2016 Statistics Graduate Bootcamp
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1 Day 2 - Calculus Review

1.1 Limits

1. Evaluate the limit:
   (a) $\lim_{n\to\infty}(1-\frac{t}{n})^n$
   (b) $\lim_{x\to\infty} 1-e^{-\lambda x}$ for $\lambda > 0$
   (c) $\lim_{x\to 0}\frac{\sin x}{x}$
   (d) $\lim_{x\to\infty} x-\sqrt{x^2+x+1}$
   (e) $\lim_{x\to 1}\frac{x^n-1}{x-1}, n > 1$.

2. The *Gamma Function* is defined as
$$\Gamma(k) = \int_0^\infty x^{k-1}e^{-x}\,dx,$$
and can be roughly thought of as a generalization of the factorial. Stirling's formula states that
$$\Gamma(k+1) \approx \sqrt{2\pi}k^{k+1/2}e^{-k}.$$
Use Stirling's formula to evaluate the following limit:
$$\lim_{n\to\infty}\frac{\Gamma[(n+1)/2]}{\sqrt{n/2}\Gamma(n/2)}.$$

3. **Little "o" Notation:** For arbitrary functions f and g, we say f is $o(g)$ ("little o of g") if
$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = 0.$$
Intuitively $f \in o(g)$ if g grows much faster than f.
True or False:
   (a) $x^2 \in o(x^3)$
   (b) $x^2 \in o(x)$
   (c) $x^2 \in o(x^2)$
   (d) $x \in o(\ln x)$
   (e) $e^x \in o(x!)$
   (f) $\frac{1}{t} \in o(\frac{1}{t^2})$

1.2 Derivatives

1. Find the derivative.

   (a) $\frac{d^2}{dx^2} x \ln x$

   (b) $\frac{d}{dy} \frac{1}{\theta} e^{-\frac{y}{\theta}}$

   (c) $\frac{d}{d\mu} \exp\{\frac{-(x-\mu)^2}{2\sigma^2}\}$

2. Find the maximum (with respect to θ) of $\ln(\mathcal{L}(\theta))$ for

$$\mathcal{L}(\theta) = \prod_{i=1}^{n} \theta y_i^{\theta-1}.$$

   Assume the y_i are fixed numbers in $[0, 1]$.

3. Find the absolute maximum and minimum values of the function $f(x) = x^3 - 3x^2 + 1$ on the interval $[-1, 4]$.

4. Find the maximum of the function on the interval $0 \leq y \leq \theta$. Assume the y_i are fixed real numbers.

$$\mathcal{L}(\theta) = \prod_{i=1}^{n} \frac{2y_i}{\theta^2} \mathbb{1}_{[0,\theta]}(y_i).$$

5. Find the maximum of the function on the interval $(0, \infty)$. Assume $\alpha, \beta > 0$.

$$f(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}.$$

6. Find the maximum of the function $\mathcal{L}$ by finding the maximum of $\log(\mathcal{L})$ (assume the x_i are fixed positive real numbers, assume $n \in \mathbb{N}$). Verify it's a maximum using the second derivative test.

$$\mathcal{L}(\mu) = \frac{1}{\mu^n} \exp\{-\frac{1}{\mu} \sum_{i=1}^{n} x_i\}$$

   .

7. **Lagrange Multipliers:** The method of *Lagrange Multipliers* can be used to solve constrained optimization problems. The method is as follows. To find the max and min values of $f(x, y)$ subject to the constraint $g(x, y) = k$ first find all values of x, y and λ such that

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

$$g(x, y) = k.$$

   Next evaluate f at all points found in the first step; the largest of these values is the maximum, the smallest is the minimum.

   - Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.
   - Consider two fixed values x_1 and x_2 and two numbers $a_1, a_2 > 0$ such that $a_1 + a_2 = 1$. Define the linear combination $m = a_1 x_1 + a_2 x_2$. Find the values of a_1 and a_2 that minimize

$$C(a_1, a_2) = (x_1 - m)^2 + (x_2 - m)^2.$$

   - Consider n fixed values $x_1, \cdots, x_n$ and weights $a_1, \cdots, a_n > 0$ such that $\sum_{i=1}^{n} a_i = 1$. Define the linear combination $m = \sum_{i=1}^{n} a_i x_i$. Find the values of $a_1, \cdots, a_n$ that minimize

$$C(a_1, a_2, \cdots, a_n) = \sum_{i=1}^{n} (x_i - m)^2.$$

1.3 Integrals

1. $\int_0^\infty e^{3x}\,dx$

2. $\int_0^\infty xe^{3x}\,dx$

3. $\int x(1-x)^{999}\,dx$

4. $\int_0^\infty e^{tx}\left(\beta e^{-\beta x}\right)\,dx$

5. Find c satisfying
$$1 = \int_0^2 \int_0^1 c(2x+y)\,dx\,dy.$$

6. (a) Let $p(x) = ab^{-b|x|}$ be defined on $\mathbb{R}$. Solve for a in terms of b:
$$1 = \int_{-\infty}^{\infty} p(x)\,dx.$$

   (b) Simplify $p(x)$ using the relationship in the previous part. Sketch the graph of $p(x)$ using the simplified form.

