MTH 442

Homework 4

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May 4th 2015

1 Problem 1

2 Problem 2

Let $R = \mathbb{F}[x_1, \ldots, x_n], \mathcal{F} = \{f_1, f_2, \ldots, f_n\}$ where $f_i \in R, I = \langle \mathcal{F} \rangle$, and $\mathcal{F}$ be a Gröbner basis for $\langle \mathcal{F} \rangle$.

2.1 a

If $f \in \mathcal{F}$ such that $lt(f) \in Lt(\mathcal{F} \setminus \{f\})$ then prove $\mathcal{F} \setminus \{f\}$ is also a Gröbner basis for I.

From $lt(f) \in Lt(\mathcal{F} \setminus \{f\})$ we can view $lt(f) = \sum r_i f_i$ where $r_i \in R, f_i \in \mathcal{F} \setminus \{f\}$. From that we see $lp(f) = \sum r_i lp(f_i)$, since each f_i is monic; so $\exists f_f \in \mathcal{F}$ such that $lp(f_f) | lp(f)$.

Since $\mathcal{F}$ is a Gröbner basis, $\forall g \in I, \exists h \in \mathcal{F}$ such that $lp(h)|lp(g)$. If $h \neq f$ then we know that $\mathcal{F} \setminus \{f\}$ is almost a Gröbner basis, but we need to consider if $h = f$. The fact that $f \in Lt(\mathcal{F} \setminus \{f\})$ means that $\exists f' \in \mathcal{F} \setminus \{f\}$ such that $lp(f')|lp(f)$, so if we have the case where $lp(f)|lp(g)$ we get $lp(f')|lp(g)$. $\therefore \forall g \in I, \exists e \in \mathcal{F} \setminus \{f\}$ such that $lp(e)|lp(g)$, so $\mathcal{F} \setminus \{f\}$ is a Gröbner basis for $I = \langle \mathcal{F} \rangle$.

2.2 b

Suppose each element of $\mathcal{F}$ is monic. Prove: $\mathcal{F}$ is a minimal Gröbner basis for I if and only if no proper subset of $\mathcal{F}$ is a Gröbner basis for I.

Suppose $\mathcal{F}$ is minimal, so $\forall f_i, f_j \in \mathcal{F}, lp(f_i)|lp(f_j)$ iff $i = j$.

Looking at the subsets $\mathcal{F}_a = \mathcal{F} \setminus \{f_a\} \subset \mathcal{F}$ for $f_a \in \mathcal{F}$, we know that $f_a \in I$ and from $\mathcal{F}$ being minimal we see that no other $f_i \in \mathcal{F}_a$ divides f_i in regards to the leading powers. This produces a nonzero remainder for each $f_a \in I$, therefore $\mathcal{F}_a$ is not a Gröbner basis for $\mathcal{F}$ if $\mathcal{F}$ is minimal. We can continue this arguement to look at all proper subsets of $\mathcal{F}$ to see that no further subsets will be Gröbner bases for I.

I only finished the forward direction.