EECS 16A Designing Information Devices and Systems I
Fall 2016 Babak Ayazifar, Vladimir Stojanovic Homework 6

This homework is due October 11, 2016, at Noon.

1. **Homework process and study group**

   Who else did you work with on this homework? List names and student ID's. (In case of hw party, you can also just describe the group.) How did you work on this homework?
   Working in groups of 3-5 will earn credit for your participation grade.

   Solution: I worked on this homework with...

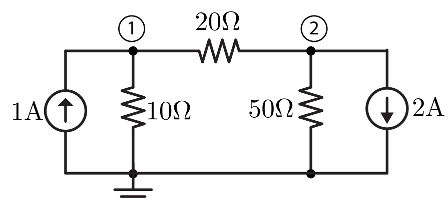
   I first worked by myself for 2 hours, but got stuck on Problem 5 so I went to office hours on...

   Then I went to homework party for a few hours, where I finished the homework.

2. **Nodal Analysis**

   Using techniques presented in class, label all unknown node voltages and apply KCL to each node to find all the node voltages.

   (a) Solve for all node voltages using nodal analysis. Verify with superposition.



Solution:
Method 1: Nodal Analysis
Applying KCL at Node 1, we get

$$\frac{V_1 - 0}{10} + \frac{V_1 - V_2}{20} - 1 = 0 \quad (1)$$

which gives

$$2V_1 + V_1 - V_2 - 20 = 0 \quad (2)$$

implying

$$3V_1 - V_2 = 20 \quad (3)$$

Applying KCL at Node 2, we get

$$\frac{V_2 - V_1}{20} + \frac{V_2 - 0}{50} + 2 = 0 \quad (4)$$

which gives

$$5V_2 - 5V_1 + 2V_2 + 200 = 0 \quad (5)$$

implying

$$-5V_1 + 7V_2 = -200 \tag{6}$$

Writing equations 3 and 6 in matrix form, we get

$$\begin{bmatrix} 3 & -1 \\ -5 & 7 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 20 \\ -200 \end{bmatrix} \tag{7}$$

Solving the system of equations, we will get $V_1 = -3.75V$ and $V_2 = -31.25V$.

Method 2 (verification): Superposition First, consider the effect of only $1A$ current source on. Using current divider rule, we have

$$i_1 = \frac{70}{10+70} \times 1A = 0.875A \tag{8}$$

and

$$i_2 = 1A - 0.875A = 0.125A \tag{9}$$

Therefore,

$$V_1^a = i_1 \times 10\Omega = 0.875A \times 10\Omega = 8.75V \tag{10}$$

and

$$V_2^a = i_2 \times 50\Omega = 0.125A \times 50\Omega = 6.25V \tag{11}$$

Second, consider the effect of only $2A$ current source on. Using current divider rule, we have

$$i_1 = \frac{50}{50+30} \times 2A = 1.25A \tag{12}$$

and

$$i_2 = 2A - 1.25A = 0.75A \tag{13}$$

Therefore,

$$V_1^b = 0 - i_1 \times 10\Omega = -1.25A \times 10\Omega = -12.5V \tag{14}$$

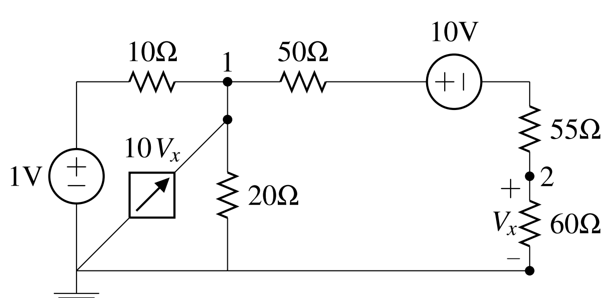
and

$$V_2^b = 0 - i_2 \times 50\Omega = -0.75A \times 50\Omega = -37.5V \tag{15}$$

Since the circuit is linear (i.e. we have linear elements and sources), we can use the *principle of superposition* to get $V_1 = V_1^a + V_1^b$ and $V_2 = V_2^a + V_2^b$. So, we get $V_1 = 8.75V - 12.5V$ and $V_2 = 6.25V - 37.5V$. Finally, $V_1 = -3.75V$ and $V_2 = -31.25V$.

This solution agrees with the solution we obtained using nodal analysis :)

(b) Solve for all node voltages using nodal analysis.



Solution: We assign the two nodes 1 and 2, and note that $V_x = V_2$ (because we have set the bottom wire as ground). Applying KCL at node 1 (ensuring that currents flowing into node 1 sum to zero), we get

$$\frac{V_1 - 1}{10} + \frac{V_1 - 0}{20} + \frac{V_1 - (10 + V_2)}{105} - 10V_2 = 0 \tag{16}$$

which gives

$$\frac{V_1 - 1}{2} + \frac{V_1 - 0}{4} + \frac{V_1 - V_2 - 10}{21} - 50V_2 = 0 \tag{17}$$

implying

$$67V_1 - 4204V_2 = 82 \tag{18}$$

Applying KCL to Node 2 (ensuring that currents flowing into node 2 sum to zero), we get

$$\frac{V_2 - 0}{60} + \frac{V_2 - (V_1 - 10)}{105} = 0 \tag{19}$$

which gives

$$\frac{V_2}{4} + \frac{V_2 + 10 - V_1}{7} = 0 \tag{20}$$

implying

$$-4V_1 + 11V_2 = -40 \tag{21}$$

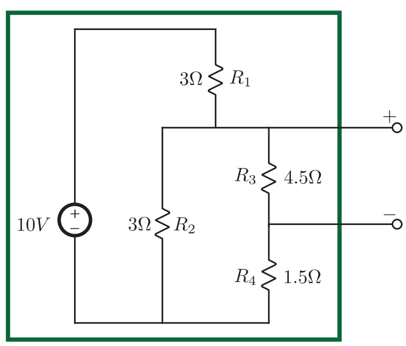
Writing the equations in matrix form, we get

$$\begin{bmatrix} 67 & -4204 \\ -4 & 11 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 82 \\ -40 \end{bmatrix} \tag{22}$$

Solving the system of equations, we will get $V_1 = 10.4023V$ and $V_2 = 0.1463V$.