7. Assume $f(x)$ is a real-valued function such that $f(x) > 0\,\forall x \in \mathbb{R}$ and $\int_{\mathbb{R}} f(x)\,dx = 1$. Define $F(x) = \int_{-\infty}^{x} f(t)\,dt$. For a fixed $x_0 \in \mathbb{R}$, define
$$g(x) = \begin{cases} f(x)/[1-F(x_0] & x \geq x_0 \\ 0 & x < x_0. \end{cases}.$$
Prove that $g(x)$ is a nonnegative function such that $\int_{\mathbb{R}} g(x)\,dx = 1$.

8. Use the fact that
$$\int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}\,dx = 1$$
to evaluate
$$\int_0^\infty x\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}\,dx.$$

9. A function g is said to be *concave* if $g''(x) < 0\ \forall x \in \mathbb{R}$. *Jensen's Inequality* states that for a concave function g and a function f such that $f(x) > 0$ and $\int_{\mathbb{R}} f(x)\,dx = 1$, the following inequality holds
$$\int_{-\infty}^{\infty} g(x)f(x)\,dx \leq g\left(\int_{-\infty}^{\infty} xf(x)\,dx\right).$$
Prove that if $g(x) = ax + b$ is a linear function, then equality holds. The converse is also true but is harder to prove.

10. (Optional) Let $f(x) = \frac{a}{b^2+x^2}$ be defined on $\mathbb{R}$. Solve for b in terms of a:
$$1 = \int_{\infty}^{\infty} f(x)\,dx.$$

1.4 Series

1. Compute the series in terms of N:

$$\sum_{x=1}^{N} \frac{1}{N} x.$$

2. Compute the series in terms of p:

$$\sum_{x=1}^{n} p(1-p)^{x-1}.$$

3. Use the binomial theorem to compute the series:

$$\sum_{x=0}^{n} p^x (1-p)^{n-x}.$$

4. Evaluate the series:

$$\sum_{x=1}^{\infty} e^{tx} \left(\frac{1}{2}\right)^x.$$

5. Evaluate the series:

$$\sum_{k=1}^{\infty} kp(1-p)^{k-1}.$$

Hint: Integrate the summands.

1.5 Taylor Series

Taylor Remainder Theorem Suppose a real-valued function $f(x)$ has $n+1$ derivatives in an interval $I \subseteq \mathbb{R}$. Then f can be expressed as $f(x) = T_n(x) + R_n(x)$ where T_n is the nth degree Taylor polynomial of f at $a \in I$ and where, for all $x \in I$, the remainder term R_n is equal to

$$R_n(x) = \frac{f^{n+1}(z)}{(n+1)!}(x-a)^{n+1},$$

for some z strictly between a and x.

1. (a) Find the Taylor series expansion about $a = 0$ for $f(x) = e^x$.
   (b) Find the Taylor series expansion about $a = 1$ for $g(x) = \log x$.
   (c) Find the MacLaurin Series expansion for $h(x) = \frac{1}{1-x}$.
2. Compute the series:

$$\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!}.$$

3. Use the result of the previous problem to compute the series:

$$\sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}.$$

4. Approximate the integral using a 2nd order Taylor series and bound the error of the approximation

$$\int_0^1 \frac{1}{\sqrt{2\pi}} \exp\{-\frac{x^2}{2}\}\, dx.$$

1.6 The Newton-Raphson Method

The Newton-Raphson Method (also called Newton's Method) is a root-finding algorithm for differentiable functions. Starting from an initial guess x_1 as a root for $f(x)$, the algorithm iteratively updates by the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

until convergence. The method is commonly used in statistical estimation procedures, which are optimization problems.

1. Use Newton's method with the specified initial approximation x_1 to find x_2 and x_3, the third approximation to the root of the given equation.

$$x^3 + 2x - 4 = 0, \quad x_1 = 2.$$

2. Explain why Newton's method fails when applied to the equation $\sqrt[3]{x} = 0$ with any initial approximation $x_1 \neq 0$. Illustrate your explanation with a sketch.

3. (a) Adapt the Newton-Raphson method to find the maximum or minimum of a function $f(x)$. Clearly lay out the steps of the algorithm. State any conditions that $f(x)$ needs to satisfy for the algorithm to be used. Comment on the convergence behavior of the algorithm.
   (b) Use the Newton-Raphson method to find the maximum of $f(x) = x^3 e^{-x/2}$.

4. Use the Taylor Remainder Theorem with $n = 1, a = x_n$, and $x = r$ to show that if $f''(x)$ exists on an interval I containing r, x_n and x_{n+1} and $|f''(x)| \leq M, |f'(x)| \geq K$ for all $x \in I$ then

$$|x_{n+1} - r| \leq \frac{M}{2K}|x_n - r|^2.$$

Note: Observe that the Newton-Raphson Method is the direct application of a first order Taylor Series, which means that the Remainder Theorem is applicable. At each iteration the Newton-Raphson method guesses x_{n+1} to be the root of the (Taylor) linearization of f at x_n.