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EE HW 04

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I worked alone to finish, then met up with Matt and Will to compare answers.

2) a) The maximum possible # of lin. independent vectors from the column vectors of any 3x5 matrix is $\boxed{3}$, because it is easy to see that the basis contains 3 vectors.

b) $\boxed{2}$ $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ spans A

c) vectors that span $N(A)$:

$$A\vec{x} = b = 0$$

Find all possible scalars $x_1 \cdots x_5$ s.t.

$$x_2\begin{bmatrix}1\\0\\0\end{bmatrix} + x_4\begin{bmatrix}-2\\-1\\0\end{bmatrix} + x_5\begin{bmatrix}3\\1\\0\end{bmatrix} + x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_3\begin{bmatrix}0\\1\\0\end{bmatrix} = \vec{0}$$

let $x_2, x_4, x_5 = 1, 0, 0$:
$x_1, x_3 = -1, 0$ $\therefore \begin{bmatrix}-1\\1\\0\\0\\0\end{bmatrix}$ is in $N(A)$

let $x_2, x_4, x_5 = 0, 1, 0$:
$x_1, x_3 = 2, 1$ $\therefore \begin{bmatrix}2\\0\\1\\1\\0\end{bmatrix}$ is in $N(A)$

let $x_2, x_4, x_5 = 0, 0, 1$:
$x_1, x_3 = -3, -1$ $\therefore \begin{bmatrix}-3\\0\\-1\\0\\1\end{bmatrix}$ is in $N(A)$

$$\boxed{\text{span}\left\{\begin{bmatrix}-1\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}2\\0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\-1\\0\\1\end{bmatrix}\right\} = N(A)}$$

Dimension of $N(A)$ is $\boxed{3}$, the number of dependent column vectors of A.

d) $B\vec{x} = 0$.

Row Reduce B:

$$\begin{bmatrix} 1 & -2 & 2 & 4 \\ 1 & -2 & 3 & 5 \\ 2 & -4 & 5 & 9 \\ 3 & -6 & 7 & 13 \end{bmatrix} \Rightarrow \left\{\begin{matrix} R_2 - R_1 \\ R_3 - 2(R_1) \\ R_4 - 3(R_1) \end{matrix}\right\} \Rightarrow \begin{bmatrix} 1 & -2 & 2 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \Rightarrow \left\{\begin{matrix} R_3 - R_2 \\ R_4 - R_2 \\ R_1 - 2(R_2) \end{matrix}\right\} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} 1 & -2 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow \text{RR form.}$$

Colums 1 & 3 are independent.

$$x_2\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = -x_1\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + -x_3\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

let $x_2, x_4 = 1, 0$:

$-x_1, -x_3 = -2, 0$ $\quad \therefore \begin{bmatrix} +2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is in $N(A)$

let $x_2, x_4 = 0, 1$:

$-x_1, x_3 = 2, 1$ $\quad \therefore \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ is in $N(A)$

$$\boxed{\operatorname{span}\left\{\begin{bmatrix} +2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}\right\} = N(A)}$$

3) a) Yes, from eq. 2, $t_2 - t_1 = 0$, $t_2 = t_1 = 10$,
and from eq 1, $t_3 = -t_1 = -10$,
~~and from eq. 3,~~

b) It is only possible with the Berkely students claim.
With data of t_1 & t_4, t_2, t_3, and t_5 can be figured out because the system of equations is:

① $t_3 + t_1 - t_4 = 0$
② $t_2 - t_1 = 0$
③ $t_5 - t_2 - t_3 = 0$
④ $t_4 - t_5 = 0$

t_2 and t_5 can be solved by Eq ② and Eq ④, respectively, and t_3 can be solved by resubstituting t_2 and t_5.
The stanford students claim does not work because with just information about t_1 and t_2, there is not enough information to calculate the other three.

c) From the set of equations above:

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix},$$

and let $\vec{t}_1$ & $\vec{t}_2$ be elements of the set of valid flows.

Then $A\vec{t}_1 = 0$ and $A\vec{t}_2 = 0$

$A t_1 + A t_2 = A(\vec{t}_1 + \vec{t}_2) = 0$ ← Closed under vector addition,

$\alpha(A\vec{t}_1) = A(\alpha \vec{t}_1) = 0$ ← closed under scalar multiplication.

∴ the set of valid flows forms a subspace.

