Adaptives:

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Abstract

1 Introduction

2 The Monoidal Double Category $\mathbb{O}\mathbf{rg}$

3 $\mathbb{O}\mathbf{rg}$-Enrichment as Dynamical Structure

A monoidal double category is a viable setting for enriching various categorical structures (using the notions of enrichment in [Lei99] and [Sha22]). Generally speaking, enrichment in $\mathbb{O}\mathbf{rg}$ replaces the usual set of arrows between two objects in a category or similar structure with a $[p, q]$-coalgebra for some choice of polynomials p, q. Therefore not only can the arrows be realized as maps of polynomials $p \to q$, but these maps carry dynamics that encode how a position in p and a direction in q determine a transition to one arrow to another.

3.1 $\mathbb{O}\mathbf{rg}$-enriched categories

Enrichment of categories only makes use of the double category structure of $\mathbb{O}\mathbf{rg}$, as any double category forms an fc-multicategory (also known as a virtual double category) in the sense of [Lei99]. The following definition of enrichment in $\mathbb{O}\mathbf{rg}$ is an unwinded version of [Lei99], which defines categories enriched in any fc-multicategory.

Definition 3.1. An $\mathbb{O}\mathbf{rg}$-enriched category A consists of

- A set A_0 of objects
- For each $a \in A_0$, a polynomial p_a
- For each $a, b \in A_0$, a $[p_a, p_b]$-coalgebra $S_{a,b}$
- For each $a \in A_0$, a square in $\mathbb{O}\mathbf{rg}$ as below left

- For each $a, b, c \in A_0$, a square in $\mathbb{O}\mathbf{rg}$ as below right

$$
\begin{tikzcd}
p_a \arrow[r, "\mathrm{id}_{p_a}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & p_a \arrow[d, equal] \\
p_a \arrow[r, "S_{a,a}"'] & p_a
\end{tikzcd}
\qquad
\begin{tikzcd}
p_a \arrow[r, "S_{a,b}"] \arrow[d, equal] & p_b \arrow[r, "S_{b,c}"] \arrow[d, phantom, "\Downarrow"] & p_c \arrow[d, equal] \\
p_a \arrow[rr, "S_{a,c}"'] & & p_c
\end{tikzcd}
$$

such that these squares satisfy unit and associativity equations. ◊

The underlying sets $S_{a,b}$ of the coalgebras form an underlying ordinary category of A.

3.2 $\mathbb{O}$rg-enriched operads

A monoidal double category also gives rise to an fm-multicategory in the sense of [Lei99], so it makes sense to talk about multicategories enriched in $\mathbb{O}\mathbf{rg}$ as in [Lei99].

Definition 3.2. An $\mathbb{O}\mathbf{rg}$-enriched multicategory A consists of

- A set A_0 of objects
- For each $a \in A_0$, a polynomial p_a
- For each $a_1, \ldots, a_n, b \in A_0$, a $[p_{a_1} \otimes \cdots \otimes p_{a_n}, p_b]$-coalgebra $S_{a_1,\ldots,a_n;b}$
- For each $a \in A_0$, a square in $\mathbb{O}\mathbf{rg}$ as below left
- For each $a_{1,1}, \ldots, a_{1,m_1}, a_{2,1}, \ldots, a_{n,m_n}, b_1, \ldots, b_n, c \in A_0$, a square in $\mathbb{O}\mathbf{rg}$ as below right

$$
\begin{tikzcd}
p_a \arrow[r, "\mathrm{id}_{p_a}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & p_a \arrow[d, equal] \\
p_a \arrow[r, "S_{a;a}"'] & p_a
\end{tikzcd}
\qquad
\begin{tikzcd}
p_{a_{1,1}} \otimes \cdots \otimes p_{a_{n,m_n}} \arrow[r, "\otimes_i S_{a_{i,1},\ldots,a_{i,m_i};b_i}"] \arrow[d, equal] & p_{b_1} \otimes \cdots \otimes p_{b_n} \arrow[r, "S_{b_1,\ldots,b_n;c}"] \arrow[d, phantom, "\Downarrow"] & p_c \arrow[d, equal] \\
p_{a_{1,1}} \otimes \cdots \otimes p_{a_{n,m_n}} \arrow[rr, "S_{a_{1,1},\ldots,a_{n,m_n};c}"'] & & p_c
\end{tikzcd}
$$

such that these squares satisfy unit and associativity equations. ◊

There is similarly an underlying multicategory of A given by the sets $S_{a_1,\ldots,a_n;b}$. We will mostly be interested in the case when A has only one object, in which case we call it an $\mathbb{O}\mathbf{rg}$-enriched operad.

