Nonexistence of Wandering domain for strongly dissipative infinite renormalizable Hénon-like maps

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Wandering Interval

Definition (Wandering interval)

Let $I \subset \mathbb{R}$ (or $I = \mathbb{S}^1$) and $f : I \to I$ be continuous. A nonempty connected open subset $J \subset I$ is a wandering interval of f if

1. $f^m(J) \cap f^n(J) = \phi$ for all $m \neq n$
2. the orbit of J does not tends to a periodic orbit

Characteristic:

- The orbit wanders around in the domain
- Does not tend to go back to itself

Circle Rotation

$f : \mathbb{S}^1 \to \mathbb{S}^1$ orientation preserving homeomorphism

- Rotation number

$$\rho(f) = \lim_{n\to\infty} \frac{F^n(x) - x}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \left[F^j(x) - F^{j-1}(x)\right]$$

where $x \in \mathbb{R}$ and F is the lift of f to $\mathbb{R}$.

Circle Rotation

$f : \mathbb{S}^1 \to \mathbb{S}^1$ orientation preserving homeomorphism

- Rigid rotation: $R_\theta(x) = x + \theta (\mod \mathbb{S}^1)$
- [1932, Denjoy]: $f \in C^{1+bv} \Longrightarrow$ no wandering interval

Theorem (Poincaré)

Assume f has irrational rotation number. Then f is conjugate to a rigid rotation iff it has no wandering interval

- R_θ is a universal model for circle rotation

Real Dimension One Results

1932 Denjoy	$\mathbb{S}^1$	C^{1+bv} diffeomorphism (non-existence) (exist: C^1)
1979 Gackerheimer	unimodal	negative Schwarzian derivative & no inflection points
1984 Yoccoz	$\mathbb{S}^1$	C^∞ diffeomorphisms & non-flat critical points
1988 de Melo, 1989 Van Strien	unimodal	C^∞ & non-flat critical points or Misiurenicz condition
1989 Lyubich, 1989 Blokh and Lyubich	multi modal	C^∞ & all critical points are turning points
1992 Martens, de Melo, and Van Strien	$\mathbb{S}^1$ or I cpt	C^2 & finite number of critical points & non-flat

(may not be complete)

Sullivan's No-Wandering-Domain Theorem

Definition

A Fatou component is wandering if it is not eventually periodic.

Sullivan's No-Wandering-Domain Theorem

Theorem (1985 Sullivan)

Assume f is a rational map on the Riemann sphere with $deg(f) \geq 2$. Every Fatou component is eventually periodic.

Classification of periodic Fatou components U

1. U contains an attracting periodic point
2. U is parabolic
3. U is a Siegel disc
4. U is a Herman ring

Complex Dimension One Results

Non-existence:

1985 Sullivan	$\hat{\mathbb{C}}$	rational map with degree ≥ 2
1992 Eremenko, Lyubich 1986 Goldberg, Keen	$\mathbb{C}$	entire map with finitely many singular values

Existence:

1976 Baker	$\mathbb{C}$	entire map
1984 Herman	$\mathbb{C}$	entire map
1985 Sullivan	$\mathbb{C}$	entire map
1987 Eremenko, Lyubich	$\mathbb{C}$	entire map
2015 Bishop	$\mathbb{C}$	entire map with bounded singular set

Higher Dimension Results

1989 Norton	cpt $\mathbb{C}^2$ manifold	non-existence: area non-contracting on boundary
1993 McSwiggen	2-torus	existence: $C^{3-\epsilon}$ diffeomorphism
1994 Bonatti, Gambaudo, Lion, Tresser	2-disk	non-existence: $C^{1+\alpha}$ infinitely renormalizable diffeomorphism existence: C^1
2001 Colli, Vargas	cpt $\mathbb{C}^1$ surface	existence: C^∞ diffeomorphisms
2016 Astorg, Buff, Dujardin, Peters, Raissy	$\mathbb{C}^2$	existence: polynomial affine type
2016 Kiriki, Soma (arxiv)	$\mathbb{R}^2$	C^r-existence: diffeomorphisms, C^r-close to some Hénon maps ($r < \infty$)

(may not be complete)

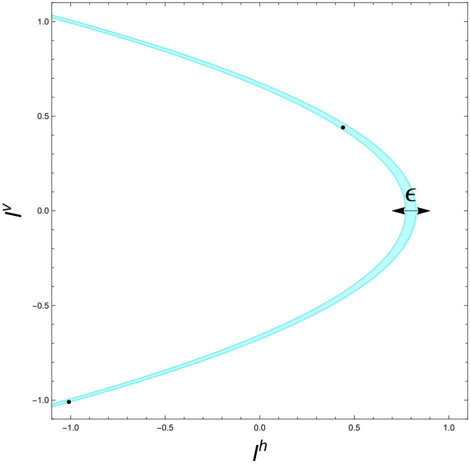
Hénon-like Maps

Hénon-like map = Unimodal map + perturbation

A Hénon-like map is a continuous map $F : I^h \times I^v \to \mathbb{R}^2$ with the form

$$\begin{aligned} F(x,y) &= (h_y(x), x) \\ &= (\underbrace{f(x)}_{\text{unimodal}} - \underbrace{\epsilon(x,y)}_{\text{pertubation}}, x) \end{aligned}$$

In this talk, F satisfies some regularity conditions that allows us to do renormalization



Examples

- A Hénon-like map

$$F(x,y) = (\underbrace{f(x)}_{\text{unimodal}} - \underbrace{\epsilon(x,y)}_{\text{pertubation}}, x)$$

- **Degenerate (unimodal) case**: $\epsilon = 0$
  Reduces to the one-dimension unimodal map.
  The behavior is fully described by the dynamics of the unimodal map.
- **Classical Hénon family**:
  $f(x)$: quadratic polynomial
  $\epsilon(x,y) = by$, b is a constant

Hénon-like Maps

Theorem (2011, M. Lyubich and M. Martens)

F: strongly dissipative infinite (period-doubling) renormalizable Hénon-like map. Then $\omega(x)$ is one of the following

1. *periodic orbit of period 2^n*
2. *attracting Cantor set*

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  1. periodic point attractor, or
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  - Size of the stable sets of the periodic points?

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Theorem (2011, M. Lyubich and M. Martens)

There exist Hénon-like maps such that the stable lamination is nowhere laminar

Hénon-like Maps

Definition

Assume that $F : D \to D$ is a non-degenerate Hénon-like map. A nonempty connected open set $J \subset D$ is a **wandering domain** of F if J does not intersect any of the stable sets of periodic points.

This means that a wandering domain is an open subset of the Cantor attractor.

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Theorem (2016, Ou)

A strongly dissipative infinite renormalizable Hénon-like map does not have wandering domains.

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Theorem (2016, Ou)

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Corollary

The union of the stable set of the periodic points is dense.

Dynamics on the Partition

- A Hénon-like map $F : I^h \times I^v \to \mathbb{R}^2$

$$F(x,y) = (\underbrace{f(x)}_{\text{unimodal}} - \underbrace{\epsilon(x,y)}_{\text{pertubation}}, x)$$

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- Study the dynamics: need a domain D that makes F a self-map
- Renormalization: First returning map on a subset $C \subset D$
- Define a partition on a subset of $I^h \times I^v$ by using the local stable manifolds of the two saddle fixed points

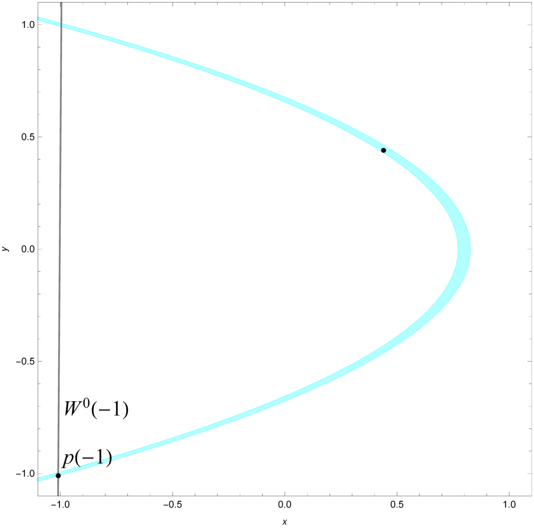
Partition for Hénon-like maps

Assume that ϵ is small

$F(x,y) = (f(x) - \epsilon(x,y), x)$

$W^0(-1)$	connected component of the stable set for $p(-1)$ that contains $p(-1)$
$W^2(-1)$	preimage of $W^0(-1)$
D	$W^0(-1) \sim W^2(-1)$

[2011, M. Lyubich and M. Martens]



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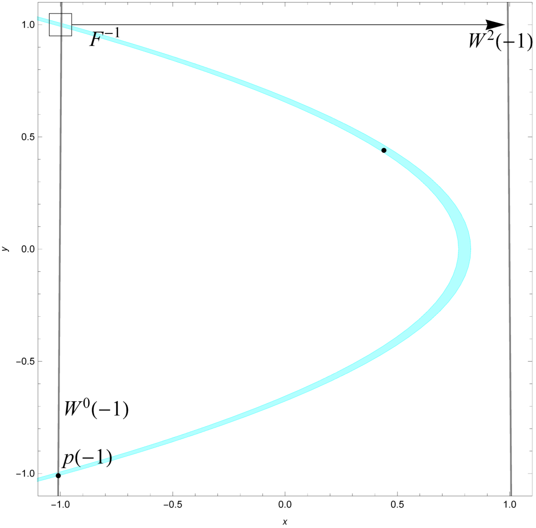

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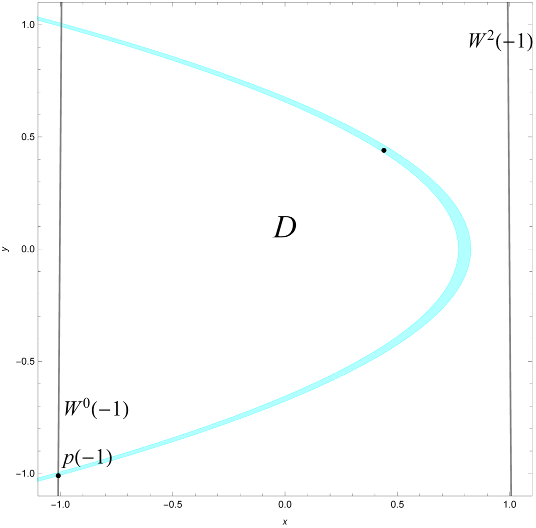
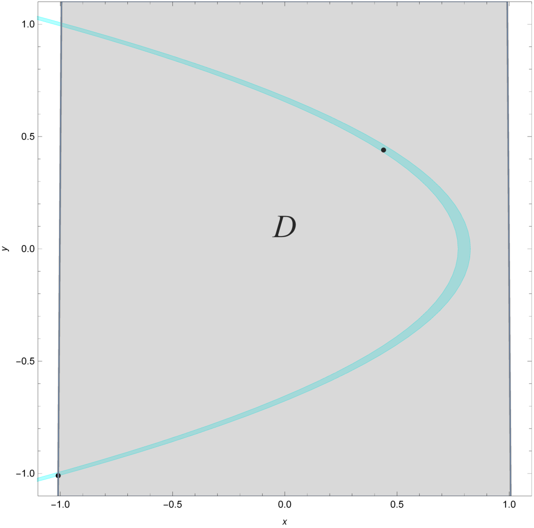


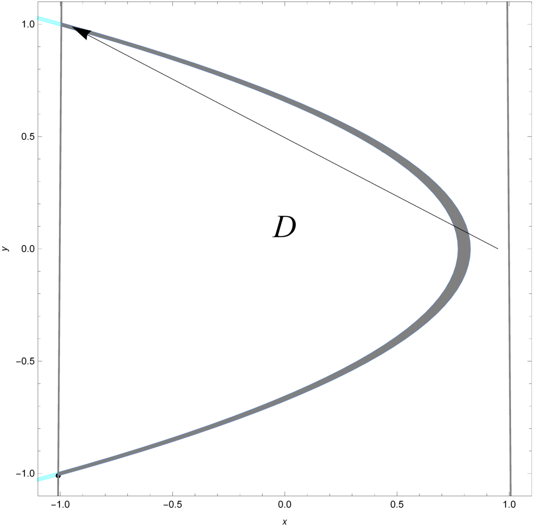
Dynamics on the Partition

D



Dynamics on the Partition

$\boldsymbol{D} \to \boldsymbol{D}$
F is a self map on D



Partition for Hénon-like maps

$W^0(0)$	connected component of the stable set for $p(0)$ that contains $p(0)$
$W^1(0)$	preimage of $W^0(0)$
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A	$W^0(-1) \sim W^1(0)$
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B	$W^1(0) \sim W^0(0)$
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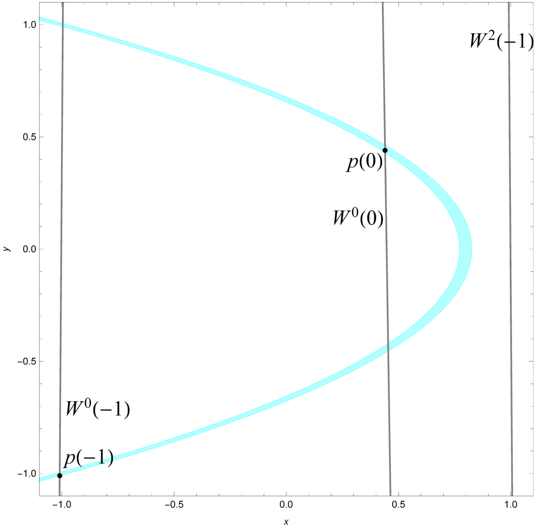

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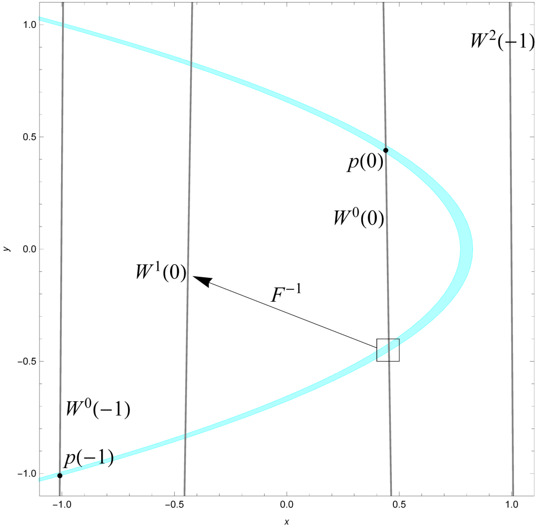

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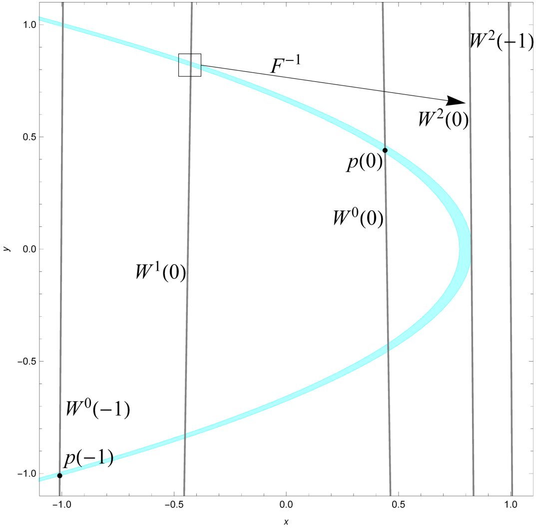