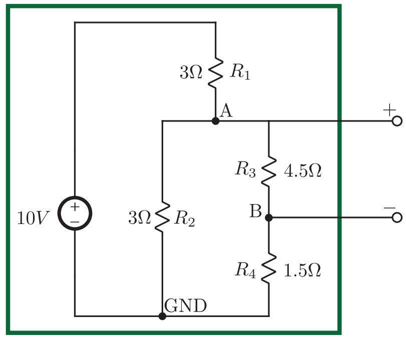
3. Thévenin and Norton equivalent circuits

(a) Find the Thévenin and Norton equivalent circuits seen from the outside the box.



Solution: To find the Thévenin and Norton equivalent circuits we are going to find the open circuit voltage between the output ports and the current flowing through the output ports when the ports are shorted.

For finding the open circuit voltage between the output ports, let us label the nodes as shown in the figure below.



First, let us begin by calculating the effective resistance between nodes A and GND. We have 3Ω resistor in parallel to $4.5\Omega + 1.5\Omega$ resistance. This gives an equivalent resistance of

$$\frac{1}{\frac{1}{3\Omega} + \frac{1}{4.5\Omega+1.5\Omega}} = 2\Omega.$$

Then we see that we have a voltage divider from the positive terminal of 10V supply. Voltage divider is made up of two resistances in series, where the resistances are 3Ω and 2Ω. This gives the voltage at node A equal to

$$V_{\text{A}} = 10V \frac{2\Omega}{3\Omega + 2\Omega} = 4V.$$

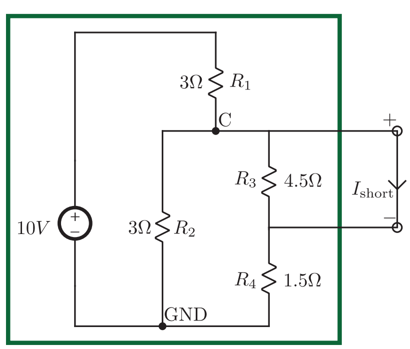
To find the voltage at node B, note that we have another voltage divider between nodes A and GND. Hence, we can find the voltage at node B as

$$V_{\text{A}} \frac{1.5\Omega}{4.5\Omega + 1.5\Omega} = 4V \frac{1}{4} = 1V.$$

Hence the open circuit voltage seen between the output ports is equal to

$$\begin{aligned} V_{\text{open}} &= V_A - V_B \\ &= 4V - 1V \\ &= 3V. \end{aligned}$$

Now let us find the short circuit current flowing through the output ports. When doing this, we get the following circuit.



Note that when we short the output terminals, the voltages at the nodes change, this is why we changed the label of the node below the resistor R_1. Since there is a short circuit parallel to the resistor R_3, there will be no current flowing through it, hence we have

$$I_{\text{short}} = I_{R_4}. \tag{23}$$

To find this current, let us find the equivalent resistance due to R_2 being connected parallel to R_4 when we short the output ports. We have 3Ω parellel to 1.5Ω, which gives an equivalent resistance

$$\frac{1}{\frac{1}{3\Omega}+\frac{1}{1.5\Omega}} = 1\Omega.$$

We again have a voltage divider between the positive side of the 10V supply and the ground. Using this voltage divider, we calculate the voltage at node C as

$$V_C = 10V\frac{1\Omega}{3\Omega+1\Omega} = 2.5V.$$

Hence, we see that the voltage across the resistor R_4 is equal to $2.5V$. Using Ohm's law, we get

$$I_{R_4} = \frac{2.5V}{1.5\Omega} = \frac{5}{3}A.$$

Since we have $I_{\text{short}} = I_{R_4}$, we have

$$I_{\text{short}} = I_{R_4}.$$

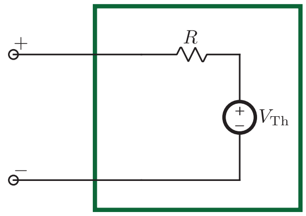
Summarizing the results, we have

$$V_{\text{open}} = 3V,$$
$$I_{\text{short}} = \frac{5}{3}A.$$

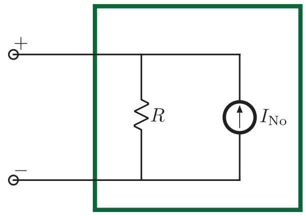
This gives

$$R_{\text{Th}} = \frac{V_{\text{open}}}{I_{\text{short}}} = \frac{9}{5}\Omega.$$

Hence the Thévenin equivalent circuit is given by

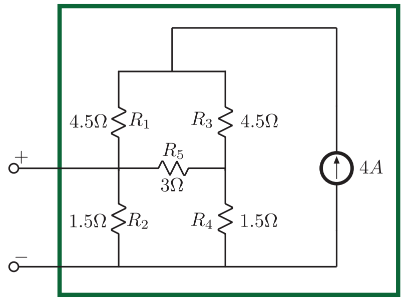


where $R = R_{\text{Th}}$ and $V_{\text{Th}} = V_{\text{open}}$; and the Norton equivalent circuit is given by



where $R = R_{\text{Th}}$ and $I_{\text{No}} = I_{\text{short}}$.