Now, solve for $At = 0$. Since we know we just need t_1 & t_4,

$$\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix} = \begin{bmatrix} t_1 \\ t_1 \\ t_4 - t_1 \\ t_4 \\ t_5 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + t_4 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

d) $B\vec{t} = \vec{0}$:

$$B = \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

Each row represents traffic flows @ one intersection

Each column represents a particular flow @ all intersections

e) With GE, get RREF of B:

$$\begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \Rightarrow R_3 + R_2 \rightarrow \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \Rightarrow R_4 + R_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow R_3 \times -1 \Rightarrow \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

dim(N(A)) = # of lin. dependent cols of B, $\boxed{\text{dim} = 2}$

$$t_3\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_4\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = -t_1\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - t_2\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - t_5\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

if $t_3, t_4 = 1, 0$:

$t_1, t_2, t_5 = -1, -1, 0$ $\rightarrow \therefore \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in N(A)$

if $t_3, t_4 = 0, 1$:

$t_1, t_2, t_5 = 1, 1, +1$ $\rightarrow \therefore \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ +1 \end{bmatrix} \in N(A)$

$\therefore$ Basis of N(A) : $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

No, this doesn't exactly match my answer from (a), but it is the same basis that span the same Nullspace.

The dimensions of the nullspaces for figures 1 & 2 tell us how many cameras we need to install in our network.

f) No, and a counterexample can be seen with the stanford student's suggestion in part (b). Although the stanford student proposed the same number of cameras as the berkeley student, the bad placements led to insufficient information to calculate exact traffic flows.

3) ~~a~~ Suppose we have $\vec{t}_1$ & $\vec{t}_2$ vectors such that

$$M\vec{t}_1 = 0 \quad \text{and} \quad M\vec{t}_2 = 0$$

That means $M\vec{t}_1 = M\vec{t}_2$

$$M\vec{t}_1 - M\vec{t}_2 = 0$$

$$M(\vec{t}_1 - \vec{t}_2) = 0$$

$(\vec{t}_1 - \vec{t}_2)$ belongs to the ~~subspace of~~ $N(M)$,
while $\vec{t}_1$ & $\vec{t}_2$ belong to the $N(B_G^T)$.

$\therefore$, the exact traffic flow can only be recovered when there is no $(\vec{t}_1 - \vec{t}_2)$ in $N(M)$.

4) MDTM Prob 3 : True / False

a) True

b) False, if $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $A \neq 0_{2\times 2}$

c) ~~False, $(AB)C \neq A(BC)$~~

True

d) True

e) False, if $A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $AB \neq BA$

f) ~~True~~ False

5) MDTM Prob 4 : Proof

a) if $A\vec{x} = 0$ for some nonzero x,
by definition, the components of A are linearly dependent.
Since we can represent A as a row of its column vectors, $A = [a_1, a_2, \dots a_n]$,
the column vectors of A are linearly dependent.

b) Proof by contradiction:
Assume the columns of A are linearly independent.
then A ~~has~~ is invertible and has a unique inverse, A^{-1}.

Apply A^{-1} : $A^{-1}(AA)A^{-1} = (A^{-1}A)(AA^{-1})$
$= I \neq 0$.

$\therefore$ columns of A cannot be lin. independent

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6) MDTM Prob 5 : Inverse of a Matrix

Check invertibility :

$$\left[\begin{array}{ccc|ccc} 5 & 4 & 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 9 & 6 & 3 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left\{\begin{array}{l} \text{Swap } R_1, R_2 \\ \frac{1}{3} R_3 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 5 & 4 & 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right] \Rightarrow \left\{\begin{array}{l} R_2 - 5(R_1) \\ R_3 - 3(R_1) \end{array}\right\} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & -6 & -3 & 1 & -5 & 0 \\ 0 & -4 & -2 & 0 & -3 & \frac{1}{3} \end{array}\right] \Rightarrow \begin{array}{l} (-\frac{1}{3} R_2) \\ -\frac{1}{2}(R_3) \end{array} \Rightarrow$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & -\frac{1}{3} & \frac{5}{3} & 0 \\ 0 & 2 & 1 & 0 & \frac{3}{2} & -\frac{1}{6} \end{array}\right]$$

redundant rows ⇒ row of 0s,

~~Cannot~~ $\boxed{\text{Not Invertible}}$

Cannot turn the left side into an identity matrix.

