MTH 443 - Homework 1

Daniel Takamori

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1 Problem 1

Let $V = \{(a_1, a_2, a_3) \in \mathbb{R}^3 | \, a_1 + a_2 + a_3 = 0\}$.

a) Show that V is a subspace of $\mathbb{R}^3$

b) Find a basis for V

1.1 a

Let u, v be arbitrary vectors $\in V$. The sum $u + v = (u_1, u_2, u_3) + (v_1, v_2, v_3) = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$. The condition of vectors being in V means that $u_1 + u_2 + u_3 = 0 = v_1 + v_2 + v_3$ and this new vector $u+v$ has the same property from commutativity of it's real components: $u + v \to (u_1 + v_1) + (u_2 + v_2) + (u_3 + v_3 = (u_1 + u_2 + u_3) + (v_1 + v_2 + v_3) = 0 + 0 = 0$.

To show that V is closed under scalar multiplication we need to look at $\forall r \in \mathbb{R}$ and $\forall v \in V, rv = r(v_1, v_2, v_3) = (rv_1, rv_2, rv_3) \to rv_1 + rv_2 + rv_3 = r(v_1 + v_2 + v_3) = r(0) = 0$.

And of course the zero vector is in V, so V is a vector subspace of $\mathbb{R}^3$.

$\square$

1.2 b

Since $(1, 1, 1) \notin V$, we know that $V \neq \mathbb{R}^3$, similarly we can see by inspection that V isn't 0 or 1 dimensional, so intuitively $dim(V) = 2$. That means we need to find 2 vectors in V which are linearly independent to form a basis.

$(1, 0, -1)$ and $(0, 1, -1)$ are linearly independent vectors in V so they can form a basis for the 2 dimensional subspace of $\mathbb{R}^3$. To see this more clearly we can show containment of $span(\{(1, 0, -1), (0, 1, -1)\}) \subseteq V$ and vice versa.

$\square$

2 Problem 2

Let V be a real vector space and let $u, v, w \in V$. Show that the set $\{u, v, w\}$ is linearly independent if and only if the set $\{u + v, u + w, v + w\}$ is linearly independent.

2.1

Let $S = \{u, v, w\}$ be a set of linearly independent vectors in V. $span(S)$ clearly contains $T = \{u + v, u + w, v + w\}$. And $span(T)$ contains $u = \frac{1}{2}[(u + v) + (u + w) - (v + w)]$, $v = \frac{1}{2}[(u + v) - (u + w) + (v + w)]$ and $w = \frac{1}{2}[-1(u + v) + (u + w) + (v + w)]$. So T is a generating set for the subspace $span(S)$, and we know that if it contains the same number of elements then it is also a basis, thus linearly independent.

$\square$

3 Problem 3

Let V be a vector space over a field F. Show that if $V = W_1 \oplus W_2$, then each element in V can be written *uniquely* as the sum of an element in W_1 and an element in W_2.

3.1

Suppose there exist 2 representations of $v \in V$ for $\alpha_1, \alpha_2 \in W_1$ and $\beta_1, \beta_2 \in W_2$.

$$v = \alpha_1 + \beta_1 = \alpha_2 + \beta_2 \tag{1}$$
$$v - v = (\alpha_1 + \beta_1) - (\alpha_2 + \beta_2) \tag{2}$$
$$0 = \alpha_1 + \beta_1 - \alpha_2 - \beta_2 \tag{3}$$
$$0 = \alpha_1 - \alpha_1 + \beta_1 - \beta_2 \tag{4}$$
$$\alpha_2 - \alpha_1 = \beta_1 - \beta_2 \tag{5}$$

We know the difference of the α's live in W_1 and the difference of the β's live in W_2. From this we deduce that each lives in the other so they both live in the intersection which is just the zero vector. So $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.

□

4 Problem 4

Let $\mathcal{F}(\mathbb{R}, \mathbb{R})$ denote the real vector space consisting of all functions $f : \mathbb{R} \to \mathbb{R}$. We say $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ is *even* if $f(-r) = f(r)$ for all $r\mathbb{R}$, and we say f is *odd* if $f(-r) = -f(r)$ for all $r \in \mathbb{R}$. Let $\mathcal{E}$ and $\mathcal{O}$ be the set of all even functions and off functions in $\mathcal{F}(\mathbb{R}, \mathbb{R})$, respectively.

1. Show that $\mathcal{E}$ and $\mathcal{O}$ are subspaces of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.
2. Show that $\mathcal{F}(\mathbb{R}, \mathbb{R}) = \mathcal{E} \oplus \mathcal{O}$.

4.1 a

If $f_e, g_e \in \mathcal{E}$ then $(f_e + g_e)(x) = f_e(x) + g_e(x) = f_e(-x) + g_e(-x) = (f_e + g_e)(-x)$. So $(f_e + g_e) \in \mathcal{E}$.

$\forall r \in \mathbb{R}$ and $f_e \in \mathcal{E}, rf_e(-x) = r(-f_e(x)) = -(rf_e(x))$ so $rf_e \in \mathcal{E}$.

Similarly $f_o, g_o \in \mathcal{O}$ then $(f_o+g_o)(x) = f_o(x)+g_o(x) = -f_o(x)-g_o(x) = -(f_o+g_o)(x)$. Thus $(f_o+g_o)(x) \in \mathcal{O}$.

$\forall r \in \mathbb{R}$ and $f_o \in \mathcal{O}, rf_o(x) = r(-f_o(x)) = -(rf_e(x))$, so we get the product $rf_o \in \mathcal{O}$.

Consider a function $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ such that $f(x) = f(-x) = -f(x)$. This is clearly the zero function and uniquely the only thing in the intersection $\mathcal{E} \cap \mathcal{O}$.

□

4.2 b

$\forall f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ we can take $f_e(x) = \frac{1}{2}(f(x) + f(-x))$ and $f_o(x) = \frac{1}{2}(f(x) - f(-x))$. Clearly $f_e(x) = -f_e(x)$ and $f_o(-x) = -f_o(x)$. Moreover the sum of $f_e + f_o = \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x)) = f$, so this is the unique combination of an even and odd function for all functions $\mathbb{R} \to \mathbb{R}$.

□

I discussed the problem set with Sam Kowash and James Rekow.