(b) Find the Thévenin and Norton equivalent circuits seen from the outside the box.



Solution: As with the previous part of this question, to find the Thévenin and Norton equivalent circuits we are going to find the open circuit voltage between the output ports and the current flowing through the output ports when the ports are shorted.

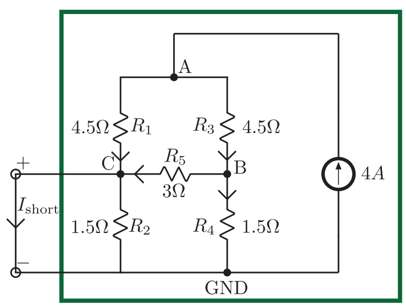
Let us first find the open circuit voltage between the output ports. In the solutions of homework 6, we have seen 3 different approaches for finding the voltages at each node of this same circuit. One of those approaches was to use the symmetry in the circuit to see there will be no current flowing through R_5. Then, we have a current divider where each branch has the same resistance, hence the current $4A$ divides equally between the left and right branches. Hence we have

$$I_{R_1} = I_{R_2} = I_{R_3} = I_{R_4} = 2A.$$

This gives us the voltage across R_2, equivalently the open circuit voltage between the output terminals, equal to

$$V_{\text{open}} = 2A \times 1.5\Omega = 3V.$$

Now let us find the short circuit current across the output terminals. Let us find this using nodal analysis on the resulting circuit when we short the output ports. To help do the analysis, let us label the nodes as shown in the figure below.



Now what are the unkown node voltages? We do not know the voltage at node A and B. On the other hand, because node C is connected by a short circuit to GND, we know its voltage is equal to the GND which we set as 0; hence voltage at node C is not an unkown. Next, because there is a short circuit across resistor R_2, there will be no current flowing through it.

Let us write KCL at the nodes

$$\begin{aligned} 4A &= I_{R_1} + I_{R_3} && \text{(Node A)} \\ I_{R_3} &= I_{R_4} + I_{R_5} && \text{(Node B)} \\ I_{\text{short}} &= I_{R_1} + I_{R_5}. && \text{(Node C)} \end{aligned} \tag{24}$$

Now let us relate the currents I_{R_1}, I_{R_2}, I_{R_3}, I_{R_4} and I_{R_5} to node voltages using Ohm's law. We have

$$I_{R_1} = \frac{V_A - V_C}{R_1} = \frac{V_A}{4.5\Omega}, \tag{25}$$

since $V_C = 0$ because it is connected to the ground by short circuit. Furthermore, we have

$$I_{R_2} = 0, \tag{26}$$

$$I_{R_3} = \frac{V_A - V_B}{R_3} = \frac{V_A - V_B}{4.5\Omega}, \tag{27}$$

$$I_{R_4} = \frac{V_B}{R_4} = \frac{V_B}{1.5\Omega}, \tag{28}$$

$$I_{R_5} = \frac{V_B - V_C}{R_5} = \frac{V_B}{3\Omega}. \tag{29}$$

Plugging these into the first two KCL equations, we get

$$\begin{aligned} 4A &= \frac{V_A}{4.5\Omega} + \frac{V_A - V_B}{4.5\Omega}, \\ \frac{V_A - V_B}{4.5\Omega} &= \frac{V_B}{1.5\Omega} + \frac{V_B}{3\Omega}. \end{aligned}$$

These equations are solved by

$$\begin{aligned} V_A &= 9.9V, \\ V_B &= 1.8V. \end{aligned}$$

Using the KCL at node C, we get

$$\begin{aligned} I_{\text{short}} &= I_{R_1} + I_{R_5} \\ &= \frac{V_A}{4.5\Omega} + \frac{V_B}{3\Omega} \\ &= \frac{9.9}{4.5\Omega} + \frac{1.8}{3\Omega} \\ &= 2.8A. \end{aligned}$$

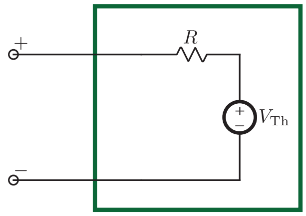
Summarizing the results, we have

$$\begin{aligned} V_{\text{open}} &= 3V, \\ I_{\text{short}} &= 2.8A. \end{aligned}$$

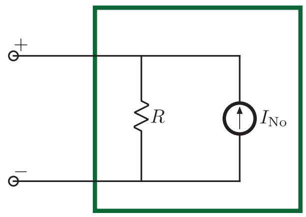
This gives

$$\begin{aligned} R_{\text{Th}} &= \frac{V_{\text{open}}}{I_{\text{short}}} \\ &= \frac{15}{14}\Omega. \end{aligned}$$

Hence the Thévenin equivalent circuit is given by



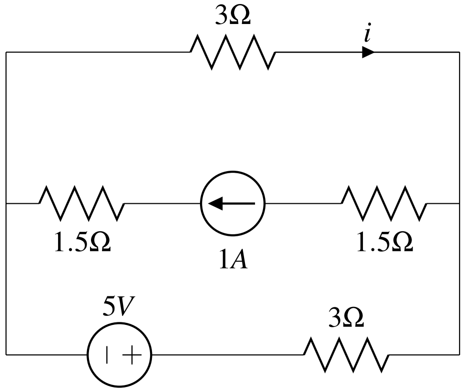
where $R = R_{\mathrm{Th}}$ and $V_{\mathrm{Th}} = V_{\mathrm{open}}$; and the Norton equivalent circuit is given by



where $R = R_{\mathrm{Th}}$ and $I_{\mathrm{No}} = I_{\mathrm{short}}$.

