General Relativity

Installment XV.

October 22, 2003

1 The Exterior Schwarzschild Metric

On second thought, we had better look a little more into the derivation of the Schwarzschild metric, even though there is a good bit of typing involved. Some authors do this by putting the metric in the form

$$d\tau^2 = e^A dt^2 - e^B dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{1}$$

When we go into index notation, we set $x^0 = t, x^1 = r, x^2 = \theta, x^3 = \phi$ so that, for example,

$$d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{2}$$

For a diagonal metric like this, one may denote it as $g_{\mu\nu} = \mathrm{diag}(e^A, -e^B, -r^2, -r^2\sin^2\theta)$. Similarly, we write $g^{\mu\nu} = \mathrm{diag}(e^{-A}, -e^{-B}, -r^{-2}, -r^{-2}\sin^{-2}\theta)$. Recall that the Christoffel symbol is

$$\Gamma^{\ \sigma}_{\mu\nu} = \frac{1}{2} g\sigma\rho \left(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}\right) \tag{3}$$

We also recall the notation

$$[\mu\nu, \lambda] = \frac{1}{2}\left(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}\right) \tag{4}$$

Let us now try a funny notation. The index (σ) is not σ but it has the same value as σ. We do this to make the next step easier to visualize (with any luck). We may then write

$$\Gamma^{\ \sigma}_{\mu\nu} = \frac{1}{2} g^{(\sigma)\sigma}[\mu\nu, (\sigma)] \tag{5}$$

In this expression, (σ) is summed over. The notation says that unless σ has the same value as (σ), there is no contribution since $g^{\mu\nu}$ is diagonal. What this is supposed to help you see is that if μ, ν and σ are all different, the Christoffel symbol is zero.

We then proceed to evaluate the nonzero terms. First we note that, since the solution is supposed to be stationary,

$$g_{\mu\nu,0} = 0 \tag{6}$$

This is just an assumption that would be proven in a more thorough discussion than this one.

Moreover,

$$g_{00,1} = g_{00}A', \qquad g_{11,1} = g_{11}B', \qquad g_{22,1} = -2r, \qquad g_{33,1} = -2r\sin^2\theta \tag{7}$$

and so on, where the prime means differentiation with respect to r. Now we see that

$$[11,1] = (g_{11}B') \tag{8}$$

and so on. We then go to

$$\Gamma = \frac{1}{2}g^{1\mu}[11,\mu] \tag{9}$$

Since the metric does not depend on t, θ or ϕ, we get

$$\Gamma = -\frac{1}{2}e^{-B}\left(-e^{B}B'\right) = \frac{1}{2}B' \tag{10}$$

In this manner, we find

$$\Gamma^{\,1}_{11} = \frac{1}{2}B'; \qquad \Gamma^{\,1}_{22} = -re^{-B}; \qquad \Gamma^{\,1}_{33} = -r\sin^2\theta\, e^{-B}; \qquad \Gamma^{\,1}_{00} = \frac{1}{2}e^{A-B}A' \tag{11}$$

$$\Gamma^{\,0}_{10} = \frac{1}{2}A'; \qquad \Gamma^{\,2}_{12} = \frac{1}{r}; \qquad \Gamma^{\,3}_{13} = \frac{1}{r}; \qquad \Gamma^{\,3}_{23} = \cot\theta; \qquad \Gamma^{\,2}_{33} = -\sin\theta\cos\theta \tag{12}$$

So much for the simple part.

The curvature tensor is

$$R^{\lambda}_{\cdot\mu\nu\sigma} = \Gamma^{\,\alpha}_{\mu\sigma}\Gamma^{\,\lambda}_{\alpha\nu} - \Gamma^{\,\alpha}_{\mu\nu}\Gamma^{\,\lambda}_{\alpha\sigma} + \Gamma^{\,\lambda}_{\mu\sigma,\nu} - \Gamma^{\,\lambda}_{\mu\nu,\sigma} \tag{13}$$

We contract it to

$$R^{\sigma}_{\cdot\mu\nu\sigma} = \Gamma^{\,\alpha}_{\mu\sigma}\Gamma^{\,\sigma}_{\alpha\nu} - \Gamma^{\,\alpha}_{\mu\nu}\Gamma^{\,\sigma}_{\alpha\sigma} + \Gamma^{\,\sigma}_{\mu\sigma,\nu} - \Gamma^{\,\sigma}_{\mu\nu,\sigma} = R_{\mu\nu} \tag{14}$$