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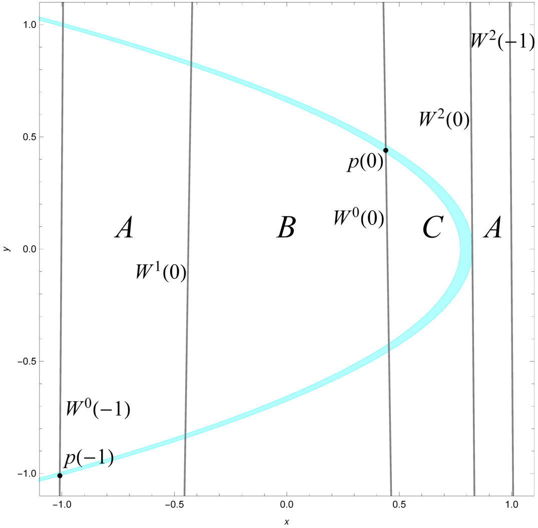
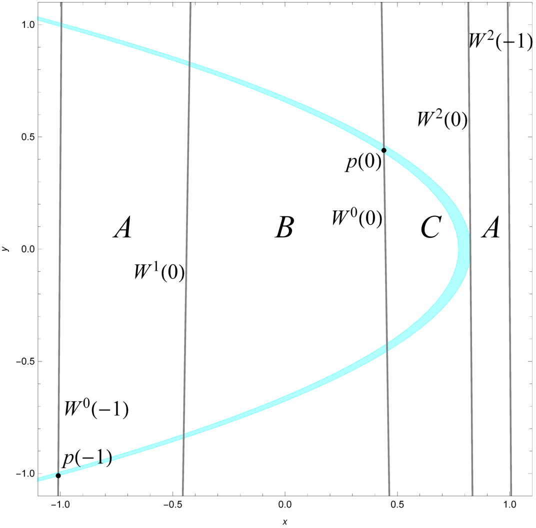


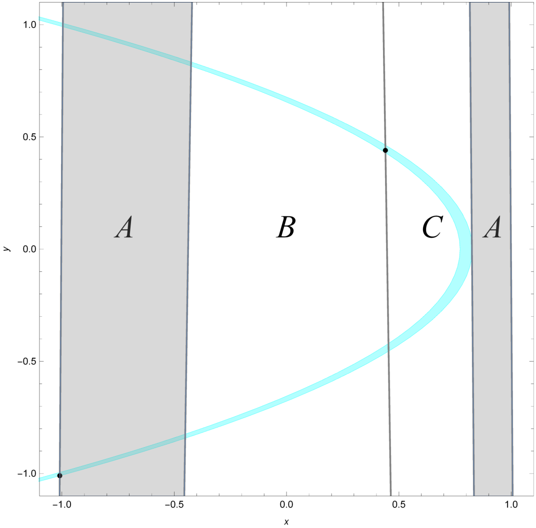
Partition for Hénon-like maps

- unimodal (degenerate) case: $\epsilon = 0$
  - $F(x, y) = (f(x), x)$
  - preimages of the fixed points
  - local stable manifolds are vertical lines determined by the preimages



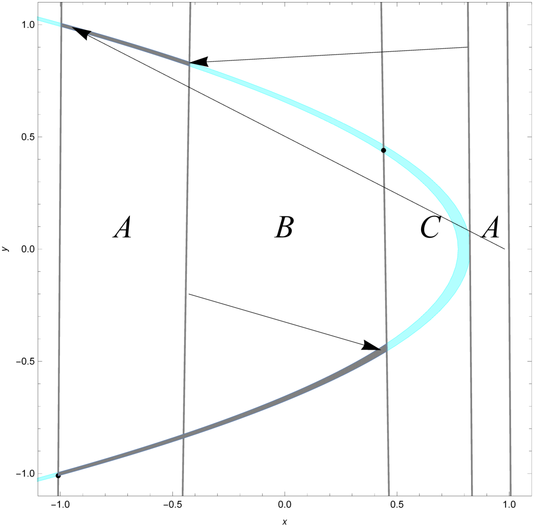
Dynamics on the Partition

A



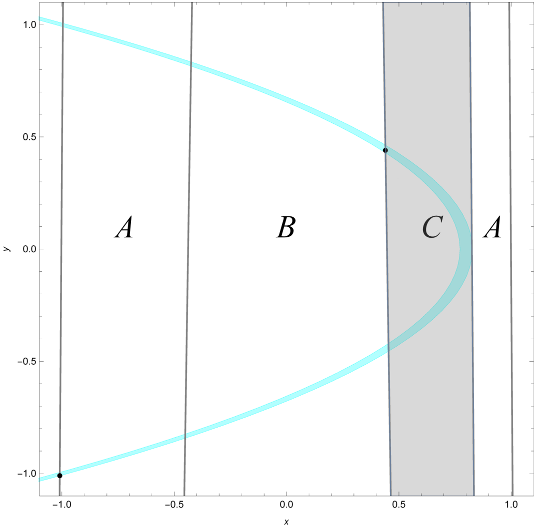
Dynamics on the Partition

$$\begin{tikzcd} \boldsymbol{A} \arrow[r] \arrow[loop below] & \boldsymbol{B} \end{tikzcd}$$



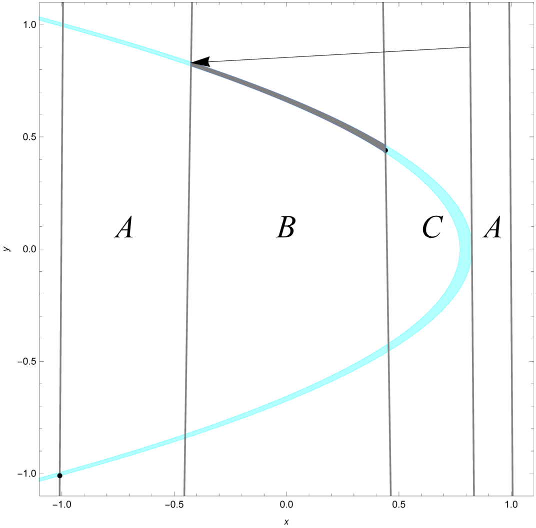
Dynamics on the Partition

$$\begin{tikzcd} A \arrow[loop below] \arrow[r] & B & \boldsymbol{C} \end{tikzcd}$$



Dynamics on the Partition

$$\begin{tikzcd} A \arrow[r] \arrow[loop below] & \boldsymbol{B} & \boldsymbol{C} \arrow[l] \end{tikzcd}$$



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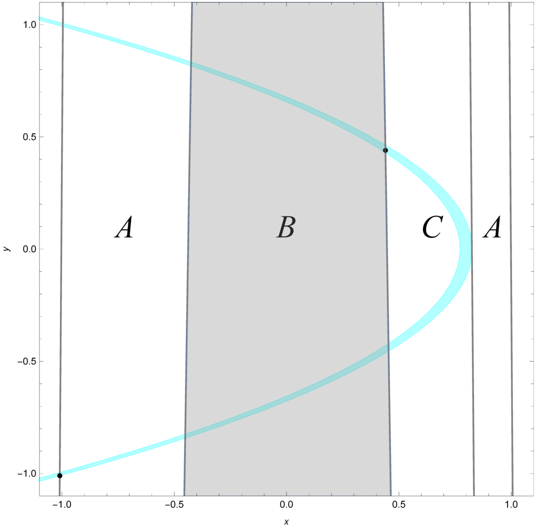
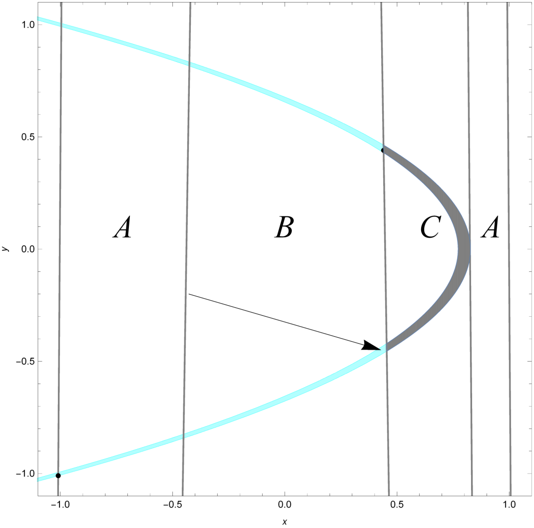


$$A \longrightarrow \boldsymbol{B} \longleftrightarrow \boldsymbol{C}$$

A Hénon-like map F is (periodic doubling) *renormalizable* if

$$F(B) \subset C.$$



Nonlinear rescaling and the Renormalization operator

- For a Hénon-like map

$$F(x, y) = (h_y(x), x) = (f(x) - \epsilon(x, y), x)$$

- Unfortunately, the second iterate

$$F^2(x, y) = (h_x(h_y(x)), \underbrace{h_y(x)}_{\text{not } x})$$

is not a Hénon-like map

Nonlinear rescaling and the Renormalization operator

[2005, A. De Carvalho, M. Lyubich, and M. Martens]

1. For small ϵ, CLM introduced a **nonlinear rescaling** ϕ that turns the first return map on C into a Hénon-like map

   x-direction preserves orientation

   y-direction reverses orientation & affine

[2005, A. De Carvalho, M. Lyubich, and M. Martens]

2. The **renormalization** $\hat{F} = RF = \phi \circ F^2 \circ \phi^{-1}$ is a Hénon-like map

$$\hat{F}(x,y) = (\underbrace{\hat{f}(x)}_{\text{unimodal}} - \underbrace{\hat{\epsilon}(x,y)}_{\text{perturbation}}, x)$$

   1. Unimodal: ϵ-close to the renormalized unimodal map

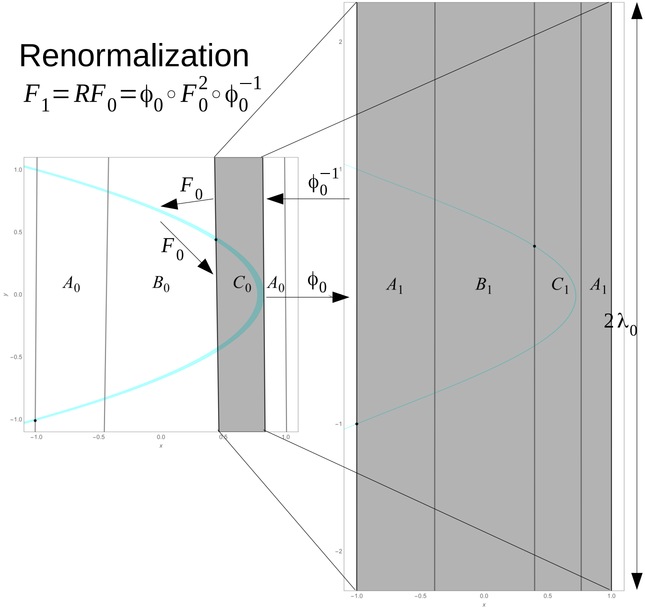
   $$\left\| \hat{f} - R_c f \right\| \leq c \left\| \epsilon \right\|$$

   2. Perturbation: has order ϵ^2

   $$\left\| \hat{\epsilon} \right\| \leq c \left\| \epsilon \right\|^2$$

Renormalization

$$
\begin{tikzcd}
D_1 \arrow[r, "F_1"] & D_1 \\
C_0 \arrow[u, "\phi_0"] \arrow[r, "F_0^2"] & C_0 \arrow[u, "\phi_0"']
\end{tikzcd}
$$



Infinite Renormalizable Hénon-like maps

- $F \overset{\text{renormalizable}}{\rightarrow} RF \overset{\text{renormalizable}}{\rightarrow} R^2F \rightarrow \cdots$
- If F can be renormalized infinite many times, we say that F is infinite renormalizable

Properties for Infinite renormalizable Hénon-like maps

[2005, A. De Carvalho, M. Lyubich, and M. Martens]
Summary for infinite renormalizable Hénon-like maps with **small** ϵ

- $F_n = R^n F$ = unimodal part f_n + perturbation part ϵ_n

unimodal $f_n \to g$ geometrically

$$\|f_n - g\| < c\rho^n \|F - G\|$$

- g: unimodal map, fixed point for R_c (unimodal renormalization)
- $G(x, y) = (g(x), x)$

pertubation $\epsilon_n \to 0$ super exponentially

$$\|\epsilon_{n+t}\| < (c\|\epsilon_n\|)^{2^t}$$

Scope

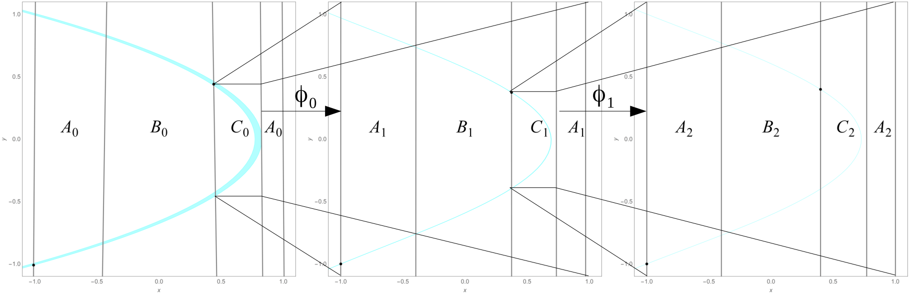
In this talk, we focus on Hénon-like maps that are:

1. non-degenerate ($\epsilon \neq 0$, $\frac{\partial \epsilon}{\partial y} \neq 0$)
2. strongly dissipative (ϵ: small)
3. infinite (periodic doubling) renormalizable

Rescaling level (skip)

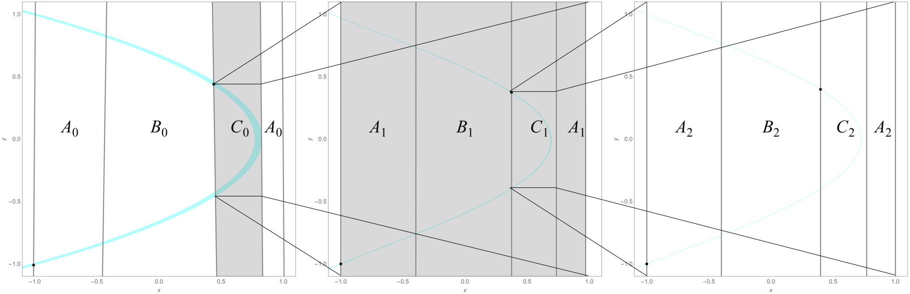
The partitions and local stable manifolds for F_0, F_1, and F_2.

