General Relativity

Installment I.

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Prologue

Consider a bead that slides freely on a stationary wire. Imagine that there is no friction, no air resistance, in this highly idealized world consisting of only the wire and the bead. This world is not quantum mechanical either.

If we let the bead be, it will sit there in a fixed place. If the wire is straight, and we give the bead an impulse, the bead will slide along at constant velocity. That is, the bead resists changes of its state of motion, as Galileo and some of his contemporaries realized it would more than three hundred years ago and as Newton implied in his first law of motion.

Now what happens to the motion of the bead as it moves along the wire if it comes to a place where the wire bends? Since we have assumed that the wire is stationary, we must conclude that the bead will stay on the wire — it corners nicely. Why? What keeps the bead on the wire? From our three-dimensional perspective, the bead has changed its velocity and so, if it is obeying Newtons' s second law, a force must have been exerted. This force, we normally say, is exerted by the wire on the bead. If I tell you the shape of the wire, you should be able to figure out what that force must be.

But there is a simpler way to think about this problem. We may assume that the bead lives in a world that is defined by the wire and, even it if were sentient, it would know nothing of our three-dimensional existence. In that formulation of the bead problem, we simply postulate that the bead is *constrained* to remain in the one-dimensional world defined by the wire. It is we, looking down on the bead and wire from our three-dimensional perspective, who worry about why the bead stays on the wire in order to be able to describe what is happening in our world. But may we not then wonder whether beings in higher dimensions than three are looking down on us (so to say), perhaps only in their imaginations, and deciding what the reason for our remaining in our own space may be? Is there a force constraining us to remain in our three-dimensional world? At any rate, we are seemingly confined to that world of three (or four, if we include time) dimensions and there is nothing to do about it for now. Like the bead on the wire, we move along in our own world, so here we begin by trying to determine what this world is like.

As most of us of have observed, the geometry of the nearby world is pretty nearly the geometry of Euclid. That is why the axioms of Euclidean geometry seem so natural to us and it is why many people regard geometry as a triumph of physics as well as of mathematical reasoning. But like the hypothetical wire, our world has bends in it, though they are hard to imagine for us living in this world. As objects such as the moon and the planets (and falling apples) move in our space, they follow paths that do not seem straight to us who think in terms of Euclidean geometry. To explain this behavior, Newton (and others) proposed that there is a gravitational force that forces them to behave in this way. We shall review briefly Newton's force and some of its features. However, here we are more concerned with the suggestion made by Einstein in the twentieth century that the planets are not driven by forces but that their seemingly bent paths are the analogues of straight lines in a curved geometry. The motions of the planets are like those of the bead on the bent wire wire or a ship circumnavigating the earth.

To see how Einstein's vision can be expressed in quantitative terms, we need to go beyond the geometry of Euclid. To learn what geometry we need for this, we shall go over classical mechanics, looking for clues. We shall also want to make the outcome compatible with the world as it is imagined in relativity, which we shall review as well. When we put it all together, we shall have made a semblance of Einstein's theory of gravity, a.k.a. general relativity. Along the way, we shall take a few digressions following the advice of Yogi Berra: "When you come to a fork in the road, take it."

So here goes.

1 Classical Mechanics

1.1 Newton's second law

Galileo knew that if a body in motion was not subjected to any external influences, it would keep on in its state of motion. This property of material objects is called inertia and it is expressed mathematically by Newton's first law of motion. Even more remarkably, Galileo stated what we may call the first statement of relativity in which he suggested that the laws of nature are not modified by uniform motion. This is what he wrote about it:

Shut yourself and a friend below deck in the largest room of a great ship, and have there some flies, butterflies, and similar small flying animals; take along also a large vessel of water with little fish inside it; fit up also a tall vase that shall drip water into another narrow-necked receptacle below. Now, with the ship at rest, observe diligently how those little flying animals go in all direction; you will see the fish wandering indifferently to every part of the vessel and the falling drops will enter into the receptacle placed below .... When you have observed these things, set the ship moving with any speed you like (so long as the motion is uniform and not variable); you will perceive not the slightest change in any of the things named, nor will you be able to determine whether the ship moves or stands still by events pertaining to your person.