4. Nodal Analysis Or Superposition?

Solve for the current through the 3Ω resistor, marked as i, using superposition. Verify using nodal analysis. You can use IPython to solve the system of equations if you wish. Where did you place your ground, and why?

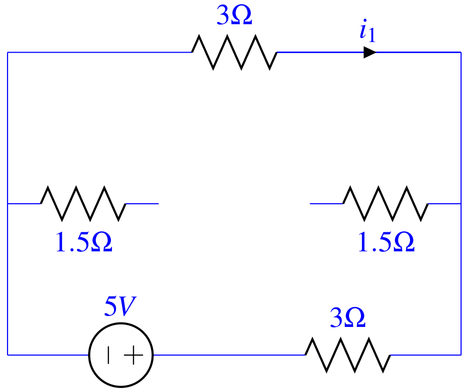


Solution: $i = \frac{-1}{3}A$.

Method 1: Superposition

Consider the circuits obtained by:

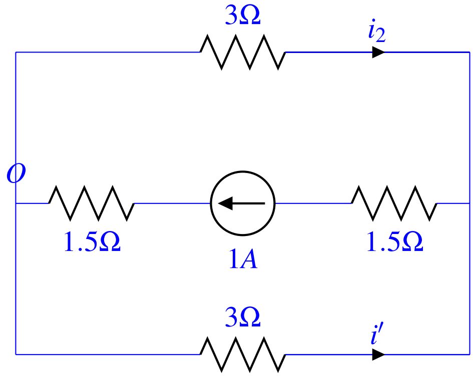
(a) Turning off the $1A$ current source:



In the above circuit, no current is going to flow through the middle branch, as it is an open circuit. Thus this is just a $5V$ voltage source connected to two 3Ω resistors in series so

$$i_1 = -\frac{5}{6}A \tag{30}$$

(b) Turning off the $5V$ voltage source:



In the above circuit, notice that the 3Ω resistors are in parallel and therefore form a current divider. Since the values of the resistances are equal, the current flowing through them will also be equal, that is $i_2 = i'$. Applying KCL to node O, we get

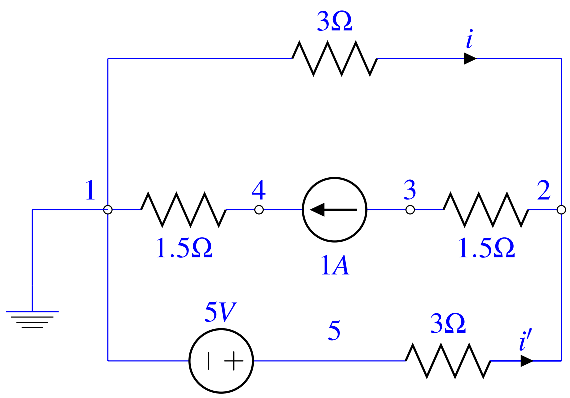
$$1 - i_2 - i' = 0 \tag{31}$$

which gives us

$$i_2 = \frac{1}{2}A \tag{32}$$

Now, applying the principle of superposition, we have $i = i_1 + i_2 = -\frac{5}{6} + \frac{1}{2} = -\frac{1}{3}A$.

Method 2: Nodal Analysis First, let's identify and label the nodes on the circuit. (Note that the numbers are arbitrary.) We also ground node 1. (Our choice of ground is arbitrary (whatever you chose is fine) but node 1 is a convenient choice because node 5 will turn out to be 5V from the voltage source, the voltage at node 4 can be calculated quickly from the current source and the 1.5Ω resistor using Ohm's law, and i can be calculated quickly once we know the voltage at node 2.)



First, we write KCL at each node. At nodes 1 and 2, we get the same equation.

$$i + i' = 1A \tag{33}$$

At nodes 3 and 4, we get the trivial equation

$$1A = 1A. \tag{34}$$

We now write the voltage drops across the circuit elements in terms of the currents using Ohm's law or in terms of known voltages

$$V_5 - V_1 = 5 \tag{35}$$
$$V_5 - V_2 = i'(3\Omega) \tag{36}$$
$$V_2 - V_3 = 1A(1.5\Omega) \tag{37}$$
$$V_4 - V_1 = 1A(1.5\Omega) \tag{38}$$
$$V_2 - V_1 = -i(3\Omega) \tag{39}$$

Since we've chosen node 1 as ground ($V_1 = 0$), we can rewrite the equations involving V_1 which gives us values for V_5 and V_4.

$$V_5 = 5 \tag{40}$$
$$V_4 = 1A(1.5\Omega) \tag{41}$$
$$V_2 = -i(3\Omega) \tag{42}$$

