General Relativity

Installment XIV.

October 20, 2003

1 The Einstein Equation

In his book, A First Course in General Relativity', Bernard Schutz gives these useful guidlines:

"If two tensors of the same type have equal components in a given basis, they have equal components in all bases and are said to be identical (or equal, or the same). In particular, if a tensor's components are all zero in one basis they are zero in all, and the tensor is said to be zero.

If an equation is formed using components of tensors combined only by the permissible tensor operations, and if the equation is true in one basis, then it is true in any other. This is a very useful result. It comes from the fact that the equation ... is simply an equality between components of two tensors of the same type, which [by the preceding remark] is then true in any system."

A fluid with four-velocity u^μ and stress-energy tensor $T^{\mu\nu}$ has a rest frame density given by the scalar $u_\mu u_\nu T^{\mu\nu}$. We could try to generalize Poisson's equation on the basis of this information, but that would be giving preference to the rest frame of the fluid. We want to make all frames equal (though it is sometimes true that some frames are more equal than others). So we shall seek a tensor formulation of the basic equation of gravity with $T^{\mu\nu}$ as the source of the gravitational field in the classical (nonrelativistic) limit. On the left side of Poisson's equation we can try to replace the Laplacian of the potential by second derivatives of the metric tensor. So our thoughts naturally turn to the curvature tensor, a beast with four indices. But $T^{\mu\nu}$ has only two, so a little trimming down needs to be done.

As we saw in Installment XII, the curvature tensor is

$$R^{\kappa}_{\cdot\lambda\mu\nu} = -\Gamma^{\ \kappa}_{\lambda\mu,\nu} + \Gamma^{\ \kappa}_{\lambda\nu,\mu} + \Gamma^{\ \sigma}_{\nu\lambda}\Gamma^{\ \kappa}_{\sigma\mu} - \Gamma^{\ \sigma}_{\mu\lambda}\Gamma^{\ \kappa}_{\nu\sigma} \tag{1}$$

Since the Christoffel symbols involve derivatives of $g_{\mu\nu}$, the Riemann tensor is made up of second derivatives. It is closely connected with the curvature of space. Indeed, it can be shown that a necssary and sufficient condition for the space to be flat is that all the components of the Riemann tensor should be zero. That may sound like a lot of components since there are $4^4 = 256$ components of the Riemann tensor. But because of the symmetries of this tensor, many of the components are not independent. In fact, as we see on inspection,

$$R_{\kappa\lambda\mu\nu} = -R_{\lambda\kappa\mu\nu} = -R_{\kappa\lambda\mu\nu} = R_{\mu\nu\kappa\lambda} \tag{2}$$

When we work through the combinatorics implied here, we find that, in four dimensions, $R_{\kappa\lambda\mu\nu}$ has a mere twenty independent components.

We need a second rank tensor that carries much of the geometrical information about the space we are working in. So a natural thing to try would be a contracted form of the Riemann tensor such as the Ricci tensor

$$R_{\mu\nu} = R^{\kappa}_{\cdot\mu\kappa\nu} \tag{3}$$

It is natural to wonder how the choice was made to contract that particular pair of indices. If you try contracting with the first index and any of the other three, you will find that you get the same result because of the symmetry. Another pleasant exercise is to show that

$$R_{\mu\nu} = R_{\nu\mu} \tag{4}$$

This is a good thing since the stress tensor is also symmetric in its two indices.

Now, one's first thought might be to try setting the Ricci proportional to the stress tensor. But by our construction of the stress tensor, it has a zero divergence:

$$T^{\mu\nu}_{\quad;\nu} = 0 \tag{5}$$

The divergence of $R_{\mu\nu}$ does not vanish. But Einstein realized that the divergence of

$$G^{\mu\nu} = R^{\mu\nu} - Rg^{\mu\nu} \tag{6}$$

is zero, where

$$R = R^{\mu}_{\mu} \tag{7}$$

This contraction of the Ricci is called the curvature scalar; $G^{\mu\nu}$ is the famous Einstein tensor. Einstein's generalization of the Poisson equation is then

$$G^{\mu\nu} = \kappa T^{\mu\nu} \tag{8}$$

where κ is a constant whose value needs to be specified. It is basically the Newton constant G; we shall see in the weak field limit that there is also a factor or 4π that comes in.