- Dynamic property:
  $A \to A$ or B
  $B \to C$



Rescaling level (skip)

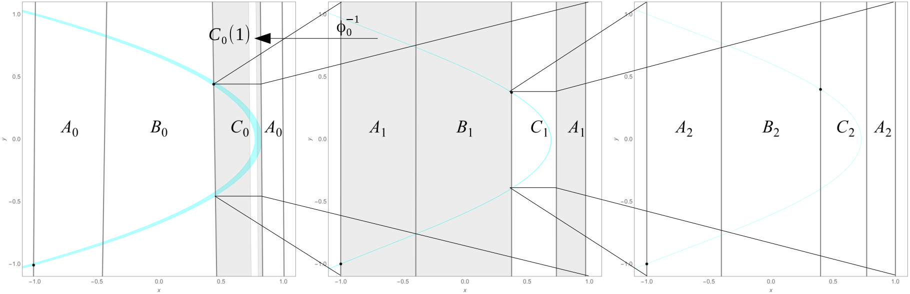
- $C_0 \to D_1$ (A_1 or B_1 or C_1)
- A_1 or B_1: Stop rescale
- C_1: Continue rescale



Rescaling level (skip)

First rescaling level $C_0(1)$ for F_0

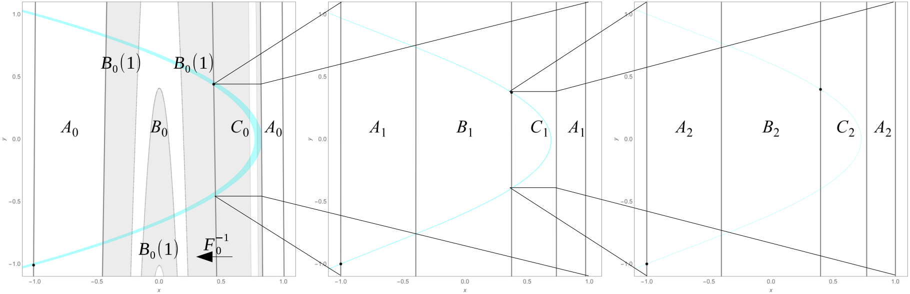
- $\boldsymbol{C_0(1) \xrightarrow{\phi_0} A_1 + B_1(\text{or } C_1(0))}$
  points in C that can be rescaled at most 1 time
- Remarks:
  - $C_0(1)$ has two components
  - Boundaries are local stable manifold of some periodic points



Rescaling level (skip)

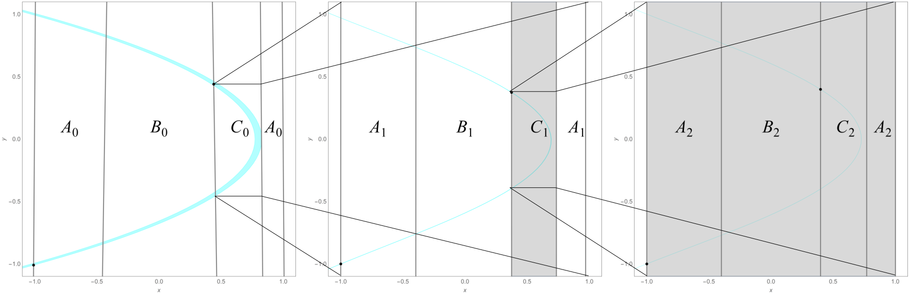
First rescaling level $B_0(1)$ for F_0

- $B_0(1) \xrightarrow{F_0} C_0(1)$



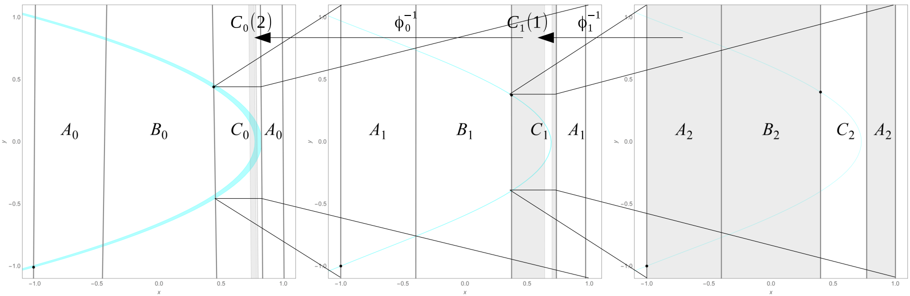
Rescaling level (skip)

- $C_1 \to D_2$ (A_2 or B_2 or C_2)
- A_2 or B_2: Stop rescale
- C_2: Continue rescale



Second rescaling level $C_0(2)$ for F_0

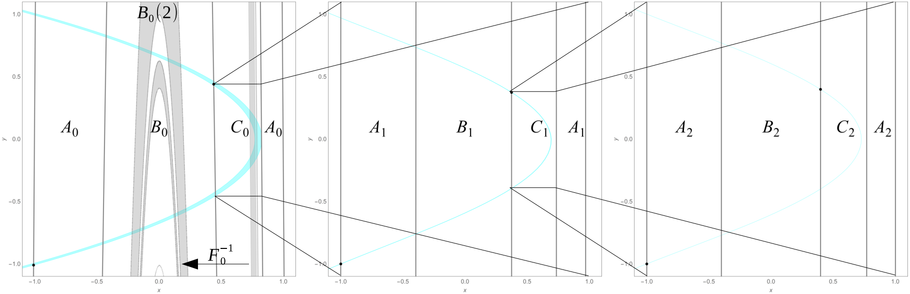
- $\boldsymbol{C_0(2) \xrightarrow{\phi_0} C_1(1) \xrightarrow{\phi_1} A_2 + B_2}$
  points in C that can be rescaled at most 2 times



Rescaling level (skip)

Second rescaling level $B_0(2)$ for F_0

- $B_0(2) \xrightarrow{F_0} C_0(2)$

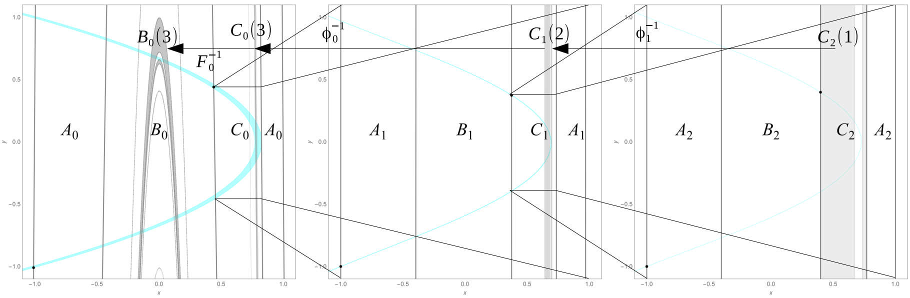


Rescaling level (skip)

Third rescaling level $B_0(3)$ and $C_0(3)$ for F_0

... and so on

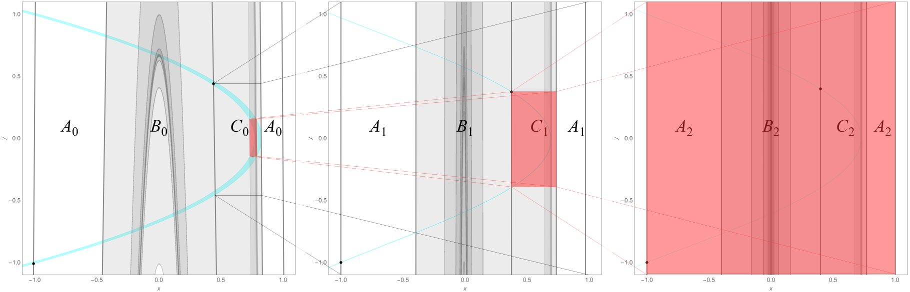
- Rescaling Level j (in C)
  Points in C that can be rescaled at most j times



Rescaling level (skip)

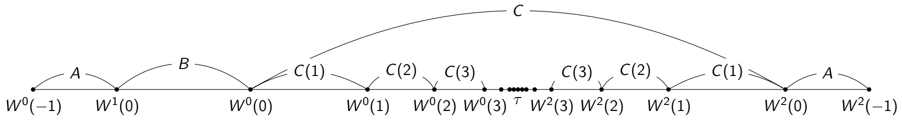
Tip τ: $\{\tau_n\} = \cap_{j=0}^{\infty} \left(\Phi_n^j\right)^{-1} (C_{n+j} \cap I^h \times I^h)$
(intersection of the shrinking boxes)

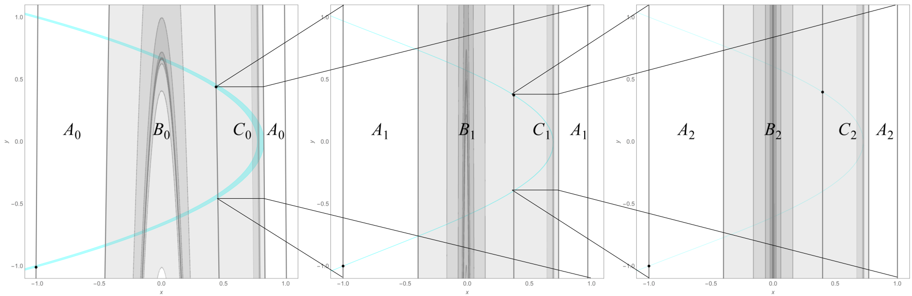
- Analog of the critical value
- Generates the attracting Cantor set



Rescaling level (skip)

Horizontal cross-section that intersects the tip





Wandering domain

- Assume that $F : D \to D$ is a Hénon-like map.
- We say that a nonempty connected open set $J \subset D$ is a **wandering domain** of F if J does not intersect any of the stable sets of periodic points.
- Properties:
  1. J: wandering domain $\Longrightarrow F(J)$: wandering domain
  2. $J \subset C(F)$: wandering domain $\Longrightarrow \phi(J) \subset D(RF)$: wandering domain
  3. nonempty open subset of a wandering domain is also a wandering domain

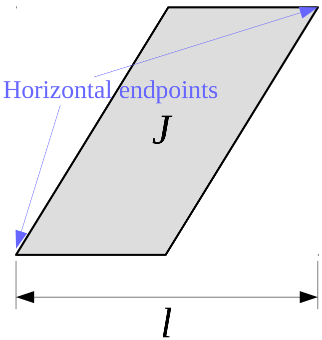
Horizontal size

- Horizontal size l of J

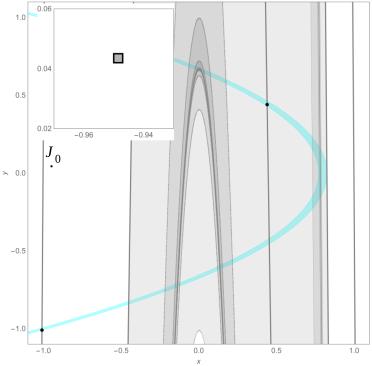
$$l(J) = \sup \{|x_1 - x_2|\}$$

for $(x_1, y_1), (x_2, y_2) \in J$.

- Goal: Prove that $l \to \infty$ after iterating and rescaling a wandering domain.

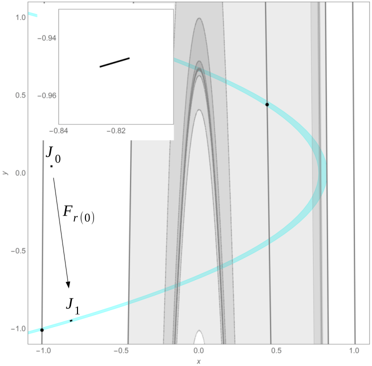


Prove by contradiction.
Assume that F has an wandering domain $J_0 \subset D$

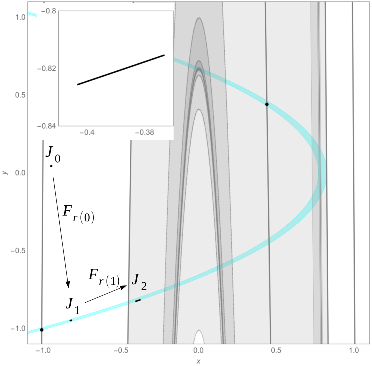


General Idea of the Proof

$J_0 \subset A_{r(0)}$: iterate $J_0 \xrightarrow{F_{r(0)}} J_1$



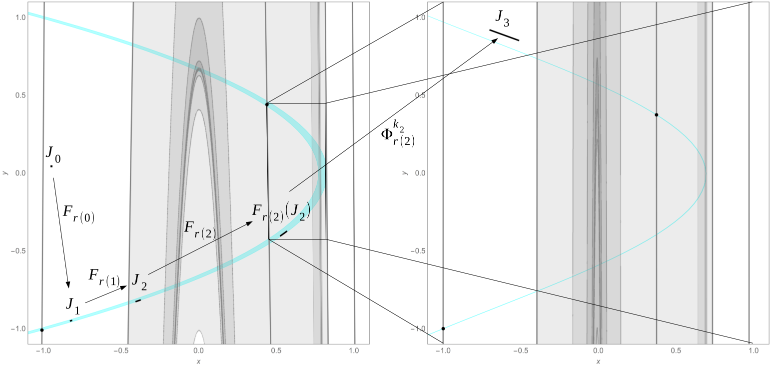
$J_1 \subset A_{r(1)}$: iterate $J_1 \overset{F_{r(1)}}{\Longrightarrow} J_2$



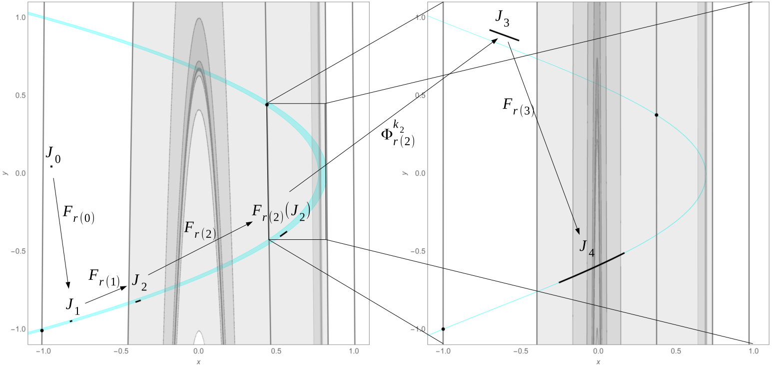
$J_2 \subset B_{r(2)}$: iterate then rescale as many times as possible

$$J_2 \xRightarrow{F_{r(2)}} F_{r(2)}(J_2) \xRightarrow{\Phi_{r(2)}^{k_2}} J_3$$

1 step=iteration+rescaling (if rescaling happens)



$J_3 \subset A_{r(3)}$: iterate $J_3 \xrightarrow{F_{r(3)}} J_4$
The horizontal size is too large that it intersects with some stable manifolds!
Contradiction!