To evaluate this it is useful to note that

$$\Gamma^{\,\sigma}_{\mu\sigma} = \frac{1}{2}g^{\sigma\rho}\left(g_{\mu\rho,\sigma} + g_{\sigma\rho,\mu} - g_{\mu\sigma,\rho}\right) = \frac{1}{2}g^{\sigma\rho}g_{\sigma\rho.\mu} \tag{15}$$

Then we simply start evaluating the components of $R_{\mu\nu}$. For instance

$$R_{11} = \Gamma^{\,\alpha}_{1\sigma}\Gamma^{\,\sigma}_{\alpha 1} - \Gamma^{\,\alpha}_{11}\Gamma^{\,\sigma}_{\alpha\sigma} + \Gamma^{\,\sigma}_{1\sigma,1} - \Gamma^{\,\sigma}_{11,\sigma} \tag{16}$$

But

$$\Gamma^{\,\sigma}_{1\sigma} = \frac{1}{2}g^{\sigma\rho}g_{\sigma\rho,1} \tag{17}$$

and

$$\Gamma^{\sigma}_{1\sigma,1} = \frac{1}{2} g^{\sigma\rho} g_{\sigma\rho,1,1} + \frac{1}{2} g^{\sigma\rho}{}_{,1}\, g_{\sigma\rho,1} \tag{18}$$

Plugging everything in, we get

$$R_{00} = -e^{A-B}\left(\frac{1}{2}A'' - \frac{1}{4}A'B' + \frac{1}{4}\left(A'\right)^2 + \frac{A'}{r}\right) \tag{19}$$

$$R_{11} = \frac{1}{2}A'' - \frac{1}{4}A'B' + \frac{1}{4}\left(A'\right)^2 + \frac{B'}{r} \tag{20}$$

$$R_{22} = e^{-B}[1 + \frac{1}{2}r(A' - B')] - 1 \tag{21}$$

$$R_{33} = R_{22}\sin^2\theta \tag{22}$$

A nice exercise that is left for you is to show that the curvature (or Ricci) scalar vanishes. Then, since we are in the external case where there is no matter, the Einstein equation (recall Laplace's equation) is $R_{\mu\nu} = 0$. If we combine $R_{00} = 0$ and $R_{11} = 0$, we get

$$e^{A-B}\left(\frac{A'}{r} + \frac{B'}{r}\right) = 0 \tag{23}$$

Then $A' = -B'$ and $A = -B+$constant. In the metric, since A and B are in the exponents, the integration constant merely introduces a change in the scale of time, which is anyway arbitrary so far. We can set the constant equal to zero.

Now, with $R_{22} = 0$, we find

$$e^A(rA' + 1) = 1 \tag{24}$$

or

$$(re^A)' = 1 \tag{25}$$

so that

$$g_{00} = e^A = 1 + \frac{const.}{r} \tag{26}$$

But, we found that $g_{00} = 1+2\phi$, where ϕ is the gravitational potential, so that $const. = 2Gm$ (really $2Gm/c^2$) and this is then Schwarzschild's famous exterior solution. We may also choose units so that $G = 1$ and then $const. = 2m$. In these units, time, distance and mass are all measured in units of length. For example, $m_\odot = 1.5\,km$.

The Schwarzschild exterior metric is then

$$d\tau^2 = (1 - \frac{2m}{r})dt^2 - (1 - \frac{2m}{r})^{-1}\,dr^2 - r^2\,d\Omega \tag{27}$$

with

$$d\Omega = d\theta^2 + \sin^2\theta\,d\phi^2 \tag{28}$$

2 Tests of General Relativity

2.1 Prerelativity

Kepler found that the orbit of each planet around the sun is an ellipse with the sun at one focus. The point in the ellipse of closest approach to the sun is called the perihelion. Observations revealed that the perihelia of the planets closest to the sun move slowly around the sun as if the ellipses those planets were trying to follow were themselves slowly turning in space. This curious behavior is mainly due to perturbations from the gravitational fields of other planets. However, long and extensive calculations failed to produce the right answer for the perihelion of Mercury whose calculated precession rate was off the observed value by some 43"/century. This was not so bad since the full precessional rate was more like 5558"/century. Still, the discrepancy was outside the estimated errors and so it seemed to be pointing to an interesting effect.

For a time, some thought that the solution lay in the influence of an unseen planet within the orbit of Mercury. They were pretty sure of this and even gave the mysterious object a name: Vulcan. Another possibility seriously considered was that the sun is not perfectly round and has a quadrapole moment. This would produce a bit of inverse cube force and could cause the observed precession. But attempts to detect a significant oblateness were not successful. This problem was a fly in the Newtonian ointment and was becoming worrisome. Indeed, people were right to worry since the discrepancy pointed to a limitation of Newton's theory. Einstein's theory correctly predicted the discrepancy.