2 The Schwarzschild Solution

Immediately after the Einstein equations were published, Karl Schwarzschild gave the solution for the metric around a spherical mass. This has been the starting point of much of the excitement in the subject to do with strong gravitational fields. In fact, the interest in this sort of thing actually goes back to the end of the 18th century when John Michel in England Pierre Laplace in France wondered about the possible existence of bodies whose escape velocities were greater than the speed of light. Such bodies would be dark, they reckoned, and would be detectable only through their obscuration of stars. Thus they were, in effect, thinking about Newtonian black holes over two hundred years ago.

The Einstein equations are not so easy to solve as Newton's. Instead of attacking them frontally, one normally looks for metrics with various simple properties and, by putting them into the equations, derives the full solution. In this way, Schwarzschild imagined a sphere of mass m whose center is at the origin of spherical coordinates, r, θ, ϕ such that

$$x = r \sin\theta \cos\phi \ ; \qquad y = r \sin\theta \sin\phi \ ; \qquad z = r\cos\theta \tag{9}$$

The coordinate differentials are

$$dx = dr \sin\theta \cos\phi + r\cos\theta\cos\phi \, d\theta - r\sin\theta\sin\phi \, d\phi \tag{10}$$

$$dy = dr \sin\theta \sin\phi + r\cos\theta\sin\phi \, d\theta - r\sin\theta\cos\phi \, d\phi \tag{11}$$

$$dz = dr\cos\theta - r\sin\theta \, d\theta \tag{12}$$

From this we find that

$$dx^2 + dy^2 + dz^2 = dr^2 + dr^2 + r^2 \sin^2\theta \, d\phi^2 \tag{13}$$

We recognize the differential of solid angle

$$d\Omega = d\theta^2 + \sin^2\theta d\phi^2 \tag{14}$$

Schwarzshild sought a stationary, spherically symmetric solution with a metric of the form

$$d\tau^2 = F(r) \, dt^2 - G(r) \, dr^2 - r^2 D\Omega \tag{15}$$

The minus signs are there so that we get back to the right signs for Minkowski space in the limit of weak gravitational fields.

Now, it is possible to insert these forms into the expression for the Christoffel symbols, get them and go on to find the components of the Ricci tensor and the curvature scalar. There are arbitrary constants in the outcome and they are chosen to give back the weak field limit that we found earlier. This is a calculation that we shall not go through since it is not conceptually informative, though it does involve some tricks of the trade that you ought to learn someday if you want be involved in this subject. If the units are selected so that $G = 1$ and $c = 1$, the outcome is the famous Schwarzschild metric

$$d\tau^2 = (1 - \frac{2m}{r})dt^2 - \frac{1}{1 - \frac{2m}{r}}\,dr^2 - r^2\,d\Omega \tag{16}$$

The g_{00} is consistent with what we knew but the singularity in the g_{11} at $r = 2m$ alerts us to the fact that something interesting is going on. In fact, when we use ordinary units, this singularity occurs at

$$r = \frac{2Gm}{c^2}, \tag{17}$$

which is called the Schwarzschild radius. (This was in fact the radius found for the Newtonian black holes.)

3 Precession of the Perihelion of Mercury

3.1 Prerelativity

Kepler found that the orbit of each planet around the sun is an ellipse with the sun at one focus. The point in the ellipse of closest approach to the sun is called the perihelion. Observations revealed that the perihelia of the planets closest to the sun move slowly around the sun as if the ellipses those planets were trying to follow were themselves slowly turning in space. This curious behavior was for a time attributed to perturbations from the gravitational fields of other planets. However, long and extensive calculations failed to produce the right answer for the perihelion of Mercury whose calculated precession rate was off the observed value by some 43"/century. This was not so bad since the full precessional rate was more like 5558"/century. Still, the discrepancy was outside the estimated errors and so it seemed to be pointing to an interesting effect.

For a time, some thought that the solution lay in the influence of an unseen planet within the orbit of Mercury. They were pretty sure of this and even gave the mysterious object a name: Vulcan. Another possibility was that the sun was not perfectly round and had a quadrapole moment. This would produce a bit of inverse cube force and could cause the observed precession. But attempts to detect an oblateness were not successful. This problem was a fly in the Newtonian ointment and was becoming worrisome. Indeed, people were right to worry since the discrepancy pointed to a limitation of Newton's theory that Einstein's theory dealt with.