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Problem 1

We consider a population with true model $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}\boldsymbol{\gamma} + \boldsymbol{\varepsilon}$, for which we fit the model $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$.

(a)

For the fitted model, the estimated regression coefficients are given by the usual SLR formula:

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y}.$$

From this, the expected value of the regression coefficient estimates is

$$\begin{aligned}
\mathbb{E}\hat{\boldsymbol{\beta}} &= \mathbb{E}\Big[(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y}\Big] \\
&= (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\mathbb{E}\Big[\boldsymbol{Y}\Big] \\
&= (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\mathbb{E}\Big[\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}\boldsymbol{\gamma} + \varepsilon\Big] \\
&= (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\Big(\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}\boldsymbol{\gamma}\Big) \\
&= \boldsymbol{\beta} + (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Z}\boldsymbol{\gamma}
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbb{E}\hat{\boldsymbol{Y}} &= \mathbb{E}\Big[\boldsymbol{H}\boldsymbol{Y}\Big] = \boldsymbol{H}\mathbb{E}\boldsymbol{Y} \\
&= \boldsymbol{H}(\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}\boldsymbol{\gamma}) \\
&= \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{X} + \boldsymbol{H}\boldsymbol{Z}\boldsymbol{\gamma} \\
&= \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{H}\boldsymbol{Z}\boldsymbol{\gamma}
\end{aligned}$$

(c)

For bias $\boldsymbol{\delta} = \mathbb{E}(\hat{\boldsymbol{y}}) - \mathbb{E}(\boldsymbol{Y})$, the expected bias is

$$\begin{aligned}
\mathbb{E}\boldsymbol{\delta} &= \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{H}\boldsymbol{Z}\boldsymbol{\gamma} - (\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}\boldsymbol{\gamma}) \\
&= \boldsymbol{H}\boldsymbol{Z}\boldsymbol{\gamma} - \boldsymbol{Z}\boldsymbol{\gamma} \\
&= (\boldsymbol{H} - \boldsymbol{I})\boldsymbol{Z}\boldsymbol{\gamma}
\end{aligned}$$