You can read about this in "Galileo Galilei" by Ludovic Geymonat (McGraw-Hill 1965, pp.119-20). Galileo adds

And if you should ask me the reason for all these effects, I shall tell you now: Because the general motion of the ship is communicated to the air and everything else contained in it, and is contrary to their natural tendencies, but is indelibly conserved in them.

One of the most important of these laws of Nature is Newton's second law of motion. According to Newton, an external action or force is needed to change the state of motion of a body. A force produces an acceleration proportional to its strength and in the same direction as the force:

$$\mathbf{f} = m\mathbf{a} \tag{1}$$

where $\mathbf{f}$ is the force and $\mathbf{a}$ is the acceleration. The constant of proportionality, m, measures the body's resistance to the change of its state of motion and it is called the mass, or the inertia.

Of course, both Galileo and Newton were thinking that all this activity and its descriptions were taking place in a space that fulfills the axioms of Euclid to very good approximation. In such a space, we can install Cartesian coordinates with a conveniently chosen origin. To go from that origin to any point with coordinates (x, y, z) we make a displacement of a given amount and in a given direction. Such a displacement is traditionally indicated by an arrow of the right length and direction. This arrow is often called a vector and is denoted by $\mathbf{x}$. The association between a point in space, designated by its coordinates, (x, y, z), and a displacement vector, $\mathbf{x}$, is a wonderful convenience of Euclidean space that simplifies the expression of Newton's second law of motion.

Suppose we idealize the body obeying the law as just a point endowed with inertial mass. We call such a body a particle, from the Latin word *pars* meaning part with the addition of the diminutive ending *icle*. We may describe the particle's motion by associating to each point along the trajectory a vector $\mathbf{x}$ that varies with time, t. The dependence of the position $\mathbf{x}$ on time t describes the motion and we express this dependence by writing $\mathbf{x}(t)$. The information embodied in this description is what astronomers call an ephemeris, though in practice it can be provided only at a finite number of discrete times.

The quantity t is mysterious. St. Augustine said (roughly) "I know what time is until someone asks me to define it." Indeed, basic quantities such as mass, time and coordinates are exceedingly hard to define. The more elementary something is, the harder it is to define since there is nothing more elementary in terms of which to define or explain them. Such things are perhaps best defined by prescriptions for their determination. Thus, there is somewhere stored some well-conserved object used as a standard of mass and also some variable object that is used to define units of time. So we have prescriptions for measuring the basic quanitities even though it is not easy to know how to define them abstractly. One way to deal with these deep things is to suggest you will learn what these are when you read a more advanced treatment. Experience shows, however, that when you get there you will be assumed to have learned what these quantities are by reading elementary books. Such dilemmas are not unique to physics. In his book called "Money, " J.K. Galbraith says

It will be asked in this connection if a book on the history of money should not begin with some definition of what money really is. ... The precedents for such effort are not encouraging. Television interviewers with a reputation for penetrating thought regularly begin interviews with economists with the question: "Now tell me, just what is money anyway?" The answers are invariably incoherent.

Here we shall call on your experience of basic things and not try to go more deeply into them as we move onto more advanced topics.

2 Trajectories

The situation then is that we are relying on our daily experience to guide us in thinking of the motion of a particle in space and its trajectory. We still ought to make a more precise statement of what such a trajectory is. For this we call on the ideas of mathematicians, without striving for their rigor.

Consider a straight line lying in Euclidean space. Mathematicians call this one-dimensional space $\mathbf{R}$ since to every point on the line one can associate a real number, s (say), that increases monotonically and continuously along the line. Now if this line is rigid, we can pick it up and toss it into space. When we snap our finers, the line goes flaccid and develops bends, loops and similar complications as it flies out there. When we snap our fingers a second time, $\mathbf{R}$ goes rigid and stays put. So now we have a curve in space. Each point on the curve has carried with it its value of s and each point also has acquired coordinates (x, y, z) (assuming that the space has three dimensions). So we have automatically obtained an association between the coordinates in space and position along the curve. That is, the coordinates of the points along the curve are functions of s and we write this as $x(s), y(s), z(s)$, or $\mathbf{x}(\mathbf{s})$. The motion of our particle takes place along one such curve, or trajectory.