General Idea of the Proof

- Goal: Want to show $(E > 1)$

$$l_{n+1} \geq E l_n$$

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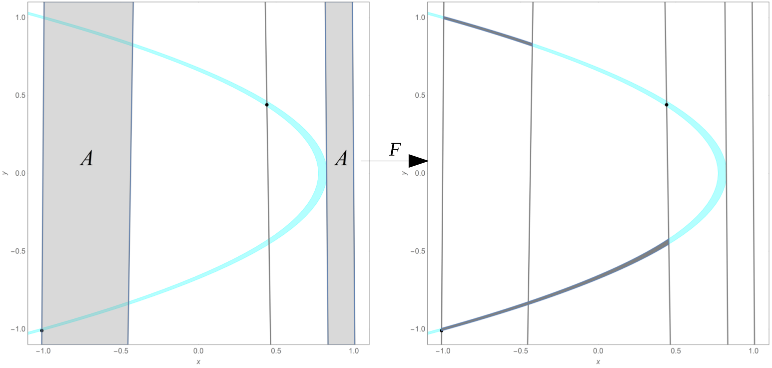
$$l_{n+1} \geq E l_n$$

- $A \to A$ or B: ✓

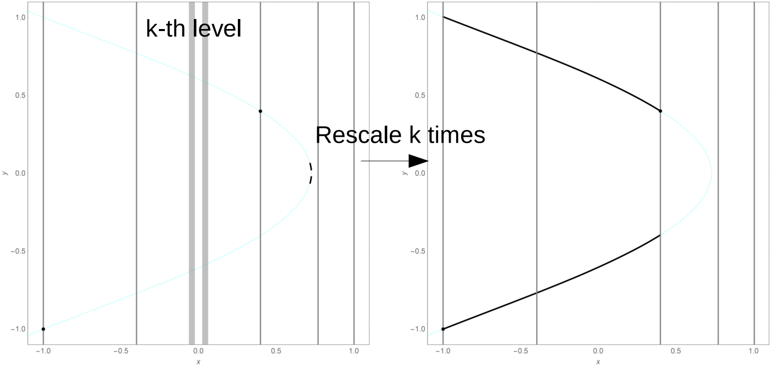


General Idea of the Proof

- Goal: Want to show ($E > 1$)

$$I_{n+1} \geq E I_n$$

- $A \to A$ or B: ✓
- $B \overset{\text{contract}}{\to} C \overset{\text{rescale}}{\to} A$ or B: ✓ in the unimodal case



General Idea of the Proof

- Goal: Want to show ($E > 1$)

$$I_{n+1} \geq E I_n$$

- $A \to A$ or B: ✓
- $B \overset{\text{contract}}{\to} C \overset{\text{rescale}}{\to} A$ or B: ✓ in the unimodal case
- $B \overset{\text{contract}}{\to} C \overset{\text{rescale}}{\to} A$ or B: **False** in the Hénon case

Difficulty

- What makes it different between the **Hénon** and **unimodal** case?
  - Why does the expansion argument false in the Hénon case?
  - How large/small will the horizontal size become when the expansion fails?

Difficulty

- What makes it different between the **Hénon** and **unimodal** case?
  - Why does the expansion argument false in the Hénon case?
  - How large/small will the horizontal size become when the expansion fails?
- Two features make Hénon-like maps different from the unimodal maps
  - **Good and Bad regions**
  - **Thickness**

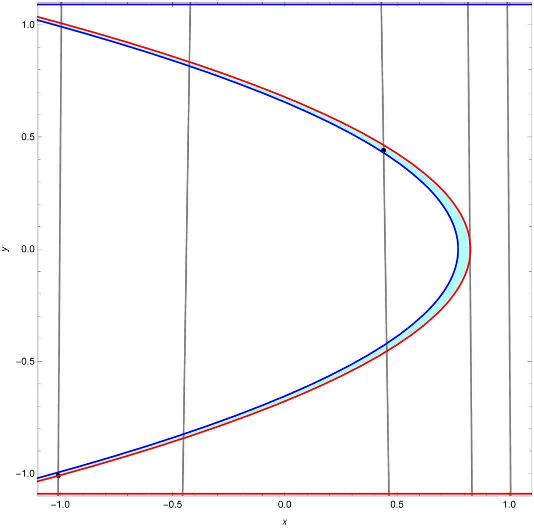
Bad News

- Assume the simple case

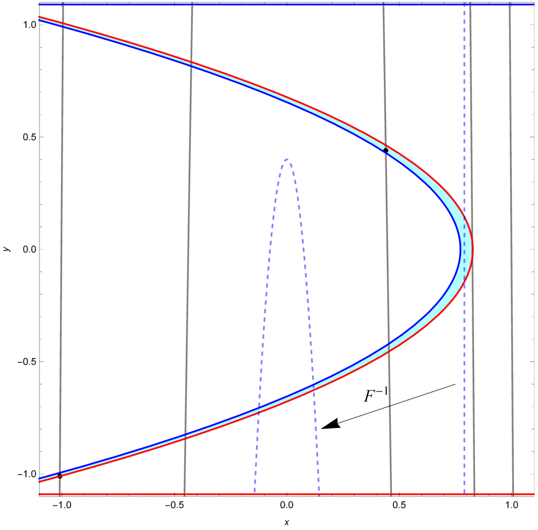
$$\epsilon(x, y) = ay$$

for $a > 0$. i.e.

$$F(x, y) = (f(x) - ay, x)$$

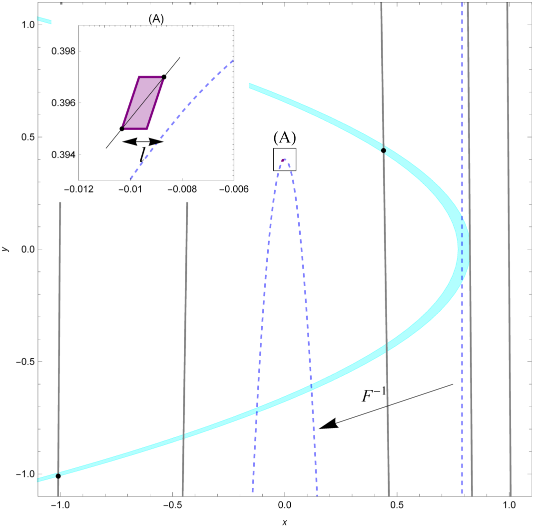


- Draw a vertical line close to the tip that intersects the image once
- Take the preimage of the vertical line



Bad News

- Draw a vertical line close to the tip that intersects the image once
- Take the preimage of the vertical line
- If the horizontal endpoints of a wandering domain is parallel to the preimage, then the image of the horizontal endpoints ~ 0.



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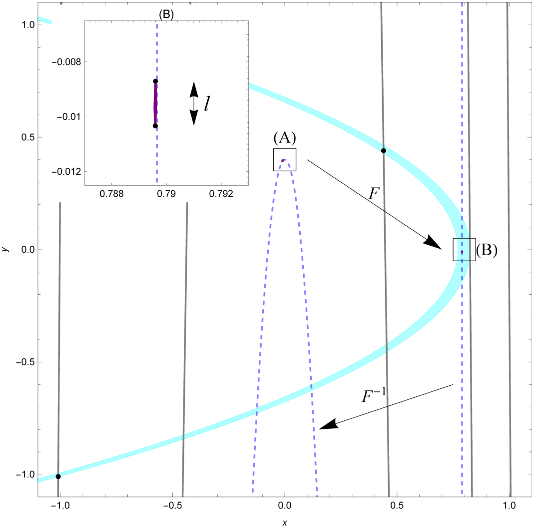
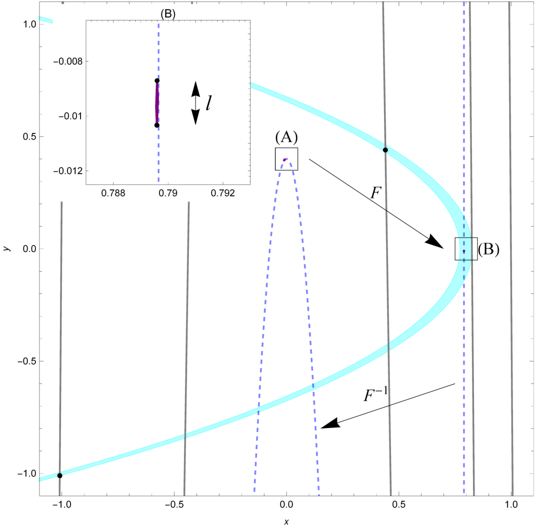


Bad News

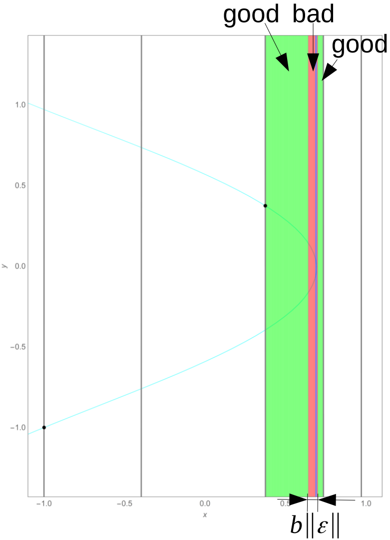
- Want to avoid "vertical line intersects the image once"
- Size of the image is $\|\epsilon_n\|$



Good and Bad regions

Let $b > 0$ be a large number and $F_n = R^n F$.

- **Good region** in C_n:
  The levels $C_n(j)$ that are $b \|\epsilon_n\|$ **away** from the tip
- **Bad region** in C_n:
  The levels $C_n(j)$ that are $b \|\epsilon_n\|$ **close** to the tip



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- **Good region** in C_n:
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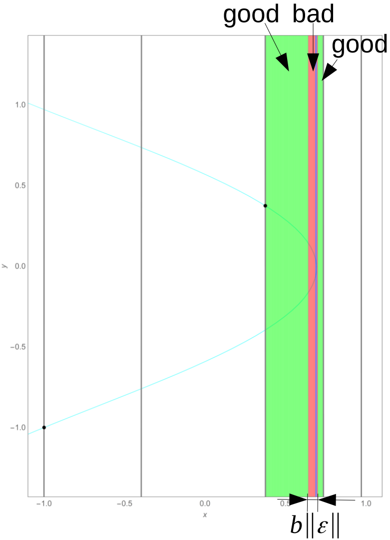
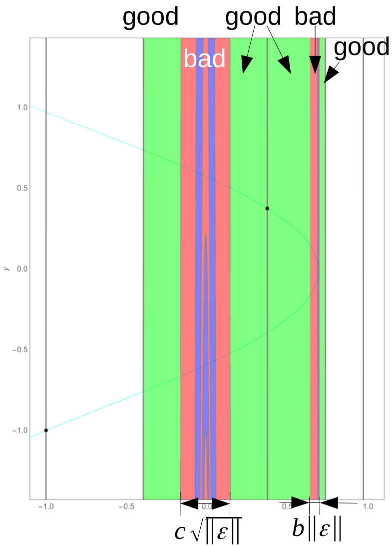
- **Bad region** in C_n:
  The levels $C_n(j)$ that are $b\,\|\epsilon_n\|$ **close** to the tip
- "Good and Bad regions" is a two dimensional feature
  Degenerate Hénon-like maps does **NOT** have bad region



Good and Bad regions

- Good region in B_n:
  $B_n(j)$ good $\Longleftrightarrow$ $C_n(j)$ good
- Bad region in B_n:
  $B_n(j)$ bad $\Longleftrightarrow$ $C_n(j)$ good

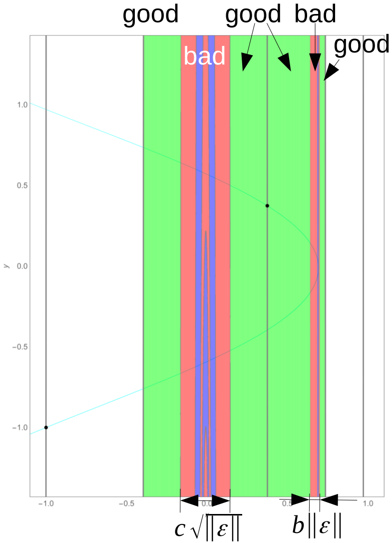


Good and Bad regions

Size of bad region

in C_n $b\|\epsilon_n\|$

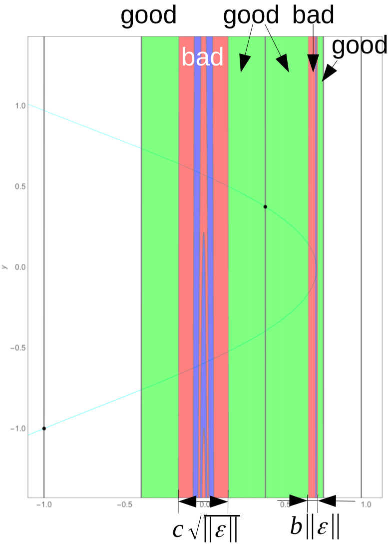
in B_n $c\sqrt{b\|\epsilon_n\|}$



Properties for the Good and Bad regions

- Good region is GOOD (**like** unimodal)
  - Geometry: Boundary stable manifolds of $B_n(j)$ are vertical graph
  - Dynamics: Horizontal size expands

$$l_{n+1} \geq E l_n$$



Properties for the Good and Bad regions

- Good region is GOOD (**like** unimodal)
  - Geometry: Boundary stable manifolds of $B_n(j)$ are vertical graph
  - Dynamics: Horizontal size expands

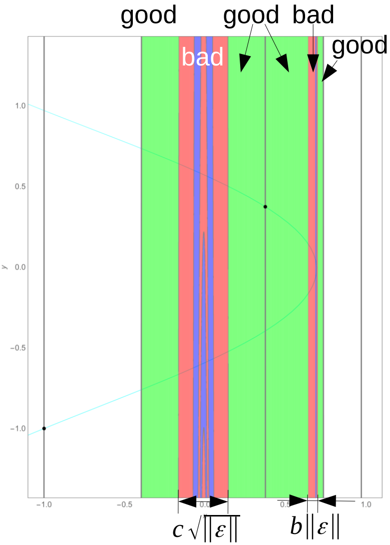
$$l_{n+1} \geq E l_n$$

- Bad region is BAD (**unlike** unimodal)
  - Geometry: Boundary stable manifolds of $B_n(j)$ are curved
  - Dynamics: Horizontal size may NOT expand

$$l_{n+1} \not\geq E l_n$$

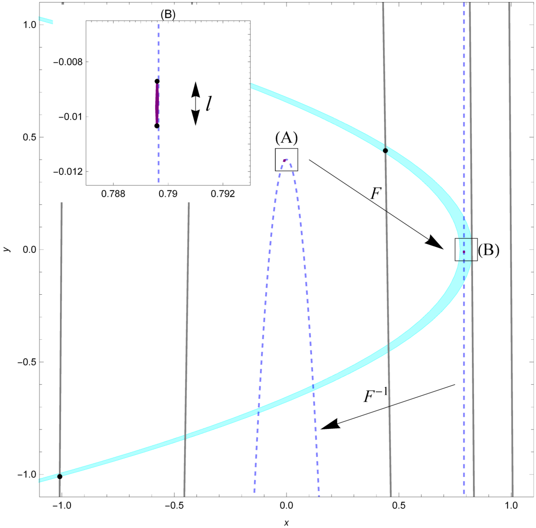
The ratio can be as small as possible!

$$\frac{l_{n+1}}{l_n} \searrow 0$$



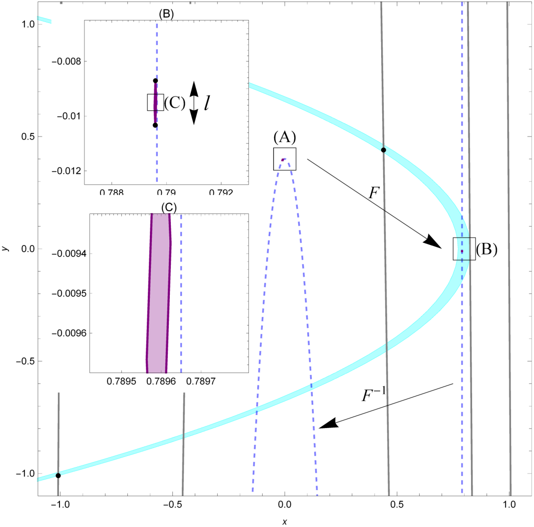
Bad News

- How **small** can the horizontal size be after the wandering domain enters the bad region?



