Optimal observers

Check on room for EOrl

AE353

Spring 2025

Bretl

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

$\leftarrow$ n_x number of states
n_u number of inputs
n_y number of outputs

OPTIMAL CONTROLLER

$u = -Kx$ where $K = lqr(A, B, Q_c, R_c)$

diagonal w/ positive numbers $\begin{cases} Q_c \text{ is } n_x \times n_x & \leftarrow \text{error} \\ R_c \text{ is } n_u \times n_u & \leftarrow \text{effort} \end{cases}$

OPTIMAL OBSERVER

$\dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} - y)$ where $L = lqr(A^T, C^T, R_o^{-1}, Q_o^{-1})^T$

diagonal w/ positive numbers $\begin{cases} Q_o \text{ is } n_y \times n_y & \leftarrow \text{noise} \\ R_o \text{ is } n_x \times n_x & \leftarrow \text{disturbance} \end{cases}$

EXAMPLE - estimate distance of drone to ground

ONE MEASUREMENT

$y = x$ $\qquad y = 1$ $\qquad \hat{x} = 1$

TWO MEASUREMENTS

$y_1 = x + n_1$ $\qquad y_1 = 1$ $\qquad \hat{x} = \cancel{1.5}$

$y_2 = x + n_2$ $\qquad y_2 = 2$

$1 = 1\,y_1 + 0\,y_2$

$1.2 = 0.8\,y_1 + 0.2\,y_2$

$1.3 = 0.7\,y_1 + 0.3\,y_2$

MEASUREMENT ERROR (SENSOR NOISE)

x	$n_1 = y_1 - x$	$n_2 = y_2 - x$	n_1^2	n_2^2
1	0	1	0	1
1.2	-0.2	0.8	0.04	.64
1.3	-0.3	0.7	.09	.49

THE "COST" OF AN ESTIMATE

$$\left.\begin{array}{ll} \underset{x}{\text{minimize}} & q_1 n_1^2 + q_2 n_2^2 \\ \text{subject to} & y_1 = x + n_1 \\ & y_2 = x + n_2 \end{array}\right\} \quad \underset{x}{\text{minimize}} \; \underbrace{q_1 (y_1 - x)^2 + q_2 (y_2 - x)^2}_{h(x)}$$

$$\left.\begin{aligned} &\underset{x}{\text{minimize}} \quad q_1 n_1^2 + q_2 n_2^2 \\ &\text{subject to} \quad y_1 = x + n_1 \\ &\qquad\qquad\quad\;\; y_2 = x + n_2 \end{aligned}\right\} \quad \underset{x}{\text{minimize}} \; \underbrace{q_1 (y_1 - x)^2 + q_2 (y_2 - x)^2}_{h(x)}$$

$$0 = \frac{\partial h}{\partial x} = -2 q_1 (y_1 - x) - 2 q_2 (y_2 - x)$$

$$= -2 \left(q_1 y_1 - q_1 x + q_2 y_2 - q_2 x \right)$$

$$= -2 \left((q_1 y_1 + q_2 y_2) - (q_1 + q_2) x \right)$$

$$\Rightarrow \quad x = \frac{q_1 y_1 + q_2 y_2}{q_1 + q_2} = \left(\frac{q_1}{q_1 + q_2} \right) y_1 + \left(\frac{q_2}{q_1 + q_2} \right) y_2$$

PROBLEM

$$\underset{x}{\text{minimize}} \quad q_1 n_1^2 + q_2 n_2^2$$

weights

$$\text{subject to} \quad y_1 = x + n_1, \quad y_2 = x + n_2$$

SOLUTION

$$\hat{x} = \left(\frac{q_1}{q_1 + q_2} \right) y_1 + \left(\frac{q_2}{q_1 + q_2} \right) y_2$$

weighted average

q_1	q_2	(q_1 / q_1+q_2)	(q_2 / q_1+q_2)	$\hat{x}$
9	1	9/10	1/10	1.1
1	9	1/10	9/10	1.9
5	5	5/10	5/10	1.5

$$\underset{u[t_0,\infty)}{\text{minimize}} \quad \int_{t_0}^{\infty} \Big(\underbrace{x(t)^T Q_c\, x(t)}_{\text{error}} + \underbrace{u(t)^T R_c\, u(t)}_{\text{effort}} \Big)\, dt$$

$$\boxed{u = -K_c x}$$

$$\text{subject to} \quad \dot{x}(t) = Ax(t) + Bu(t) \quad \text{for } t \in [t_0, \infty)$$