We next combine the KCL equations and the Ohm's Law equations to solve for V_2 and V_3. (We don't actually need to solve for V_3 once we know V_2 but the calculation is easy.)

$$-\frac{V_2}{3\Omega} + \frac{5V - V_2}{3\Omega} = 1A \tag{43}$$
$$\Rightarrow \qquad V_2 = 1V \tag{44}$$

and

$$V_3 = V_2 - 1A(1.5\Omega) \tag{45}$$
$$V_3 = -0.5V \tag{46}$$

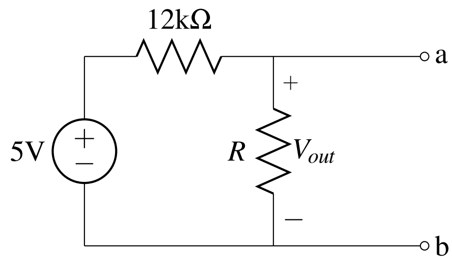
Finally, we use V_2 to solve for i

$$i = -\frac{V_2}{3\Omega} = -\frac{1}{3}A \tag{47}$$

5. (OPTIONAL) Resistive Voltage "Regulator"

In this problem, we will design a circuit that provides an approximately constant voltage divider across a range of loads. We will use a resistor divider circuit as seen in discussion. The goal is to design a circuit that from a source voltage of 5V would yield an output voltage within 5% of 4V for loads in the range of 1kΩ to 100kΩ.

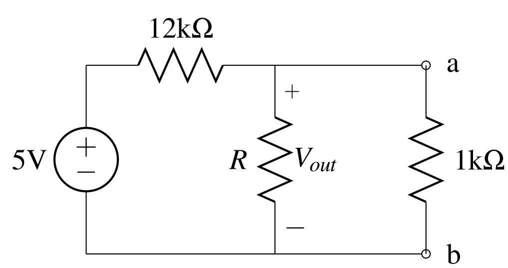
(a) First, consider the resistive voltage divider in the following circuit. What value of the resistor R would achieve a voltage V_{out} of 4V?



Solution: The current in the circuit is $I = \frac{5\text{V}}{R+12\text{k}\Omega}$ which means that the output voltage can be calculated as $V_{out} = I \cdot R = 5\text{V}\frac{R}{R+12\text{k}\Omega}$. By constraining $V_{out} = 4\text{V}$, we get

$$\begin{aligned}
\frac{4\text{V}}{5\text{V}} &= \frac{R}{R+12\text{k}\Omega} \\
\frac{4\text{V}}{5\text{V}}(R+12\text{k}\Omega) &= R \\
\frac{4\text{V}}{5\text{V}}12\text{k}\Omega &= R\left(1-\frac{4\text{V}}{5\text{V}}\right) \\
\frac{4\text{V}}{5\text{V}\left(1-\frac{4\text{V}}{5\text{V}}\right)}12\text{k}\Omega &= R \\
\frac{4\text{V}}{5\text{V}-4\text{V}}12\text{k}\Omega &= R \\
48.00\text{k}\Omega &= R
\end{aligned}$$

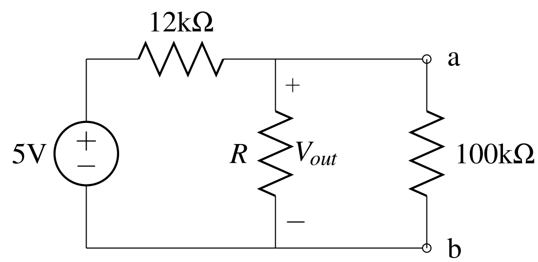
(b) Now consider loading the circuit with a resistor of 1kΩ as depicted in the following circuit with the same value of the resistor R as calculated in part (a). What is the voltage V_{out} now?



Solution: The value from part (a) is $R = 48.00\text{k}\Omega$. This means that the effective resistor $(48.00\text{k}\Omega \,||\, 1\text{k}\Omega) = \frac{48.00 \cdot 1}{48.00+1} = 0.98\text{k}\Omega$. The current is therefore $I = \frac{5\text{V}}{12\text{k}\Omega + 0.98\text{k}\Omega}$, which yields

$$V_{out} = \frac{5\text{V} \cdot 0.98\text{k}\Omega}{12\text{k}\Omega + 0.98\text{k}\Omega}$$
$$V_{out} = 0.38\text{V}$$

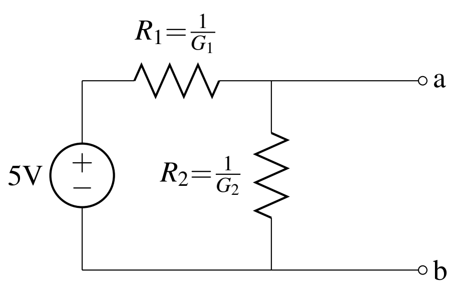
(c) Now consider loading the circuit with a resistor of $100\text{k}\Omega$, instead, as depicted in the following circuit with the same value of the resistor R as calculated in part (a). What is the voltage V_{out} now?