2.2 Orbital Equations

A body moving in the Schwarzschild geometry will naturally follow the geodesics there if there are no (nongravitational) forces acting and if we may neglect the distortions of the geometry produced by the body itself. A body with such a negligible effect is called a test particle. The equation of motion is then

$$\ddot{x}^\alpha + \Gamma^{\,\alpha}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0 \tag{29}$$

For $\alpha = 2$ we have

$$\ddot{x}^2 + 2\Gamma^{\,2}_{12}\dot{x}^1\dot{x}^2 + \Gamma^{\,2}_{33}(\dot{x}^3)^2 = 0 \tag{30}$$

or

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\,\cos\theta\,\dot{\phi}^2 = 0 \tag{31}$$

That is,

$$(r^2\dot{\theta}\dot{)} - r^2\sin\theta\,\cos\theta\,\dot{\phi}^2 = 0 \tag{32}$$

To make life simple, we let the particle at $t = 0$ be at $\theta = \pi/2$ with $\dot{\theta} = 0$. Then, we see that $\ddot{\theta} = 0$ at $t = 0$, so that $\dot{\theta}$ remains zero for all time and θ stays at the value $\pi/2$. The orbit is in a plane.

Now, for the other components of the geodesic equation, we get

$$\ddot{r} + \frac{1}{2}B'\dot{r}^2 - re^{-B}\dot{\phi}^2 + \frac{1}{2}e^{A-B}A'\dot{t}^2 = 0 \tag{33}$$

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} = 0 \tag{34}$$

$$\ddot{t} + A'\dot{r}\dot{t} = 0 \tag{35}$$

where $A = 1 - 2m/r$. The ϕ equation is just

$$\left(r^2\,\dot{\phi}\right)^{\cdot} = 0 \tag{36}$$

which gives us

$$r^2\dot{\phi} = h \tag{37}$$

where h is a constant of integration (whose meaning you should surmise). The t equation becomes

$$\left(\ln \dot{t}\right)^{\cdot} + \dot{A} = 0 \tag{38}$$

which integrates to

$$\dot{t} = Ce^{-A} \tag{39}$$

where C is another integration constant.

Finally, instead of using the (forbidding) r equation explicitly, we sneak around it by using the metric statement that

$$e^{A}\dot{t}^2 - e^{-A}\dot{r}^2 - r^2\dot{\phi}^2 = 1 \tag{40}$$

into which we substitute our expression for $\dot{t}$ to get

$$C^2e^{-A} - e^{-A}\dot{r}^2 - \frac{h^2}{r^2} = 1 \tag{41}$$

If we merely want the orbit itself but not (for the moment) the ephemeris, we can look for $r(\phi)$, that is, we examine

$$\dot{r} = \frac{dr}{d\phi}\dot{\phi} \tag{42}$$

Then, with

$$\dot{r}^2 = \left(\frac{dr}{d\phi}\dot{\phi}\right)^2 = \frac{h^2}{r^2}\left(\frac{dr}{d\phi}\right)^2 \tag{43}$$

we get

$$\frac{h^2}{r^4}e^{-A}\left(\frac{dr}{d\phi}\right)^2 + \frac{h^2}{r^2} - C^2e^{-A} = -1 \tag{44}$$

or

$$e^{-A}\left(\frac{h}{r^2}\frac{dr}{d\phi}\right)^2+\frac{h^2}{r^2}-C^2e^{-A}=-1 \tag{45}$$

Then

$$\left(\frac{h}{r^2}\frac{dr}{d\phi}\right)^2+(1-\frac{2m}{r})\frac{h^2}{r^2}=C^2-(1-\frac{2m}{r}) \tag{46}$$

Now (as is commonly done in the Newtonian case) we set $u=1/r$ and we find the orbital equation

$$\left(\frac{du}{d\phi}\right)^2+u^2=\frac{C^2-1}{h^2}+\frac{2m}{h^2}u+2mu^3 \tag{47}$$

In the Newtonian case, the analogous equation turns out to be

$$\left(\frac{du}{d\phi}\right)^2+u^2=\frac{C^2-1}{h^2}+\frac{2m}{h^2}u \tag{48}$$

So the extra term of relativity is like that from an inverse cube term in the classical case. You can see why people in the nineteenth century looked for an oblateness of the sun.