EE 16A: Homework 6

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   I worked alone for 5 hours then met up with William and Matt to discuss how to approach and solve each problem.

2. **Nodal Analysis**

   (a) *Solve for all node voltages using nodal analysis. Verify with superposition.*

   We would like to compute the node potentials at V_1 and V_2. So, let us perform nodal analysis by summing the current (flow) that goes in and out of these two nodes.

   For V_1, we have:

$$-1 + \frac{1}{10}(V_1 - 0) + \frac{1}{20}(V_1 - V_2) = 0 \tag{1}$$

   which simplifies to

$$\frac{3}{20}V_1 - \frac{1}{20}V_2 = 1. \tag{2}$$

   For V_2, we have:

$$2 + \frac{1}{50}(V_2 - 0) + \frac{1}{20}(V_2 - V_1) = 0 \tag{3}$$

   which simplifies to

$$\frac{1}{20}V_1 - \frac{7}{100}V_2 = 2. \tag{4}$$

   By putting equations (2) and (4) into the matrix-vector form $Av = b$, we get

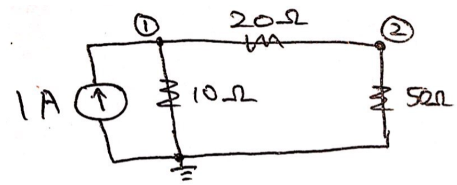
$$\begin{bmatrix} \frac{3}{20} & -\frac{1}{20} \\ \frac{1}{20} & -\frac{7}{100} \end{bmatrix} v = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{5}$$

   Using ipython notebook to solve for V_1 and V_2 using numpy (code provided in the back), we get

$$\boxed{V_1 = -3.75V} \tag{6}$$

$$\boxed{V_2 = -31.25V} \tag{7}$$

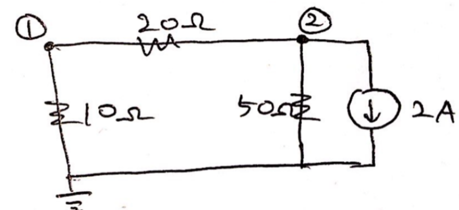
   Verifying with superposition: first we consider the left current source ($1A$):



$$V_1 = IR_{eq} = (1A)(\frac{1}{\frac{1}{10\Omega} + \frac{1}{70\Omega}}) = \frac{70}{8}V \tag{8}$$

$$V_2 = (\frac{50\Omega}{20\Omega + 50\Omega})V_1 = \frac{50}{8}V \tag{9}$$

next, the right current source (2A):



$$V_2 = IR_{eq} = (-2A)(\frac{1}{\frac{1}{30\Omega} + \frac{1}{50\Omega}}) = -\frac{150}{4}V \tag{10}$$

$$V_1 = (\frac{10\Omega}{10\Omega + 20\Omega})V_2 = -\frac{50}{4}V \tag{11}$$

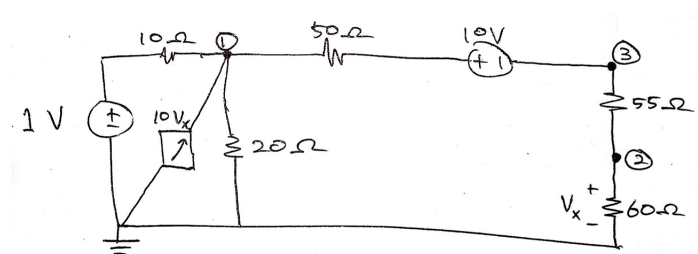
Thus, by adding equations (8) + (11) and (9) + (10) we get

$$V_1 = \frac{70}{8}V - \frac{50}{4}V \implies \boxed{V_1 = -\frac{15}{4}V} \tag{12}$$

$$V_2 = \frac{50}{8}V - \frac{150}{4}V \implies \boxed{V_2 = -\frac{125}{4}V} \tag{13}$$

(b) *Solve for all node voltages using nodal analysis.*

Redraw the circuit with an added node 3:



Notice that $V_2 = V_x$ since V_2 is connected to the ground through the 60Ω resistor. Keeping this in mind, we proceed with nodal analysis on each node (1,2,3):

For V_1, we have:

$$\frac{1}{10\Omega}(V_1 - 1V) - 10V_2 + \frac{1}{20\Omega}(V_1 - 0) + \frac{1}{50\Omega}(V_1 - 10V - v_3) = 0 \tag{14}$$

For V_2, we have:

$$\frac{1}{60\Omega}(V_2 - 0) + \frac{1}{55\Omega}(V_2 - V_3) = 0 \tag{15}$$

For V_3, we have:

$$\frac{1}{50\Omega}(V_3 + 10V - V_1) + \frac{1}{55\Omega}(V_3 - V_2) = 0 \tag{16}$$