We could have associated the parameter s to points on the curve in a number of ways. If the curve is indeed a trajectory of our particle, we could complete the description of the motion by specifying the time t at which the particle passes through each point $\mathbf{x}(\mathbf{s})$. That is, we give $s(t)$ or $\mathbf{x}(\mathbf{s}(\mathbf{t}))$, written in the mildly inconsistent but suggestive way of some physicists as $\mathbf{x}(t)$. By assigning a value of t to each point along the curve, we have given a new parametrization of the curve, one that is significant in the description of the motion of the particle. The velocity of the particle is then the time derivative of the position.

Let $\mathbf{x}_0$ be the position of the particle at time t_0 and $\mathbf{x}_1 = \mathbf{x}_0 + \delta\mathbf{x}$ be the position at time $t_1 = t_0 + \delta t$. Draw a straight line through $\mathbf{x}_0$ and $\mathbf{x}_1$. When δt approaches zero, this line becomes the tangent line to the curve at the point $\mathbf{x}$. The instantaneous velocity of the particle at t_0 is along this line and it is given by

$$\mathbf{v}(t) = \lim_{\delta t \to 0} \frac{\delta \mathbf{x}}{\delta t} \tag{2}$$

The right side of this expression is the derivative of position with respect to time and we write the velocity vector as

$$\mathbf{v}(t) = \frac{d\mathbf{x}}{dt}. \tag{3}$$

(We shall be looking at vectors more carefully, but let us for now continue in this casual manner meant to provide an intuitively helpful entrée to the kinematics of particle motion.) The velocity of the particle is then the tangent vector to the trajectory when the points along it are labeled by the time at which the particle passes through them. Naturally, we need to figure out how to do that labelling, namely how to deal with the dynamics of the particle, but that too will come later.

Let us think back now to the illustration of the bead on the hoop. The curve we have been talking about is the wire that the bead slides along; this wire is the world that the bead inhabits. The vector that we have just introduced to describe the velocity of the bead does not lie in that world but in the tangent line to the bead's world, or wire. In Euclidean space, we do not suffer such an inconvenience, but for someone living in the wire, the velocity vector cannot be part of heir world — it is confined to the tangent line. The tangent line at any point $\mathbf{x}$ on the trajectory is called the tangent space of the wire and is denoted (in mathematical language) by $\mathbf{T}_\mathbf{x}$. The tangent space is the vector space inhabited by the velocity of the bead and there is such a tangent space at every point on the wire.

As we shall see, in Einstein's description of our world, we give up the Euclidean vision of our geometry and so we must give up the luxury of drawing vectors, such as velocities, in the space itself. As for the wire, we shall need to introduce a tangent space analogous to $\mathbf{T}_\mathbf{x}$. This is already evident when we think of the velocity of a ship sailing across the Pacific Ocean. We shall not only need a vector space at each point in our world, we shall also have to devise a suitable vector calculus for dealing with dynamical issues that arise. For now though let us close this section by stating that we can compute the derivative of $\mathbf{v}$ just as we computed the derivative of $\mathbf{x}$. In this way, we get the acceleration,

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{x}} \tag{4}$$

Newton's second law is then written as

$$\mathbf{f} = m\ddot{\mathbf{x}} \tag{5}$$

That part of the force that keeps the bead on the wire is neither in the wire nor in $\mathbf{T}_\mathbf{x}$. That is why we said that it is of no real concern to the bead, even though we do worry about it ourselves.

