1 Derivation

To solve this minimization we rewrite the energy as follows ($Id|_{constr}$ was replaced by C for more readability):

$$\begin{aligned} E_{warp} &= \|Lx' - Lx\|^2 + \lambda\|Cx' - c\|^2 \\ &= (Lx' - Lx)^T(Lx' - Lx) + \lambda(Cx' - c)^T(Cx' - c) \\ &= x'^T L^T Lx' - 2x'^T L^T Lx + x^T L^T Lx + \lambda x'^T C^T Cx' - 2\lambda x'^T C^T c + c^T c. \end{aligned}$$

Using $\frac{\partial x^T a}{\partial x} = a$ and $\frac{\partial x^T Bx}{\partial x} = (B + B^T)x$ from the Matrix Cookbook and the equality $A^T A + (A^T A)^T = 2A^T A$, we derive

$$\frac{\partial}{\partial x'} E_{warp} = 2L^T Lx' - 2L^T Lx + 2\lambda C^T Cx' - 2\lambda C^T c.$$

To minimize the energy, we set the result to 0, then we divide by 2 and rearrange the terms to obtain the system of equations

$$(L^T L + \lambda C^T C)x' = L^T Lx + \lambda C^T c.$$

2 Constraints

We have three types of constraints:

- Boundary: We fix the vertices on the boundary of the template, meaning that the targets are their current positions.
- Landmarks: The template landmarks are constrained to the corresponding scan landmarks.
- Vertices close to the scan mesh: For each vertex of the template mesh we determine the closest vertex of the scanned mesh and if their distance is below a threshold provided by the user, we add the template vertex to the constraint vertices and the corresponding scan vertex as its target. To make the search for the closest scan vertex more efficient, we used a modified version of the spatial index implemented in Assignment 2.

We made the use of the landmark constraints optional. The default is set to not using them as they sometimes seem to cause small artifacts in the warped mesh. Depending on the parameters and inputs, there can be small areas around the landmarks which stick outward or inward of the face.

Finally, we decided to prioritize constraints in the case that one vertex applied to multiple constraint types to keep the computation simple. The priority is the above list in descending order, i.e. the boundary has the highest priority.