EE16A MIDTERM 1 STUDY GUIDE

WK1 - VECTORS AND SYSTEMS OF EQUATIONS

Intro

Linear Algebra—the EECS Way: What is Lin Alg and why is it important?

- Linear algebra is the study of vectors and their transformations
- A lot of objects in EECS can be treated as vectors and studied with linear algebra.
- Linearity is a good first-order approximation to the complicated real world
- There exist good fast algorithms to do many of these manipulations in computers.

Vector and Matrix

Definition 1.1 (Vector): A vector is a collection of numbers. Suppose we have a collection of n real numbers, $x_1, x_2, \cdots, x_n$. We can represent this collection as a single point in an n-dimensional space, denoted:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \tag{12}$$

We call $\vec{x}$ a **vector**. Each x_i (for i between 1 and n) is called a **component**, or **element**, of the vector. The **size** of a vector is the number of components it contains.

Example 2.1 (3-D Vector): For $n = 3$, we could have $x_1 = -1, x_2 = 3.5, x_3 = 0$, and $\vec{x} = \begin{bmatrix} -1 \\ 3.5 \\ 0 \end{bmatrix}$. This vector represents a point in 3-D space.

Definition 1.2 (Matrix): A **matrix** is a rectangular array of numbers, written as:

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} \tag{13}$$

Each A_{ij} (where i is the row index and j is the column index) is a **component, or element** of the matrix A.

Special Vectors: **Zero Vector**, **Standard Unit Vector**

We can denote the 3 standard unit vectors in $\mathbb{R}^3$ as:

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{6}$$

Special Matrices: **Zero Matrix**, **Identity Matrix** (square matrix w diagonal elements 1)

Systems of Linear Equations

In general, we can represent any system of m linear equations with n variables in the form

$$A\vec{x} = \vec{b}$$

Gaussian Elimination

First we eliminate the first unknown from all but one equation, then among the remaining equations eliminate the second unknown from all but one equation. Repeat this until you reach one of three situations:

1. 1. We have no all-zero rows and an equation with one unknown. This means that the system of equations have a **unique solution**. We can solve for that unknown and eliminate it from every equation. Repeat this until every equation has one unknown left and the system of equations is solved.
2. 2. We have a all-zero row in the matrix corresponding to the equation 0 = 0. Then there are fewer equations than unknowns and the system of linear equations is **underdetermined**. There are an infinite number of solutions.
3. 3. We have a all-zero row in the matrix corresponding to the equation 0 = a where a/= 0. This means that the system of linear equations is **inconsistent and there are no solutions**.

Vector Addition / Scalar Multiplication

Properties of Vector Addition: for $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$

$$\begin{array}{ll} \text{commutative} & \vec{x}+\vec{y}=\vec{y}+\vec{x} \\ \text{associative} & (\vec{x}+\vec{y})+\vec{z}=\vec{x}+(\vec{y}+\vec{z}) \\ \text{zero} & \vec{x}+\vec{0}=\vec{x} \\ \text{additive inverse} & \vec{x}+(-\vec{x})=\vec{0} \end{array}$$

Properties of Scalar Multiplication

$$\begin{array}{ll} \text{associative} & (\alpha\beta)\vec{x}=\alpha(\beta\vec{x}) \\ \text{distributive} & (\alpha+\beta)\vec{x}=\alpha\vec{x}+\beta\vec{x} \\ \text{identity} & 1\vec{x}=\vec{x} \end{array} \tag{8}$$

Vector / Matrix Transpose

The transpose of a vector is crucial for vector and matrix calculations. We write the transpose of vector $\vec{x}$ as $\vec{x}^T$. If $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, then $\vec{x}^T = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$. This is sometimes used so that writing down long column vectors doesn't take up vertical space.

The **transpose** of a column vector is a row vector and vice versa.

The **transpose** of an $m \times n$ matrix A, denoted A^T is the $n \times m$ matrix given by $(A^T)_{ij} = A_{ji}$.

