MacCready speed-to-fly theory including airmass velocity

Definitions

t	= total time
t_1	= cruise time
t_2	= climb time
d	= total distance flown
d_1	= cruise distance
d_2	= climb distance
h	= height gain in climb
v	= airspeed
v_1	= ground speed during cruise
v_2	= net ground speed during climb
A_x, A_z	= Airmass horizontal and vertical speed (crosswind ignored)
c	= climb rate when thermalling
$s(v)$	= sink during glide
	= sink from glider polar plus A_z

In cruise, the glider's ground speed is its airspeed plus the horizontal wind speed

$$v_1 = v + A_x$$

The vertical airmass speed A_z only applies to cruise. When climbing, c is the total climb rate. In a thermal, the usual assumption is that the glider's net ground speed is equal to the horizontal wind speed. Per G Dale, a thermal has significant mass, so as it ascends from ground level it does not immediately accelerate to the speed of the wind around it. So, we will take glider's net ground speed in the thermal to be

$$v_2 = f \cdot A_x$$

where f is an adjustable parameter between 0 and 1. This is only included out of curiosity.

The height gained while climbing is equal to the height lost while cruising.

$$h = c \cdot t_2 = -s(v) \cdot t_1$$

$$t_2 = \frac{-s(v)}{c} t_1 \tag{1}$$

Average speed is a function of the chosen cruise speed v and is the total distance divided by the total time. Use Equation (1) to eliminate t_2.

$$\begin{aligned}
V_{avg}(v) &= \frac{d}{t} \\
&= \frac{d_1 + d_2}{t_1 + t_2} \\
&= \frac{v_1 \cdot t_1 + v_2 \cdot t_2}{t_1 + t_2} \\
&= \frac{(v + A_x)t_1 + f \cdot A_x \left(\frac{-s(v)}{c} t_1\right)}{t_1 + \left(\frac{-s(v)}{c} t_1\right)} \\
&= \frac{c(v + A_x) - f \cdot A_x s(v)}{c - s(v)}
\end{aligned}$$

The optimal cruise speed v is the speed that maximizes $V_{avg}(v)$. To find this maximum, take the derivative and find its zeroes.

$$\begin{aligned}
0 &= \frac{d}{dv} V_{avg}(v) \\
&= \frac{c(c + \big((1 - f)A_x + v\big)s'(v) - s(v)}{(c - s(v))^2}
\end{aligned}$$

At the risk of discarding some zeroes, we can simplify this to

$$0 = c + \big((1 - f)A_x + v\big)s'(v) - s(v)$$

Notice that the A_x term goes away when we assume that thermals drift at the speed of the surrounding wind ($f = 1$), and we are left with Reichmann's* equation II, though our notation is different:

$$\frac{d}{dv} s(v) \cdot v = s(v) - c$$

Reichmann's derivation is different. He disregards distance and minimizes t, which we have now shown to be the same as maximizing the average speed.

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* Helmut Reichmann, *Cross Country Soaring*, 7th Edition, 1993, Soaring Society of America, page 122.