General Relativity

Installment XIII.

October 15, 2003

1 The Stress-Energy Tensor

1.1 Conservation Laws

Let $\mathcal{D}$ be the density of some stuff (scalar, vector or tensor). The total amount of that stuff in a given volume $\bullet$ is $\int_\bullet \mathcal{D} dV$ and the conservation law for the stuff is

$$\frac{\partial}{\partial t}\int_\bullet \mathcal{D} dV = -\int_\circ (\mathcal{D}\mathbf{v})\cdot \mathbf{n} dS + \int_\bullet \mathcal{S} dV \tag{1}$$

where $\mathbf{v}$ is the flow velocity of the stuff. The first integral on the left is a surface integral and the outward normal on the surface is $\mathbf{n}$. The quantity $\mathcal{S}$ is the net rate of creation and destruction of the stuff per unit volume.

The volume is fixed, so we can move the time derivative under the integral. We use the divergence theorem on surface integral and we obtain

$$\int_\bullet \left[\frac{\partial}{\partial t}\mathcal{D} + \nabla\cdot(\mathcal{D}\mathbf{v}) - \mathcal{S}\right] dV = 0 \tag{2}$$

Since the volume is arbitrary, this implies that

$$\frac{\partial}{\partial t}\mathcal{D} + \nabla\cdot(\mathcal{D}\mathbf{v}) = \mathcal{S} \tag{3}$$

1.2 Mass and Momentum

If the conserved substance is mass, whose density we denote as ρ, we write

$$\frac{\partial}{\partial t}\rho + \nabla\cdot(\rho\mathbf{v}) = 0 \tag{4}$$

since we are not allowing matter to be created or destroyed.

The momentum density is $\rho\mathbf{v}$ and it is generated by a force per unit volume as in

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla\cdot(\rho\mathbf{v}\mathbf{v}) = \mathbf{force}/\text{volume} \tag{5}$$

The force on a fluid or other continuous medium, besides external body forces, comes from the pressure. A pressure force on an element of fluid is the pressure difference across it. For example, in flat space with Cartesian coordinates, the pressure difference in the x-direction across a gap of dx is

$$p(x) - p(x+dx) \approx -\frac{\partial p}{\partial x} \tag{6}$$

Since pressure is force per unit area, the force on a small element of fluid with transverse area $dydz$ and thickness dx is $-(\partial p/\partial x)dx\,dy\,dz$ so the pressure force per unit volume in the x-direction is $-(\partial p/\partial x)$. Then the equation of momentum conservation is

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla\cdot(\rho\mathbf{v}\mathbf{v}) + \nabla p = \mathbf{0}. \tag{7}$$

These are the equations of motion of the perfect fluid. You need not attempt to follow this derivation; it is intended only that you should get an impressionistic feeling for it.

1.3 Preparing for the Jump to Relativity

Let $x^0 = t$ and let the spatial coordinates x^i be Cartesion for the first steps in the following rearrangements. Let us introduce a quantity v^0 whose numerical value is unity. Then the equation mass conservation may be written as

$$\frac{\partial(\rho v^0)}{\partial x^0} + \frac{\partial(\rho\, v^i)}{\partial x^i} = 0 \tag{8}$$

where v^i are the components of $\mathbf{v}$. Now let $\mu = 0, 1, 2, 3$. The conservation equation for mass becomes

$$(\rho v^\mu)_{,\mu} = 0 \tag{9}$$

For the momentum equation, we proceed along similar lines. We rewrite (7) as

$$\frac{\partial(\rho v^i v^0)}{\partial x^0} + \frac{\partial(\rho v^i v^j)}{\partial x^j} + \delta^{ij}\frac{\partial p}{\partial x^j} = 0. \tag{10}$$

This can be written as

$$\frac{\partial(\rho v^\mu v^i)}{\partial x^\mu} + \delta^{ij}\frac{\partial p}{\partial x^j} = 0 \tag{11}$$

or

$$\left(\rho v^\mu v^i\right)_{,\mu} + \delta^{ij}\frac{\partial p}{\partial x^j} = 0 \tag{12}$$

If we then rewrite (9) as

$$\left(\rho v^\mu v^0\right)_{,\mu} = 0 \tag{13}$$

we see that (13) and (12) together have a nice look. But what to do with the pressure term?