A square matrix is said to be **symmetric** if $A = A^T$, which means that $A_{ij} = A_{ji}$ for all i and j.

Matrix Addition / Scalar Matrix Multiplication

Properties of Addition:

Commutativity $A + B = B + A$
Associativity $(A + B) + C = A + (B + C)$
Additive Identity (Zero) $A + 0 = A$
Additive Inverse $A + (-A) = 0$

Properties of Scalar Multip:

Associativity $(\alpha\beta)A = \alpha(\beta A)$
Distributivity $(\alpha + \beta)A = \alpha A + \beta A$ and $\alpha(A + B) = \alpha A + \alpha B$
Multiplicative Identity (One) $(1)A = A$

Matrix Transformation *T*

To find the matrix *A* that applies a transformation *T*, find columns of *A* s.t. they're *T* applied to the standard unit vectors $\{e_1, e_2, \dots, e_N\}$. Then, combine the columns to get *A*.

Ex) T_1: Reflects a vector about the line $y = -x$.

Sol)

$$\vec{a}_1 = T_1\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\-1\end{bmatrix}$$

, solve for a_2 similarly.

$$A_1 = [\vec{a}_1 \quad \vec{a}_2] = \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$$

Special: Rotation Matrix in R^2:

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

where Θ is the angle of rotation.

Ex) clockwise about the origin by 45° is $\Theta = -\pi/4$.

WK1 - PROOFS

A mathematical proof provides a means for guaranteeing that a statement is true.

So what proof is? — a finite sequence of **logical deductions** which establishes the truth of a desired statement. (**logical deductions**: simple steps that apply the rules of logic)

The power of proof lies in the fact that using finite means, we can guarantee the truth of a statement with infinitely many cases.

When encountering a proof problem, we understand the problem by generally asking:
- "What can we assume based on the problem statement?"
- "What is it we want to show?"

Ans to 1st Q gives us the condition we're working under; Ans to 2nd Q gives us a clear goal.

Then we ask:
- "How can we use what we know to get to goal under the conditions?"

Examples using this technique:

1. Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a set of linearly dependent vectors in $\mathbb{R}^n$. Take any matrix $A \in \mathbb{R}^{m\times n}$. Prove that the set of vectors $\{A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_n\}$ is linearly dependent.

Proof: (1) What do we know? Based on the problem statement, we know that $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a set of linearly dependent vectors. How do we translate this into mathematical form? Recall one of the two definitions of linear dependence we introduced in the last lecture – the set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly dependent if there exist an index i and scalars α_j's such that

$$\vec{v}_i = \sum_{j\neq i} \alpha_j \vec{v}_j. \tag{1}$$

(2) What would we like to show? We would like to show that the set of vectors $\{A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_n\}$ is linearly dependent. Again using the definition of linear dependence, we can translate it into a mathematical statement – we would like to show that there exist index k and scalars β_l's such that

$$A\vec{v}_k = \sum_{l\neq k} \beta_l \left(A\vec{v}_l\right). \tag{2}$$

(3) Now, how do we use what we know mathematically from (1) to prove the mathematical statement in (2)? We somehow would like to get vectors of the form $A\vec{v}$. How could we do that? Let's multiply both sides of equation (1) by the matrix A and see what happens:

$$A\vec{v}_i = A\left(\sum_{j\neq i} \alpha_j \vec{v}_j\right). \tag{3}$$

By distributivity of matrix-vector multiplication, we know that

$$A\left(\sum_{j\neq i} \alpha_j \vec{v}_j\right) = \sum_{j\neq i} A\left(\alpha_j \vec{v}_j\right) = \sum_{j\neq i} \alpha_j \left(A\vec{v}_j\right). \tag{4}$$

Now, we have that

$$A\vec{v}_i = \sum_{j\neq i} \alpha_j \left(A\vec{v}_j\right), \tag{5}$$

which is in exactly the mathematical form we would like to show according to step (2). Hence, we have completed our proof!

