Wiring operads

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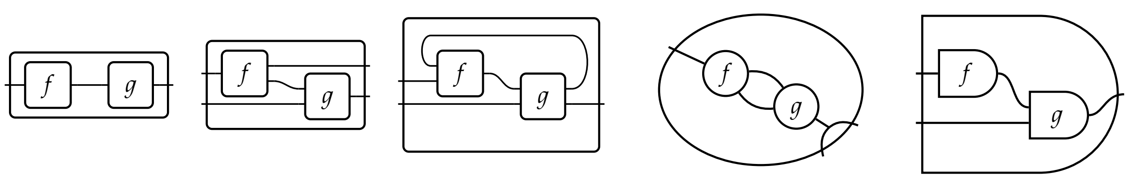
1 Introduction

In this paper we introduce the notion of *wiring operad*, a category-theoretic framework for organizing theories of composition, e.g. those that arise in various categorical structures.

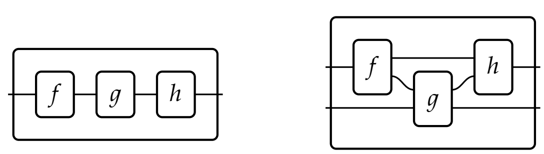
1.1 Operads associated to various categorical structures

There are many sorts of categorical structures—categories, monoidal categories, traced monoidal categories, hypergraph categories, operads, etc.—and each has an associated sort of wiring diagram. A category with said structure, call it C, can "interpret" such a wiring diagram as a way of assembling new morphisms from old in C. For example, the five sorts of categorical structures listed above can interpret the following sorts of wiring diagrams:[1]



category | monoidal category | traced monoidal category | hypergraph category | operad

In each case, a morphism in C is being assembled from two morphisms $f, g \in C$, which we might call *components*. String diagrams in general can assemble a composite morphism from any number of component morphisms; for example the following build a composite morphism from three components:



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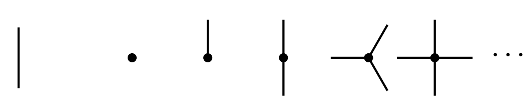
[1]What we call wiring diagrams are strongly related to much more well-known concept called *string diagrams*. The difference is that string diagrams do not include the exterior box, the composite, as an explicit part. In the work that follows, this difference is important, because the exterior box and the interior boxes have the same status: they are all simply objects in the wiring operad. We also prefer the term "wiring", because the term "string" also has other interpretations in the intersection of math and physics.

The first composite can be denoted $h \circ g \circ f$; technically it should be either $(h \circ g) \circ f$ or $h \circ (g \circ f)$, but they are equal in any category. The wiring diagram "abstracts away the difference". The second composite is more complicated to write as a wiring of text—because it is inherently 2-dimensional compared to text which is 1-dimensional—but it can be denoted $(h \otimes \mathrm{id}) \circ (\mathrm{id} \otimes g) \circ (f \otimes \mathrm{id})$.

Operads have been proposed as a way of organizing the various sorts of wiring diagrams [**Spivak:2013b**; **Rupel.Spivak:2013a**]. An operad morphism has many inputs and one output, and these correspond to a wiring diagram assembling many component morphisms to make one composite morphism. For example, it was shown in [**Spivak.Schultz.Rupel:2016a**] that the operad for traced monoidal categories is **Cob**, the operad of oriented 1-dimensional cobordisms, and it was shown in [**Fong?**] that the operad for hypergraph categories is **Cospan**.

While the operad formalism neatly captures wiring composition—and can thus be used to define the above sorts of categorical structures as operad-algebras—there is a technical annoyance that emerges in each case: the operad handles morphisms, but not objects in these categorical structures. The objects must be "baked in" to the operad as a varying set of string labels. Thus it is not the case that traced monoidal categories are algebras on **Cob** as our informal statement in the previous paragraph may have suggested. Instead, it is the case that for any generating set Λ of objects, traced-monoidal-categories-whose-object-set-is-generated-by-Λ are algebras on $\mathbf{Cob}/\Lambda$. The latter is a mouthful, and is not entirely pleasing.

In this paper we remedy this by following an idea from the work of [**Joyal.Kock**] that has been further developed by [**Raynor**]. Joyal and Kock defined a category elGr of *elementary graphs*. The objects are connected graphs without internal edges. Up to isomorphism, these are the *stick graph* $(\shortmid)$ with no vertices, and for $n \geq 0$, the *n-corollas* with one vertex and n "legs".