3.3 $\mathbb{O}$rg-enriched monoidal categories: adaptives

A monoidal double category is precisely a representable MC-multicategory in the sense of [Sha22], so we can also enrich strict monoidal categories in $\mathbb{O}\mathbf{rg}$.[1] These will be similar to $\mathbb{O}\mathbf{rg}$-enriched multicategories but allow for many-to-many coalgebras rather than just many-to-1.

Definition 3.3. An $\mathbb{O}\mathbf{rg}$-enriched strict monoidal category A consists of

- A monoid A_0 of objects
- For each $a \in A_0$, a polynomial p_a

[1] We use throughout the notion *strong* enrichment in a monoidal double category from [Sha22].

- An isomorphism of polynomials $y \cong p_e$ for e the unit of A_0
- For each $a, a' \in A_0$, an isomorphism of polynomials $p_a \otimes p_{a'} \cong p_{aa'}$
- For each $a, b \in A_0$, a $[p_a, p_b]$-coalgebra $S_{a,b}$
- For each $a \in A_0$, a square in $\mathbb{O}\mathbf{rg}$ as below left
- For each $a, b, c \in A_0$, a square in $\mathbb{O}\mathbf{rg}$ as below center
- For each $a, a', b, b' \in A_0$, a square in $\mathbb{O}\mathbf{rg}$ as below right

$$
\begin{tikzcd}
p_a \arrow[r, "\mathrm{id}_{p_a}", "\shortmid"{marking}] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow"] & p_a \arrow[d, equal] \\
p_a \arrow[r, "S_{a;a}"', "\shortmid"{marking}] & p_a
\end{tikzcd}
\qquad
\begin{tikzcd}
p_a \arrow[r, "S_{a,b}", "\shortmid"{marking}] \arrow[d, equal] & p_b \arrow[r, "S_{b,c}", "\shortmid"{marking}] \arrow[d, phantom, "\Downarrow"] & p_c \arrow[d, equal] \\
p_a \arrow[rr, "S_{a,c}"', "\shortmid"{marking}] & & p_c
\end{tikzcd}
\qquad
\begin{tikzcd}
p_a \otimes p_{a'} \arrow[r, "S_{a,b} \otimes S_{a',b'}", "\shortmid"{marking}] \arrow[d, equal, "\wr"'] \arrow[dr, phantom, "\Downarrow"] & p_b \otimes p_{b'} \arrow[d, equal, "\wr"'] \\
p_{aa'} \arrow[r, "S_{aa',bb'}"', "\shortmid"{marking}] & p_{bb'}
\end{tikzcd}
$$

such that these isomorphisms and squares satisfy unit, associativity, and interchange equations. ◊

Here the sets $S_{a,b}$ for the arrows in the underlying strict monoidal category of A.

Definition 3.4. An *adaptive* is an $\mathbb{O}\mathbf{rg}$-enriched strict monoidal category with object monoid $\mathbb{N}$. ◊

- Collectives
- Multi-collectives
- Dynamical systems
- Multi-categories
- Initial and terminal

4 Basic theory of adaptives

4.1 Change of base adjunction

4.2 Populating adaptives

5 Gradient descent example

A Proofs

References

[Lei99] Tom Leinster. *Generalized enrichment for categories and multicategories*. arXiv:9901139. 1999. arXiv: `9901139 [math.CT]` (cit. on pp. 1, 2).

[Sha22] Brandon Shapiro. *Enrichment of Algebraic Higher Categories*. *In Preparation*. 2022 (cit. on pp. 1, 2).