2. $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_1+\vec{v}_2$ are all solutions to the system of linear equation $A\vec{x}=\vec{b}$. Prove that $\vec{b}$ must be a zero vector.

Proof: What does it mean for $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_1+\vec{v}_2$ to be the solutions to $A\vec{x}=\vec{b}$? It means these vectors must satisfy the following equations:

$$A\vec{v}_1=\vec{b} \tag{6}$$
$$A\vec{v}_2=\vec{b} \tag{7}$$
$$A(\vec{v}_1+\vec{v}_2)=\vec{b} \tag{8}$$

Notice that using distributivity of matrix-vector multiplication, equation (8) can be rewritten as

$$A\vec{v}_1+A\vec{v}_2=\vec{b}. \tag{9}$$

Now from equation (6) and (7), we can substitute $A\vec{v}_1$ and $A\vec{v}_2$ with the vector $\vec{b}$, which leads us to

$$\vec{b}+\vec{b}=\vec{b}. \tag{10}$$

Subtracting $\vec{b}$ from both sides of the equation above, we have

$$\vec{b}=\vec{0}. \tag{11}$$

Hence $\vec{b}$ is the zero vector, as desired.

Wk2 - Linear Dependence

Lin Dependence — very useful concept often used to characterize “redundancy” in information

Definition 3.1 (Linear Dependence (I)): A set of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly dependent if there exist scalars $\alpha_1, \dots, \alpha_n$ such that $\alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n = \vec{0}$ and not all α_i's are equal to zero.

Definition 3.2 (Linear Dependence (II)): A set of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly dependent if there exist an index i and scalars α_j's such that $\vec{v}_i = \sum_{j \neq i} \alpha_j \vec{v}_j$. In words, a set of vectors is linearly dependent if one of the vectors could be written as a linear combination of the rest of the vectors.

Definition 3.3 (Lin InD): Set of vectors is **linearly independent** if it’s not **linearly dependent**.

Basically all components in an independent set are unique.

Connecting Concept of LD w Systems of Linear Equations

(1) If the system of linear equations $\vec{A}x \vec{=} b$ has infinite number of solutions, then the columns of A are linearly dependent.

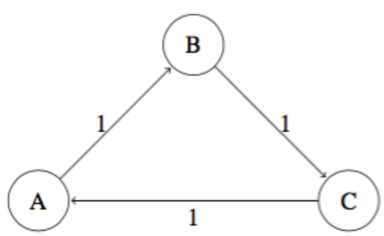
(2) If the columns of A in the system of linear equations $\vec{A}x = b$ are linearly dependent, then the system does not have a unique solution.

(3) If the system of linear equations $\vec{A}x \vec{=} b$ has infinite number of solutions and the number of rows of A is greater or equal to the number of columns (A is a square or a tall matrix), then the rows of A are linearly dependent.

Visualizing Matrix-Vector Multiplication

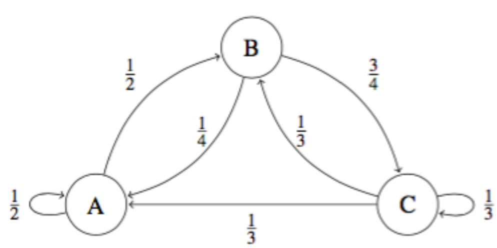
It is vital as an engineer that you understand the ideas that we are talking about in an intuitive way. This will often come from having a series of examples that make sense.

Basic Pump system and the corresponding matrix-vector multiplication:



$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

Conservation of Water:



$$\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{3}{4} & \frac{1}{3} \end{bmatrix}$$

WK2 - RANK, SPAN, INVERSES

The **dimension** of a space is determined by the minimum number of parameters that we need to describe a vector in that space.