3 Work and Energy

Another convenience of working with Cartesian coordinates in Euclidean space is that the square of the length of the position vector $\mathbf{x}$ is the sum of the squares of the coordinates. We call this quantity the scalar product of $\mathbf{x}$ with itself. This product is denoted by a dot, so it is also called the dot product:

$$\mathbf{x} \cdot \mathbf{x} = x^2 + y^2 + z^2 \tag{6}$$

This is sometimes written in the suggestive form $\mathbf{x} \cdot \mathbf{x} = \mathbf{x}^2$. Similarly, if we denote the components of $\mathbf{v}$ by (u, v, w) we can write the dot product of the velocity with itself as

$$\mathbf{v}^2 = u^2 + v^2 + w^2 \tag{7}$$

Now if we take the dot product of $\mathbf{v} = \dot{\mathbf{x}}$ with the law of motion (5), we find that

$$m(\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z}) = \mathbf{v} \cdot \mathbf{f} \tag{8}$$

The quantity on the right of this equation is called the rate of working of the force $\mathbf{f}$. The quantity on the left is the rate of change (or time derivative) of $\frac{1}{2}m(\dot{x})^2 + \frac{1}{2}m(\dot{y})^2 + \frac{1}{2}m(\dot{z})^2$, the kinetic energy. So we have

$$\mathbf{f} \cdot \dot{\mathbf{v}} = \frac{1}{2}m\frac{d}{dt}(\mathbf{v})^2 \,. \tag{9}$$

The rate of working of $\mathbf{f}$ is the rate of change of the kinetic energy of the particle.

We know that certain forces, such as Newton's force of gravity, can be expressed as the derivative (or gradient) of a function of space and time called the potential. Without discussing what there is about $\mathbf{f}$ that lets us do this, let us just assume that we may, for now, write that

$$\mathbf{f} = -\nabla\phi \tag{10}$$

where the function ϕ is assigned a value at every point in the space for every time; we write this as $\phi(\mathbf{x}, t)$. The minus sign in (10) is included by convention and, though it is not necessary, its absence would make some people uncomfortable.

When we introduce (10) into (9), we can integrate the resulting equation and we find that

$$\frac{1}{2}m\mathbf{v}^2 + \phi = E \tag{11}$$

where E is a constant of integration called the energy. The fact that E is constant expresses expresses the law of conservation of energy for this simple example of Newton's second law. It says that the total energy, kinetic energy ($\frac{1}{2}m\mathbf{v}^2$) plus potential energy ($\phi(\mathbf{x}, t)$) is a constant.

If we keep to the simple case where the force is the gradient of a potential, we may write Newton's second law as

$$m\dot{\mathbf{v}} + \nabla\phi = \mathbf{0}\,. \tag{12}$$

When the two terms in this equation are together like that on the same side of the equal sign, we feel the temptation to think they have something in common. Some people even call $m\dot{\mathbf{v}}$ a force — the intertial force. We may then define a total force as

$$\mathbf{F} = m\dot{\mathbf{v}} + \nabla\phi\,. \tag{13}$$

It is sometimes useful to go beyond Galileian relativity and transform ourselves into a coordinate system moving so that $\mathbf{F} = \mathbf{0}$ and to make Newton's second law look like a version of the first law. This is a hint of what is coming, but let us not get ahead of our story.

But here is a last thought on these matters before going on to the next section. Suppose we use (11) to define the energy without worrying about where that expression for E came from. Then we would not know that E is supposed to be a constant and we might innocently ask how it changes in time. That is, we might compute

$$\frac{dE}{dt} = m\mathbf{v}\cdot\dot{\mathbf{v}} + \mathbf{v}\cdot\nabla\phi\,. \tag{14}$$

(This was done by noting that $d\phi/dt = \mathbf{v}\cdot\nabla\phi$.) Then (14) may be written as

$$\frac{dE}{dt} = \mathbf{v}\cdot\mathbf{F}. \tag{15}$$

So if we want to conserve energy ($dE/dt = 0$) we need only to require that the total force, $\mathbf{F}$, be perpendicular to the velocity. This is a weaker condition than the demand that Newton's second law be satisfied and is merely the demand that the total force does no work. In our example of a bead on the wire, the velocity of the bead is tangent to the wire. For energy to be conserved, the total force must therefore be perpendicular to the wire, as is the force needed to keep the bead on the wire. Thus, keeping an object in its space does not interfere with the conservation of energy, which is a lesson to think about.

