EE 16A: Homework 13

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1. **Worked With...**

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   I worked alone on Friday morning, then met up with Ilya and James to discuss on Saturday afternoon.

2. **Mechanical Problem** Compute eigenvalues + eigenvectors.

   (a)

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$$
$$A - \lambda I_2 = \begin{bmatrix} 3-\lambda & 0 \\ 0 & 5-\lambda \end{bmatrix} = 0$$

   Set the determinant to zero:

$$(3-\lambda)(5-\lambda) = 0$$

   Implies that $\lambda = 3, \lambda = 5$. Plugging those values into the original equation above, we can see that
   The eigenvectors associated with $\lambda = 3$ are all of the form

$$\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \alpha \in \mathbb{R}.$$

   The eigenvectors associated with $\lambda = 5$ are all of the form

$$\alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \alpha \in \mathbb{R}.$$

   (b)

$$A = \begin{bmatrix} 22 & 6 \\ 6 & 13 \end{bmatrix}$$
$$(22-\lambda)(13-\lambda) - 36 = 0$$
$$250 - 35\lambda + \lambda^2 = 0$$
$$(\lambda - 25)(\lambda - 10) = 0$$

   Implies $\lambda = 25, 10$. Plugging those values into the partial lambda matrix we get coressponding eigenparis

$$e_1 = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}, 25 \right), \quad e_2 = \left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}, 10 \right).$$

   (c)

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$
$$(1-\lambda)(4-\lambda) - 4 = 0$$
$$4 - 5\lambda + \lambda^2 - 4 = 0$$
$$(5-\lambda)\lambda = 0$$

   Implies that $\lambda = 5, 0$. Plugging those values into the partial lambda matrix we get coressponding eigenparis

$$e_1 = \left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}, 0 \right), \quad e_2 = \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, 5 \right).$$

(d)

$$A = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\left(\frac{\sqrt{3}}{2} - \lambda\right)\left(\frac{\sqrt{3}}{2} - \lambda\right) - \frac{1}{4} = 0$$

$$\frac{1}{2} - 2\sqrt{3}\lambda + \lambda^2 = 0$$

Implies by the quadratic formula $\lambda = \frac{1}{2}(\sqrt{3} \pm i)$.
Plugging those values into the partial lambda matrix we get coressponding eigen-paris

$$e_1 = \left(\begin{bmatrix} i \\ 1 \end{bmatrix}, \frac{1}{2}(\sqrt{3} + i)\right), \quad e_2 = \left(\begin{bmatrix} -i \\ 1 \end{bmatrix}, \frac{1}{2}(\sqrt{3} - i)\right).$$

3. **Mechanical Diagonlization** Diagonalize the matrix

$$A = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 3/2 & 1/2 \\ -1 & 1 & 1 \end{bmatrix}$$

given that A has eigenvalues 1, 2, and 0.

For $\lambda = 1$, we find $\vec{v}$ such that

$$\begin{bmatrix} -1 & 1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} -2 & 2 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \vec{v} = 0$$

Therefore

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Repeating for $\lambda = 2$ we get

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda = 0$ we get

$$v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Putting everything together we define

$$P_A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad D_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P_A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

Then A can be diagonalized as $A = P_A D_A P_A^{-1}$.

4. **Spectral Mapping and the Fibonacci Sequence**

   (a) A^N in terms of PDP^{-1}:

   $$A^N = \prod_{i=1}^{N} PDP^{-1} = PD^N P^{-1}$$

   It says that polynomial functions of A are easily calculated by raising D, the eigenvalues of A, to certain powers.

   (b) The ratio of fibbonacci numbers tends towards the golden ratio.

   (c) Find the eigen system for the matrix, with $\lambda_1 = \phi, \lambda_2 = \Phi$.
   Eigen vectors are

   $$v_1 = \begin{bmatrix} \frac{1}{2}(1+\sqrt{5}) \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} \frac{1}{2}(1-\sqrt{5}) \\ 1 \end{bmatrix}$$

   Now we solve for P^{-1}.

   $$P^{-1} = \begin{bmatrix} \frac{-1}{\sqrt{5}} & \frac{1}{10}(5+\sqrt{5}) \\ \frac{1}{\sqrt{5}} & \frac{1}{10}(5-\sqrt{5}) \end{bmatrix}$$

   Putting it all together we get

   $$F = \begin{bmatrix} 1/2(1-\sqrt{5}) & \phi \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/2(1-\sqrt{5})^{N-1} & 0 \\ 0 & \phi^{N-2} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{5}} & \frac{1}{10}(5+\sqrt{5}) \\ \frac{1}{\sqrt{5}} & \frac{1}{10}(5-\sqrt{5}) \end{bmatrix}$$

   $$F_N = \frac{1}{\sqrt{5}}\phi^{N-1} - \frac{-1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^N$$

5. **Image Compression**

(a) Yes.
Let $U = [v_1 \ \cdots \ v_k \ \ 0 \ \cdots]^T$ and $V = U^T$. Then let $\Lambda = diag(\lambda_1 \ \ldots \ \lambda_k \ \ 0 \ \ldots)$.

(b) To **fully** capture the information you'll need all of them.

(c) *see iPython*

(d) 20 is lowest

6. **Counting the paths of a Random Surfer**

(a) *Write out the adjacency matrix for graph A*

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(b) There is 1 one-hop path.
There are 0 two-hop path.
There is 1 three hop path.

(c) The importance scores are: $(1, 1)$.

(d) *Write out the adjacency matrix for graph B*

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

(e) By squaring the adjacency matrix we get all of the paths of length 2.

$$A^2 = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

From 1 to 3 there is 1 two-hop path.
From 1 to 2 there is 1 two-hop path.