Range & Span

so we can conclude that the range of the operator A is the space of all possible linear combinations of its columns, another name for this is the **span** of (the columns of) A, which we can write as

$$span(A) = \{\vec{v} \mid \vec{v} = \sum_{i=1}^{m} x_i \vec{a}_i, \text{where } x_i\text{'s are scalars}\} \tag{4}$$

Basically, **Span(Range)** of a set of vectors is the set of all **linear combos** of the vectors.

Dimension of the Span, the Rank

Since the dimension is given by the fewest number of parameters that you need to identify any element in the space, it turns out that the dimension of $span(A)$ is equal to the number of linearly independent columns of A, which will be less than or equal to the $min(m,n)$.

$$\dim(span(A)) \leq min(m,n). \tag{9}$$

Now let's introduce the term **rank**. The rank of a matrix is the dimension of the span of its columns, i.e., $rank(A) = \dim(span(A))$. For example, the rank of the matrix

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 3 & 2 & 5 \\ 5 & 1 & 6 \\ 2 & 2 & 4 \end{bmatrix} \tag{10}$$

defined previously is equal to 2 since it has two linearly independent columns.

Basically, the **rank** of a matrix is the number of linearly independent columns.

or, the **rank** of a matrix is the number of pivots in its reduced row-echelon form.

Matrix Inversion

Definition 6.1 (Inverse): A square matrix A is said to be invertible if there exists an matrix B such that

$$AB = BA = I. \tag{11}$$

Important property: **If A is a nonsingular (invertible) matrix, then its inverse must be unique**

Use Gaussian Elimination to check invertibility / find inverse:

For any *n x n* matrix *M*, perform Gauss on the **augmented matrix**.
If we don't end up with an identity matrix on the left after running Gauss, not invertible.

$$\left[\begin{array}{c|c} M & I_n \end{array}\right] \Rightarrow \left[\begin{array}{c|c} I_n & M^{-1} \end{array}\right]$$

Invertibility of a Matrix vs Its Columns

Matrix *A* is invertible iff its columns are linearly independent.

Properties of Inverses

Properties of Inverses. For a matrix A, if its inverse exist, then:

$$A^{-1}A = AA^{-1} = I$$

$$(A^{-1})^{-1} = A$$

$$(kA)^{-1} = k^{-1}A^{-1} \qquad \text{for a nonzero scalar } k$$

$$(A^T)^{-1} = (A^{-1})^T$$

$$(AB)^{-1} = B^{-1}A^{-1} \qquad \text{assuming } A, B \text{ are both invertible}$$

Wk3 - Vector Spaces, Basis

"Vector" Spaces

A vector space $(\mathbb{V}, \mathbb{F})$ is a set of vectors $\mathbb{V}$, a set of scalars $\mathbb{F}$, and two operators that satisfy

the **Axioms** of Vector space: 1) Closure, 2) Vector Addition, 3) Vector Scalar Multiplication.

To check if a subset of a Vector Space is also a **Subspace**, just verify the **Closure Axioms**.

Bases

Definition 7.1 (Basis):

Given a vector space $(\mathbf{V},\mathbf{F})$, a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a **basis** of the vector space if it satisfies the following two properties:

- $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent vectors
- For any vector $\mathbf{v} \in \mathbb{V}$, there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}$ such that $\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n$.

^Second bullet basically means the set "spans" the vector space.

Basically, a basis of a vector space is the minimum set of vectors needed to represent all the vectors in the vector space.

Definition 7.2 (Dimension): The dimension of a vector space is the number of basis vectors.

Note that a vector space can have many bases, but all its bases must have the same number of vectors in it.

