A method for providing sufficient strict individual rankings' consistency level while group decision-making with feedback

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Abstract

A way of determining sufficient consistency level of strict individual rankings is suggested. It is assumed that absence of cycles in the final preference relationship built according to Condorcet method based on individual ordinal pair comparison matrices can be considered an adequate sufficient consistency indicator. A method is developed for achieving sufficient estimate consistency, targeted at minimizing the number of times the experts in the group are addressed.

Keywords: ordinal expert estimation, sufficient rankings' consistency, feedback with the expert

1. Introduction

Ordinal (rank) expert evaluation is used for decision-making in many weakly-structured human activity scopes (business, state government, scientific planning etc). Ordinal expert estimates are extremely useful when experts, building pair comparisons, find it difficult to estimate the degree of dominance of one alternative over another. In such cases instead of forcing experts to provide any estimates we can offer them to input values of ordinal preferences between alternatives, i.e. build alternative rankings. Presumably, experts possess sufficient information for differentiating the relative importance of any pair of alternatives, so, individual expert alternative rankings will be strict. The problem of adequate and "fair" individual rankings' aggregation remains topical in the context of group expert alternative evaluation. Sufficient consistency level of individual rankings is the decisive criterion, enabling the decision-maker to aggregate individual rankings into generalized group ranking, influencing the final choice of decision variants.

It is shown in (Tsyganok and Kadenko, 2010) that concordation and rank correlation coefficients, introduced by (Kendall, 1962), do not depend monotonously on the minimal number of permutations in a ranking, necessary for bringing it to the group ranking given a priori, so they cannot be used as sufficient individual rankings' consistency measures; individual rankings' set is considered consistent enough for their aggregation if aggregate preference relation built on their basis using (Condorcet, 1785) method is transitive, i.e. represents a rank order relation. The aggregate relation is built as a dominance matrix (see example below). If a matrix is intransitive, individual rankings' aggregation is considered inadmissible, and experts should be offered to change their alternative rankings, so the aggregate relation becomes transitive.

Obtaining a strict aggregate ranking can be an additional (but non-compulsory) condition of organizing feedback with experts and individual expert estimates' aggregation. This requirement can arise from the specificity of a given problem or from the fact that given individual rankings are also strict.

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We should stress, that any expert must be free to agree or disagree with suggestions concerning changes in his\her ranking.

2. Problem Statement

What is given: A set of n alternatives evaluated by m experts; strict expert alternative rankings; an aggregate relation (in general case, intransitive) given as a matrix built through aggregation of individual rankings using (Condorcet, 1785) method.

We should find a way of organizing feedback with experts in order to transform the aggregate relation into a transitive one, i.e. define successively, which experts should be addressed and which alternatives they should swap in individual rankings. Minimizing the number of times the experts are addressed should be chosen as feedback quality criterion.

3. Solution Idea

Since the number of questions posed to the experts, regarding changes of their initial opinions, depends on their answers (consent or refusal), it would be impossible to minimize this number using traditional approaches. So, beside the general number of questions posed to the expert group, we suggest taking the probability of their positive answers into consideration, since during feedback negative answers (refusals to change previously given opinions) lead to further search of other variants of transforming the aggregate relation into a transitive one and result in new suggestions concerning changes of initial individual rankings.

Let's take the following presumptions, characterizing the expert's logic, as guidance:

1) An expert more willingly changes previous estimates, which, to his mind, are less important. For example, an expert will more likely swap 5th and 6th alternatives than 1st and 2nd ones, in his\her ranking (providing, alternatives are ranked in order of their importance decrement).

2) The possibility of getting a negative answer from the expert grows proportionally to the number of suggestions made. So, the more changes an expert is offered to make, the less chances (s)he will agree with all of them. Thus, an expert will more probably agree to swap the 3rd alternative with the 4th, than with the 6th, because in the second case the "former" 4th and 5th alternatives will also have to change places.

Using these points, we shall form the further feedback organization strategy to minimize the number of suggestions made to experts, during transforming the set of individual expert rankings, which do not allow building a strict aggregate ranking, in such a way that this condition is fulfilled.

Thus, we suggest organizing the search among all strict expert ranking variants. The search aims to find the set of rankings, closest to the given one in terms of changes' quantity and quality (and, consequently, maximize the probability of experts' agreement to such changes), which will form a strict rank order relation after their aggregation. We suggest using genetic algorithm (GA) (Holland, 1994), allowing to find acceptable solutions of multi-variable function extreme values' search problems.

To solve problem using GA, a specific utility function must be formed and solution variants from the acceptability area must be coded.

The suggested algorithm envisions searching a solution variant, corresponding to the minimum of utility function. In this case solution variant represents a set of strict expert rankings, whose aggregation also presents a certain strict ranking. Moreover, it is possible to transform the initial expert rankings' set into this solution variant, using the minimal number of change suggestions to the experts. We suggest building the utility function, based on the aforementioned presumptions about expert's logic and equivalence of permutations in the rankings.

