General Relativity

Installment X.

October 6, 2003

1 Review of Coordinate Transformations

We require that the properties of spacetime and the contents of our statements of the physical laws should not depend on our choice of coordinate systems. We still do want to have the convenience of coordinates to formulate our statements about these things. So let us look again at the business of changing coordinates. (There is a nice discussion of this matter in Stephani's 'General Relativity.') Suppose that we have coordinates x^i, $i = 1, 2, ..., n$ and we wish to change to new coordinates, $x^{i'}$. The transformation can be indicated by

$$x^i = x^i(x^{i'}) \tag{1}$$

To simplify the discussion we may sometimes work with the coordinate differentials

$$dx^{i'} = \frac{\partial x^{i'}}{\partial x^i} dx^i \tag{2}$$

This is a set of simultaneous linear equations for the dx^i in terms of the $dx^{i'}$. The coefficients of the equations are in the transformation matrix

$$\Lambda^{i'}{}_i = \frac{\partial x^{i'}}{\partial x^i} \tag{3}$$

and we require that det $\left(\Lambda^{i'}{}_i\right) \neq 0$. For those not familiar with this condition in the solution of simulataneous linear equations a brief example may serve to explain or recall the situation.

Suppose that you have two simultaneous equations in the unknowns x and y,

$$Ax + By = a; \qquad\qquad Cx + DY = b. \tag{4}$$

This can be written as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{5}$$

Define the row vectors $\mathbf{A} = (A, B)$ and $\mathbf{C} = (C, D)$ and the column vector

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{6}$$

Then the simultaneous equations are

$$\mathbf{A} \cdot \mathbf{x} = a \qquad\qquad \mathbf{C} \cdot \mathbf{x} = b \tag{7}$$

Now observe that the determinant of the matrix can be written as $\mathbf{A} \times \mathbf{C}$ which has only one component (and not really in the plane either). So if the determinant did vanish, this would mean that $\mathbf{A}$ and $\mathbf{C}$ are colinear and that we could write that

$$\mathbf{A} = \lambda \mathbf{C} \tag{8}$$

where λ is a constant. Then, the equations (9) become

$$\lambda \mathbf{C} \cdot \mathbf{x} = a \qquad\qquad \mathbf{C} \cdot \mathbf{x} = b \tag{9}$$

These are compatible only if $a = \lambda b$, otherwise we cannot find a solution. That is why we require the condition $\det\left(\Lambda^{i'}{}_{i}\right) \neq 0$ for the differential transformations that we work with.

4000 homework for Oct. 13: Give a similar geometric argument for the three-dimensional case.

Going back to the inverse transformation, we have the transformation on differentials

$$dx^j = \frac{\partial x^j}{\partial xj'} dx^{j'} = \Lambda^j{}_{j'} dxj' \tag{10}$$

where

$$\Lambda^j{}_{j'} = \frac{\partial x^j}{\partial xj'} \tag{11}$$

Then

$$\Lambda^j{}_{j'} \Lambda^{j'}{}_k = \delta^j{}_k \ , \qquad\qquad \Lambda^{i'}{}_i \Lambda^i{}_{j'} = \delta^{i'}{}_{j'} \tag{12}$$

It may sometimes happen that you are given the differential transformation, but that there is no integrated transformation of the form of (1). Such a situation is called anholonomic. I doubt that you will ever see such thing here but I mention it because something like this arises in mechanics and in thermodynamics, and I wanted to get you a little prepared for that.

2 Spacetime

2.1 The Galilean Transformation

Figure 1 shows a section of Manhattan. The x-coordinate labels the Avenues and the y-coordinate labels the streets. The unit of length in x is the av and the unit of y is the st.