Subspace, and the Orthogonalities of Subspaces

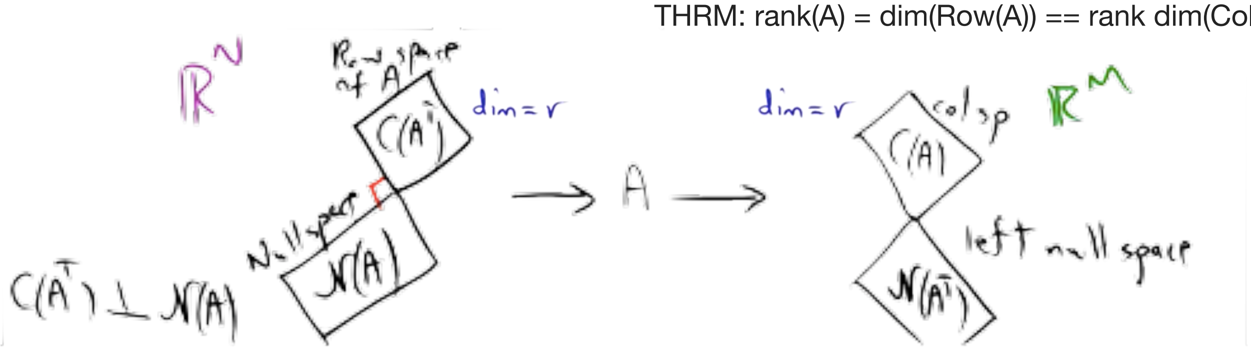
Four fundamental subspaces are: Column Space, Row Space, N(A), L N(A)

Column Space is defined as the span of the columns of a matrix, or the range of the matrix.

Row Space is defined as the span of the rows of a matrix.
- any vector in the $N(A)$ is orthogonal to any vector in $Row(A)$ [think of A in terms of its rows, dot product].
- the $Row(A)$ can be written as $Col(A^T)$

$L\ N(A)$, the **Left Null-space**, = $N(A^T)$

THRM: rank(A) = dim(Row(A)) == rank dim(Col(A))



Wk3 - Nullspaces

Loss of Dimensionality / Nullspace

The nullspace of A consists of all vectors $\vec{x}$ in $\mathbb{R}^m$, such that $A\vec{x} = \vec{0}$,

$$N(A) = \{\vec{x} \mid A\vec{x} = \vec{0},\ \vec{x} \in \mathbb{R}^m\}.$$

The **nullspace** of A is the set of vectors that get mapped to zero by *A*.

What is the dim(*N*(*A*))? We need to ask how many independent ways can we create the zero vector from the linear combos of the columns of *A*.

dim(*N*(*A*)) is equal to the number of **linearly dependent** columns of *A*.

$$m - \dim(range(A)) = \dim(N(A)),$$

So, the loss of dimensionality from the input space to the output space is the nullspace, and this is called the **rank-nullity theorem**.

Computing the Nullspace

In solving for nullspace, we are fundamentally trying to solve the system of equations $\vec{A}x = 0$.

Basically, find each lin dependent cols as combination of lin independent cols. Any scalar multiples and the sums of these found vectors are in *N(A)*.

Get *A* into Row Reduced Echelon Form (RREF) to make life easier—helps easily find linearly independent columns (the columns with pivots).

Example:

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

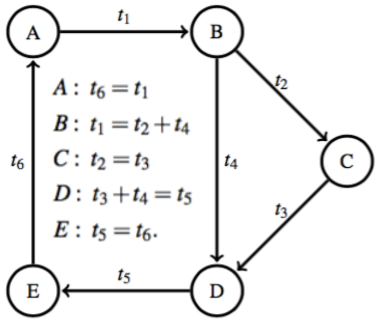
First, identify the linearly dependent / independent columns. Then,

We would want to find all possible scalars x_1, x_2, x_3, x_4, x_5 such that

$$x_2 \times \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \times \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_1 \times \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \times \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \times \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \vec{0}.$$

$$N(A) = \{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \alpha + \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \beta \mid \alpha, \beta \in \mathbb{R}\}.$$

Network Flow — Traffic



$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \end{bmatrix} = \vec{0}$$

Represent the above matrix-vector system as $B\vec{t} = \vec{0}$. The matrix B is called the **incidence matrix** of the network.

We know that any valid flow $\vec{t}$ needs to satisfy the equation $B\vec{t} = \vec{0}$. What does this mean? It means that the set of valid flows is just the nullspace of B.

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