Basic Statistical Tests

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Test of Proportions I

- The first test we will discuss considers a basic test of proportions.
- Consider a sample $X_1, X_2, \ldots, X_n$ of Bernoulli trials and we are interested in testing whether or not the parameter of the distribution is some p_0.
- That is

$$\begin{aligned} H_0 &: p = p_0 \\ H_1 &: p \neq p_0 \end{aligned}$$

- Our first concern is finding a test statistic that will allow us to answer this questions

Test of Proportions II

- We know that $\bar{X}$ is a consistent estimator for p and $Var(X) = p(1-p)$ so by the CLT and under the null hypothesis

$$Z = \sqrt{n}\frac{\bar{X} - p_0}{\sqrt{p_0(1-p_0)}} \xrightarrow{L} N(0,1)$$

- Now we will want to reject H_0 for values of Z that are too large and will want to reject if $|Z| \geq c$
- In order to choose c we decide the significance level of test, that is how often we are willing to make a type I error (rejecting H_0 when it is true)
- For a normal distribution this becomes $|Z| \geq \Phi^{-1}(1-\alpha/2)$ for an equal tail test.

Test of Proportions III

- Our rule then is

$$\text{Reject } H_0 \text{ if } |Z| \geq \Phi^{-1}(1 - \alpha/2)$$

- If we are interested in finding the p-value under the null hypothesis for this test we can $\Phi(Z)$.
- Only now can we calculate z and find perform our test now that we have constructed our test statistic, the rejection region, and set up a rule for deciding to reject the null.

Testing the Mean with Known Variance I

- Consider a population where we believe the data is distributed normally.
- One test we could perform is to test if the mean value of this population is equal to some value μ_0.
- Recall from the Central Limit Theorem

$$\sqrt{n}\frac{\bar{X}-\mu}{\sigma} \xrightarrow{L} N(0,1)$$

Testing the Mean with Known Variance II

- Notice that the normal distribution is a parameterized by both it's mean and variance, thus to use this result we must assume that know the population variance.
- Assuming that the variance is known, this proceeds very similarly to our proportion example.

Testing the Mean with Known Variance III

i) Construct a Null and Alternative Hypothesis

$$\begin{aligned} H_0 &: \mu = \mu_0 \\ H_1 &: \mu \neq \mu_0 \end{aligned}$$

ii) Under the null hypothesis find a test statistic

$$Z = \sqrt{n}\frac{\bar{X} - \mu_0}{\sigma} \xrightarrow{L} N(0,1)$$

iii) Decide on a critical region consistent with an appropriate significance level

$$|Z| > c = \phi^{-1}(1 - \alpha/2)$$

iv) Obtain a sample and calculate the observed test statistic and compare with the critical region

v) Report your conclusions in the context of the problem

Testing the Mean with Known Variance IV

- If we know the variance this is a very simple procedure which can allow us to decide if the observed data agrees with our hypothesis of the population mean
- Additionally this procedure could allow us to provide p-values and other summaries to our collaborators based on the normal distribution
- Unfortunately, we do not always know the variance of the population beforehand and must estimate it
- This necessitates understanding the distribution of the sample variance.

Distribution of Sums of Normal Random Variables I

- Provided are some results about sums of normally distributed random variables which are useful for hypothesis testing.
- The proofs of these results should be investigated at your own pace and will come up throughout the year as you investigate transformations of random variables

Distribution of Sums of Normal Random Variables II

- Consider a random sample $X_1, X_2, \ldots, X_n$ where each $X_i \sim N(\mu, \sigma^2)$ and let

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

  Then

  a) $\bar{X}$ and S^2 are independent random variables
  b) $\bar{X} \sim N(\mu, \sigma^2)$
  c) $\frac{(n-1)}{\sigma^2} S^2 \sim \chi^2_{n-1}$ that is a chi-squared random variable with $n-1$ degrees of freedom

- Notice that the above holds for **normally distributed random variables**.
- Additionally notice that we do not need to rely on asymptotics for the previous hypothesis test!

The t-Distribution I

- Again we provide some unproven relationships which define special random variables.
- You will most likely show this, but it will be important that you at least know the relationship between the normal distribution and the chi-square distribution and how they are used to create t random variable.

The t-Distribution II

- Consider two independent random variables Y and Z such that $Y \sim \chi^2_n$ and $Z \sim N(0,1)$.
- We define a transformation of these random variables T such that

$$T = \frac{Z}{\sqrt{Y/n}}$$

- The distribution of T is called the t distribution with n degrees of freedom.