4 More Mechanics

We have seen that the sum of the kinetic energy and the potential energy is conserved in the special case we looked at. We might ask if there is anything interesting about the difference between the two kinds of energy in similarly simple circumstances. That is, is there anything about

$$L = \frac{1}{2}m\mathbf{v}^2 - \phi(\mathbf{x}, t). \tag{16}$$

that we ought to know? This quantity is called the Lagrangian (or Lagrangean) after Luigi Lagrange, as he is known in Turin, his birthplace. The derivative of L is

$$\frac{dL}{dt} = m\mathbf{v} \cdot \dot{\mathbf{v}} - \mathbf{v} \cdot \nabla\phi\,. \tag{17}$$

There does not seem to be much to say about this. It however turns out that what is interesting about L is not so much its dervative as its integral.

In our three-dimensional space, with no wire to guide it, a particle must choose its route to go from P_1 to P_2. In the simplest vision of mechanics, this choice is dictated by the forces acting. Remarkably, this is just the route that gives the minimum value of A among possible trajectories between the two points. A trajectory is specified by giving the coordinates locating the particle at every instant t between t_1, the time of departure from P_1, and t_2, the arrival time at P_2. Let us designate one such trajectory as $\mathbf{x}(t)$, where $\mathbf{x}(t_1) = P_1$ and $\mathbf{x}(t_2) = P_2$. The action is then defined as

$$A = \int_{t_1}^{t_2} L dt, \tag{18}$$

which is taken between any two points in space, P_1 and P_2.

To think about how to find the minimum of A, let us imagine a space in which each point corresponds to a particular trajectory from P_1 to P_2. To every point in that space, we may associate a value of A, which is then a function defined on the space of trajectories. This is a rather abstract notion. A trajectory is itself a function of time, so that A is a function of functions. Such things are called functionals.

For any particular trajectory $\mathbf{x}(t)$, we can evaluate A. And for a trajectory close to that one, say $\mathbf{x}(t) + \delta\mathbf{x}(t)$, we can also make that evaluation and obtain a the new value, $A + \delta A$. Here we are assuming that at every time t, $\delta\mathbf{x}(t)$ is a small separation between corresponding points on the two trajectories. We may then construct the difference

$$\delta A = A[\mathbf{x}(t) + \delta\mathbf{x}(t)] - A[\mathbf{x}(t)] \tag{19}$$

To evaluate δA approximately, we note that, along the perturbed trajectory, we have

$$L = \frac{1}{2}m[\dot{\mathbf{x}}(t) + \delta\dot{\mathbf{x}}(t)]^2 - \phi(\mathbf{x} + \delta\mathbf{x}, \mathbf{t}) \tag{20}$$

Then, for small $|\delta\mathbf{x}|$, we can estimate that,

$$L \approx \frac{1}{2}m\dot{\mathbf{x}}(t) \cdot \delta\dot{\mathbf{x}}(t) - (\nabla\phi) \cdot \delta\mathbf{x}(t) \tag{21}$$

where the last term comes from taking the leading term in the Taylor expansion for ϕ. We see then that

$$\delta A = \int_{t_1}^{t_2} \left[\frac{1}{2} m\dot{\mathbf{x}}(t) \cdot \delta\dot{\mathbf{x}}(t) - \delta\mathbf{x} \cdot \nabla\phi\right] dt \tag{22}$$

We integrate the first term on the right by parts and note that there is no contribution from the end points since all trajectories go from P_1 to P_2. Then we find

$$\delta A = -\int_{t_1}^{t_2} [\ddot{\mathbf{x}} + \nabla\phi] \cdot \delta\mathbf{x} dt$$

As for functions, functionals like A are stationary at their extrema, their minima and maxima. If we want A to be an extremum for the preferred trajectory, we should expect δA to be zero on that trajectory. We want $\delta A = 0$ on the correct trajectory and this should be true for any $\delta\mathbf{x}$. This latter stipulation ensures that we must have

$$\ddot{\mathbf{x}} + \nabla\phi = \mathbf{0} \tag{23}$$

This is Newton's law of motion when the force is $\mathbf{f} = \nabla\phi$.

Notice that all that we have really determined is that A is stationary on the correct path (as given by Newton), but this is remarkable enough. More importantly, we have seen an example of how to find the condition for a functional to have a stationary value. This is an important process that it will be useful for us to know and that is really why we have taken the trouble to look at this example in a bit of detail.