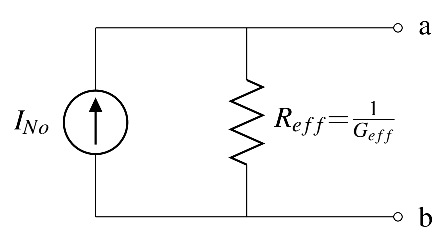


Solution: The value from part (a) is $R = 48.00\text{k}\Omega$. This means that the effective resistor $(48.00\text{k}\Omega \,||\, 100\text{k}\Omega) = \frac{48.00 \cdot 100}{48.00+100} = 32.43\text{k}\Omega$. The current is therefore $I = \frac{5\text{V}}{12\text{k}\Omega + 32.43\text{k}\Omega}$, which yields

$$V_{out} = \frac{5\text{V} \cdot 32.43\text{k}\Omega}{12\text{k}\Omega + 32.43\text{k}\Omega}$$
$$V_{out} = 3.65\text{V}$$

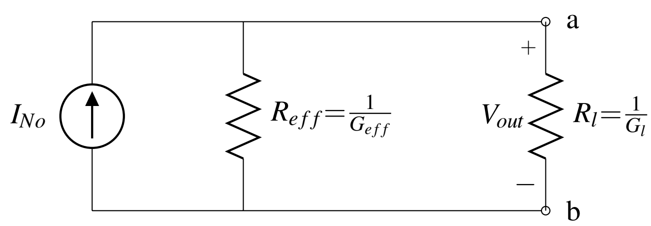
(d) Now we would like to design a divider that would keep the voltage V_{out} regulated for loads for a range of loads R_l. By that, we would like the voltage to remain within 5% window of 4V. That is, we would like to design the following circuit such that $3.80\text{V} \leq V_{out} \leq 4.20\text{V}$ for a range of loads R_l. As a first step, what is the Norton equivalent of the circuit on the left? Write I_{No} and G_{eff} it in terms of conductance values $G_1 = \frac{1}{R_1}$ and $G_2 = \frac{1}{R_2}$.





Solution: In open circuit, the current through the resistors is $I = \frac{5\text{V}}{R_1+R_2}$. This means that the open circuit voltage is $V_{Th} = I \cdot R_2 = 5\text{V} \cdot \frac{R_2}{R_1+R_2}$. We now calculate the short circuit current I_{No}. In short circuit, the current is simply $I = \frac{5\text{V}}{R_1}$. Therefore, the effective resistance is $R_{eff} = \frac{V_{Th}}{I_{No}} = \frac{5\text{V} \cdot \frac{R_2}{R_1+R_2}}{\frac{5\text{V}}{R_1}} = \frac{R_1 \cdot R_2}{R_1+R_2} = (R_1 \,||\, R_2)$. From this we can write $I_{No} = 5\text{V} \cdot G_1$ and $G_{eff} = G_1 + G_2$.

(e) The second step, using the Norton equivalent circuit you found in part (d), what is the range of G_{eff} that achieves $3.80\text{V} \le V_{out} \le 4.20\text{V}$ in terms of I_{No} and G_l?



Solution: From Ohm's law, $V_{out} \cdot (G_{eff} + G_l) = I_{No}$. This means that $V_{out} = \frac{I_{No}}{G_{eff}+G_l}$. From the bound $3.80\text{V} \le V_{out} \le 4.20\text{V}$ we have $3.80\text{V} \le \frac{I_{No}}{G_{eff}+G_l} \le 4.20\text{V}$. This yields the following two inequalities:

$$3.80\text{V} \le \frac{I_{No}}{G_{eff}+G_l}$$
$$\frac{I_{No}}{G_{eff}+G_l} \le 4.20\text{V}$$

which is equivalent to

$$G_{eff} + G_l \le \frac{I_{No}}{3.80\text{V}}$$
$$\frac{I_{No}}{4.20\text{V}} \le G_{eff} + G_l$$

which is equivalent to

$$G_{eff} \le \frac{I_{No}}{3.80\text{V}} - G_l$$
$$\frac{I_{No}}{4.20\text{V}} - G_l \le G_{eff}$$

or concisely $\frac{I_{No}}{4.20\text{V}} - G_l \le G_{eff} \le \frac{I_{No}}{3.80\text{V}} - G_l$

(f) Translate the range of G_{eff} in terms of I_{No} and G_l (that you found in part (e)) into a range on G_2 in terms of G_1 and G_l.

Solution: By plugging in the values of G_{eff} and I_{No} into $\frac{I_{No}}{4.20\text{V}} - G_l \le G_{eff} \le \frac{I_{No}}{3.80\text{V}} - G_l$ we get $\frac{5\text{V}G_1}{4.20\text{V}} - G_l \le G_1 + G_2 \le \frac{5\text{V}G_1}{3.80\text{V}} - G_l$ or equivalently $\frac{5\text{V}G_1}{4.20\text{V}} - G_l - G_1 \le G_2 \le \frac{5\text{V}G_1}{3.80\text{V}} - G_l - G_1$ which is equivalent to $G_1\left(\frac{5\text{V}}{4.20\text{V}} - 1\right) - G_l \le G_2 \le G_1\left(\frac{5\text{V}}{3.80\text{V}} - 1\right) - G_l$

(g) Say we want to support loads in the range $1\text{k}\Omega \le R_l \le 100\text{k}\Omega$ with approximately constant voltage as described above (that is, $3.80\text{V} \le V_{out} \le 4.20\text{V}$). What is the range of G_2 in terms of G_1 now? Translate the range of G_2 in terms of G_1 into a range of R_2 in terms of R_1.

Solution: Note: the unit of conductance is Siemens and is denoted by S.