7) MT1 Prob 6 : Shovel

a) $\boxed{\text{Yes}}$, $\begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$ is a lin. combination of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$$3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (-2) \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad \boxed{3, -2}$$

b) All three vectors for the Pickaxe can be represented as linear combinations of the two vectors of shovel.

i) $\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

ii) $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

iii) $\begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

8) MT1 Prob #7 : Graph Majors

a)

$$A = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ K_1 & 0.4 & K_2 \\ K_3 & 0.3 & K_4 \end{bmatrix}$$

b) From $\vec{x}[n+1] = A\vec{x}[n]$

$$K_1 x_e[n] + 0.4 x_c[n] + K_2 x_m[n] = x_c[n+1] \quad E_1$$

$$K_3 x_e[n] + 0.3 x_c[n] + K_4 x_m[n] = x_m[n+1] \quad E_2$$

From the A matrix,

$$K_1 + K_3 = 0.5 \quad E_3$$

$$K_2 + K_4 = 0.8 \quad E_4$$

c) Let n = 10, $\vec{x}[n] = \vec{x}[10] = \begin{bmatrix} x_e[10] \\ x_c[10] \\ x_m[10] \end{bmatrix} = \begin{bmatrix} 100 \\ 200 \\ 200 \end{bmatrix}$,

$$\vec{x}[n+1] = \vec{x}[11] = \begin{bmatrix} x_e[11] \\ x_c[11] \\ x_m[11] \end{bmatrix} = \begin{bmatrix} 150 \\ 100 \\ 250 \end{bmatrix}.$$

$\therefore T\vec{k} = \vec{b}$:

$$\begin{bmatrix} 100 & 200 & 0 & 0 \\ 0 & 0 & 100 & 200 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{bmatrix} = \begin{bmatrix} 20 \\ 190 \\ 0.5 \\ 0.8 \end{bmatrix}$$

d) It is possible to find $\vec{x}[2]$ if A is invertible.

Check:
$$\left[\begin{array}{ccc|ccc} 1 & 4 & 5 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 3 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 4 & 5 & 1 & 0 & 0 \\ 0 & -10 & -14 & -3 & 1 & 0 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right]$$

With the establishment of the pivots, we can already easily tell that A^{-1} exists.

Now to find $\vec{x}[2]$,

$$\vec{x}[2] = (A^{-1})^{921} \vec{x}[923]$$

e) $\boxed{\text{No}}$, The first and third rows in A are the same. This means, at any positive timestamp, $\vec{x}$ would be forced to have the same number ~~of~~ in its first and third entries. Therefore, the given $\vec{x}[5]$ is not in the range of this matrix.

9) MT1 Prob 8 : Basketball

a) $\begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} -4 \\ 10 \end{bmatrix} \Rightarrow \begin{matrix} 7a + 3 = -4 \rightarrow \underline{a = -1} \\ 3b + 7 = 10 \rightarrow \underline{b = 1} \end{matrix}$

$\therefore \boxed{A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}$

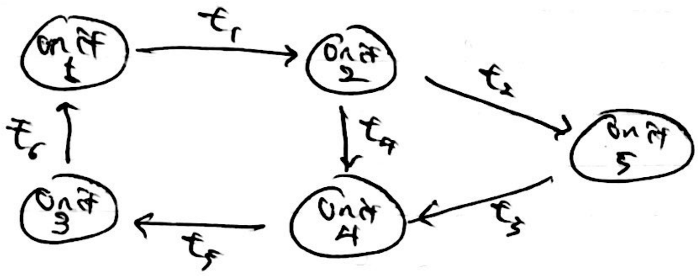
b) This transformation ~~turns~~ transforms

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ into $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ into $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\therefore$ We can graphically see that this transformation rotates $45°$ clock-wise and scales by $2\sqrt{2}$.

c) No, because $[0, 0]^T$ cannot be mapped into another vector.

(a)



Represent the above student flow as $B\vec{t} = \vec{0}$, where B is the incident matrix. Find the set of valid flows.

Answer:

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_6 \end{bmatrix} = \vec{0}$$

$$B \qquad \vec{t} = \vec{0}$$

Transform B into RREF:

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From the RREF, we can easily see that

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ +1 \end{bmatrix} \right\} \text{ spans } N(B).$$

Since nullspace is a valid subspace of B,

the set of valid flows can be represented as:

$$\boxed{\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \alpha + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \beta \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}}$$