

# Communicating with convexity

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“Nothing takes place in the world whose meaning is not that of some maximum or minimum.”

LEONARD EULER (1707-1783)

## INTRODUCTION

Convexity is an important property in optimisation. This is because if a problem is convex then the task of finding a global minimum is reduced to that of finding a local minimum. The importance of finding these minima efficiently in science and engineering has driven the development of software packages, such as `cvxpy`.

Here we show that an interesting problem in communications has a convex formulation that can be easily implemented computationally using the `cvxpy` *Python* module. This demonstrates `cvxpy` as an invaluable tool for both students and researchers in many areas of science and engineering. First let us introduce what convexity means.

### Convex functions

A function  $f$ , where  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , is *convex* if  $\forall x, y \in \text{dom} f$  and  $0 \leq \theta \leq 1$  [1, p.67]:

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y). \quad (1)$$

This means that the function is less than or equal to linear, as shown in Fig. 1.

### Convex optimisation

A *convex optimisation problem* has three components:

- a convex objective function  $f_0(x)$ ,
- $m$  convex inequality constraint functions  $f_i(x)$ ,
- $k$  convex equality constraint functions  $g_j(x)$ ,

where  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$  [1, p.141]. We seek an optimum,  $x^* \in \mathbb{R}^n$ , where  $f_0(x^*)$  is a minimum. Formally the problem is defined as:

$$\begin{aligned} &\text{minimise} && f_0(x) \\ &\text{subject to} && f_i(x) \leq 0, && i \in \{1, \dots, m\} \\ &&& g_j(x) = 0, && j \in \{1, \dots, k\}. \end{aligned} \quad (2)$$

Any local optimum  $x^*$  for a convex optimisation problem is also a global optimum [1, pp.138-139], however an optimum may not be unique.

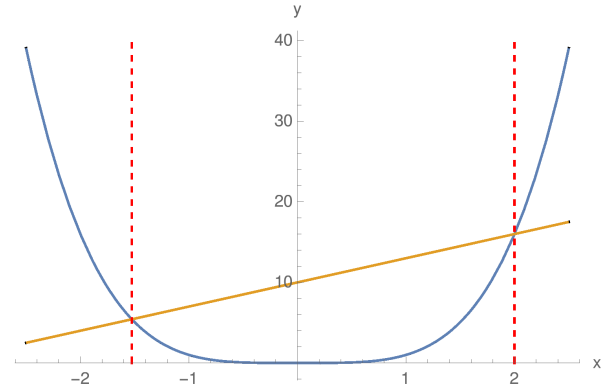


Figure 1. A chord showing convexity of the function  $x^4$ .

### Disciplined convex programming

In order to solve convex optimisation problems, we used `cvxpy`, a symbolic programming module for *Python*[3]. It uses a set of rules, called *disciplined convex programming* (DCP), to determine whether a function is convex. This is implemented using predefined classes containing functions with their curvature and sign, and using general composition theorems from convex analysis [2]. If DCP rules can be applied then interior point methods guarantee an optimal solution.

### EXAMPLE: POWER MINIMISATION IN COMMUNICATIONS

To demonstrate the applicability of `cvxpy` to a real-world example, we consider a system of  $n$  transmitters each of power  $p_i$  and  $m$  receivers, all in 2D Euclidean space[5]. In this problem we are constrained to having a minimum signal-to-interference-plus-noise ratio (SINR) at each receiver; the strength of the desired signal,  $S$ , relative to

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the interference power,  $I$ , plus the background noise,  $\sigma$ , at a receiver. How can we minimise the total power consumption,  $P$ , of transmitters, yet achieve this minimum SINR,  $\gamma_0$ , for all receivers? This question is relevant to telecoms companies who want to offer a service at a minimum quality standard. We formulate the problem with a given square path gain matrix,  $G$ , background noise level  $\sigma$  and a maximum power constraint  $P_{\max}$  of each transmitter:

$$\begin{aligned} & \underset{p}{\text{minimise}} && \sum_j p_j \\ & \text{subject to} && p_j \leq P_{\max} \\ & && p_j \geq 0 \\ & && \gamma_i \geq \gamma_0. \end{aligned}$$