$$x(t_0) = x_0$$

OPTIMAL CONTROLLER

OPTIMAL OBSERVER

$$\underset{x(t_1)}{\text{minimize}} \quad \int_{-\infty}^{t_1} \Big(\underbrace{n(t)^T Q_o\, n(t)}_{\text{sensor noise}} + \underbrace{d(t)^T R_o\, d(t)}_{\text{process disturbance}} \Big)\, dt$$

$$\text{subject to} \quad \left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + d(t) \\ y(t) &= Cx(t) + n(t) \end{aligned}\right\} \text{for } t \in (-\infty, t_1]$$

SUPPOSE:

$x(t_1) = \alpha$ if $t_1 = t_\alpha$

$x(t_1) = \beta$ if $t_1 = t_\beta$

THEN:

the solution to

$$\boxed{\dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} - y)}, \quad \hat{x}(t_\alpha) = \alpha$$

is $\hat{x}(t_\beta) = \beta$

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

$\leftarrow$ n_x number of states

n_u number of inputs

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OPTIMAL CONTROLLER

$$u = -Kx \qquad \text{where} \qquad K = \mathrm{lqr}(A, B, Q_c, R_c)$$

diagonal w/ positive numbers $\left\{ \begin{array}{l} Q_c \text{ is } n_x \times n_x \\ R_c \text{ is } n_u \times n_u \end{array} \right.$

$\leftarrow$ error

$\leftarrow$ effort

OPTIMAL OBSERVER

$$\dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} - y) \qquad \text{where} \qquad L = \mathrm{lqr}(A^T, C^T, R_o^{-1}, Q_o^{-1})^T$$

diagonal w/ positive numbers $\left\{ \begin{array}{l} Q_o \text{ is } n_y \times n_y \\ R_o \text{ is } n_x \times n_x \end{array} \right.$

$\leftarrow$ noise

$\leftarrow$ disturbance

$$\underset{x(t_1)}{\text{minimize}} \int_{-\infty}^{t_1} \left(n(t)^T Q_o n(t) + d(t)^T R_o d(t) \right) dt$$

$$\text{subject to} \quad \left. \begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + d(t) \\ y(t) &= Cx(t) + n(t) \end{aligned} \right\} \text{ for } t \in (-\infty, t_1]$$

↑ IN GENERAL

↓ SPECIFIC EXAMPLE IN CLASS (scalar system, $A = 0$, $B = 1$, $C = 1$, $D = 0$)

$$\underset{x(t_1)}{\text{minimize}} \int_{-\infty}^{t_1} \left(q\, n(t)^2 + r\, d(t)^2 \right) dt$$

$$\text{subject to} \quad \left. \begin{aligned} \dot{x}(t) &= u(t) + d(t) \\ y(t) &= x(t) + n(t) \end{aligned} \right\} \text{ for } t \in (-\infty, t_1]$$

How *exactly* should we choose Q_0 and R_0?

Define:

$$n = y - (Cx + Du)$$
$$Q = \mathrm{diag}(q_1, q_2, \ldots)$$

Suppose σ_i is the standard deviation of n_i. Then a good choice is:

$$q_i = (1/\sigma_i)^2$$ ← inverse of error variance

Why? If σ_i is small, then q_i will be big, and so you are saying you trust the ith sensor a lot — which is exactly what you should do if your model of this sensor has low noise.

← this is just intuition — it is possible to prove that this choice of Q_0 is, in fact, optimal

LQR is a standard way to choose L

$$\boxed{\begin{aligned} &\underset{x(t_1)}{\text{minimize}} \quad \int_{-\infty}^{t_1} \left(n(t)^T Q_0 n(t) + d(t)^T R_0 d(t) \right) dt \\ &\text{subject to} \quad \begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + d(t) \\ y(t) &= Cx(t) + Du(t) + n(t) \end{aligned} \end{aligned}}$$

The solution is:

$$\dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} + Du - y)$$

where

$$L = \mathrm{lqr}(A^T, C^T, R_0^{-1}, Q_0^{-1})^T$$

Q_0 big ⇒ n small ⇒ TRUST SENSORS

R_0 big ⇒ d small ⇒ TRUST DYNAMIC MODEL

Choose R_0 in exactly the same way, by looking at the error

$$\dot{x} - (Ax + Bu)$$

in the linearized EOMs.