MTH 443 - Homework 3

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1 Problem 13

Does there exist a linear transformation $T : V \to V$ such that $N(T) = R(T)$, when...

1.1 $V = \mathbb{R}^2$

Yes! Let $T : V \to V$ be defined by the $T((x, y)) = (0, x)$. This is a linear transformation because $T(c(x, y)+(a, b)) = T((cx+a, y+b)) = (0, cx+a) = (0, cx)+(0, a) = c(0, x)+(0, a) = cT((x, y) + T((a, b))$. An element of the kernel is anything of the form $(0, t)$ and clearly the image is everything of the form $(0, s)$ so the subspaces are the same!

1.2 $V = \mathbb{R}^3$

Nope! This is easy to see from the Rank-Nullity theorem. $dim(V) = dim(\mathbb{R}^3) = 3 = dim(N(T)) + dim(R(T))$, because 3 is odd $dim(N(T)) \neq dim(R(T))$.

This exercise extends to all even and odd dimensioned spaces, where for even spaces you just project onto some 'half' of the space. In the infinite dimensional case the result is unclear to me, but I suspect it can be done essentially by alternating the projected coordinates.

2 Problem 14

Recall that $\mathcal{P}_n(\mathbb{R})$ is the vector space of all real polunomials in one variable of degree at most n. Show that if $f(x)$ is a polynomial of degree n over $\mathbb{R}$, then the set $\mathcal{F} = \{f(x), f'(x), f''(x), \ldots, f^{(n)}(x)\}$ is a basis for $\mathcal{P}_n(\mathbb{R})$.

2.1

Since f is a polynomial of degree n, the degree of the i^{th} derivative of f is exactly $n-i$ since f has degree n and differentiating lowers the degree of each monomial by exactly 1. This set is clearly linearly independent from the fact that the field of scalars cannot increase the exponent of a polynomial. To see this set spans the space of $\mathcal{P}_n(\mathbb{R})$ we can look at the span of the subsets of $\mathcal{F}$.

The span of $f^{(n)}(x)$ is the span of the scalar $a_n n!$, where a_n is the nonzero coefficient of x^n of f. Since $a_n n! \in \mathbb{R}$ it has an inverse in $\mathbb{R}$; $span(a_n n!) = span(\frac{a_n n!}{a_n n!}) = span(1)$, so this span is all of $\mathbb{R} = \mathcal{P}_0(\mathbb{R})$. The next subset is $\mathcal{F}_1 = \{f^{(n)}(x), f^{(n-1)}(x)\}$ which are degree 0 and 1 polynomials respectively. The span of these 2 are all of $\mathcal{P}_1(\mathbb{R})$ since $\mathcal{P}_0(\mathbb{R}) \subset \mathcal{P}_1(\mathbb{R})$ and $(a_n n!)^{-1} f^{(n-1)} - (a_{n-1}(n-1)!)^{-1}(a_n n!)^{-1} f^{(n)} = x$. So $span(\mathcal{F}_1) = span(1, x) = \mathcal{P}_1(\mathbb{R})$. Using this process of subtracting off the lower degree terms from higher degree polynomials we can see that each x^i appears in the span of $\mathcal{F}_i$. Since $\mathcal{F} = \mathcal{F}_0 \cap \mathcal{F}_1 \cap \ldots \cap \mathcal{F}_n$ the span can be seen to be all of $\mathcal{P}_n(\mathbb{R})$. From linear independence and span we see that the set $\mathcal{F}$ is basis for $\mathcal{P}_n(\mathbb{R})$.

3 Problem 15

Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(a, b) = (2a + b, a - 3b)$, let β be the standard ordered basis for $\mathbb{R}^2$, and let $\beta' = \{(1, 1), (1, 2)\}$.

3.1 Find $[T]_\beta$

$[T]_\beta$ is just the matrix which maps the standard basis to the standard basis under the transformation by T. The columns of the matrix are where the linear transformation maps the standard basis, so it looks like:

$$[T]_\beta = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix} \tag{1}$$

3.2 Find Q the change of basis matrix from $\beta' \to \beta$

This matrix is represented as it's columns being what β' looks like under the standard basis. So it maps $(1, 0) \mapsto (1, 1)$ and $(0, 1) \mapsto (1, 2)$.

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \tag{2}$$

3.3 Find Q^{-1} the change of basis matrix from $\beta \to \beta'$

Similarly to before this matrix's columns are what the standard basis maps to under β', so $\beta_1 = 2\beta'_1 - \beta'_2$ and $\beta_2 = -\beta'_1 + \beta'_2$.

$$Q^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \tag{3}$$

3.4 Find $[T]_{\beta'}$ using $[T]_{\beta'}, Q$ and Q^{-1}

Theorem 2.23 in Friedberg states: Let T be a linear operator on a finite dimensional vector space V, and let β and β' be ordered bases for V. Suppose that Q is the change of coordinate matrix that changes β'-coordinates into β-coordinates. Then:

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q \tag{4}$$

$$[T]_{\beta'} = Q^{-1}[T]_\beta Q = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -2 & -5 \end{bmatrix} = \begin{bmatrix} 8 & 13 \\ -5 & -9 \end{bmatrix} \tag{5}$$

4 Problem 16

Given a linear transformation $T : V \to V$ from a vector space V to itself, define Fix(T) $= \{v \in V | T(v) = v\}$.

4.1 Show that Fix(T) is a subspace of V

For arbitrary $u, v \in$ Fix(T) and $c \in F$ the image of $T(cu+v) = T(cu)+T(v) = cT(u)+T(v) = cu + v$ so this is in Fix(T). Also clearly $0 \in$ Fix(T) since T is a linear transformation, so Fix(T) is a subspace of V.

4.2 Show that $T^2 = T$ iff $V =$Fix(T)$\oplus N(T)$

4.2.1 Suppose $T^2 = T$

To show that $V =$ Fix(T)$\oplus N(T)$ we first need to see their intersection is exactly the zero vector. So suppose $\exists v \in$ Fix(T)$\cap N(T)$, $T(v) = v$ and $T(v) = 0$ so $v = 0$. Now we need to show that the subspaces span V.

From $T^2 = T$ we can see that the range of T is Fix(T) since for $v \in$ Fix(T) $T(v) = v = T(T(v))$. In the finite dimensional case we can just apply the Rank-Nullity Theorem to see $dim(V) = dim(R(T)) + dim(N(T)) = dim(\text{Fix}(T)) + dim(N(T))$. Since Fix($T$) $\subset V$ and $N(T) \subset V$ and Fix(T)$\cap N(T) = \{0\}$ we know they must span all of V.

For the infinite dimensional case we cannot use the Rank-Nullity Theorem and instead look at the arbitrary vector $v = v - T(v) + T(v)$. $T(v) \in$ Fix(T) and $v - T(v) \in N(T)$ since $T(v - T(v)) = T(v) - T(T(v)) = T(v) - T(v) = 0$. So an arbitrary $v \in V$ can be written as a unique combination of vectors, one in Fix(T) and one in $N(T)$.

4.2.2 Suppose $V =$ Fix(T)$\oplus N(T)$

Let v be an arbitrary vector in V, then v can be written as combination of vectors $v = f + n, f \in$ Fix(T) and $n \in N(T)$. To see the image of this vector $T(v) = T(f + n) = T(f) + T(n) = f + 0 = f$ and also $T(T(v)) = T(f) = f$ so $T^2 = T$.