Adaptives:

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Abstract

1 Introduction

2 Categorical preliminaries

2.1 PROPS

Definition 2.1. A PROP P consists of a set of objects $\mathrm{Ob}(P)$ and a symmetric strict monoidal category whose monoid of objects is freely generated by $\mathrm{Ob}(P)$. A morphism of PROPs is a strict monoidal functor that preserves generating objects. ◊

While it is technically convenient to regard a PROP as a monoidal category, it can also be thought of as a structure similar to a polycategory, with a set of objects $\mathrm{Ob}(P)$ and sets of many-to-many morphisms $\mathrm{Hom}(x_1, ..., x_m; x'_1, ..., x'_n)$ equipped with composition, units, and permutations of the domain and codomain objects. These many-to-many morphisms correspond to $\mathrm{Hom}(x_1 \otimes \cdots \otimes x_m, x'_1 \otimes \cdots \otimes x'_n)$ in the corresponding monoidal category.

When P has only one object x, we can write $\mathrm{Hom}(m, n)$ for the set of morphisms $x^{\otimes m} \to x^{\otimes n}$. In this case P is simply a monoidal category with object monoid $\mathbb{N}$.

Example 2.2.
- The monoid $\mathbb{N}$ regarded as a discrete monoidal category is a PROP with one object and $\mathrm{Hom}(m, n)$ either empty when $m \neq n$ or containing only the identity when $m = n$.
- The terminal PROP has one object and a unique morphism in each $\mathrm{Hom}(m, n)$.
- A skeleton of the category of finite sets forms a prop. The one generator is the singleton set and the monoidal structure is given by the empty set and disjoint union.

◊

2.2 The monoidal double category $\mathbb{O}$rg

2.3 Lax functors to monoidal double categories

In order to state the definition of an adaptive, we recall various notions of lax functors out of monoidal categories.

Definition 2.3. For $\mathbf{C}$ a category and $\mathbf{D}$ a bicategory, a lax functor $F : \mathbf{C} \to \mathbf{D}$ consists of

- an assignment $F : \mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{D})$
- for each $c, c' \in \mathrm{Ob}(\mathbf{C})$, functions

$$F : \mathrm{Hom}_{\mathbf{C}}(c, c') \to \mathrm{Ob}(\mathbf{D}(F(c), F(c')))$$

- for each $c \in \mathrm{Ob}(\mathbf{C})$, an "identitor" 2-cell in $\mathbf{D}(F(c), F(c))$

$$F_{\mathrm{id}_c} : \mathrm{id}_{F(c)} \Rightarrow F(\mathrm{id}_c)$$

- for each $f : c \to c'$ and $f' : c' \to c''$ in $\mathbf{C}$, a "compositor" 2-cell in $\mathbf{D}(F(c), F(c''))$

$$F_{f \circ f'} : F(f') \circ F(f) \Rightarrow F(f' \circ f)$$

such that the identitors and compositors satisfy unit and associativity equations. ◊

The equations here ensure a unique lax structure map

$$F_{f_1 \circ \cdots \circ f_n} : F(f_n) \circ \cdots \circ F(f_1) \Rightarrow F(f_n \circ \cdots \circ f_1)$$

built out of composites of whiskerings of the identitors and compositors, for all $n \geq 0$ and composable morphisms $f_1, \ldots, f_n$.

Definition 2.4. For $\mathbf{C}$ a monoidal category and $\mathbf{D}$ a monoidal double category, a lax monoidal lax functor $F : \mathbf{C} \to \mathbf{D}$ consists of

- a lax functor $F : \mathbf{C} \to \mathbf{D}$
- an "identitor" vertical morphism in $\mathbf{D}$

$$F_I : I_{\mathbf{D}} \to F(I_{\mathbf{C}})$$

- for each $c_1, c_2 \in \mathrm{Ob}(\mathbf{C})$, a "productor" vertical morphism in $\mathbf{D}$

$$F_{c_1, c_2} : F(c_1) \otimes F(c_2) \to F(c_1 \otimes c_2)$$

- for each $f_1 : c_1 \to c_1'$ and $f_2 : c_2 \to c_2'$ in $\mathbf{C}$, a "productor" square $F_{f_1 \otimes f_2}$ in $\mathbf{D}$

$$\begin{tikzcd}
F(c_1) \otimes F(c_2) \arrow[r, "F_{c_1,c_2}"] \arrow[d, "F(f_1)\otimes F(f_2)"'] \arrow[dr, phantom, "\Downarrow"] & F(c_1 \otimes c_2) \arrow[d, "F(f_1 \otimes f_2)"] \\
F(c_1') \otimes F(c_2') \arrow[r, "F_{c_1',c_2'}"'] & F(c_1' \otimes c_2')
\end{tikzcd}$$

such that the identitor and productors satisfy unit and associativity equations up to coherent isomorphisms, along with interchange equations with respect to the identitors and compositors coming from the lax functoriality of F. ◊

These equations guarantee that for any m strings of n composable morphisms in $\mathbf{C}$, there is a unique coherence square from the result of first applying F on each morphism then taking products and composites in $\mathbf{D}$, to the result of first taking products and composites in $\mathbf{C}$ then applying F.

Proposition 2.5. *For $\mathbf{C}$ a monoidal category and $\mathbf{D}$ a monoidal double category, the following are equivalent.*

1. *A lax monoidal lax functor $F : \mathbf{C} \to \mathbf{D}$ such that F_{c_1,c_2} is the identity for all $c_1, c_2 \in \mathrm{Ob}(\mathbf{C})$*[1]
2. *A lax functor F from $\mathbf{C}$ to the horizontal bicategory of $\mathbf{D}$ such that $F(c_1 \otimes c_2) = F(c_1) \otimes F(c_2)$ for all $c_1, c_2 \in \mathrm{Ob}(\mathbf{C})$, and the following diagram commutes for all*

3 Definition and first examples

- Collectives
- Multi-collectives
- Dynamical systems
- Multi-categories
- Initial and terminal

3.1 Adaptives

4 Basic theory of adaptives

4.1 Change of base adjunction

4.2 Populating adaptives

5 Gradient descent example

A Proofs

[1] Note that this strict monoidality condition is not assumed for the squares $F_{f_1 \otimes f_2}$