Midterm: Theory of Linear Models

Fall 2024

Due: Monday, 10/21 at 10:00am, hard-copy handed-in

Directions:

- Type your solutions in RMarkdown or latex
- Everyone must work independently: no consulting with other students, professors, or posting questions online
- You may use any textbook and your class notes for reference.
- **No ChatGPT**
- Along with your exam, provide a list of references that you used to complete the problems

1. Consider the variance components model $Y = g + \varepsilon$, where Y is an $n \times 1$ response vector, g is a $n \times 1$ random effects vector with distribution $g \sim \mathcal{N}(0, \sigma_g^2 K)$, where K is an $n \times n$ symmetric positive definite matrix; assume $\varepsilon \sim \mathcal{N}(0, \sigma^2 I_n)$ and that g and ε are independent.

   a. What is $\mathrm{Var}(Y)$?
   b. Compute the maximum likelihood estimate for σ_g^2.
   c. Suppose X is an $n \times p$ matrix of covariates such that $K = XX^T$. Provide an interpretation of the elements of K with respect to the covariates measured in X.
   d. Provide an interpretation of the quantity $\frac{\sigma_g^2}{\sigma_g^2 + \sigma_\varepsilon^2}$.
   e. Compute the score test statistic for hypothesis $H_0 : \sigma_g^2 = 0$.
   f. Consider now the model $Y = X\beta + \eta$, $\eta \sim \mathcal{N}(0, \sigma^2 I_n)$. Compute the score test statistic for $H_0 : \beta = 0$.
   g. Are the two score test null hypotheses always equivalent? Why or why not?
   h. Assuming the null hypotheses are equivalent, identify a situation for which the two tests will have different power.

2. A random variable $X = \sum_{i=1}^{k} Z_i^2$, for Z_i independent standard normal random variables, follows a *Chi-Squared* distribution with k degrees of freedom. Show that $Y^T \Sigma^{-1} Y \sim \chi_k^2$ for $Y \sim \mathcal{N}(0, \Sigma)$. Show your work and carefully explain your reasoning for full credit.

3. Let P_X and P_W be the projection matrices corresponding respectively to the column spaces of matrices X and W, respectively. Assume X is $n \times r$ with $\mathrm{rank}(X) = r$, and W is $n \times q$ with $\mathrm{rank}(W) = q$. Under what conditions will the projections commute? I.e., when is $P_X P_W = P_W P_X$ true?

4. Consider the regression model $Y = X\beta + \varepsilon$ with $\varepsilon \sim \mathcal{N}(0, \Sigma)$, where Σ is a positive definite covariance matrix that is otherwise unspecified.

   a. Verify that the generalized least squares estimator $\hat{\beta}_G = (X^T \Sigma^{-1} X)^{-1} X^T \Sigma^{-1/2} Y$ is unbiased.
   b. Compute the variance of $\hat{\beta}_G$.
   c. In this setting, calculate the bias of the usual OLS estimator $\hat{\beta} = (X^T X)^{-1} X^T Y$.
   d. Compute the variance of $\hat{\beta}$.
   e. What is the statistical advantage of using $\hat{\beta}_G$ instead of $\hat{\beta}$ when we know that $\Sigma \neq \sigma^2 I_n$? Explain your answer in rigorous statistical terms referring to concepts such as power, variance, mean squared error, bias, etc.

5. Consider the regression model $Y = X\beta + \varepsilon$, with $\mathbb{E}[\varepsilon] = 0$. Assume X is full column rank so that

$\hat{Y} = X(X^TX){-}1X^TY.$

a. Show that $\mathrm{Cov}(\hat{Y}, (Y - \hat{Y})) = 0$.
b. Show that if $\varepsilon \sim \mathcal{N}(0, \sigma^2 I_n)$, then $\hat{Y}$ and $Y - \hat{Y}$ are independent.
c. Prove or Disprove: if $\varepsilon \sim \mathcal{N}(0, \Sigma)$, then $\hat{Y}$ and $Y - \hat{Y}$ are independent.

6. Let $Y_i = \beta \cdot i + \varepsilon_i$, $i = 1, \ldots, n$, $\mathbb{E}\varepsilon_i = 0$, $\mathrm{Var}(\varepsilon_i) = i^2\sigma^2$.

   a. Compute the usual Least Squares Estimate for β.
   b. Compute the best linear unbiased estimator for β.
   c. What is the difference in variance for the estimators from parts (a) and (b)?

7. Consider the model $Y_{ij} = \alpha_i + \beta_j + \varepsilon_{ij}$, where $i = 1, \ldots, a; j = 1, \ldots, b$, and $\varepsilon_{ij} \overset{iid}{\sim} (0, \sigma^2)$.

   a. Clearly describe the structure of the design matrix for this model. That is, we wish to write the model in matrix form as $Y = X\gamma + \varepsilon$ for some matrix X.
   b. Recall that a function $\sum c_i\alpha_i + \sum d_j\beta_j$ is estimable iff the linear contrast vector $u^T = (c_1, \ldots, c_a, d_1, \ldots, d_b)$ is in $\mathcal{C}(X^T)$. Identify a condition of $c_1, \ldots, c_a, d_1, \ldots, d_b$ that is necessary and sufficient for the estimability of $u^T\gamma$.