(f) The importance scores are given by the eigen vector of eigenvalue 1,

$$I = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{8} \\ \frac{3}{8} \end{bmatrix}$$

(g) *Write out the adjacency matrix for graph C*

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

(h) THere are no paths from one to three.

(i) We get eigenvalue 1 with multiplicity two. It is clear that the first two web pages have 0.5, 0.5 importance score from the first eigenvector, but in the first eigenvector there is no importance to the second cluster. The second eigen vector gives 3 of inportance 0.4, four of importance 0.2 and 5 of importance 0.4. We can use this to give us relative importance scores by adding the eigenvectors and normalizing:

$$I = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.24 \\ 0.12 \\ 0.24 \end{bmatrix}$$

This inherently puts more weight to the larger cluster, but to be precise we consider the importance scores independently and use the score

$$I_1 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 0 \\ 0 \\ 0.4 \\ 0.2 \\ 0.4 \end{bmatrix}$$

7. **Sports Rank**

(a)

$$Q = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$$

(b) Okay, let us now develop a method for finding the dominant eigenvector for a matrix when it is unique.

(c)

$$Q^n c\vec{v} = c\prod_{k=1}^{n-1} \lambda\vec{v} = c\prod_{k=1}^{n-2} \lambda^2\vec{v} = \cdots = c\lambda^n\vec{v}$$

(d) Using the above derivation,

$$Q^n \left(\sum_{i=1}^{m} c_i\vec{v}_i\right) = \sum_{i=1}^{m} c_i\lambda_i^n\vec{v}_i.$$

(e) Assuming that $|\lambda_1| > |\lambda_i|$ for $i = 2, \ldots, m$, prove that

$$\lim_{n\to\infty} \frac{1}{\lambda_1^n} Q^n \left(\sum_{i=1}^{m} c_i\vec{v}_i\right) = c_1\vec{v}_1$$

Proof. Using the limit properties,

$$\lim_{n\to\infty} \frac{1}{\lambda_1^n} Q^n \left(\sum_{i=1}^{m} c_i v_i\right) = \lim_{n\to\infty} \left(\sum_{i=1}^{m} \frac{\lambda_i^n c_i v_i}{\lambda_1^n}\right).$$

Then we examine the sequence itself and show absolute convergence; that is in the absolute case

$$\lim_{n\to\infty} \left(\sum_{i=1}^{m} \frac{|\lambda_i^n| c_i v_i}{|\lambda_1^n|}\right) = \sum_{i=1}^{m} \lim_{n\to\infty} \frac{|\lambda_i^n| c_i v_i}{|\lambda_1^n|}.$$

Since $|\lambda_i^n| \leq |\lambda_1^n|$, we have that $b_i^n = |\lambda_i^n|/|\lambda_1^n| \leq 1$ for all n and therefore $b_i^n \to 0, n \to \infty$, when $i \neq 1$. Otherwise, $b_1 = 1$ for all n and we have

$$\sum_{i=1}^{m} \lim_{n\to\infty} \frac{|\lambda_i^n| c_i\vec{v}_i}{|\lambda_1^n|} = c_1\vec{v}_1.$$

□

(f) Assuming that λ_1 is positive, prove that

$$\lim_{n\to\infty} \frac{Q^n \left(\sum_{i=1}^{m} c_i v_i\right)}{\|Q^n \left(\sum_{i=1}^{m} c_i v_i\right)\|} = \frac{c_1 v_1}{\|c_1 v_1\|}$$

Proof. By the previous theorem

$$\lim_{n\to\infty} \frac{Q^n\left(\sum_{i=1}^m c_i v_i\right)}{\left\|Q^n\left(\sum_{i=1}^m c_i v_i\right)\right\|} = \lim_{n\to\infty} \frac{\frac{1}{\lambda_1^n} Q^n\left(\sum_{i=1}^m c_i v_i\right)}{\left\|\frac{1}{\lambda_1^n} Q^n\left(\sum_{i=1}^m c_i v_i\right)\right\|} = \frac{c_1 v_1}{\|c_1 v_1\|}.$$

□

(g) The top 5 are ORE, ALA, ARIZ MISS, UCLA.
Then the fourteenth and the seventeenth are LSU and USC, respectively.

8. **Dynamics of Romeo and Juliet's Love Affair**

(a)

$$Av_1 = \begin{bmatrix} a+b \\ c+d \end{bmatrix} = (a+b)v_1.$$

Thus $(a+b)$ is an eigenvalue. To solve for the other eigenvalues we set the determinent zero.

$$\begin{aligned} \lambda^2 - (a+d)\lambda + ad - bc = 0 \\ \lambda = \frac{a+d \pm \sqrt{(a+d)^2 - 4ad + bc}}{2} = a+b \\ 2a + 2b = a + d \pm \sqrt{(a+d)^2 - 4ad + bc} \\ \implies (a+d)^2 - 4ad + bc = (\pm(a - d + 2b))^2 \\ \implies \lambda_2 = \frac{a + d - a + d - 2b}{2} = d - b. \end{aligned}$$

Plugging in the eigenvalues we get

$$v_2 = \begin{bmatrix} -\frac{b}{c} \\ 1 \end{bmatrix}$$

For the new system the eigen pairs are $((1,1),1)$ and $((-1,1),0.5)$.

All *fixed points* are on the linear space with basis $(1,1)$.

The reprasentative points are

$$s[n] = 0.5^n s[0] \to 0$$

For the next scenario we know that $c_1 = 4$, $c_2 = 1$. We have

$$s[n] = 4v_1 + 0.5^n v_2 \to 4v_1 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$