We plug $R_l = 1\text{k}\Omega$ (equivalently $G_l = \frac{1}{1}\text{mS}$) into the bound found above and get

$$G_1\left(\frac{5\text{V}}{4.20\text{V}} - 1\right) - \frac{1}{1}\text{mS} \le G_2 \le G_1\left(\frac{5\text{V}}{3.80\text{V}} - 1\right) - \frac{1}{1}\text{mS}$$

Similarly, we plug $R_l = 100\text{k}\Omega$ (equivalently $G_l = \frac{1}{100}\text{mS}$) into the bound found above and get

$$G_1\left(\frac{5\text{V}}{4.20\text{V}} - 1\right) - \frac{1}{100}\text{mS} \le G_2 \le G_1\left(\frac{5\text{V}}{3.80\text{V}} - 1\right) - \frac{1}{100}\text{mS}$$

The intersection of the above two sets of inequalities is

$$G_1\left(\frac{5\text{V}}{4.20\text{V}}-1\right)-\frac{1}{100}\text{mS} \le G_2 \le G_1\left(\frac{5\text{V}}{3.80\text{V}}-1\right)-\frac{1}{1}\text{mS}$$

In terms of R_2 that range is (by taking the reciprocal of *all* sides and using $G = \frac{1}{R}$)

$$\frac{1}{\frac{1}{R_1}\left(\frac{5\text{V}}{4.20\text{V}}-1\right)-\frac{1}{100\text{k}\Omega}} \ge R_2 \ge \frac{1}{\frac{1}{R_1}\left(\frac{5\text{V}}{3.80\text{V}}-1\right)-\frac{1}{1\text{k}\Omega}}$$

(h) Note that conductance is always non-negative. From the bounds on G_2 you found in the previous part, derive a bound on G_1 that ensures that G_2 is always non-negative and non-empty (that is, the whole range of possible G_2 values is non-negative and is not empty). Translate this range into a range of possible R_1 values. (Hint: In addition to the conductance being non-negative, also make sure that the range for G_2 is non-empty.)

Solution: Since conductance has to be positive, we need the range above to consist of positive values (that is, the lower bound has to be positive and the upper bound is larger than the lower bound) which yields:

$$0 \le G_1\left(\frac{5\text{V}}{4.20\text{V}}-1\right)-\frac{1}{100}\text{mS}$$

and

$$G_1\left(\frac{5\text{V}}{4.20\text{V}}-1\right)-\frac{1}{100}\text{mS} \le G_1\left(\frac{5\text{V}}{3.80\text{V}}-1\right)-\frac{1}{1}\text{mS}$$

which is equivalent to

$$\frac{1}{100}\text{mS} \le G_1\left(\frac{5\text{V}}{4.20\text{V}}-1\right)$$

and

$$\frac{1}{1}\text{mS}-\frac{1}{100}\text{mS} \le G_1\left(\frac{5\text{V}}{3.80\text{V}}-\frac{5\text{V}}{4.20\text{V}}\right)$$

which is equivalent to

$$0.0525\text{mS} = \frac{1}{\frac{5\text{V}}{4.20\text{V}}-1}\cdot\frac{1}{100}\text{mS} \le G_1$$

and

$$7.9002\text{mS} = \frac{\frac{1}{1}\text{mS}-\frac{1}{100}\text{mS}}{\frac{5\text{V}}{3.80\text{V}}-\frac{5\text{V}}{4.20\text{V}}} \le G_1$$

The intersection of both ranges yields

$$7.9002\text{mS} \le G_1$$

which in terms of R_1 is equivalent to

$$R_1 \le 0.12657\text{k}\Omega$$

(i) Pick the values of R_1 and R_2 that achieve $3.80\text{V} \le V_{out} \le 4.20\text{V}$ for $1\text{k}\Omega \le R_l \le 100\text{k}\Omega$ while minimizing the power consumed by the voltage divider circuit in open circuit (when there is no load attached to the output). What are these values R_1 and R_2? How much power is consumed in this case? Calculate and report this power consumption using both the original circuit and the Norton equivalent circuit.

Are the power you calculated using the original circuit and the power you calculated using the Norton equivalent circuit equal?

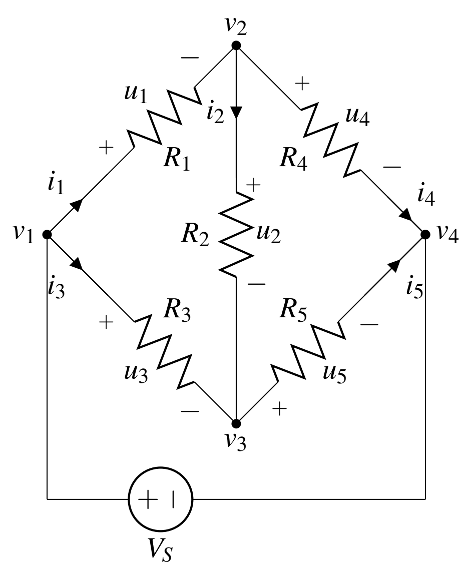
Solution: The power in the original circuit (no load) can be calculated as $\frac{V^2}{R_1+R_2}$. In order to minimize the power, we should pick the largest possible R_1+R_2. This can be achieved by choosing $R_1 = 0.12657\text{k}\Omega$ (the highest possible) and $R_2 = \frac{1}{\frac{1}{0.12657\text{k}\Omega}\left(\frac{5\text{V}}{4.20\text{V}}-1\right)-\frac{1}{100\text{k}\Omega}} = 0.66894\text{k}\Omega$ (the upper bound). The power in this case is $P = \frac{(5\text{V})^2}{0.12657\text{k}\Omega+0.66894\text{k}\Omega} = 31.4264\text{mW}$. If we use the Norton equivalent circuit, we have $R_{eff} = 0.1064\text{k}\Omega$ which yields the power consumption $P = I_{No}^2 \cdot R_{eff} = 166.0928\text{mW}$. The powers are not equal since Thévenin and Norton equivalents don't preserve power consumption of the circuit (think about the Thévenin equivalent in open circuit, the power consumption is always 0W).