Bad News

- How **small** can the horizontal size be after the wandering domain enters the bad region?
- area/cross-section determines the horizontal size

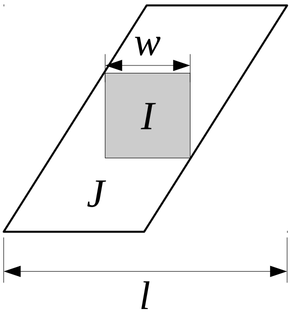


Thickness

square a closed set $I \subset \mathbb{R}^2$ with horizontal and vertical sides of the same length

thickness $w(J)$ the side of the "largest" square subset of J

- Thickness measures the size of horizontal cross-section



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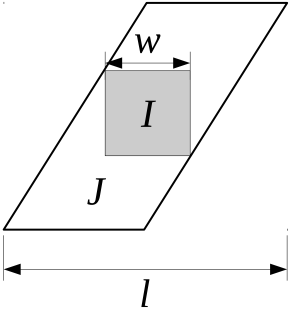

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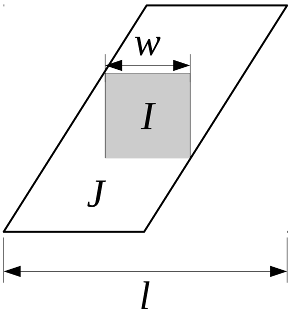
- Thickness measures the size of horizontal cross-section
- Thickness determines the horizontal size after a wandering domain enters the bad region
- Thickness is also a two dimensional feature



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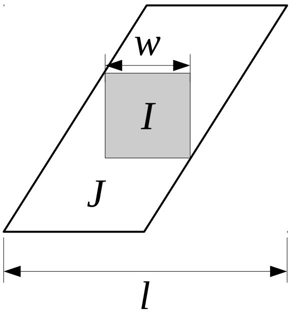
thickness $w(J)$ the side of the "largest" square subset of J

- Thickness measures the size of horizontal cross-section
- Thickness determines the horizontal size after a wandering domain enters the bad region
- Thickness is also a two dimensional feature
- Contraction $\cong$ Jacobian of $F \cong \|\epsilon\|$



Relation Between Horizontal Size with Thickness

Good	Bad	
$J_0^{(0)}$ $l_0^{(0)} = w_0^{(0)}$		

Relation Between Horizontal Size with Thickness

Good			Bad	
$J_0^{(0)}$	$\rightarrow$	$J_1^{(0)}$		
$I_0^{(0)} = w_0^{(0)}$	$\nearrow$	$I_1^{(0)}$		

Relation Between Horizontal Size with Thickness

Good	Bad	
$J_0^{(0)} \to J_1^{(0)} \to \cdots \to J_{n^{(0)}-1}^{(0)}$ $I_0^{(0)} = w_0^{(0)} \nearrow I_1^{(0)} \nearrow I_{n^{(0)}-1}^{(0)}$		

Relation Between Horizontal Size with Thickness

Good								Bad	
$J_0^{(0)}$	$\rightarrow$	$J_1^{(0)}$	$\rightarrow$	$\cdots$	$\rightarrow$	$J_{n^{(0)}-1}^{(0)}$	$\rightarrow$	$J_{n^{(0)}}^{(0)}$	
$I_0^{(0)} = w_0^{(0)}$	$\nearrow$	$I_1^{(0)}$			$\nearrow$	$I_{n^{(0)}-1}^{(0)}$	$\nearrow$	$I_{n^{(0)}}^{(0)}$	

Relation Between Horizontal Size with Thickness

Good							Bad		
$J_0^{(0)}$	$\rightarrow$	$J_1^{(0)}$	$\rightarrow$	$\cdots$	$\rightarrow$	$J_{n^{(0)}-1}^{(0)}$ $\rightarrow$	$J_{n^{(0)}}^{(0)}$	$\rightarrow$	$J_{n^{(0)}+1}^{(0)}$
$I_0^{(0)} = w_0^{(0)}$	$\nearrow$	$I_1^{(0)}$			$\nearrow$	$I_{n^{(0)}-1}^{(0)}$ $\nearrow$	$I_{n^{(0)}}^{(0)}$	$\times$	

Relation Between Horizontal Size with Thickness

Good							Bad		
$J_0^{(0)}$ $l_0^{(0)} = w_0^{(0)}$ $w_0^{(0)}$	$\to$	$J_1^{(0)}$	$\to$	$\cdots$	$\to$	$J_{n^{(0)}-1}^{(0)}$ $\to$	$J_{n^{(0)}}^{(0)}$	$\to$ $\times$	$J_{n^{(0)}+1}^{(0)}$

Relation Between Horizontal Size with Thickness

Good	Bad	
$J_0^{(0)} \to J_1^{(0)} \to \cdots \to J_{n^{(0)}-1}^{(0)} \to$	$J_{n^{(0)}}^{(0)}$	$\to \quad J_{n^{(0)}+1}^{(0)}$
$I_0^{(0)} = w_0^{(0)}$		$\times$
$w_0^{(0)} \searrow w_1^{(0)}$		

Relation Between Horizontal Size with Thickness

Good								Bad		
$J_0^{(0)}$ $I_0^{(0)} = w_0^{(0)}$ $w_0^{(0)}$	$\rightarrow$ $\searrow$	$J_1^{(0)}$ $w_1^{(0)}$	$\rightarrow$	$\cdots$	$\rightarrow$ $\searrow$	$J_{n^{(0)}-1}^{(0)}$ $w_{n^{(0)}-1}^{(0)}$	$\rightarrow$	$J_{n^{(0)}}^{(0)}$	$\rightarrow$ $\times$	$J_{n^{(0)}+1}^{(0)}$

Relation Between Horizontal Size with Thickness

Good							Bad		
$J_0^{(0)}$	$\rightarrow$	$J_1^{(0)}$	$\rightarrow$	$\cdots$	$\rightarrow$	$J_{n^{(0)}-1}^{(0)}$ $\rightarrow$	$J_{n^{(0)}}^{(0)}$	$\rightarrow$	$J_{n^{(0)}+1}^{(0)}$
$l_0^{(0)} = w_0^{(0)}$								$\times$	
$w_0^{(0)}$	$\searrow$	$w_1^{(0)}$			$\searrow$	$w_{n^{(0)}-1}^{(0)}$ $\searrow$	$w_{n^{(0)}}^{(0)}$		

Relation Between Horizontal Size with Thickness

Good								Bad		
$J_0^{(0)}$	$\to$	$J_1^{(0)}$	$\to$	$\cdots$	$\to$	$J_{n^{(0)}-1}^{(0)}$	$\to$	$J_{n^{(0)}}^{(0)}$	$\to$	$J_{n^{(0)}+1}^{(0)}$
$l_0^{(0)} = w_0^{(0)}$									$\times$	
$w_0^{(0)}$	$\searrow$	$w_1^{(0)}$			$\searrow$	$w_{n^{(0)}-1}^{(0)}$	$\searrow$	$w_{n^{(0)}}^{(0)}$	$\searrow$	$l_{n^{(0)}+1}^{(0)} = w_{n^{(0)}+1}^{(0)}$

Relation Between Horizontal Size with Thickness

Good								Bad		
$J_0^{(0)}$	$\to$	$J_1^{(0)}$	$\to$	$\cdots$	$\to$	$J_{n^{(0)}-1}^{(0)}$	$\to$	$J_{n^{(0)}}^{(0)}$	$\to$	$J_{n^{(0)}+1}^{(0)}$
$l_0^{(0)} = w_0^{(0)}$									$\times$	
$w_0^{(0)}$	$\searrow$	$w_1^{(0)}$			$\searrow$	$w_{n^{(0)}-1}^{(0)}$	$\searrow$	$w_{n^{(0)}}^{(0)}$	$\searrow$	$l_{n^{(0)}+1}^{(0)} = w_{n^{(0)}+1}^{(0)}$
$J_0^{(1)}$										
$l_0^{(1)} = w_0^{(1)}$										

Relation Between Horizontal Size with Thickness

Good								Bad		
$J_0^{(0)}$	$\rightarrow$	$J_1^{(0)}$	$\rightarrow$	$\cdots$	$\rightarrow$	$J_{n^{(0)}-1}^{(0)}$	$\rightarrow$	$J_{n^{(0)}}^{(0)}$	$\rightarrow$	$J_{n^{(0)}+1}^{(0)}$
$l_0^{(0)} = w_0^{(0)}$									$\times$	
$w_0^{(0)}$	$\searrow$	$w_1^{(0)}$			$\searrow$	$w_{n^{(0)}-1}^{(0)}$	$\searrow$	$w_{n^{(0)}}^{(0)}$	$\searrow$	$l_{n^{(0)}+1}^{(0)} = w_{n^{(0)}+1}^{(0)}$
$J_0^{(1)}$	$\rightarrow$	$J_1^{(1)}$								
$l_0^{(1)} = w_0^{(1)}$	$\nearrow$	$l_1^{(1)}$								

Relation Between Horizontal Size with Thickness

Good										Bad		
$J_0^{(0)}$ $l_0^{(0)} = w_0^{(0)}$ $w_0^{(0)}$	$\rightarrow$ $\searrow$	$J_1^{(0)}$ $w_1^{(0)}$	$\rightarrow$	$\cdots$			$\rightarrow$ $\searrow$	$J_{n^{(0)}-1}^{(0)}$ $w_{n^{(0)}-1}^{(0)}$	$\rightarrow$ $\searrow$	$J_{n^{(0)}}^{(0)}$ $w_{n^{(0)}}^{(0)}$	$\rightarrow$ $\times$ $\searrow$	$J_{n^{(0)}+1}^{(0)}$ $l_{n^{(0)}+1}^{(0)} = w_{n^{(0)}+1}^{(0)}$
$J_0^{(1)}$ $l_0^{(1)} = w_0^{(1)}$	$\rightarrow$ $\nearrow$	$J_1^{(1)}$ $l_1^{(1)}$	$\rightarrow$	$\cdots$	$\rightarrow$ $\nearrow$	$J_{n^{(1)}-1}^{(1)}$ $l_{n^{(1)}-1}^{(1)}$						

Relation Between Horizontal Size with Thickness

Good										Bad		
$J_0^{(0)}$	$\to$	$J_1^{(0)}$	$\to$	$\cdots$			$\to$	$J_{n^{(0)}-1}^{(0)}$	$\to$	$J_{n^{(0)}}^{(0)}$	$\to$	$J_{n^{(0)}+1}^{(0)}$
$I_0^{(0)} = w_0^{(0)}$											$\times$	
$w_0^{(0)}$	$\searrow$	$w_1^{(0)}$					$\searrow$	$w_{n^{(0)}-1}^{(0)}$	$\searrow$	$w_{n^{(0)}}^{(0)}$	$\searrow$	$I_{n^{(0)}+1}^{(0)} = w_{n^{(0)}+1}^{(0)}$
$J_0^{(1)}$	$\to$	$J_1^{(1)}$	$\to$	$\cdots$	$\to$	$J_{n^{(1)}-1}^{(1)}$	$\to$			$J_{n^{(1)}}^{(1)}$		
$I_0^{(1)} = w_0^{(1)}$	$\nearrow$	$I_1^{(1)}$			$\nearrow$	$I_{n^{(1)}-1}^{(1)}$	$\nearrow$			$I_{n^{(1)}}^{(1)}$		

Relation Between Horizontal Size with Thickness

Good										Bad		
$J_0^{(0)}$	$\rightarrow$	$J_1^{(0)}$	$\rightarrow$	$\cdots$			$\rightarrow$	$J_{n^{(0)}-1}^{(0)}$	$\rightarrow$	$J_{n^{(0)}}^{(0)}$	$\rightarrow$	$J_{n^{(0)}+1}^{(0)}$
$I_0^{(0)} = w_0^{(0)}$											$\times$	
$w_0^{(0)}$	$\searrow$	$w_1^{(0)}$					$\searrow$	$w_{n^{(0)}-1}^{(0)}$	$\searrow$	$w_{n^{(0)}}^{(0)}$	$\searrow$	$I_{n^{(0)}+1}^{(0)} = w_{n^{(0)}+1}^{(0)}$
$J_0^{(1)}$	$\rightarrow$	$J_1^{(1)}$	$\rightarrow$	$\cdots$	$\rightarrow$	$J_{n^{(1)}-1}^{(1)}$	$\rightarrow$			$J_{n^{(1)}}^{(1)}$	$\rightarrow$	$J_{n^{(1)}+1}^{(1)}$
$I_0^{(1)} = w_0^{(1)}$	$\nearrow$	$I_1^{(1)}$			$\nearrow$	$I_{n^{(1)}-1}^{(1)}$	$\nearrow$			$I_{n^{(1)}}^{(1)}$	$\times$	

Relation Between Horizontal Size with Thickness

Good										Bad		
$J_0^{(0)}$	$\to$	$J_1^{(0)}$	$\to$	$\cdots$			$\to$	$J_{n^{(0)}-1}^{(0)}$	$\to$	$J_{n^{(0)}}^{(0)}$	$\to$	$J_{n^{(0)}+1}^{(0)}$
$l_0^{(0)} = w_0^{(0)}$											$\times$	
$w_0^{(0)}$	$\searrow$	$w_1^{(0)}$					$\searrow$	$w_{n^{(0)}-1}^{(0)}$	$\searrow$	$w_{n^{(0)}}^{(0)}$	$\searrow$	$l_{n^{(0)}+1}^{(0)} = w_{n^{(0)}+1}^{(0)}$
$J_0^{(1)}$	$\to$	$J_1^{(1)}$	$\to$	$\cdots$	$\to$	$J_{n^{(1)}-1}^{(1)}$	$\to$			$J_{n^{(1)}}^{(1)}$	$\to$	$J_{n^{(1)}+1}^{(1)}$
$l_0^{(1)} = w_0^{(1)}$											$\times$	
$w_0^{(1)}$												