Test of the Mean with Unknown Variance I

- We'll use those facts to create a test for the population mean when the variance is unknown
- This test is often referred to as a t-test and you should recognize it from your basic statistics courses
- We outline a bit more of the technicalities then you may have seen in your intro stats course

Test of the Mean with Unknown Variance II

- Consider a random sample $X_1, X_2, \ldots, X_n$ where each $X_i \sim N(\mu, \sigma^2)$.
- We would like to test the hypothesis that the mean μ is some value μ_0, without knowing the population variance
- As with our last example, the null and alternative hypotheses remain the same

$$
\begin{aligned}
H_0 &: \mu = \mu_0 \\
H_1 &: \mu \neq \mu_0
\end{aligned}
$$

- We now must find an appropriate test statistic

Test of the Mean with Unknown Variance III

- Based on previous results, we know that

$$Z = \sqrt{n}\frac{\bar{X} - \mu}{\sqrt{\sigma^2}} \sim N(0,1) \quad \text{and} \quad W = \frac{(n-1)}{\sigma^2}S^2 \sim \chi^2_{n-1}$$

  and we want to create a statistic that does not contain σ^2.

- Therefore if we divide $\sqrt{\frac{S^2}{\sigma^2}}$, we have that

$$T = \frac{\sqrt{n}\frac{\bar{X}-\mu}{\sqrt{\sigma^2}}}{\sqrt{\frac{S^2}{\sigma^2}}} = \sqrt{n}\frac{\bar{X} - \mu}{\sqrt{S^2}}$$

Test of the Mean with Unknown Variance IV

- Considering T, we notice that it is actually the ratio of two random variables

$$T = \frac{X}{\sqrt{\frac{W}{n-1}}}$$

- Therefore by our previous results we know that

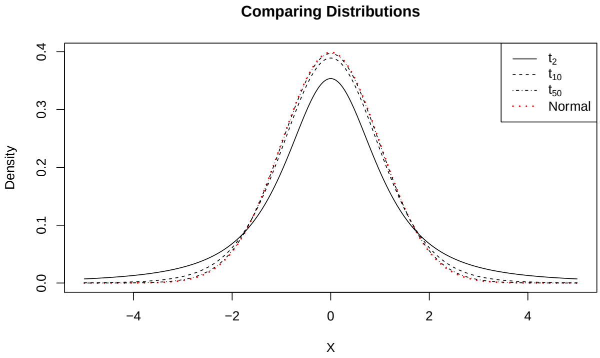
$$T \sim t_{n-1}$$

Test of the Mean with Unknown Variance V

- This implies that we can use the test statistic T for testing the population mean when the various is unknown.
- Now we must define a critical region for the test statistic for an appropriate significance level
- Let's investigate the t-Distribution a little further

Test of the Mean with Unknown Variance VI

- We see that t-distribution is very similar to a normal distribution in shape and spread
- Also it appears that for large degrees of freedom the t distribution approaches a normal distribution



Test of the Mean with Unknown Variance VII

- Therefore we can choose symmetric points from the t distribution such that

$$|T| > c$$

- For a specific α level we will define these points as $T_{\alpha/2}$ and $T_{1-a\alpha/2}$.
- Now that we have defined the critical region for a specific significance level, we can create our testing procedure.

Test of the Mean with Unknown Variance VIII

i) Construct a Null and Alternative Hypothesis

$$
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\end{aligned}
$$

ii) Under the null hypothesis find a test statistic

$$T = \sqrt{n}\frac{\bar{X} - \mu_0}{\sqrt{S^2}} \sim t_{n-1}$$

iii) Decide on a critical region consistent with an appropriate significance level

$$|T| > T_{\alpha/2} = T_{1-\alpha/2}$$

iv) Obtain a sample and calculate the observed test statistic and compare with the critical region

v) Report your conclusions in the context of the problem

Comparing Multiple Populations

- All three of these tests can be modified to consider testing the population expectation between two groups under study
- I won't review all them in the interest of time, but it is worth picking up a basics statistics textbook and reviewing
    - Two-Proportion Test
    - Two-Sample Test of the Mean with Variance Known
    - Two-Sample Test of the Mean with Unknown Variance (Two-Sample t-Test)
    - Paired Tests

References