Introduce a quantity $h^{\mu\nu}$ such that $h^{ij} = \delta^{ij}$ with $h^{00} = 0$ and $h^{i0} = 0$. We'll see more about this in a minute, but first notice that (13) and (12) combine into

$$\left(\rho v^\mu v^\nu\right)_{,\mu} + h^{\mu\nu}\frac{\partial p}{\partial x^\mu} = 0 \tag{14}$$

The quantity $h^{\mu\nu}$, is called a projection operator. In this case, it projects onto the spatial components of a vector. In order to have the pressure term fit into the rest of the scheme, we really want to write it as something like $(h^{\mu\nu}p)_{,\mu}$. Indeed, it fits in fairly well if we write $h^{00} = v^0 v^0 - \eta^{00}$. This of course all seems off the wall. In fact it comes from long experience with these basic equations and some educated guesses about what the relativistic generalization must be. In going over this sort of thing carefully, you may get a strong impression about the transition between the transition between classical and relativistic theory. Though it must be admitted that it is easier to see through in the other direction, you need a good familiarity with the tensor calculus in that case.

Finally, we can write the equations for the conservation of mass and momentum in a classical perfect fluid as

$$T^{\mu\nu}_{\ ,\mu} = 0 \tag{15}$$

where

$$T^{\mu\nu} = \rho v^\mu v^\nu + h^{\mu\nu}\, p \tag{16}$$

This version of the equations has nothing to do with relativity except that it was constructed with relativity in mind.

1.4 Relativistic Fluid Dynamics

We need now to compare $v^i = (v^0, \mathbf{v})$, where $v^0 = 1$, to the four velocity. In fact, the v^i are just the conventional space velocity components, given by dx^i/dt. But in the frame of the moving observer (particle, or whatever) the four velocity is

$$u^\mu = \frac{dx^\mu}{d\tau} \tag{17}$$

where τ is the proper time. Thus, $u^\mu = \gamma v^\mu$ where $\gamma = \sqrt{1 - \mathbf{v}^2}$. For $\mathbf{v}^2 << 1$ — the classical limit — u^μ and v^μ are equal to good approximation and we have no hesitation in writing (9) as

$$\left(\rho u^\mu\right)_{,\mu} = 0 \tag{18}$$

and so on. For special relativity, we surmise then that with u^μ as the four-velocity field of the fluid, the full equations of motion are

$$T^{\mu\nu}_{,\mu} = 0 \tag{19}$$

where now

$$T^{\mu\nu} = \rho u^\mu u^\nu + h^{\mu\nu}\, p \tag{20}$$

If you are moving with the fluid, you perceive a four-velocity $(1, 0, 0, 0)$; that is, your motion is entirely through time. We want this equation to be equally valid in all inertial frames, so $T^{\mu\nu}$ had better be a tensor. But a careful examination of the $h^{\mu\nu}$ as we have so far defined it, shows that it will not do. So having dared thus far, we generalize it to

$$h^{\mu\nu} = u^\mu u^\nu - \eta^{\mu\nu} \tag{21}$$

This expression boils down to the former h in the frame moving with the fluid, so it is a sensible generalization. It is the simplest tensor extension that has been thought of and there are other reasons for adopting it, but you could not be blamed for regarding it with suspicion.

By now, you have guessed the next step: we go into curved space and maintain our demand that our equation be expressed as a tensor. The conservation of mass and momentum should be expressed as

$$T^{\mu\nu}{}_{;\mu} = 0 \tag{22}$$

Of course, this being relativity, energy is also involved and this we see when we reexpress the stress (or stress-energy) tensor as

$$T^{\mu\nu} = (p + \rho) u^\mu u^\nu - \eta^{\mu\nu}\, p \tag{23}$$

Since p is a measure of the internal energy of the fluid (that is the kinetic energy of its constituent particles) we see that it contributes to the mass and so the kinetic energy term has become $(p+\rho)u^\mu u\nu$. But we must skip all that this entails as it would going too far afield to get into the thermodynamic issues. Suffice it to say that $T^{\mu\nu}$ is what Einstein decided as the surrogate for ρ in his generalization of the Poisson equation to the case of relativity.