Good										Bad		
$J_0^{(0)}$	$\rightarrow$	$J_1^{(0)}$	$\rightarrow$	$\cdots$			$\rightarrow$	$J_{n^{(0)}-1}^{(0)}$	$\rightarrow$	$J_{n^{(0)}}^{(0)}$	$\rightarrow$	$J_{n^{(0)}+1}^{(0)}$
$l_0^{(0)} = w_0^{(0)}$											$\times$	
$w_0^{(0)}$	$\searrow$	$w_1^{(0)}$					$\searrow$	$w_{n^{(0)}-1}^{(0)}$	$\searrow$	$w_{n^{(0)}}^{(0)}$	$\searrow$	$l_{n^{(0)}+1}^{(0)} = w_{n^{(0)}+1}^{(0)}$
$J_0^{(1)}$	$\rightarrow$	$J_1^{(1)}$	$\rightarrow$	$\cdots$	$\rightarrow$	$J_{n^{(1)}-1}^{(1)}$	$\rightarrow$			$J_{n^{(1)}}^{(1)}$	$\rightarrow$	$J_{n^{(1)}+1}^{(1)}$
$l_0^{(1)} = w_0^{(1)}$											$\times$	
$w_0^{(1)}$	$\searrow$	$w_1^{(1)}$										

Relation Between Horizontal Size with Thickness

Good										Bad		
$J_0^{(0)}$	$\to$	$J_1^{(0)}$	$\to$	$\cdots$			$\to$	$J_{n^{(0)}-1}^{(0)}$	$\to$	$J_{n^{(0)}}^{(0)}$	$\to$	$J_{n^{(0)}+1}^{(0)}$
$l_0^{(0)} = w_0^{(0)}$											$\times$	
$w_0^{(0)}$	$\searrow$	$w_1^{(0)}$					$\searrow$	$w_{n^{(0)}-1}^{(0)}$	$\searrow$	$w_{n^{(0)}}^{(0)}$	$\searrow$	$l_{n^{(0)}+1}^{(0)} = w_{n^{(0)}+1}^{(0)}$
$J_0^{(1)}$	$\to$	$J_1^{(1)}$	$\to$	$\cdots$	$\to$	$J_{n^{(1)}-1}^{(1)}$	$\to$			$J_{n^{(1)}}^{(1)}$	$\to$	$J_{n^{(1)}+1}^{(1)}$
$l_0^{(1)} = w_0^{(1)}$											$\times$	
$w_0^{(1)}$	$\searrow$	$w_1^{(1)}$			$\searrow$	$w_{n^{(1)}-1}^{(1)}$						

Relation Between Horizontal Size with Thickness

Good										Bad		
$J_0^{(0)}$	$\rightarrow$	$J_1^{(0)}$	$\rightarrow$	$\cdots$			$\rightarrow$	$J^{(0)}_{n^{(0)}-1}$	$\rightarrow$	$J^{(0)}_{n^{(0)}}$	$\rightarrow$	$J^{(0)}_{n^{(0)}+1}$
$l_0^{(0)} = w_0^{(0)}$											$\times$	
$w_0^{(0)}$	$\searrow$	$w_1^{(0)}$					$\searrow$	$w^{(0)}_{n^{(0)}-1}$	$\searrow$	$w^{(0)}_{n^{(0)}}$	$\searrow$	$l^{(0)}_{n^{(0)}+1} = w^{(0)}_{n^{(0)}+1}$
$J_0^{(1)}$	$\rightarrow$	$J_1^{(1)}$	$\rightarrow$	$\cdots$	$\rightarrow$	$J^{(1)}_{n^{(1)}-1}$	$\rightarrow$			$J^{(1)}_{n^{(1)}}$	$\rightarrow$	$J^{(1)}_{n^{(1)}+1}$
$l_0^{(1)} = w_0^{(1)}$											$\times$	
$w_0^{(1)}$	$\searrow$	$w_1^{(1)}$			$\searrow$	$w^{(1)}_{n^{(1)}-1}$	$\searrow$			$w^{(1)}_{n^{(1)}}$		

Relation Between Horizontal Size with Thickness

Good										Bad		
$J_0^{(0)}$	$\rightarrow$	$J_1^{(0)}$	$\rightarrow$	$\cdots$			$\rightarrow$	$J_{n^{(0)}-1}^{(0)}$	$\rightarrow$	$J_{n^{(0)}}^{(0)}$	$\rightarrow$	$J_{n^{(0)}+1}^{(0)}$
$l_0^{(0)} = w_0^{(0)}$											$\times$	
$w_0^{(0)}$	$\searrow$	$w_1^{(0)}$					$\searrow$	$w_{n^{(0)}-1}^{(0)}$	$\searrow$	$w_{n^{(0)}}^{(0)}$	$\searrow$	$l_{n^{(0)}+1}^{(0)} = w_{n^{(0)}+1}^{(0)}$
$J_0^{(1)}$	$\rightarrow$	$J_1^{(1)}$	$\rightarrow$	$\cdots$	$\rightarrow$	$J_{n^{(1)}-1}^{(1)}$	$\rightarrow$			$J_{n^{(1)}}^{(1)}$	$\rightarrow$	$J_{n^{(1)}+1}^{(1)}$
$l_0^{(1)} = w_0^{(1)}$											$\times$	
$w_0^{(1)}$	$\searrow$	$w_1^{(1)}$			$\searrow$	$w_{n^{(1)}-1}^{(1)}$	$\searrow$			$w_{n^{(1)}}^{(1)}$	$\searrow$	$l_{n^{(1)}+1}^{(1)} = w_{n^{(1)}+1}^{(1)}$

Relation Between Horizontal Size with Thickness

Good										Bad		
$J_0^{(0)}$ $I_0^{(0)} = w_0^{(0)}$ $w_0^{(0)}$	$\rightarrow$ $\searrow$	$J_1^{(0)}$ $w_1^{(0)}$	$\rightarrow$	$\cdots$			$\rightarrow$ $\searrow$	$J_{n^{(0)}-1}^{(0)}$ $w_{n^{(0)}-1}^{(0)}$	$\rightarrow$ $\searrow$	$J_{n^{(0)}}^{(0)}$ $w_{n^{(0)}}^{(0)}$	$\rightarrow$ $\times$ $\searrow$	$J_{n^{(0)}+1}^{(0)}$ $I_{n^{(0)}+1}^{(0)} = w_{n^{(0)}+1}^{(0)}$
$J_0^{(1)}$ $I_0^{(1)} = w_0^{(1)}$ $w_0^{(1)}$	$\rightarrow$ $\searrow$	$J_1^{(1)}$ $w_1^{(1)}$	$\rightarrow$	$\cdots$	$\rightarrow$ $\searrow$	$J_{n^{(1)}-1}^{(1)}$ $w_{n^{(1)}-1}^{(1)}$	$\rightarrow$ $\searrow$			$J_{n^{(1)}}^{(1)}$ $w_{n^{(1)}}^{(1)}$	$\rightarrow$ $\times$ $\searrow$	$J_{n^{(1)}+1}^{(1)}$ $I_{n^{(1)}+1}^{(1)} = w_{n^{(1)}+1}^{(1)}$
$J_0^{(2)}$ $I_0^{(2)} = w_0^{(2)}$ $w_0^{(2)}$												

Relation Between Horizontal Size with Thickness

Good										Bad		
$J_0^{(0)}$	$\to$	$J_1^{(0)}$	$\to$	$\cdots$			$\to$	$J_{n^{(0)}-1}^{(0)}$	$\to$	$J_{n^{(0)}}^{(0)}$	$\to$	$J_{n^{(0)}+1}^{(0)}$
$I_0^{(0)} = w_0^{(0)}$											$\times$	
$w_0^{(0)}$	$\searrow$	$w_1^{(0)}$					$\searrow$	$w_{n^{(0)}-1}^{(0)}$	$\searrow$	$w_{n^{(0)}}^{(0)}$	$\searrow$	$I_{n^{(0)}+1}^{(0)} = w_{n^{(0)}+1}^{(0)}$
$J_0^{(1)}$	$\to$	$J_1^{(1)}$	$\to$	$\cdots$	$\to$	$J_{n^{(1)}-1}^{(1)}$	$\to$			$J_{n^{(1)}}^{(1)}$	$\to$	$J_{n^{(1)}+1}^{(1)}$
$I_0^{(1)} = w_0^{(1)}$											$\times$	
$w_0^{(1)}$	$\searrow$	$w_1^{(1)}$			$\searrow$	$w_{n^{(1)}-1}^{(1)}$	$\searrow$			$w_{n^{(1)}}^{(1)}$	$\searrow$	$I_{n^{(1)}+1}^{(1)} = w_{n^{(1)}+1}^{(1)}$
$J_0^{(2)}$	$\to$	$J_1^{(2)}$	$\to$	$\cdots$		$\cdots$	$\to$	$J_{n^{(2)}-1}^{(2)}$	$\to$	$J_{n^{(2)}}^{(2)}$		
$I_0^{(2)} = w_0^{(2)}$	$\nearrow$	$I_1^{(2)}$					$\nearrow$	$I_{n^{(2)}-1}^{(2)}$	$\nearrow$	$I_{n^{(2)}}^{(2)}$		
$w_0^{(2)}$												

- Continue to add new rows until the wandering domain does not enter the bad region again

Relation Between Horizontal Size with Thickness

Good										Bad		
$J_0^{(0)}$	$\rightarrow$	$J_1^{(0)}$	$\rightarrow$	$\cdots$			$\rightarrow$	$J_{n^{(0)}-1}^{(0)}$	$\rightarrow$	$J_{n^{(0)}}^{(0)}$	$\rightarrow$	$J_{n^{(0)}+1}^{(0)}$
$I_0^{(0)} = w_0^{(0)}$											$\times$	
$w_0^{(0)}$	$\searrow$	$w_1^{(0)}$					$\searrow$	$w_{n^{(0)}-1}^{(0)}$	$\searrow$	$w_{n^{(0)}}^{(0)}$	$\searrow$	$I_{n^{(0)}+1}^{(0)} = w_{n^{(0)}+1}^{(0)}$
$J_0^{(1)}$	$\rightarrow$	$J_1^{(1)}$	$\rightarrow$	$\cdots$	$\rightarrow$	$J_{n^{(1)}-1}^{(1)}$	$\rightarrow$			$J_{n^{(1)}}^{(1)}$	$\rightarrow$	$J_{n^{(1)}+1}^{(1)}$
$I_0^{(1)} = w_0^{(1)}$											$\times$	
$w_0^{(1)}$	$\searrow$	$w_1^{(1)}$			$\searrow$	$w_{n^{(1)}-1}^{(1)}$	$\searrow$			$w_{n^{(1)}}^{(1)}$	$\searrow$	$I_{n^{(1)}+1}^{(1)} = w_{n^{(1)}+1}^{(1)}$
$J_0^{(2)}$	$\rightarrow$	$J_1^{(2)}$	$\rightarrow$	$\cdots$		$\cdots$	$\rightarrow$	$J_{n^{(2)}-1}^{(2)}$	$\rightarrow$	$J_{n^{(2)}}^{(2)}$		
$I_0^{(2)} = w_0^{(2)}$	$\nearrow$	$I_1^{(2)}$					$\nearrow$	$I_{n^{(2)}-1}^{(2)}$	$\nearrow$	$I_{n^{(2)}}^{(2)}$		
$w_0^{(2)}$												

- Continue to add new rows until the wandering domain does not enter the bad region again
- Each row corresponds to entering the bad region once

Relation Between Horizontal Size with Thickness

Good										Bad		
$J_0^{(0)}$ $I_0^{(0)}=w_0^{(0)}$ $w_0^{(0)}$	$\to$ $\searrow$	$J_1^{(0)}$ $w_1^{(0)}$	$\to$	$\cdots$			$\to$ $\searrow$	$J_{n^{(0)}-1}^{(0)}$ $w_{n^{(0)}-1}^{(0)}$	$\to$ $\searrow$	$J_{n^{(0)}}^{(0)}$ $w_{n^{(0)}}^{(0)}$	$\to$ $\times$ $\searrow$	$J_{n^{(0)}+1}^{(0)}$ $I_{n^{(0)}+1}^{(0)}=w_{n^{(0)}+1}^{(0)}$
$J_0^{(1)}$ $I_0^{(1)}=w_0^{(1)}$ $w_0^{(1)}$	$\to$ $\searrow$	$J_1^{(1)}$ $w_1^{(1)}$	$\to$	$\cdots$	$\to$ $\searrow$	$J_{n^{(1)}-1}^{(1)}$ $w_{n^{(1)}-1}^{(1)}$	$\to$ $\searrow$			$J_{n^{(1)}}^{(1)}$ $w_{n^{(1)}}^{(1)}$	$\to$ $\times$ $\searrow$	$J_{n^{(1)}+1}^{(1)}$ $I_{n^{(1)}+1}^{(1)}=w_{n^{(1)}+1}^{(1)}$
$J_0^{(2)}$ $I_0^{(2)}=w_0^{(2)}$ $w_0^{(2)}$	$\to$ $\nearrow$	$J_1^{(2)}$ $I_1^{(2)}$	$\to$	$\cdots$		$\cdots$	$\to$ $\nearrow$	$J_{n^{(2)}-1}^{(2)}$ $I_{n^{(2)}-1}^{(2)}$	$\to$ $\nearrow$	$J_{n^{(2)}}^{(2)}$ $I_{n^{(2)}}^{(2)}$		

- Continue to add new rows until the wandering domain does not enter the bad region again
- Each row corresponds to entering the bad region once
- $n^{(j)}$: time span in good region for row j

Relation Between Horizontal Size with Thickness

Good										Bad		
$J_0^{(0)}$	$\to$	$J_1^{(0)}$	$\to$	$\cdots$			$\to$	$J^{(0)}_{n^{(0)}-1}$	$\to$	$J^{(0)}_{n^{(0)}}$	$\to$	$J^{(0)}_{n^{(0)}+1}$
$l_0^{(0)} = w_0^{(0)}$											$\times$	
$w_0^{(0)}$	$\searrow$	$w_1^{(0)}$					$\searrow$	$w^{(0)}_{n^{(0)}-1}$	$\searrow$	$w^{(0)}_{n^{(0)}}$	$\searrow$	$l^{(0)}_{n^{(0)}+1} = w^{(0)}_{n^{(0)}+1}$
$J_0^{(1)}$	$\to$	$J_1^{(1)}$	$\to$	$\cdots$	$\to$	$J^{(1)}_{n^{(1)}-1}$	$\to$			$J^{(1)}_{n^{(1)}}$	$\to$	$J^{(1)}_{n^{(1)}+1}$
$l_0^{(1)} = w_0^{(1)}$											$\times$	
$w_0^{(1)}$	$\searrow$	$w_1^{(1)}$			$\searrow$	$w^{(1)}_{n^{(1)}-1}$	$\searrow$			$w^{(1)}_{n^{(1)}}$	$\searrow$	$l^{(1)}_{n^{(1)}+1} = w^{(1)}_{n^{(1)}+1}$
$J_0^{(2)}$	$\to$	$J_1^{(2)}$	$\to$	$\cdots$		$\cdots$	$\to$	$J^{(2)}_{n^{(2)}-1}$	$\to$	$J^{(2)}_{n^{(2)}}$		
$l_0^{(2)} = w_0^{(2)}$	$\nearrow$	$l_1^{(2)}$					$\nearrow$	$l^{(2)}_{n^{(2)}-1}$	$\nearrow$	$l^{(2)}_{n^{(2)}}$		
$w_0^{(2)}$												

