General Relativity

Installment IX.

October 1, 2003

1 Geodesic Motion

We specify a parameterized curve by giving the coordinates of the points on the curve as functions of the parameter along the curve, say the distance on the curve from selected point. That is, we specify

$$x^i = x^i(s) \tag{1}$$

where s is the arclength along the curve. The components of the tangent vector to the curve at any point x^i are

$$v^i = \frac{d\,x^i}{ds} \tag{2}$$

and we may write the tangent vector as

$$\mathbf{v} = v^i \mathbf{e}_i \tag{3}$$

The derivative along the curve of the velocity is

$$\frac{d\,\mathbf{v}}{ds} = \frac{d\,v^i}{ds}\mathbf{e}_i + v^i \frac{d\,\mathbf{e}_i}{ds} \tag{4}$$

We saw (for example, in Installment VIII) that

$$d\mathbf{e}_i = \Gamma_{ij}^{\,k} \mathbf{e}_k d\,x^i \tag{5}$$

(However, the sign is different this time. Why is that?) We may then write that

$$\frac{d\,\mathbf{e}_i}{ds} = \Gamma_{ij}^{\,k} \mathbf{e}_k\, v^i \tag{6}$$

and so

$$\frac{d\,\mathbf{v}}{ds} = \left(\frac{d\,v^k}{ds} + v^i v^j \Gamma_{ij}^{\,k} \right) \mathbf{e}_k \tag{7}$$

If there is no external force, there is no acceleration, and we expect

$$\frac{d\,v^k}{ds} + v^i v^j \Gamma_{ij}^{\,k} = 0 \tag{8}$$

This is geodesic motion.

On the other hand, if we want to explain that the motion is not on straight lines, we might instead pretend that the deviation from Newton's first law is caused by an external force with a potential ϕ. We might guess that the force is

$$\nabla\phi = v^i v^j \Gamma_{ij}^{\,k} \mathbf{e}_k \tag{9}$$

The gradient operator is a vector operator and can be expressed as

$$\nabla = \mathbf{e}^i \frac{\partial}{\partial x^i} \tag{10}$$

Hence, if we dot $\mathbf{e}_\ell$ into (9), we get

$$\frac{\partial\phi}{\partial x^\ell} = v^i v^j \Gamma_{ij}^{\,k}\, g_{\ell k} \tag{11}$$

On the left we have first derivative of the potential. On the right we have a kind of first derivative of the metric tensor g_{ij}, for that what the Γ really is. So the (somewhat loose, but suggestive) conclusion is that we have motion that is like accelerated motion with the metric serving as the potential. The imagery however is totally different in the two cases. To complete the story, we need to replace the Newtonian outlook as a means of getting this 'potential.'

In Newtonian gravity, the potential is a scalar and is determined by the density of matter, but the metric is a tensor and we need to replace the rule for finding the metric as a result of the influence of mass and energy. But the rough idea for describing the motion is already seen here: if we can give a suitable metric, the geodesic equation can describe orbits in the way that Newton's second law does.

A complicating point is that in the term in (8) with Γ we have $\mathbf{v}$. But in fact, if s is arc length along the curve, $\mathbf{v}$ is a unit vector. That is, since

$$d\,s^2 = g_{ij} dx^i\, dx^j \tag{12}$$

we have

$$\mathbf{v}^2 = g_{ij} v^i\, v^j = 1. \tag{13}$$

This means that if the trajectory is specified, we know both Γ and $\mathbf{v}$ along it. But we don't in general know the trajectory and (8) is a sort of consistency condition to determine it. Things are really no simpler than in Newtonian mechanics and we don't even know how to pick the right g_{ij} yet.

However, if we multiply

$$\Gamma_{ij}^{\,k} = \frac{1}{2} g^{i\ell} \left(g_{\ell\, i,j} + g_{\ell\, j,i} - g_{i\, i,\ell} \right) \tag{14}$$

by $g_{k\,m}$ we find

$$g_{km} \Gamma_{ij}^{\,k} = \frac{1}{2} \left(g_{k\, i,j} + g_{k\, j,i} - g_{i\, i,k} \right) \tag{15}$$

If we interchange indices suitably and add the resulting expressions, we get a formula for the derivative of g_{ij} in terms of Γ (try this for yourself) and so can get a sort of equation for g_{ij} in terms of ϕ. It is not so easy to solve such equations, but you can see the emerging outline of the process from this.