MTH 443 - Homework 2

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1 Problem 7

Let F be a field. The set $\{u_1, u_2, u_3, u_4\}$ is a basis for F^4, where $u_1 = (1,1,1,1), u_2 = (0,1,1,1), u_3 = (0,0,1,1), u_4 = (0,0,0,1)$. Given and arbitrary vector $(a_1, a_2, a_3, a_4) \in F^4$, write down the explicitly unique expression of (a_1, a_2, a_3, a_4) as a linear combination of the u_j.

1.1

To discover the unique representation of $a = (a_1, a_2, a_3, a_4) \in F^4$ as a linear combination of the basis $\beta = \{u_1, u_2, u_3, u_4\}$ we will write a linear transformation, $T : F^4 \to F^4$, from the standard basis, denoted δ, to β as $[T]_\delta^\beta$. The matrix representation of this linear transformation will map $e_i \mapsto u_i; i \in 1, 2, 3, 4$.

$$[T]_\alpha^\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{1}$$

Now we can solve the linear system:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \tag{2}$$

$$\begin{aligned}
a_1 &= x_1 && (3) \\
a_2 &= x_1 + x_2 && (4) \\
a_3 &= x_1 + x_2 + x_3 && (5) \\
a_4 &= x_1 + x_2 + x_3 + x_4 && (6) \\
x_1 &= a_1 && (7) \\
x_2 &= a_2 - a_1 && (8) \\
x_3 &= a_3 - a_2 + a_1 && (9) \\
x_4 &= a_4 - a_3 + a_2 - a_1 && (10) \\
& && (11)
\end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 - a_1 \\ a_3 - a_2 + a_1 \\ a_4 - a_3 + a_2 - a_1 \end{bmatrix} \tag{12}$$

2 Problem 8

Let V be a finite dimensional vector space, and let W_1 be a subspace of V.

a) Prove that there exists a subspace W_2 of V such that $V = W_1 \oplus W_2$.

b) Give a concrete example showing that the space W_2 constructed in part **(a)** is not unique.

2.1 a

WLOG let $dim(V) = n$ and $dim(W) = m$. Suppose $\beta = \{\beta_1, \beta_2, \ldots, \beta_m\}$ is a basis for W_1. Because both V and W_1 are finitely generated we can use the Replacement Theorem to extend β with a set of linearly independent vectors, $\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_{n-m}\}$, where $\beta \cup \gamma$ generates V and $\beta \cap \gamma = \emptyset$. The span of any set of vectors forms a subspace and since we know the sets are linearly indepedent we see that $span(\beta \cap \gamma) = span(\beta) \cap span(\gamma) = W_1 \cap span(\gamma) = \{0\}$. The set $\beta \cup \gamma$ now forms a basis for V and as such each $v \in V$ can be written as a unique linear combination of the vectors in β and γ. $span(\gamma)$ is a subspace disjoint from W_1 besides the zero vector and thus is the complementing subspace to form $V = W_1 \oplus span(\gamma)$.

2.2 b

Let $V = \mathbb{F}^2$ and $W = span(\{(1,0)\})$. Clearly W is a subspace of V to which we could extend the set $\{(1,0)\}$ with either $\{(0,1)\}$ or $\{(1,1)\}$ to form 2 different subspaces of V, $W_2 = span(\{(0,1)\}, W_3 = span(\{(1,1)\})$. W_2, W_3 are disjoint from W besides $\{(0,0)\}$ and pair with W such that they have unique linear combinations to represent V. For example $(1,1) = 1*(1,0) + 1*(0,1) = 0*(1,0) + 1*(1,1)$.

3 Problem 9

Let α be the standard ordered basis for $\mathbb{R}^2$, and let $\beta = \{(1,1),(2,1)\}$. Let γ be the standard ordered basis for $\mathbb{R}^3$, and let $\delta = \{(1,1,0),(0,1,1),(2,2,3)\}$. Define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Calculate $[T]_\alpha^\gamma, [T]_\alpha^\delta, [T]_\beta^\gamma$.