(j) Now using the same values R_1 and R_2 from the previous part, load the circuit with load of $51\text{k}\Omega$, how much is consumed by each of the three resistors, R_1, R_2 and R_l (use the original circuit to compute the power)?

Solution: The effective resistance in the output is $(R_2 \mid\mid R_l)$ which is equal to $0.6603\text{k}\Omega$. This means that the current from the source is $I = \frac{5\text{V}}{R_1+(R_2||R_l)} = 6.3545\text{mA}$. Therefore, the power consumed by R_1 is $P_{R_1} = I^2 \cdot R_1 = 5.1108\text{mW}$. The output voltage is $V_{out} = I \cdot (R_2 \mid\mid R_l) = 4.1957\text{V}$ which means the power consumed by R_2 is $P_{R_2} = \frac{V_{out}^2}{R_2} = 26.3163\text{mW}$ and the power consumed by R_l is $P_{R_l} = \frac{V_{out}^2}{R_l} = 0.3452\text{mW}$ for a total power of $P = 31.7723\text{mW}$.

General note: try doing the whole problem using resistance directly without going through conductances. You will see that the math will not be as "pretty" as the one presented here (using conductances). Sometimes using conductance yields easier derivation than the derivation using resistance.

6. Solving Circuits with Voltage Sources



In the last homework, we implemented a circuit solver in an iPython notebook. This week we will make a small extension to allow us to solve circuits with voltage sources.

(a) What relationship does the voltage source enforce between v_1 and v_4?

Solution: $v_1 - v_4 = V_S$

(b) As you saw above, voltage sources will fix the nodes they are attached to be a constant offset from each other. We will treat v_1 and v_4 as one node, and our new vector of potentials will be $\vec{v} = \begin{bmatrix} v_2 \\ v_3 \\ v_4 \end{bmatrix}$. For the circuit above, draw the graph for the circuit where v_1 and v_4 are combined into one node. Specify a new incidence matrix for this graph.

Solution:

The incidence matrix for the circuit is as follows: $F = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$

(c) Previously we wrote Ohm's law as a matrix equation: $F\vec{v} = R\vec{i}$. Using our new incidence matrix, for every Ohm's law equation involving v_1, we will need to account for the constant offset by the voltage source. Find R and $\vec{b}$ so that Ohm's law is written $F\vec{v} + \vec{b} = R\vec{i}$.

Solution: $R = \begin{bmatrix} R_1 & 0 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 & 0 \\ 0 & 0 & R_3 & 0 & 0 \\ 0 & 0 & 0 & R_4 & 0 \\ 0 & 0 & 0 & 0 & R_5 \end{bmatrix}$

F is the incidence matrix from before, and $\vec{i} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix}$

The node v_1 is involved in Ohm's law equations for R_1 and R_3. $v_1 = v_4 + V_S$ so $\vec{b} = \begin{bmatrix} V_S \\ 0 \\ V_S \\ 0 \\ 0 \end{bmatrix}$

(d) From before, we wrote KCL as $F^T\vec{i} + \vec{f} = 0$. (What is $\vec{f}$ in this circuit?) Use this information, in addition to the previously derived equation to write $\vec{v}$ in terms of known quantities $(\vec{f}, \vec{b}, G, F, R)$. You can use G as the conductance matrix. You may need to modify several of the members of the derived equation by grounding a node and dropping a row or a column in order to give the problem a unique solution.

Solution: Drop the last column of F and call it F_{gr}. Drop the bottom row of $\vec{v}$ (setting $v_4 = 0$) as well and call this $\vec{v}_{gr}$, and we drop the bottom row of $\vec{f}$ to form $\vec{f}_{gr}$.

$F_{gr}^T G(F_{gr}\vec{v}_{gr} + b) = -\vec{f}_{gr}$. This means that:

$F_{gr}^T G F_{gr}\vec{v}_{gr} = -\vec{f}_{gr} - F_{gr}^T G b$.

We can find a solution for $\vec{v}_{gr}$ by inverting our coefficient matrix and left multiplying both sides of our equation by the inverse. We already know what v_4 is, because we set it to 0 by designating it as the ground node.

$\vec{v}_{gr} = -(F_{gr}^T G F_{gr})^{-1}(\vec{f}_{gr} + F_{gr}^T G b).$

(e) Now, use this information to write $\vec{i}$ in terms of known quantities ($\vec{f}, G, F, R, \vec{v}$) and the quantities defined the the previous part (which should also be derived from known quantities).

Solution: $\vec{i} = G(F_{gr}\vec{v}_{gr} + \vec{b})$

(f) In an iPython notebook, solve for $\vec{v}$ and $\vec{i}$ in the given circuit. Let $R_1 = 100,000\Omega$, $R_2 = 200\Omega$ $R_3 = 100\Omega$ $R_4 = 100,000\Omega$ $R_5 = 100\Omega$ and $V_S = 10V$.