5 The Equations of Lagrange

When we obtain Newton's equation of motion by demanding that the action be an extremal when evaluated on the trajectory that the Newton's law requires we are doing more than noting an interesting side line. We are discovering a way to derive governing equations in general once we state thee Lagrangian approach in general terms. We have, in other words, uncovered a powerful means to derive laws of physics. We shall do a once over lightly of this procedure in this section even though it is not essential for learning relativity. You should either just glance it over or skip it for now as it is not essential for following the rest of the rest of these notes.

In (16) we see that the Lagrangian is an explicit function of both $\mathbf{x}$ and $\dot{\mathbf{x}}$. It will also be a function of t if ϕ is. So let us we write the Lagrangian as $L(\mathbf{x}, \dot{\mathbf{x}}, t)$. The action is, as before,

$$A[\mathbf{x}(t)] = \int_{t_1}^{t_2} L(\mathbf{x}, \dot{\mathbf{x}}, t) dt \tag{24}$$

where the integral is taken along the trajectory implied by $\mathbf{x}(t)$.

Again, we take the difference between the integral over the trajectory $\mathbf{x}(t)$ and a neighboring one $\mathbf{x}(t) + \delta\mathbf{x}(t)$ to obtain

$$\delta A[\mathbf{x}(t)] = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial \mathbf{x}} \cdot \delta\mathbf{x} + \frac{\partial L}{\partial \dot{\mathbf{x}}} \cdot \delta\dot{\mathbf{x}}\right] dt \tag{25}$$

This time, we need to do some interpreting of the terms in this formula. For example, for brevity, we have written

$$\frac{\partial L}{\partial \mathbf{x}} \cdot \delta \mathbf{x} = \frac{\partial L}{\partial x} \cdot \delta x + \frac{\partial L}{\partial y} \cdot \delta y + \frac{\partial L}{\partial z} \cdot \delta z \tag{26}$$

That is, by $\partial L/\partial \mathbf{x}$ we mean what in ordinary vector calculus we would write as ∇L. We do not use the more familiar notation since we would not be able to write a similar thing for $\partial/\partial \dot{\mathbf{x}}$ in the abbreviation

$$\frac{\partial L}{\partial \dot{\mathbf{x}}} \cdot \delta \dot{\mathbf{x}} = \frac{\partial L}{\partial \dot{x}} \cdot \delta \dot{x} + \frac{\partial L}{\partial \dot{y}} \cdot \delta \dot{y} + \frac{\partial L}{\partial \dot{z}} \cdot \delta \dot{z} \tag{27}$$

We may once more integrate by parts to obtain

$$\delta A[\mathbf{x}(t)] = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial \mathbf{x}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{x}}} \right] \cdot \delta \mathbf{x} dt \tag{28}$$

For the trajectory that makes A smallest we require that $\delta A = 0$. This condition is met when the integral in (28) is zero, for small but arbitrary $\delta \mathbf{x}$. This happens when

$$\frac{\partial L}{\partial \mathbf{x}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{x}}} \right) = 0$$

This is called the Euler-Lagrange equation.

These ideas are the beginnings of a rich version of mechanics in a general sense. The basic idea of what is called Lagrangian mechanics is that we have a quantity, $L(\mathbf{x}, \dot{\mathbf{x}}, t)$, that depends on local properties of the trajectory. From L we define an action, $A = \int L dt$, where the integral is taken along a particular trajectory. The value of the action is a function of the trajectory and, in nature, the trajectory for which the functional derivative vanishes is the one followed. The condition for this is written as $\delta A/\delta \mathbf{x} = 0$ to stress that the functional derivative is intended. This gives rise to the equation of motion

$$\frac{\partial L}{\partial \mathbf{x}} - \frac{d\mathbf{p}}{dt} = \mathbf{0} \tag{29}$$

where $\mathbf{p} = \partial L/\partial \dot{\mathbf{x}}$ is identified with the momentum. As you must already suspect from this brief glimpse of advanced mechanics, there is a lot of method in this madness and we shall try to relate it to what is to come.