With some rearranging into the matrix-vector form ($Av = b$, where $v = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$) then solving through iPython(code in the back), we get

$$\boxed{V_1 = 10.402V} \tag{17}$$

$$\boxed{V_2 = 0.146V} \tag{18}$$

3. **Thvenin and Norton equivalent circuits**

(a) *Find the Thvenin and Norton equivalent circuits seen from the outside the box.*

First, calculate the effective resistance seen from the voltage source to find the current supplied by the voltage source. The resistances R_3 and R_4 are in series hence have effective resistance of 6Ω. They are connected in parallel to a R_2 resistance yielding an effective resistance of

$$(\frac{1}{6} + \frac{1}{3})^{-1} = 2\Omega. \tag{19}$$

This resistance is in series to R_1, yielding a total effective resistance of

$$R_{eq} = 3\Omega + 2\Omega = 5\Omega. \tag{20}$$

Hence the current supplied by the voltage source is

$$I = \frac{10V}{5\Omega} = 2A \tag{21}$$

Now set up these KCL / KVL equations:

$$-v_2 + v_3 + v_4 = 0 \tag{22}$$

$$v_2 = 3I_2 \tag{23}$$

$$v_3 = 4.5I_3 \tag{24}$$

$$v_4 = 1.5I_3 \tag{25}$$

$$I = I_2 + I_3 = 2A \tag{26}$$

We combine equations (4) through (7) to get

$$\implies -3I_2 + 4.5I_3 + 1.5I_3 = 0 \tag{27}$$

Solving equations (8) and (9) for I_3, we get

$$3I_3 + 6I_3 = 6A \implies I_3 = \frac{2}{3}A \tag{28}$$

Thus $V_{thvenin} = 4.5I_3 = 3$ Volts. The Norton equivalent is given by

$$I_{norton} = \frac{V_{thvenin}}{R_{thvenin}} = \frac{3}{\frac{9}{4}} = \frac{4}{3}A \tag{29}$$

(b) *Find the Thvenin and Norton equivalent circuits seen from the outside the box.*

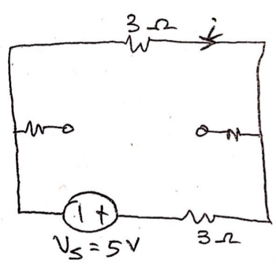
$$V_{thevenin} = IR = (2A)(1.5\Omega) \implies \boxed{V_{th} = 3V} \tag{30}$$

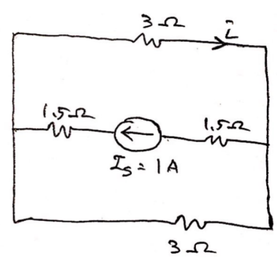
$$I_{sc} = \frac{V_{th}}{(R_1||R_3 + R_5)} \implies \boxed{I_{no} = 2.8A} \tag{31}$$

$$R_{thevenin} = \frac{V_{th}}{I_{sc}} = \frac{3V}{2.8A} \implies \boxed{R_{th} = 1.0714\Omega} \tag{32}$$

4. **Nodal Analysis Or Superposition?** *Solve for the current through the* 3Ω *resistor, marked as i, using superposition. Verify using nodal analysis. You can use IPython to solve the system of equations if you wish. Where did you place your ground, and why?*

Using superposition: We look at one independent source at a time, and in this case, V_s then I_s:





From the left superposition, we can easily see that

$$I = \frac{V}{R_{eq}} = \frac{5V}{3\Omega + 3\Omega} = \frac{5}{6}A \tag{33}$$

and since i is just in the opposing direction of the same current,

$$i = -I = -\frac{5}{6}A \tag{34}$$

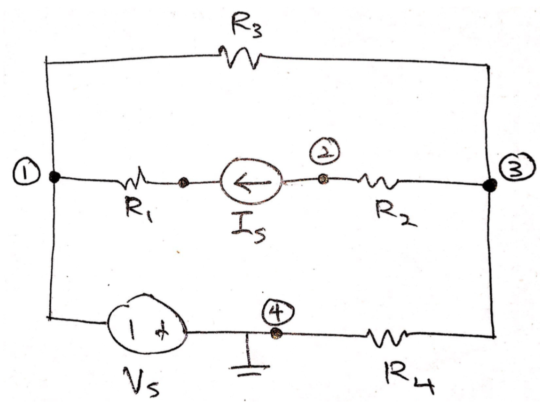
From the right superposition, we notice the symmetry so it is easy to see that the current source splits evenly:

$$i = \frac{1}{2}I_s = \frac{1}{2}A \tag{35}$$

Thus by superposition,

$$i = -\frac{5}{6}A + \frac{1}{2}A \implies \boxed{i = -\frac{1}{3}A} \tag{36}$$

Verify using nodal analysis: From all the nodes that exist in this circuit, we label nodes 1, 2, 3, and 4, and recognizing that we cannot solve for all of these potentials, we place a ground node to set the potential of $V_4 = 0$:



It is easy to see that V_1 (potential at node 1) is

$$V_1 = -V_s \tag{37}$$

Now, proceed with nodal analysis on nodes 2 and 3:

For V_2, we have:

$$I_s + G_2(V_3 - V_4) = 0 \tag{38}$$

For V_3, we have:

$$G_3(V_4 - V_1) + G_2(V_4 - V_3) + G_4(V_4 - 0) = 0 \tag{39}$$

Substituting in V_1 in equation (36) then rearranging into the matrix-vector form ($Av = b$, where $v = \begin{bmatrix} V_2 \\ V_3 \end{bmatrix}$) then solving through iPython(code in the back), we get

$$V_2 = -5.5V \tag{40}$$
$$V_3 = -4V \tag{41}$$

Finally, to solve for current i through R_3,

$$i = \frac{\Delta V}{R_3} = \frac{V_1 - V_3}{R_3} = \frac{-5V - (-4V)}{3\Omega} \implies \boxed{i = -\frac{1}{3}A} \tag{42}$$

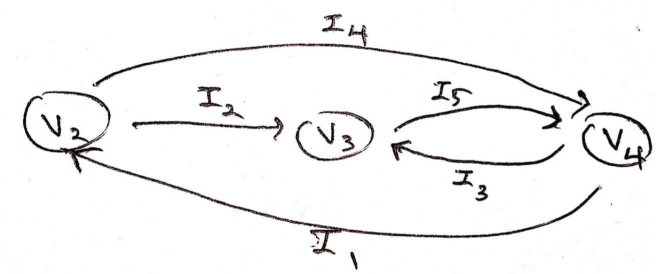
5. **Optional**
6. **Solving Circuits with Voltage Sources**

(a) *What relationship does the voltage source enforce between v_1 and v_4?*

$$\boxed{v_1 - v_4 = V_s} \tag{43}$$

The voltage source fixes the nodes to be a constant offset from each other.

(b) *Draw the graph for the circuit where v_1 and v_4 are combined into one node. Specify a new incidence matrix for this graph.*



$$F = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

(c) *Find R and $\vec{b}$ so that Ohms law is written $F\vec{v} + \vec{b} = R\vec{i}$.*

$$R = \begin{bmatrix} R_1 & 0 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 & 0 \\ 0 & 0 & R_3 & 0 & 0 \\ 0 & 0 & 0 & R_4 & 0 \\ 0 & 0 & 0 & 0 & R_5 \end{bmatrix}$$

$$b = \begin{bmatrix} V_s \\ 0 \\ V_s \\ 0 \\ 0 \end{bmatrix}$$

(d) *What is $\vec{f}$ in this circuit? write $\vec{v}$ in terms of known quantities ($\vec{f}, \vec{b}, G, F, R$). You may need to modify several of the members of the derived equation by grounding a node and dropping a row or a column in order to give the problem a unique solution.*

First let us compute the product $F^T\vec{i}$:

$$\begin{bmatrix} -1 & 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 1 & 0 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = \begin{bmatrix} -i_1 + i_2 + i_4 \\ -i_2 - i_3 + i_5 \\ i_1 + i_3 - i_4 - i_5 \end{bmatrix}$$

From this, we know the vector $\vec{f}$ has 3 elements in it. Furthermore, we know we combined the nodes 1 and 4 into a supernode. Therefore:

$$\boxed{\vec{f} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} \tag{44}$$

Solving the Ohms law equation for $\vec{i}$,

$$\vec{i} = G(F\vec{v} + \vec{b}).$$

Now plugging this expression into KCL we see

$$F^T G(F\vec{v} + \vec{b}) = -\vec{f}$$

Does this equation have a unique solution? Let us first go back to the matrix F, row-reducing we see F has a linearly dependent column.

$$F = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} -> \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This implies that the system is underdetermined. In order to make this system determined, we can drop one column of the F matrix, and ignore one potential in $\vec{v}$. We will assign this node a potential of 0. This is equivalent to grounding a node in a circuit ($\vec{v}_{gr}$).

Call the new matrix F_{gr} and the new $\vec{f}$ as $\vec{f}_{gr}$. Then F_{gr} has no null space and:

$$\vec{i} = GF_{gr}\vec{v}_{gr} + G\vec{b} \tag{45}$$

Plugging this into KCL again, we see

$$F_{gr}^T GF_{gr}\vec{v}_{gr} + F_{gr}^T G\vec{b} = -\vec{f}_{gr} \tag{46}$$

Thus,

$$\boxed{F_{gr}^T GF_{gr}\vec{v}_{gr} = -\vec{f}_{gr} - F_{gr}^T G\vec{b}} \tag{47}$$

(e) *write $\vec{i}$ in terms of known quantities*

$$\boxed{\vec{i} = G(F_{gr}\vec{v}_{gr} + \vec{b})} \tag{48}$$

(f) See iPython Notebook in the end of PDF