- If the wandering domain enters the bad region ∞ times, the contraction happens infinity times :(

Relation Between Horizontal Size with Thickness

Good										Bad		
$J_0^{(0)}$	$\to$	$J_1^{(0)}$	$\to$	$\cdots$			$\to$	$J_{n^{(0)}-1}^{(0)}$	$\to$	$J_{n^{(0)}}^{(0)}$	$\to$	$J_{n^{(0)}+1}^{(0)}$
$l_0^{(0)} = w_0^{(0)}$											$\times$	
$w_0^{(0)}$	$\searrow$	$w_1^{(0)}$					$\searrow$	$w_{n^{(0)}-1}^{(0)}$	$\searrow$	$w_{n^{(0)}}^{(0)}$	$\searrow$	$l_{n^{(0)}+1}^{(0)} = w_{n^{(0)}+1}^{(0)}$
$J_0^{(1)}$	$\to$	$J_1^{(1)}$	$\to$	$\cdots$	$\to$	$J_{n^{(1)}-1}^{(1)}$	$\to$			$J_{n^{(1)}}^{(1)}$	$\to$	$J_{n^{(1)}+1}^{(1)}$
$l_0^{(1)} = w_0^{(1)}$											$\times$	
$w_0^{(1)}$	$\searrow$	$w_1^{(1)}$			$\searrow$	$w_{n^{(1)}-1}^{(1)}$	$\searrow$			$w_{n^{(1)}}^{(1)}$	$\searrow$	$l_{n^{(1)}+1}^{(1)} = w_{n^{(1)}+1}^{(1)}$
$J_0^{(2)}$	$\to$	$J_1^{(2)}$	$\to$	$\cdots$		$\cdots$	$\to$	$J_{n^{(2)}-1}^{(2)}$	$\to$	$J_{n^{(2)}}^{(2)}$		
$l_0^{(2)} = w_0^{(2)}$	$\nearrow$	$l_1^{(2)}$					$\nearrow$	$l_{n^{(2)}-1}^{(2)}$	$\nearrow$	$l_{n^{(2)}}^{(2)}$		
$w_0^{(2)}$												

- If the wandering domain enters the bad region ∞ times, the contraction happens infinity times :(
- Key observation: wandering domain can only enter the bad region at most **finitely** many times! :)

Relation Between Horizontal Size with Thickness

Good										Bad		
$J_0^{(0)}$	$\to$	$J_1^{(0)}$	$\to$	$\cdots$			$\to$	$J_{n^{(0)}-1}^{(0)}$	$\to$	$J_{n^{(0)}}^{(0)}$	$\to$	$J_{n^{(0)}+1}^{(0)}$
$l_0^{(0)} = w_0^{(0)}$											$\times$	
$w_0^{(0)}$	$\searrow$	$w_1^{(0)}$					$\searrow$	$w_{n^{(0)}-1}^{(0)}$	$\searrow$	$w_{n^{(0)}}^{(0)}$	$\searrow$	$l_{n^{(0)}+1}^{(0)} = w_{n^{(0)}+1}^{(0)}$
$J_0^{(1)}$	$\to$	$J_1^{(1)}$	$\to$	$\cdots$	$\to$	$J_{n^{(1)}-1}^{(1)}$	$\to$			$J_{n^{(1)}}^{(1)}$	$\to$	$J_{n^{(1)}+1}^{(1)}$
$l_0^{(1)} = w_0^{(1)}$											$\times$	
$w_0^{(1)}$	$\searrow$	$w_1^{(1)}$			$\searrow$	$w_{n^{(1)}-1}^{(1)}$	$\searrow$			$w_{n^{(1)}}^{(1)}$	$\searrow$	$l_{n^{(1)}+1}^{(1)} = w_{n^{(1)}+1}^{(1)}$
$J_0^{(2)}$	$\to$	$J_1^{(2)}$	$\to$	$\cdots$		$\cdots$	$\to$	$J_{n^{(2)}-1}^{(2)}$	$\to$	$J_{n^{(2)}}^{(2)}$		
$l_0^{(2)} = w_0^{(2)}$	$\nearrow$	$l_1^{(2)}$					$\nearrow$	$l_{n^{(2)}-1}^{(2)}$	$\nearrow$	$l_{n^{(2)}}^{(2)}$		
$w_0^{(2)}$												

- Intrinsic reason: The size of bad region is way smaller than the contraction of the thickness

Relation Between Horizontal Size with Thickness

Good										Bad		
$J_0^{(0)}$	$\rightarrow$	$J_1^{(0)}$	$\rightarrow$	$\cdots$			$\rightarrow$	$J_{n^{(0)}-1}^{(0)}$	$\rightarrow$	$J_{n^{(0)}}^{(0)}$	$\rightarrow$	$J_{n^{(0)}+1}^{(0)}$
$l_0^{(0)} = w_0^{(0)}$											$\times$	
$w_0^{(0)}$	$\searrow$	$w_1^{(0)}$					$\searrow$	$w_{n^{(0)}-1}^{(0)}$	$\searrow$	$w_{n^{(0)}}^{(0)}$	$\searrow$	$l_{n^{(0)}+1}^{(0)} = w_{n^{(0)}+1}^{(0)}$
$J_0^{(1)}$	$\rightarrow$	$J_1^{(1)}$	$\rightarrow$	$\cdots$	$\rightarrow$	$J_{n^{(1)}-1}^{(1)}$	$\rightarrow$			$J_{n^{(1)}}^{(1)}$	$\rightarrow$	$J_{n^{(1)}+1}^{(1)}$
$l_0^{(1)} = w_0^{(1)}$											$\times$	
$w_0^{(1)}$	$\left\Vert\epsilon^{(1)}\right\Vert$	$w_1^{(1)}$			$\left\Vert\epsilon^{(1)}\right\Vert$	$w_{n^{(1)}-1}^{(1)}$	$\left\Vert\epsilon^{(1)}\right\Vert$			$w_{n^{(1)}}^{(1)}$	$\left\Vert\epsilon^{(1)}\right\Vert$	$l_{n^{(1)}+1}^{(1)} = w_{n^{(1)}+1}^{(1)}$
$J_0^{(2)}$	$\rightarrow$	$J_1^{(2)}$	$\rightarrow$	$\cdots$		$\cdots$	$\rightarrow$	$J_{n^{(2)}-1}^{(2)}$	$\rightarrow$	$J_{n^{(2)}}^{(2)}$		
$l_0^{(2)} = w_0^{(2)}$	$\nearrow$	$l_1^{(2)}$					$\nearrow$	$l_{n^{(2)}-1}^{(2)}$	$\nearrow$	$l_{n^{(2)}}^{(2)}$		
$w_0^{(2)}$												

- Intrinsic reason: The size of bad region is way smaller than the contraction of the thickness
  1. Contraction: $\left\Vert\epsilon^{(1)}\right\Vert$

Good										Bad		
$J_0^{(0)}$	$\rightarrow$	$J_1^{(0)}$	$\rightarrow$	$\cdots$			$\rightarrow$	$J_{n^{(0)}-1}^{(0)}$	$\rightarrow$	$J_{n^{(0)}}^{(0)}$	$\rightarrow$	$J_{n^{(0)}+1}^{(0)}$
$l_0^{(0)} = w_0^{(0)}$											$\times$	
$w_0^{(0)}$	$\searrow$	$w_1^{(0)}$					$\searrow$	$w_{n^{(0)}-1}^{(0)}$	$\searrow$	$w_{n^{(0)}}^{(0)}$	$\searrow$	$l_{n^{(0)}+1}^{(0)} = w_{n^{(0)}+1}^{(0)}$
$J_0^{(1)}$	$\rightarrow$	$J_1^{(1)}$	$\rightarrow$	$\cdots$	$\rightarrow$	$J_{n^{(1)}-1}^{(1)}$	$\rightarrow$			$J_{n^{(1)}}^{(1)}$	$\rightarrow$	$J_{n^{(1)}+1}^{(1)}$
$l_0^{(1)} = w_0^{(1)}$											$\times$	
$w_0^{(1)}$	$\left\|\epsilon^{(1)}\right\|$	$w_1^{(1)}$			$\left\|\epsilon^{(1)}\right\|$	$w_{n^{(1)}-1}^{(1)}$	$\left\|\epsilon^{(1)}\right\|$			$w_{n^{(1)}}^{(1)}$	$\left\|\epsilon^{(1)}\right\|$	$l_{n^{(1)}+1}^{(1)} = w_{n^{(1)}+1}^{(1)}$
$J_0^{(2)}$	$\rightarrow$	$J_1^{(2)}$	$\rightarrow$	$\cdots$		$\cdots$	$\rightarrow$	$J_{n^{(2)}-1}^{(2)}$	$\rightarrow$	$J_{n^{(2)}}^{(2)}$		
$l_0^{(2)} = w_0^{(2)}$	$\nearrow$	$l_1^{(2)}$					$\nearrow$	$l_{n^{(2)}-1}^{(2)}$	$\nearrow$	$l_{n^{(2)}}^{(2)}$	$< \sqrt{\left\|\epsilon^{(2)}\right\|}$	
$w_0^{(2)}$												

- Intrinsic reason: The size of bad region is way smaller than the contraction of the thickness
  1. Contraction: $\left\|\epsilon^{(1)}\right\|$
  2. Size of bad region: $\sqrt{\left\|\epsilon^{(2)}\right\|}$

Relation Between Horizontal Size with Thickness

Good										Bad		
$J_0^{(0)}$	$\to$	$J_1^{(0)}$	$\to$	$\cdots$			$\to$	$J^{(0)}_{n^{(0)}-1}$	$\to$	$J^{(0)}_{n^{(0)}}$	$\to$	$J^{(0)}_{n^{(0)}+1}$
$l_0^{(0)} = w_0^{(0)}$											$\times$	
$w_0^{(0)}$	$\searrow$	$w_1^{(0)}$					$\searrow$	$w^{(0)}_{n^{(0)}-1}$	$\searrow$	$w^{(0)}_{n^{(0)}}$	$\searrow$	$l^{(0)}_{n^{(0)}+1} = w^{(0)}_{n^{(0)}+1}$
$J_0^{(1)}$	$\to$	$J_1^{(1)}$	$\to$	$\cdots$	$\to$	$J^{(1)}_{n^{(1)}-1}$	$\to$			$J^{(1)}_{n^{(1)}}$	$\to$	$J^{(1)}_{n^{(1)}+1}$
$l_0^{(1)} = w_0^{(1)}$											$\times$	rescaled $k^{(1)}$
$w_0^{(1)}$		$w_1^{(1)}$				$w^{(1)}_{n^{(1)}-1}$				$w^{(1)}_{n^{(1)}}$		$l^{(1)}_{n^{(1)}+1} = w^{(1)}_{n^{(1)}+1}$
$J_0^{(2)}$	$\to$	$J_1^{(2)}$	$\to$	$\cdots$		$\cdots$	$\to$	$J^{(2)}_{n^{(2)}-1}$	$\to$	$J^{(2)}_{n^{(2)}}$		
$l_0^{(2)} = w_0^{(2)}$	$\nearrow$	$l_1^{(2)}$					$\nearrow$	$l^{(2)}_{n^{(2)}-1}$	$\nearrow$	$l^{(2)}_{n^{(2)}}$		
$w_0^{(2)}$												

- Enter bad region = iteration + MANY RESCALING $k^{(1)}$ $\left(2^{k^{(1)}} = \frac{1}{\left\|\epsilon^{(1)}\right\|}\right)$

Relation Between Horizontal Size with Thickness

Good										Bad		
$J_0^{(0)}$	$\to$	$J_1^{(0)}$	$\to$	$\cdots$			$\to$	$J_{n^{(0)}-1}^{(0)}$	$\to$	$J_{n^{(0)}}^{(0)}$	$\to$	$J_{n^{(0)}+1}^{(0)}$
$l_0^{(0)} = w_0^{(0)}$											$\times$	
$w_0^{(0)}$	$\searrow$	$w_1^{(0)}$					$\searrow$	$w_{n^{(0)}-1}^{(0)}$	$\searrow$	$w_{n^{(0)}}^{(0)}$	$\searrow$	$l_{n^{(0)}+1}^{(0)} = w_{n^{(0)}+1}^{(0)}$
$J_0^{(1)}$	$\to$	$J_1^{(1)}$	$\to$	$\cdots$	$\to$	$J_{n^{(1)}-1}^{(1)}$	$\to$			$J_{n^{(1)}}^{(1)}$	$\to$	$J_{n^{(1)}+1}^{(1)}$
$l_0^{(1)} = w_0^{(1)}$											$\times$	rescaled $k^{(1)}$
$w_0^{(1)}$		$w_1^{(1)}$				$w_{n^{(1)}-1}^{(1)}$				$w_{n^{(1)}}^{(1)}$		$l_{n^{(1)}+1}^{(1)} = w_{n^{(1)}+1}^{(1)}$
$J_0^{(2)}$	$\to$	$J_1^{(2)}$	$\to$	$\cdots$		$\cdots$	$\to$	$J_{n^{(2)}-1}^{(2)}$	$\to$	$J_{n^{(2)}}^{(2)}$		
$l_0^{(2)} = w_0^{(2)}$	$\nearrow$	$l_1^{(2)}$					$\nearrow$	$l_{n^{(2)}-1}^{(2)}$	$\nearrow$	$l_{n^{(2)}}^{(2)}$		
$w_0^{(2)}$												

- Enter bad region = iteration + MANY RESCALING $k^{(1)}$ ($2^{k^{(1)}} = \frac{1}{\left\|\epsilon^{(1)}\right\|}$)
- Perturbation decreases super exponentially:

$$\left\|\epsilon^{(2)}\right\| < \left\|\epsilon^{(1)}\right\|^{2^{k^{(1)}}} = \left\|\epsilon^{(1)}\right\|^{\left\|\epsilon^{(1)}\right\|^{-1}}$$