$(\shortmid)$

Figure 1: Up to isomorphism, the first few elementary graphs.

The stick graph is equipped with a non-trivial automorphism τ satisfying $\tau^2 = \mathrm{id}_{\shortmid}$. Further, elGr contains all isomorphisms of corollas and for each n-corolla C, there are $2n$ morphisms $(\shortmid) \to C$ that can be pictured as inserting $\shortmid$ into a leg of C "in either τ-direction".

A presheaf $S : \mathsf{elGr}^{op} \to \mathsf{Set}$ of elGr then consists of a set $S(C)$ for corolla C, together with the obvious symmetric group actions, and a set of *types* (elsewhere called *colours* or *objects*) $\Lambda = S(\shortmid)$ equipped with an involution. For each leg of a corolla C, there are a pair of projections $S(C) \to \Lambda$ that "commute with the involution".

We can assemble a finite graph $\mathcal{G}$ with vertex set V from a V-indexed set of corollas $\{C_v\}_{v \in V}$ "glued together" by stick graphs corresponding to the edges of $\mathcal{G}$. This gives a way of associating a unique subcategory $\mathsf{es}(\mathcal{G}) \subset \mathsf{elGr}$ to any graph $\mathcal{G}$. Then for any graph

$\mathcal{G}$, we may define

$$S(\mathcal{G}) := lim_{\mathcal{C} \in \mathsf{es}(\mathcal{G})} S(\mathcal{C}).$$

Informally, $S(\mathcal{G})$ is the set of "decorations of $\mathcal{G}$ by S". Each vertex v of $\mathcal{G}$ is decorated by an element of $\phi_v \in S(\mathcal{C}_v)$ and, by construction, each edge of $\mathcal{G}$ is decorated by a type in Λ that is suitably "compatible" with each ϕ_v under the projections $S(\mathcal{C}_v) \to \Lambda$.

Insert suitable graphic

Crucially for this paper, the object or type-labelling on the edges of a graph, just as the "morphism-labelling" on the vertices, is simply part of the data of the presheaf.

The intention of [**Joyal.Kock**] was to construct so-called *Compact symmetric multicategories (CSMs)* as algebras for a monad defined on the category of presheaves on elGr. Roughly, CSMs can be thought of as coloured modular operads [], and they generalise small compact closed categories [], as well as coloured wheeled properads []. It should be mentioned that the monad construction in [**Joyal.Kock**] does not admit a well-defined multiplication. This is rectified in [**Raynor**]. In [], an analogue construction for (non-wheeled) properads is presented, and it is clear how these methods may be further modified to handle other operad-types including (wheeled) PROPs [] and (cyclic) operads [].

Add references.

For the purposes of this article, the important point is that the monad for CSMs is constructed by evaluating elGr presheaves on graphs, as described above. CSMs are precisely those presheaves S on elGr such that for any decoration of a connected graph $\mathcal{G}$ by S, there is a unique rule for "collapsing" the edges of $\mathcal{G}$ to get a single element ("morphism") in S. In particular, the categorical structures generalised by CSMs may be described by a single monad that deals with both the morphisms and the types at the same time. Furthermore, using the obvious modifications to the (elementary) graph category we may construct monads describing the other operad-like structures mentioned above, without having to worry about the particular set of types.

1.2 Notation and terminology

For any natural number $n \in \mathbb{N} = \{0, 1, \ldots\}$, we denote the associated finite set by $\underline{n} := \{1, 2, \ldots, n\}$, so $\underline{0} = \varnothing$ and $\underline{3} = \{1, 2, 3\}$. Given a set A and a function $a \colon \underline{n} \to A$, we may denote $a(i)$ by a_i for $1 \leq i \leq n$.

Definition 1.1. To specify an operad O,

- One specifies a set $\mathrm{Typ}(O)$, elements of which will be called *types*.
- For every natural number $n \in \mathbb{N}$, function $x \colon \underline{n} \to \mathrm{Typ}(O)$, and type $y \in \mathrm{Typ}(O)$, one specifies a set $O(x; y)$, elements of which are called *n-ary operations*. An n-ary operation $f \in O(x; y)$ may be denoted $f \colon (x_1, \ldots, x_n) \to y$, so a 0-ary operation may be denoted $f \colon () \to y$.
- For types $x_1, \ldots, x_n, y \in \mathrm{Typ}(O)$ and bijection $\sigma \colon \underline{n} \xrightarrow{\cong} \underline{n}$, one specifies a bijection $O(\sigma) \colon O(x; y) \xrightarrow{\cong} O(x\sigma; y)$, called the *symmetry*.

