Adaptives:

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Abstract

- 1 Introduction
- 2 The Monoidal Double Category Org

3 Org-Enrichment as Dynamical Structure

A monoidal double category is a viable setting for enriching various categorical structures (using the notions of enrichment in [Lei99] and [Sha22]). Generally speaking, enrichment in $\mathbb{O}\mathbf{rg}$ replaces the usual set of arrows between two objects in a category or similar structure with a [p,q]-coalgebra for some choice of polynomials p,q. Therefore not only can the arrows be realized as maps of polynomials $p \to q$, but these maps carry dynamics that encode how a position in p and a direction in q determine a transition to one arrow to another.

3.1 Org-enriched categories

Enrichment of categories only makes use of the double category structure of \mathbb{O} **rg**, as any double category forms an fc-multicategory (also known as a virtual double category) in the sense of [Lei99]. The following definition of enrichment in \mathbb{O} **rg** is an unwinded version of [Lei99], which defines categories enriched in any fc-multicategory.

Definition 3.1. An Org-enriched category *A* consists of

- A set A_0 of objects
- For each $a \in A_0$, a polynomial p_a
- For each $a, b \in A_0$, a $[p_a, p_b]$ -coalgebra $S_{a,b}$
- For each $a \in A_0$, a square in \mathbb{O} **rg** as below left

• For each $a, b, c \in A_0$, a square in \mathbb{O} **rg** as below right

 \Diamond

such that these squares satisfy unit and associativity equations.

The underlying sets $S_{a,b}$ of the coalgebras form an underlying ordinary category of A.

3.2 Org-enriched operads

A monoidal double category also gives rise to an fm-multicategory in the sense of [Lei99], so it makes sense to talk about multicategories enriched in \mathbb{O} **rg** as in [Lei99].

Definition 3.2. An Org-enriched multicategory *A* consists of

- A set A_0 of objects
- For each $a \in A_0$, a polynomial p_a
- For each $a_1, ..., a_n, b \in A_0$, a $[p_{a_1} \otimes \cdots \otimes p_{a_n}, p_b]$ -coalgebra $S_{a_1,...,a_n,b}$
- For each $a \in A_0$, a square in \mathbb{O} **rg** as below left
- For each $a_{1,1}, ..., a_{1,m_1}, a_{2,1}, ..., a_{n,m_n}, b_1, ..., b_n, c \in A_0$, a square in \mathbb{O} **rg** as below right

such that these squares satisfy unit and associativity equations.

There is similarly an underlying multicategory of A given by the sets $S_{a_1,...,a_n;b}$. We will mostly be interested in the case when A has only one object, in which case we call it an \mathbb{O} **rg**-enriched operad.

3.3 Org-enriched monoidal categories: adaptives

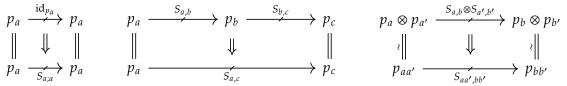
A monoidal double category is precisely a representable MC-multicategory in the sense of [Sha22], so we can also enrich strict monoidal categories in $\mathbb{O}\mathbf{rg}$. These will be similar to $\mathbb{O}\mathbf{rg}$ -enriched multicategories but allow for many-to-many coalgebras rather than just many-to-1.

Definition 3.3. An Org-enriched strict monoidal category *A* consists of

- A monoid A_0 of objects
- For each $a \in A_0$, a polynomial p_a

¹We use throughout the notion *strong* enrichment in a monoidal double category from [Sha22].

- An isomorphism of polynomials $y \cong p_e$ for e the unit of A_0
- For each $a, a' \in A_0$, an isomorphism of polynomials $p_a \otimes p_{a'} \cong p_{aa'}$
- For each $a, b \in A_0$, a $[p_a, p_b]$ -coalgebra $S_{a,b}$
- For each $a \in A_0$, a square in \mathbb{O} **rg** as below left
- For each $a, b, c \in A_0$, a square in \mathbb{O} **rg** as below center
- For each a, a', b, $b' \in A_0$, a square in \mathbb{O} **rg** as below right



such that these isomorphisms and squares satisfy unit, associativity, and interchange equations. \diamond

Here the sets $S_{a,b}$ for the arrows in the underlying strict monoidal category of A.

Definition 3.4. An *adaptive* is an \mathbb{O} **rg**-enriched strict monoidal category with object monoid \mathbb{N} .

- Collectives
- Multi-collectives
- Dynamical systems
- Multi-categories
- Initial and terminal

4 Basic theory of adaptives

4.1 Change of base adjunction

4.2 Populating adaptives

5 Gradient descent example

A Proofs

References

- [Lei99] Tom Leinster. *Generalized enrichment for categories and multicategories*. arXiv:9901139. 1999. arXiv: 9901139 [math.CT] (cit. on pp. 1, 2).
- [Sha22] Brandon Shapiro. *Enrichment of Algebraic Higher Categories*. *In Preparation*. 2022 (cit. on pp. 1, 2).