = M^{t-1}m_1 = (ADA^{-1})^{t-1}m_1 = AD^{t-1}A^{-1}m_1$$

$$= \begin{pmatrix} 1 & -1 & -1 & -1 & -1 & 1 \\ -1 & 1 & 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}^{t-1} \begin{pmatrix} 1 & -1 & -1 & -1 & -1 & 1 \\ -1 & 1 & 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}^{-1} m_1$$

$$= \begin{pmatrix} 1 & -1 & -1 & -1 & -1 & 1 \\ -1 & 1 & 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} (-2)^{t-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & (-1)^{t-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & (-1)^{t-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2^{t-1} \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{6} \begin{pmatrix} (2^t + 2)(1 + (-1)^t) \\ (2^t + 1)(1 - (-1)^t) \\ (2^t + 1)(1 - (-1)^t) \\ (2^t - 2)(1 - (-1)^t) \\ (2^t - 1)(1 + (-1)^t) \\ (2^t - 1)(1 + (-1)^t) \end{pmatrix}$$

5. The probability for a permutation.

Let $p_t(a)$ be the probabilities of making a permutation a by t bars. Similarly, we can get

Theorem 6. $m_t = M^{t-1} m_1$, where $m_1 = \left(p_1(g_1), p_1(g_2), \ldots, p_1(g_{N!})\right)^t$ is the initial condition,

and $m_t = \begin{pmatrix} p_t(g_1) \\ p_t(g_2) \\ \vdots \\ p_t(g_{N!}) \end{pmatrix}$, $M = \begin{pmatrix} p_1(g_1^{-1} \circ g_1) & p_1(g_2^{-1} \circ g_1) & \cdots & p_1(g_{N!}^{-1} \circ g_1) \\ p_1(g_1^{-1} \circ g_2) & p_1(g_2^{-1} \circ g_2) & \cdots & p_1(g_{N!}^{-1} \circ g_2) \\ \vdots & \vdots & \vdots & \vdots \\ p_1(g_1^{-1} \circ g_{N!}) & p_1(g_1^{-1} \circ g_{N!}) & \cdots & p_1(g_1^{-1} \circ g_{N!}) \end{pmatrix}$.

Example:

6. The probability for a relation.

$$r_t(j,k) = \sum_{g \in G_N, g(j)=k} p_t(g)$$

7. Generalization of this game.

Same as 4,5,6.