STAT 9120 - HW 2

Augusta University

Due: September 22nd, 2024 by 11:59pm, submitted on D2L

1. For n observations and p covariates, assume the $n \times p$ data matrix X has full column rank, $n > p$. For fixed n, describe what happens to the eigenvalues of $X^T X$ as $p \to n$. (Hint: What is true about the eigenvalues of a singular matrix?)

2. Show that $\hat{\beta} = (X^T X)^- X^T Y$ is a solution to the normal equation.

3. (a) Derive the least squares estimates for $\hat{\beta}, \hat{\sigma}^2$ from the likelihood for the linear regression model $Y = X\beta + \varepsilon, \varepsilon \sim \mathcal{N}(0, \sigma^2)$.

   (b) Derive the estimates for the variance of $\hat{\beta}$ and variance of $\hat{\sigma}^2$.

4. For a projection matrix $P \in \mathbb{R}^{p \times p}$ and vector $x \in \mathbb{R}^p$, show that $\|Px\|^2 \leq \|x\|^2$. Interpret this fact in the context of linear regression; i.e., what does this tell us about Y vs. $\hat{Y}$?

5. Suppose $\hat{\beta} = (X^T X)^- X^T Y$ is a solution to the normal equation. Show that $\tilde{\beta} = \hat{\beta} + (I - (X^T X)^-(X^T X))z$ is also a solution to the normal equation, where z is an arbitrary vector.

6. Let response vector $Y = (1, 3, -2, 2)^T$ and design matrix

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

   (a) Find $\hat{\beta}$ using Method 1.

   (b) Find $\hat{\beta}$ using Method 2.

   (c) Find $\hat{\beta}$ using Method 3.

(d) Show that β_2 and $\beta_1 + \beta_3$ are estimable. Explain why β_1 is not estimable in this case.

(e) Find the BLUEs and corresponding variance estimates of $\hat{\beta}_2$ and $\hat{\beta}_1 + \hat{\beta}_3$.

7. Consider the design matrix $X = (1, x_1, \cdots, x_{p-1})$, and the alternative design matrix $Z = (1, c_1 x_1, c_2 x_2, \cdots, c_{p-1} x_{p-1})$ for non-zero scalars $c_j, j = 1, \ldots, p-1$. Show that the fitted values for Y are equivalent under both design matrices.