Utility function F can be represented as a sum of components F_u :

$$F = \sum_{u=1}^{m} F_u . \tag{1}$$

Each component corresponds to an expert ranking, and looks as follows:

$$F_u = (2d-1)(n-h+1) - d^2 , \tag{2}$$

where n is a quantity of alternatives in the ranking, h is the smaller of two alternative ranks, involved in the permutations, d is the "distance" between swapped alternatives.

$$d = |r_1 - r_2|,$$

r_1, r_2 are the ranks of swapped alternatives in the initial expert ranking.

Equation (2) considers the number of so called elementary permutations of alternatives, i.e., swaps of neighbouring alternatives $(d = 1)$, necessary for introducing given changes into expert ranking. The equation also includes the weight of every elementary permutation, depending on alternatives' remoteness from the ranking's end (alternative with rank n). So, permutation of alternatives which are less important to the experts, has smaller weight, and, consequently, is more likely to be suggested to the expert as a necessary change. Besides, equation (2) has one more property: the weight of swapping non-neighbouring alternatives equals the sum of respective consequent elementary permutation weights.

Let's consider the issue of coding GA solution variants (called "individuals"). In the suggested GA variant, an individual represents a set of strict expert alternative rankings, whose aggregation results in a strict ranking. It is suggested to transform the initial expert rankings set into this set. It is possible to calculate a utility function value for each individual. Let's establish a unique correspondence between individuals (n alternatives' ranking variants) and respective natural values. The power of such set equals the number of combinations of n elements: $P_n = n!$.

Thus the feedback organization algorithm is a search (choice) among all possible ranking variants conducted, according to these conditions:

1) initial rankings from the set must be strict ones;
2) aggregation of these rankings must result in a strict ranking;
3) transforming the initial expert ranking set into target set must require minimal trade-offs from the experts, and aim to maximize the probability of experts' consent to introducing changes into previously built rankings.

After that experts are offered to change their initial rankings into required ones through swapping respective alternatives. The algorithm ends when experts agree to all suggested ranking changes. Presumably, experts can be addressed simultaneously.

In the process of addressing the experts the information about their answers is being stored. In case some expert refuses to sanction suggested alternative pair inversion, further questioning is terminated and new target ranking set is sought; during further questioning previous answer history is taken into consideration. So, if an expert refused to swap a certain pair of alternatives, he is not asked this same question again. If some expert has already agreed to swap a suggested alternative pair, this swapping is not suggested to him again and the weight of this given swapping is not considered while further selection criterion (utility function) formation. Besides, in case the expert agrees to swap a pair of non-neighbouring alternatives, he, presumably, agrees to all respective elementary permutations of neighbouring alternative pairs.

Thus the algorithm is designed to find a series of alternative pair permutations, allowing to achieve sufficient consistency of alternative ranking set. In case when, due to multiple refusals of experts to change their opinions, no acceptable ranking set is found, we conclude that the expert group is unable to achieve a consistent aggregate opinion and suggest conducting a new expertise.

4. Step-by-step algorithm: An example

Say, a group of 4 experts ranked 4 alternatives. The individual rankings are: $R_1 = (3, 4, 2, 1)$, $R_2 = (1, 3, 2, 4)$, $R_3 = (1, 2, 3, 4)$, $R_4 = (3, 4, 1, 2)$. This means that the first expert assigned rank "3" to the first alternative, rank "4" – to the 2nd etc. The second expert assigned rank "1" to the first alternative etc.

Steps 1-3 of the algorithm check if expert rankings' consistency level is sufficient:

Step 1. Define the domination matrices based on expert rankings: $D_1 = \begin{pmatrix} 0 & 1 & -1 & -1 \\ & 0 & -1 & -1 \\ & & 0 & -1 \\ & & & 0 \end{pmatrix}$,

$D_2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ & 0 & -1 & 1 \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$, $D_3 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ & 0 & 1 & 1 \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$, $D_4 = \begin{pmatrix} 0 & 1 & -1 & -1 \\ & 0 & -1 & -1 \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$. These matrices are built as follows: if alternative A_1 dominates over A_2 (its rank is smaller $(r_1 < r_2)$), then the respective matrix element $d_{12} = 1$, otherwise $d_{12} = -1$. Since the matrices are reciprocal $(d_{ij} = -d_{ji})$, we confine ourselves to representing only the elements above the principal diagonal.

Step 2. Build the aggregate matrix D' based on matrices, obtained on the previous step, using Condorcet's method. The method envisions the summing of initial matrices $D_i, i = (1, m)$ followed by finding the sign of respective element sums: $D' = \operatorname{sign}\left(\sum_{i=1}^{m} D_i\right)$. We should note that the aggregate matrix can include non-diagonal zero elements; these may appear if the number of experts in the group is even. Thus, $D' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ & 0 & -1 & 0 \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$.