The desired signal for receiver  $i$  is  $S_i = G_{ik}p_k$ , while the interference at  $i$  is  $I_i = \sum_{j \neq k} G_{ij}p_j$ . However DCP does not allow division of the optimisation variable  $p_i$  so

$$\gamma_i \geq \gamma_0 \iff \frac{S_i}{\sigma_i + I_i} \geq \gamma_0, \quad \forall i$$

was rearranged to,

$$S_i - \gamma_0(\sigma_i + I_i) \geq 0, \quad \forall i$$

which is affine, and hence a DCP function.

### Path gain

The path gain  $G_{ij}$  represents the proportion of power that reaches receiver  $i$  from transmitter  $j$ . Supposing that the desired signal to receiver  $i$  is from transmitter  $k$ , the power received is,

$$p_{ik}^{\text{rec}} = G_{ik}p_k. \quad (3)$$

Denoting the SINR at  $i$  by  $\gamma_i$ , we have

$$\gamma_i = \frac{G_{ik}p_k}{\sigma_i + \sum_{j \neq k} G_{ij}p_j} = \frac{S_i}{\sigma_i + I_i}. \quad (4)$$

In a physical context, the path gain from transmitter  $j$  to receiver  $i$ , as shown in Fig. 2, will depend on the distance,  $d_{ij}$  between them. Assuming isotropic propagation from each transmitter, the fraction of power from  $j$  that reaches  $i$  is:

$$G_{ij} = k_i / d_{ij}^\alpha \quad (5)$$

where  $\alpha$  is the pathloss coefficient and  $k$  is a proportionality constant specific to the receiver  $i$ . For free space  $\alpha = 2$ , while in urban environments  $\alpha \sim 3.5$  [4]. Further physical complexities, such as the stochastic effects from Rayleigh fading can be easily incorporated.

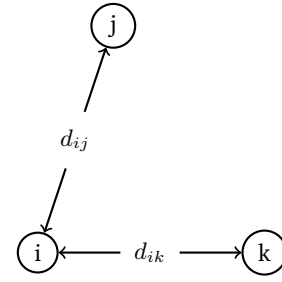


Figure 2. Path lengths between receiver  $i$  and transmitters  $j$  and  $k$ .

### Example code

To indicate the brevity of code implementation in cvxpy we show a minimal working example below:

```
import cvxpy as cvx
import numpy as np

# Define variables
n = 3
k = 1.0
d = np.random.rand(n,n)
d_s = 0.5 * (d + d.T)

G = np.zeros((n,n))
G = k / d_s ** 3.5

delta = np.identity(n)
S = G * delta
I = G - S

sigma = 0.1 * np.ones(n)
gamma = 1.0
Pmax = 1.0

# Define optimisation variable
p = cvx.Variable(n)

# Define objective function
obj = cvx.Minimize(
    cvx.sum_entries(p))

# Define constraints
constraints = [p >= 0,
               S * p - gamma * (I * p + sigma) >= 0,
               p <= Pmax]

# Solve problem and print solution
prob = cvx.Problem(obj, constraints)
prob.solve()
print("Solution status = %s"%(prob.status))
print("Optimal solution = %s"%(prob.value))
print("Power settings = %s"%(p.value))
```

### OUTLOOK

The authors hope that this article shows the utility of cvxpy for simple ‘toy’ problems of real-world relevance. We believe it is an invaluable resource for researchers wanting to test the applicability of convex optimisation in new fields.

The module could also be easily incorporated into a course on optimisation, providing students with an accessible environment to practise techniques for convex optimisation. Full code for other communication examples can be found at <https://github.com/cvxgrp/cvxpy/tree/master/examples/communications>.

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### Author Biography

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