MTH 442

Homework 1

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1 Problem 3 page 10

Consider the following polynomials in $\mathbb{Q}[x, y, z, t]$, $f_1 = x - 2y + z + t$, $f_2 = x + y + 3z + t$, $f_3 = 2x - y - z - t$, and $f_4 = 2x + 2y + z + t$. Solve the problems posed in Section 1.1 for this set of polynomials.

$$A = \begin{pmatrix} 1 & -2 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 2 & -2 & -1 & -1 \\ 2 & 2 & 1 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & -2 & 1 & 1 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & -3 & -3 \\ 0 & 6 & -1 & -1 \end{pmatrix}$$

$B_1 = A_1, B_2 = -1A_1 + A_2, B_3 = -2A_1 + A_3, B_4 = -2A_1 + A_4$

$$C = \begin{pmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & -4 & -3 \\ 0 & 0 & -4 & -1 \end{pmatrix}$$

$C_1 = A_1, C_2 = 1/4B_2, C_3 = -2C_2 + B_3, C_4 = -3/2B_2 + B_4$

$$D = \begin{pmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{3}{4} \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$D_1 = A_1, D_2 = C_2, D_3 = -\frac{1}{4}C_3, D_4 = -C_3 + C_4$

Here we let the matrix A be the representation of the linear system, where the i^{th} rows are the corresponding coefficients of f_i. For an arbitrary linear polynomial $f \in \mathbb{Q}[x, y, z, t]$ we look to see if there exists $x \in \mathbb{Q}^4$ such that the system $xA = f$ is solvable. This determines the ideal membership problem.

In the row reduced matrices, we are in fact dealing with the same ideals, since each new row represents a linear combination of the others. But the ideal is closed under multiplication by $\mathbb{Q}$ and addition of elements in the ideal.

The next problem to solve is the explicit linear combination for coefficients of $f = \sum_{i=1}^{4} q_i f_i, q_i \in \mathbb{Q}$. In this case after solving the previous problem our x vector is exactly the coefficients we are looking for!

The cosets of the ring $\mathbb{Q}[x, y, z, t]/J$ are essentially the span of each of the vectors (or linear combinations) of A.

2 Problem 2 page 10

Find a single generator for the ideal $I = (x^6 - 1, x^4 + 2x^3 + 2x^2 - 2x - 3)$. Is $x^5 + x^3 + x^2 - 7 \in I$? Show that $x^4 + 2x^2 - 3 \in I$ and write $x^4 + 2x^2 - 3$ as a linear combination of $x^6 - 1$ and $x^4 + 2x^3 + 2x^2 - 2x - 3$.

To start we need to see if $g = gcd(x^6 - 1, x^4 + 2x^3 + 2x^2 - 2x - 3)$ is non-constant. Maple tells us that $g = x^2 - 1 = (x+1)(x-1)$. That means $I = \langle x^6 - 1, x^4 + 2x^3 + 2x^2 - 2x - 3 \rangle = \langle g \rangle = \langle x^2 - 1 \rangle$.

For determining ideal membership we only need to find whether $gcd(x^2 - 1, x^5 + x^3 + x^2 - 7)$ is $x^2 - 1$, since as a generator of the ideal everything is a multiple of it. Maple shows us that in fact the $gcd = 1$ which means it is not in the ideal I.

Similarly looking at $gcd(x^2 - 1, x^4 + 2x^2 - 3) = x^2 - 1$ which means that $x^4 + 2x^2 - 3 \in I$.

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Let $f, g \in \mathbb{C}[x, y]$. Prove that if f and g have a non-constant common factor in $\mathbb{C}[x, y]$, then $V(f, g)$ is infinite. That is, show that if $h \in \mathbb{C}[x, y]$ and h is not in $\mathbb{C}$, then the equation $h = 0$ has infinitely many solutions. Generalize this exercise to the case where $h \in \mathbb{C}[x_1, \ldots, x_n]$, for $n > 2$.

The ideal $\langle x, y \rangle$ in $\mathbb{C}[x, y]$ cannot be generated by a single element since x and y are irreducible in the polynomial ring. Since there isn't a single generator for the ideal, $\mathbb{C}[x, y]$ is not a PID; all Noetherian rings are PIDs, so $\mathbb{C}[x, y]$ is not Noetherian thus the ideal $I = \langle h \rangle$ isn't finitely generated. To describe the variety $V(h)$ we consider satisfying the variety of the ideal generated by $h, V(I)$. But since $\mathbb{C}[x, y]$ is not Noetherian, there are infinitely many elements in the ideal. This ideal being infinite means that the size of the variety is infinite since it must satisfy $i = 0$ $\forall i \in I(V(h))$ $h \notin \mathbb{C}$.

Considering the $\mathbb{C}[x_1, \ldots, x_n]$ case, we need only to consider the fact that if $\mathbb{C}[x_1, \ldots, x_{n-1}]$ isn't a PID then any polynomial extension of it won't be either.