3.1 $[T]_\alpha^\gamma$

The matrix representation of T from α to β just maps the standard $\mathbb{R}^2$ basis to $\mathbb{R}^3$ so $T((1,0)) = (1,1,2)$ and $T((0,1)) = (-1,0,1)$. This matrix shows that multiplication by α under it's own representation clearly takes it to first or second column, which is the transformed vectors in $\mathbb{R}^3$.

$$[T]_\alpha^\gamma = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} \tag{13}$$

3.2 $[T]_\alpha^\delta$

The $[T]_\alpha^\gamma$ matrix can be thought of as a composition of basis changes $\alpha \xrightarrow{T} \gamma \to \delta$. This matrix takes vectors in α to their transformation by T under the δ basis. Introducting a new linear transformation $U : \mathbb{R}^3 \to \mathbb{R}^3$ we can view the composition of $[U]_\gamma^\delta [T]_\alpha^\gamma = [T]_\alpha^\delta$. The $[U]_\gamma^\delta$ matrix is the inverse of the representation of δ under γ as column vectors.

$$[T]_\alpha^\delta = [U]_\gamma^\delta [T]_\alpha^\gamma = [U]_\gamma^\delta \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 2 \\ 0 & 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 2 & -2 \\ -3 & 3 & 0 \\ 1 & -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 & -1 \\ 0 & 3 \\ 2 & 0 \end{bmatrix} \tag{14}$$

3.3 $[T]_\beta^\gamma$

This matrix similarly can be decomposed into the product of a transformation from β to α and then applying the linear transformation $[T]_\alpha^\gamma$. So similarly to the previous part we construct another linear transformation, this time $S : \mathbb{R}^2 \to \mathbb{R}^2$ and look for the matrix representation of $[S]_\beta^\alpha$.

$$[T]_\beta^\gamma = [T]_\alpha^\gamma [S]_\beta^\alpha = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} [S]_\beta^\alpha = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 3 & 5 \end{bmatrix} \tag{15}$$

4 Problem 10

Let $V = \{a_1, a_2, a_3) \in \mathbb{R}^3 | \, a_1 + a_2 + a_3 = 0\}$, let $\alpha = \{(1,0,-1),(0,1,-1)\} = \{\alpha_1, \alpha_2\}$ and $\beta = \{(1,1,-2),(1,-1,0)\} = \{\beta_1, \beta_2\}$ be an ordered bases for V, and let $T : V \to V$ be a linear transformation define by $T(a_1, a_2, a_3) = (a_2, a_3, a_1)$. Calculate $[T]_\alpha^\alpha$ and $[T]_\alpha^\beta$.

4.1 $[T]_\alpha^\alpha$

The matrix $[T]_\alpha^\alpha$ represents the vector in $\mathbb{R}^3$ under the transformation by T under the basis α. The easiest way to go about this is to look at the transformation of α under the standard basis, and then rewrite as a linear combination of the α's. In these systems, $a, b, c, d, e, f, g, h \in \mathbb{R}$.

$$T(\alpha_1) = (0,-1,1) = a * \alpha_1 + b * \alpha_2 = 0 * \alpha_1 + (-1) * \alpha_2 \to (0,-1) \tag{16}$$

$$T(\alpha_2) = (1,-1,0) = c * \alpha_1 + d * \alpha_2 = 1 * \alpha_1 + (-1) * \alpha_2 \to (1,-1) \tag{17}$$

So naturally we put the matrix together as the column vectors which shows that the matrix representation takes $\alpha_1 \mapsto (0,-1)$ and $\alpha_2 \mapsto (1,-1)$ in the alpha basis.

$$[T]_\alpha^\alpha = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \tag{18}$$

4.2 $[T]_\alpha^\beta$

This matrix should map the α basis to the transformed vectors under the β basis.

$$T(\alpha_1) = (0,-1,1) = e * \beta_1 + f * \beta_2 = -\frac{1}{2} * \beta_1 + \frac{1}{2} * \beta_2 \to (-\frac{1}{2}, \frac{1}{2}) \tag{19}$$

$$T(\alpha_2) = (1,-1,0) = g * \beta_1 + h * \beta_2 = 0 * \beta_1 + 1 * \beta_2 \to (0,1) \tag{20}$$

$$[T]_\alpha^\beta = \frac{1}{2} \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} \tag{21}$$

I talked with Sam Kowash on the last 2 problems.