Step 3. Check, if relation D' includes non-diagonal zero elements, and if there are none, check, if it is transitive. Existence of at least one non-diagonal zero element indicates that the relationship is not a strict order. Numerous sources (Kendall and Smith, 1940; Litvak, 1982; Iida, 2009) show that transitivity violation can be detected by the presence of 3-cycles (circular triads) in the relation. Three arbitrary alternatives (A_1, A_2, A_3) form a cycle if the following relation is fulfilled: $A_1 \succ A_2 \succ A_3 \succ A_1$, where "$\succ$" denotes dominance. The aggregate relation is transitive if and only if it has no cycles in it. Cycles in a relation's matrix can be easily detected algorithmically, or by checking, whether the number of cycles equals zero: $l = C_n^3 - \sum_{i-1}^{n} C_{S_i}^2 = 0$, where $S_i = \sum_{j=1}^{n} d_{ij}$ – are row sums of matrix D', and $d_{ij} = -d_{ji}$.

So, we conclude that D' does not satisfy the posed requirements (it includes non-diagonal zero elements), and proceed to the next step to organize feedback with the experts.

Step 4. Using the GA, we conduct a targeted enumeration of expert ranking variants, which are respectively coded as individuals in the population as described above. An individual represents a set of m numbers form 1 to $n!$, each individual corresponds to a strict ranking variant for n alternatives. The initial population is generated randomly and new individuals "are born", "mutate", "cross-over" and "die-off", while only the fittest ones survive. The fitness of each individual is inversely proportional to the value, calculated according to (1). GA search ends when after a given iteration (generation) number the fittest individual stabilizes. It is selected as the search result. Before calculating the value according to (1), each individual is checked as to sufficient ranking consistency level through performing steps 1-3.

Thus, using the GA, we find an individual $(12, 3, 1, 9)$ (each value is a number of respective ranking vector), corresponding to the set of rankings $R_1^* = (2, 4, 3, 1)$, $R_2^* = (1, 3, 2, 4)$, $R_3^* = (1, 2, 3, 4)$, $R_4^* = (2, 3, 1, 4)$, where the utility function $F = 5$ reaches its minimum.

Step 5. Successively offer the experts to change previously built ranking R into R^* by introducing one ore more alternative swaps. In case experts accept all suggestions, the algorithm ends, and, based on rankings R^* (obtained from R through feedback), which are sufficiently consistent, the aggregate group ranking can be built.

Thus, based on comparison of rankings R_1 and R_1^* a suggestion for expert #1 is formulated: "do you agree to swap alternatives #1 and #3 (ranked "3" and "2")?" Say, the expert agrees. Then, after making sure that after one permutation the rankings R_1 and R_1^* are already identical, we proceed to analyzing the rankings of expert #2. Since $R_2 = R_2^*$, and $R_3 = R_3^*$, we proceed to analyzing the rankings built by expert #4. Based on R_4 and R_4^* a suggestion for the 4th expert is formulated: "do you agree to swap alternatives #1 and #4 (ranked "3" and "2")?"

Say, expert #4 refuses, so we search for a new expert ranking set (step 4), taking obtained experts' answers into consideration. The result of step 4 is an individual $(12, 3, 1, 15)$, representing the rankings' set $R_1^* = (2, 4, 3, 1)$, $R_2^* = (1, 3, 2, 4)$, $R_3^* = (1, 2, 3, 4)$, $R_4^* = (3, 2, 1, 4)$, where $F = 5$ (the summand, corresponding to R_1 and R_1^* is not considered while calculating F, because the question of swapping alternatives #1 and #3 is unnecessary: the expert has already answered positively).

At step 5, while analyzing the rankings of expert #1, the question of swapping alternatives #1 and #3 is not posed, because the expert has given positive answer. While analyzing the rankings of expert #4, based on R_4 and R_4^* a new suggestion for the expert is formulated: "do you agree to swap alternatives #2 and #4 (ranked "4" and "2")?" In case the expert agrees to swap the alternatives, $R_4 = R_4^*$, and at this point the feedback algorithm ends. As a result, the set of expert rankings R is transformed into set R^*, which is sufficiently consistent for aggregating them into group ranking.

5. Conclusions

An algorithm for organizing feedback with experts while building group ranking is suggested. The problem solved, is extremely topical in the areas of expert examinations and decision-making support systems development, so the suggested algorithm can become an important element of their mathematical software. Experimental research of the algorithm testifies to its high efficiency.

The key direction of future research lies in the approach extrapolation to the cases of different expert competence and non-strict alternative rankings.

6. References

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