- For every type $x \in \mathrm{Typ}(O)$, one specifies a 1-ary operation $\mathrm{id}_x \colon (x) \to x$, called the *identity on x*.
- For operations $g \colon (y_1, \dots, y_n) \to z$ and $f_1 \colon (x_{1,1}, \dots, x_{1,m_1}) \to y_1, f_2 \colon (x_{2,1}, \dots, x_{2,m_2}) \to y_2, \dots f_n \colon (x_{n,1}, \dots, x_{n,m_n}) \to y_n$, one specifies an operation denoted $g \circ (f_1, \dots, f_n)$, called the *composite*.

These are required to satisfy well-known axioms; we refer the reader to [**Leinster:2004a**].

Remark 1.2. What we call an operad is often called a "small symmetric colored operad" or a symmetric multicategory; a definition and plenty of examples can be found in [**Leinster:2004a**].

What we call types in Definition 1.1 are often called colors or objects. What we call operations are often called morphisms.

Do we use the symmetry in the proof? Can we drop it?

2 Wiring operads and associated categorical structures

In this section we define wiring operads. A central character in this story will be **Cospan**—the category of finite sets and cospans between them—which will turn out to be the terminal wiring operad. The sort of categorical structure associated to **Cospan** is that of hypergraph categories. We begin by recalling that story.

2.1 Hypergraph categories and Cospan-algebras

Definition 2.1. Let $(\mathbf{Cospan}, 0, +)$ denote the symmetric monoidal category whose objects are natural numbers, and for which a morphism $c \colon m \to n$ consists of a finite set $\check{c}$, called the *apex*, together with functions $c^{\backprime} \colon \underline{m} \to \check{c}$ and $c' \colon \underline{n} \to \check{c}$, called the left and right leg, respectively. The notation comes from the diagram

$$\begin{tikzcd}
\underline{m} \arrow[dr, "c^{\backprime}"'] & & \underline{n} \arrow[dl, "c'"] \\
& \check{c} &
\end{tikzcd}$$

We consider two such cospans $c_1, c_2 \colon \underline{m} \to \underline{n}$ to be the same morphism if there is a bijection $e \colon \check{c}_1 \to \check{c}_2$ such that $ec_1^{\backprime} = c_2^{\backprime}$ and $ec_1' = c_2'$.

The composite of two cospans $c \colon m \to n$ and $\psi \colon n \to p$ is formed by pushout in the usual way, $\underline{m} \to (\check{c} \sqcup_{\underline{n}} \check{\psi}) \leftarrow \underline{p}$, and the identity on n is the pair of identities $\underline{n} \to \underline{n} \leftarrow \underline{n}$. The monoidal unit is 0. The monoidal product of objects m and n is their sum $m + n$; the monoidal product of cospans $c_1 \colon m_1 \to n_1$ and $c_2 \colon m_2 \to n_2$ is the disjoint union across the board: $(\underline{m_1} + \underline{m_2}) \to (\check{c}_1 + \check{c}_2) \leftarrow (\underline{n_1} + \underline{n_2})$.

To any monoidal category $(\mathcal{M}, I, \otimes)$ we can associate an operad $O_{\mathcal{M}}$, called the *operad underlying $\mathcal{M}$*: it has $\mathrm{Typ}(O_{\mathcal{M}}) \coloneqq \mathrm{Ob}(O)$ and operations $O_{\mathcal{M}}(x_1, \dots, x_n; y) \coloneqq \mathcal{M}(x_1 \otimes \cdots \otimes x_n, y)$; see [**Leinster:2004a**]. This is part of a functor $U \colon \mathbf{MonCatLax} \to \mathbf{Opd}$, from the category of monoidal categories and lax monoidal functors to the category of operads and operad functors, and this functor is fully faithful. Thus we often blur the distinction between a monoidal category and its underlying operad; for example, we will denote $\mathcal{O}_{\mathbf{Cospan}}$ simply by **Cospan**.

We can picture a type n in the operad **Cospan** as a circle with n "ports" placed around the circle in no particular order; here are pictures of the types $0, 1, 2, 3$:

3 Examples