Relation Between Horizontal Size with Thickness

Good										Bad		
$J_0^{(0)}$	$\rightarrow$	$J_1^{(0)}$	$\rightarrow$	$\cdots$			$\rightarrow$	$J_{n^{(0)}-1}^{(0)}$	$\rightarrow$	$J_{n^{(0)}}^{(0)}$	$\rightarrow$	$J_{n^{(0)}+1}^{(0)}$
$l_0^{(0)} = w_0^{(0)}$											$\times$	
$w_0^{(0)}$	$\searrow$	$w_1^{(0)}$					$\searrow$	$w_{n^{(0)}-1}^{(0)}$	$\searrow$	$w_{n^{(0)}}^{(0)}$	$\searrow$	$l_{n^{(0)}+1}^{(0)} = w_{n^{(0)}+1}^{(0)}$
$J_0^{(1)}$	$\rightarrow$	$J_1^{(1)}$	$\rightarrow$	$\cdots$	$\rightarrow$	$J_{n^{(1)}-1}^{(1)}$	$\rightarrow$			$J_{n^{(1)}}^{(1)}$	$\rightarrow$	$J_{n^{(1)}+1}^{(1)}$
$l_0^{(1)} = w_0^{(1)}$											$\times$	rescaled $k^{(1)}$
$w_0^{(1)}$	$\left\lVert\epsilon^{(1)}\right\rVert$	$w_1^{(1)}$			$\left\lVert\epsilon^{(1)}\right\rVert$	$w_{n^{(1)}-1}^{(1)}$	$\left\lVert\epsilon^{(1)}\right\rVert$			$w_{n^{(1)}}^{(1)}$	$\left\lVert\epsilon^{(1)}\right\rVert$	$l_{n^{(1)}+1}^{(1)} = w_{n^{(1)}+1}^{(1)}$
$J_0^{(2)}$	$\rightarrow$	$J_1^{(2)}$	$\rightarrow$	$\cdots$		$\cdots$	$\rightarrow$	$J_{n^{(2)}-1}^{(2)}$	$\rightarrow$	$J_{n^{(2)}}^{(2)}$		
$l_0^{(2)} = w_0^{(2)}$	$\nearrow$	$l_1^{(2)}$					$\nearrow$	$l_{n^{(2)}-1}^{(2)}$	$\nearrow$	$l_{n^{(2)}}^{(2)}$	$< \sqrt{\left\lVert\epsilon^{(2)}\right\rVert}$	
$w_0^{(2)}$												

- Intrinsic reason: The size of bad region contract is way smaller than the contraction of the thickness
  1. Contraction: $\left\lVert\epsilon^{(1)}\right\rVert$
  2. Size of bad region: $\sqrt{\left\lVert\epsilon^{(2)}\right\rVert} = \left\lVert\epsilon^{(1)}\right\rVert^{\left\lVert\epsilon^{(1)}\right\rVert^{-1}}$

Relation Between Horizontal Size with Thickness

Good										Bad		
$J_0^{(0)}$	$\rightarrow$	$J_1^{(0)}$	$\rightarrow$	$\cdots$			$\rightarrow$	$J^{(0)}_{n^{(0)}-1}$	$\rightarrow$	$J^{(0)}_{n^{(0)}}$	$\rightarrow$	$J^{(0)}_{n^{(0)}+1}$
$I_0^{(0)} = w_0^{(0)}$											$\times$	
$w_0^{(0)}$	$\searrow$	$w_1^{(0)}$					$\searrow$	$w^{(0)}_{n^{(0)}-1}$	$\searrow$	$w^{(0)}_{n^{(0)}}$	$\searrow$	$I^{(0)}_{n^{(0)}+1} = w^{(0)}_{n^{(0)}+1}$
$J_0^{(1)}$	$\rightarrow$	$J_1^{(1)}$	$\rightarrow$	$\cdots$	$\rightarrow$	$J^{(1)}_{n^{(1)}-1}$	$\rightarrow$			$J^{(1)}_{n^{(1)}}$	$\rightarrow$	$J^{(1)}_{n^{(1)}+1}$
$I_0^{(1)} = w_0^{(1)}$											$\times$	rescaled $k^{(1)}$
$w_0^{(1)}$		$w_1^{(1)}$				$w^{(1)}_{n^{(1)}-1}$				$w^{(1)}_{n^{(1)}}$		$I^{(1)}_{n^{(1)}+1} = w^{(1)}_{n^{(1)}+1}$
$J_0^{(2)}$	$\rightarrow$	$J_1^{(2)}$	$\rightarrow$	$\cdots$		$\cdots$	$\rightarrow$	$J^{(2)}_{n^{(2)}-1}$	$\rightarrow$	$J^{(2)}_{n^{(2)}}$		
$I_0^{(2)} = w_0^{(2)}$	$\nearrow$	$I_1^{(2)}$					$\nearrow$	$I^{(2)}_{n^{(2)}-1}$	$\nearrow$	$I^{(2)}_{n^{(2)}}$		
$w_0^{(2)}$												

- Intrinsic reason: The size of bad region contract is way smaller than the contraction of the thickness
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- But the proof is much more complicated because $n^{(1)}$ also involves in the contraction

Relation Between Horizontal Size with Thickness

Good										Bad		
$J_0^{(0)}$	$\to$	$J_1^{(0)}$	$\to$	$\cdots$			$\to$	$J_{n^{(0)}-1}^{(0)}$	$\to$	$J_{n^{(0)}}^{(0)}$	$\to$	$J_{n^{(0)}+1}^{(0)}$
$l_0^{(0)} = w_0^{(0)}$											$\times$	
$w_0^{(0)}$	$\searrow$	$w_1^{(0)}$					$\searrow$	$w_{n^{(0)}-1}^{(0)}$	$\searrow$	$w_{n^{(0)}}^{(0)}$	$\searrow$	$l_{n^{(0)}+1}^{(0)} = w_{n^{(0)}+1}^{(0)}$
$J_0^{(1)}$	$\to$	$J_1^{(1)}$	$\to$	$\cdots$	$\to$	$J_{n^{(1)}-1}^{(1)}$	$\to$			$J_{n^{(1)}}^{(1)}$	$\to$	$J_{n^{(1)}+1}^{(1)}$
$l_0^{(1)} = w_0^{(1)}$											$\times$	rescaled $k^{(1)}$
$w_0^{(1)}$		$w_1^{(1)}$				$w_{n^{(1)}-1}^{(1)}$				$w_{n^{(1)}}^{(1)}$		$l_{n^{(1)}+1}^{(1)} = w_{n^{(1)}+1}^{(1)}$
$J_0^{(2)}$	$\to$	$J_1^{(2)}$	$\to$	$\cdots$		$\cdots$	$\to$	$J_{n^{(2)}-1}^{(2)}$	$\to$	$J_{n^{(2)}}^{(2)}$		
$l_0^{(2)} = w_0^{(2)}$	$\nearrow$	$l_1^{(2)}$					$\nearrow$	$l_{n^{(2)}-1}^{(2)}$	$\nearrow$	$l_{n^{(2)}}^{(2)}$		
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- Intrinsic reason: The size of bad region contract is way smaller than the contraction of the thickness
  1. Contraction: $\left\|\epsilon^{(1)}\right\|$
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- But the proof is much more complicated because $n^{(1)}$ also involves in the contraction
- Instead focusing on the change of horizontal size, lets study the lower bound for the time span in the good region versus the number of

Two Rows Lemma (Key Lemma)

Lemma

The time span in the good region $n^{(j)}$ for chain j is bounded below by

$$m^{(j)} > \frac{\ln E}{-2\ln\left\|\epsilon^{(j)}\right\|} m^{(j+1)} + \frac{1}{8}\left(\frac{1}{\left\|\epsilon^{(j)}\right\|}\right)^{\alpha} + \frac{1}{-2\ln\left\|\epsilon^{(j)}\right\|}\ln l_0^{(j)}$$

The Two Row Lemma says:

Two Rows Lemma (Key Lemma)

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$$\boldsymbol{m}^{(\boldsymbol{j})} > \frac{\ln E}{-2\ln\left\|\epsilon^{(j)}\right\|}\boldsymbol{m}^{(\boldsymbol{j}+\mathbf{1})} + \frac{1}{8}\left(\frac{1}{\left\|\epsilon^{(j)}\right\|}\right)^{\alpha} + \frac{1}{-2\ln\left\|\epsilon^{(j)}\right\|}\ln l_0^{(j)}$$

The Two Row Lemma says:

- $n^{(j)}$ is large

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The Two Row Lemma says:

- $n^{(j)}$ is large
- Gives the relation of three objects in two rows:

Two Rows Lemma (Key Lemma)

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The time span in the good region $n^{(j)}$ for chain j is bounded below by

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The Two Row Lemma says:

- $n^{(j)}$ is large
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    - Time span in the good region

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$$\text{row } j \qquad \overbrace{l_0^{(j)} = w_0^{(j)} \xrightarrow{\left\|\epsilon^{(j)}\right\|} \cdots \xrightarrow{\left\|\epsilon^{(j)}\right\|} w_{n^{(j)}}^{(j)} \xrightarrow{\left\|\epsilon^{(j)}\right\|} w_{n^{(j)}+1}^{(j)}}^{\text{contraction } \left\|\epsilon^{(j)}\right\|^{n^{(j)}+1}}$$

$$\text{row } j+1 \qquad \underbrace{l_0^{(j+1)} = w_{n^{(j)}+1}^{(j)} \xrightarrow{E} \cdots \xrightarrow{E} l_{n^{(j+1)}}^{(j+1)}}_{\text{expansion } E^{n^{(j+1)}}} \qquad < \sqrt{\left\|\epsilon^{(j+1)}\right\|} \sim \left\|\epsilon^{(j)}\right\|^{\left\|\epsilon^{(j)}\right\|^{-\alpha}}$$

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row j $\quad \overbrace{l_0^{(j)} = w_0^{(j)} \xrightarrow{\left\|\epsilon^{(j)}\right\|} \cdots \xrightarrow{\left\|\epsilon^{(j)}\right\|} w_{n^{(j)}}^{(j)} \xrightarrow{\left\|\epsilon^{(j)}\right\|} w_{n^{(j)}+1}^{(j)}}^{\text{contraction } \left\|\epsilon^{(j)}\right\|^{n^{(j)}+1}}$

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row j

$$\overbrace{l_0^{(j)} = w_0^{(j)} \xrightarrow{\left\|\epsilon^{(j)}\right\|} \cdots \xrightarrow{\left\|\epsilon^{(j)}\right\|} w_{n^{(j)}}^{(j)} \xrightarrow{\left\|\epsilon^{(j)}\right\|} w_{n^{(j)}+1}^{(j)}}^{\text{contraction } \left\|\epsilon^{(j)}\right\|^{n^{(j)}+1}}$$

row $j+1$

$$\underbrace{l_0^{(j+1)} = w_{n^{(j)}+1}^{(j)} \xrightarrow{E} \cdots \xrightarrow{E} l_{n^{(j+1)}}^{(j+1)}}_{\text{expansion } E^{n^{(j+1)}}} \qquad < \sqrt{\left\|\epsilon^{(j+1)}\right\|} \sim \left\|\epsilon^{(j)}\right\|^{\left\|\epsilon^{(j)}\right\|^{-\alpha}}$$

Finite Rows

Instead of targeting on the horizontal size, we study the relation between **the time span in the good region** versus **the number of times entering the bad region.**

$$m^{(j)} > \frac{\ln E}{-2\ln\left\|\epsilon^{(j)}\right\|} m^{(j+1)} + \frac{1}{8}\left(\frac{1}{\left\|\epsilon^{(j)}\right\|}\right)^{\alpha} + \frac{1}{-2\ln\left\|\epsilon^{(j)}\right\|} \ln l_0^{(j)}$$

$$\Downarrow$$

$$m^{(j)} > \left\|\epsilon^{(j)}\right\| m^{(j+1)} + \left(\frac{1}{\left\|\epsilon^{(j)}\right\|}\right)^{\alpha}$$

Here, we assume all constants $= 1$, ignore the contribution from $l_0^{(j)}$, and apply

$$-\ln\left\|\epsilon^{(j)}\right\| = \ln\frac{1}{\left\|\epsilon^{(j)}\right\|} < \frac{1}{\left\|\epsilon^{(j)}\right\|}$$

Finite Rows

Two row lemma:

$$m^{(j)} > \left\| \epsilon^{(j)} \right\| m^{(j+1)} + \left(\frac{1}{\left\| \epsilon^{(j)} \right\|} \right)^{\alpha}$$

enters the bad region 2 times

$$m^{(0)} > 0 + \left(\frac{1}{\left\| \epsilon^{(0)} \right\|} \right)^{\alpha}$$

Finite Rows

Two row lemma:

$$m^{(j)} > \left\| \epsilon^{(j)} \right\| m^{(j+1)} + \left(\frac{1}{\left\| \epsilon^{(j)} \right\|} \right)^{\alpha}$$

enters the bad region 2 times

$$m^{(0)} \quad > 0 + \left(\frac{1}{\left\| \epsilon^{(0)} \right\|} \right)^{\alpha}$$

enters the bad region 3 times

$$m^{(1)} \quad > 0 + \left(\frac{1}{\left\| \epsilon^{(1)} \right\|} \right)^{\alpha}$$

Finite Rows

Two row lemma:

$$m^{(j)} > \left\|\epsilon^{(j)}\right\| m^{(j+1)} + \left(\frac{1}{\left\|\epsilon^{(j)}\right\|}\right)^{\alpha}$$

enters the bad region 2 times

$$m^{(0)} \quad > 0 + \left(\frac{1}{\left\|\epsilon^{(0)}\right\|}\right)^{\alpha}$$

enters the bad region 3 times

$$\begin{aligned}
m^{(1)} &> 0 + \left(\frac{1}{\left\|\epsilon^{(1)}\right\|}\right)^{\alpha} \\
m^{(0)} &> \left\|\boldsymbol{\epsilon}^{(\mathbf{0})}\right\| \left(\frac{\mathbf{1}}{\left\|\boldsymbol{\epsilon}^{(\mathbf{1})}\right\|}\right)^{\boldsymbol{\alpha}} + \left(\frac{1}{\left\|\epsilon^{(0)}\right\|}\right)^{\alpha} \\
&> \left\|\epsilon^{(0)}\right\| \left(\frac{1}{\left\|\epsilon^{(0)}\right\|}\right)^{\alpha\left\|\epsilon^{(0)}\right\|^{-\alpha}} + \left(\frac{1}{\left\|\epsilon^{(0)}\right\|}\right)^{\alpha} \\
&> \left(\frac{1}{\left\|\epsilon^{(0)}\right\|}\right)^{\alpha\left\|\epsilon^{(0)}\right\|^{-\alpha}-1}
\end{aligned}$$

Nonexistence of wandering domain

- Enter bad region $\infty \Longrightarrow$ time span in the good region $\to \infty$

Proposition (Ou, 2016)

A sequence of wandering domain cannot enter the bad region infinitely times

Nonexistence of wandering domain

Summary

- Prove by contradiction: assume a wandering domain J

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Nonexistence of wandering domain

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- Prove by contradiction: assume a wandering domain J
- Generate a sequence of wandering domain J_n
- Study the change of horizontal size l_n
  - Good region: l_n expands
  - Bad region: l_n contracts

Nonexistence of wandering domain

Summary

- Prove by contradiction: assume a wandering domain J
- Generate a sequence of wandering domain J_n
- Study the change of horizontal size I_n
  - Good region: I_n expands (∞ times)
  - Bad region: I_n contracts (finite times)

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- Prove by contradiction: assume a wandering domain J
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- Study the change of horizontal size l_n
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  - Bad region: l_n contracts (finite times)
- $l_n \to \infty$ contradiction

Nonexistence of wandering domain

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- Prove by contradiction: assume a wandering domain J
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  - Good region: I_n expands (∞ times)
  - Bad region: I_n contracts (finite times)
- $I_n \to \infty$ contradiction

Theorem (2016, Ou)

A strongly dissipative infinite renormalizable Hénon-like map does not have wandering domains