Our friends, points P and Q, are indicated on the figure. What is the distance between P and Q? To answer this question you measure Δx and Δy, the coordinate separations between the points. These are found to be 1.5 av and 1.5 st, respectively. But then how to compute the distance? The two coordinate differences are in different units and thus hard to combine into a single number such as distance. But, if I tell you that there are three sts to an av, you can do it. The answer is then $\sqrt{(3 \times 1.5)^2 + (1.5)^2} = 1.5\sqrt{10} \approx 4.7sts$. Evidentally, the two coordinates need to be in the same units if we are to use them effectively. The same is true in spacetime. These and related issues are nicely discussed in the Relativity book by Max Born and the book on Special Relativity by French and Wheeler.

In Figure 2, we show a picture of spacetime of an inertial observer. Only one spatial axis is shown and that is called x. The ordinate, t, is the time axis. This is the usual way of drawing spacetime diagrams. As with the streets and avenues, it is not practical to measure time and space in different units when working in spacetime, so we need a correction factor to convert one of them to the units of the other. Call that factor c and imagine that we have multiplied it into t. If c has the dimensions of speed, then t will have acquired the dimensions of length. In the new units, the value of c will have become unity. For now, c can be any conversion factor that is useful, though in relativity, the natural choice is the speed of light *in vacuo*. In that case, we are using light years instead of years to measure time (for example).

Suppose that there is another observer moving in the x-direction at constant velocity v. (If hey were moving in some other direction, we would have chosen that direction as the x-direction.) Heir time axis, t' is along heir worldline in spacetime, given by $x = vt$. This is arranged so that hey is stationary in heir own coordinate system. The Galileian transformation between heir system and ours is

$$t' = t \; ; \qquad\qquad x' = x - vt \tag{13}$$

as can be seen from the figure. This linear transformation of coordinates can be expressed in matrix form as

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \tag{14}$$

The inverse transformation is the same except that $-v$ is replaced by v so that

$$\begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{15}$$

Those of you who have studied matrices will recognize that the matrix in (14) shears the medium and (14) may seem to you to be a rather strange looking transformation law. It may also be surprising that no one really thought about it much until Minkowski made explicit what was implied by the theory of relativity: space and time are one and ought to be thought of together in the manner of Figure 2.

In relativity, we know the metric of space time, but what is it in classical physics where we don't worry about exceeding the speed of light? In the Euclidean plane, the locus of points equidistant from the origin is a circle. In the Minkowskian x–t plane of relativistic spacetime, the locus of points equidistant from the origin is a hyperbola whose asymptotes are $t = \pm x$, as we shall see. But the classical case is not so nice. What we want from a metric is that it should produce an invariant object or scalar that we might associate with a line element or distance. Well, we do have $dt = dt'$, so that can serve as a line element. Then time is distance, and the import of this remark will take on more significance when we go to relativity. But clearly, the classical case is not so pretty as the relativistic one.

2.2 The Lorentz Transformation

It came as a surprise that Maxwell's equations for electricity and magnetism were not invariant under the Galilean transformation that we looked at in the previous subsection. Perhaps it was a bigger surprise that there was another transformation under which Maxwell's equations *a*re invariant. Theories were constructed to rationalize this finding but they did not really explain what was going on. Then Einstein cleared away the mystery by deriving the Lorentz transformation from a couple of simple postulates: that the laws of physics are the same in any inertial frame and that the speed of light is independent of the speed of its source. (Inertial frames are those that are not accelerated.) What this meant was that the Galilean transformation is not quite the right one between moving frames of reference. Rather it is only an approximation to the correct transformation and it is applicable only when the relative speeds of the moving frames have small magnitude.

In the case of two-dimensional space time, the Lorentz transformation is

$$t' = \frac{t - vx/c^2}{\sqrt{1-(v/c)^2}} \qquad\qquad x' = \frac{x - vt/c}{\sqrt{1-(v/c)^2}}) \tag{16}$$

where c is the speed of light (and when we say "speed of light," the qualification in a vacuum is generally understood). Now, we notice that when $|v|/c << 1$, the Lorentz transformation reduces to the Galilean transformation. Since speeds comparable to c were unknown before the twentieth century (except for light itself) it is not so surpising that the Galilean transformation was good enough.

In the Lorentz transformations, as expressed in (16), space and time have different units, so we want to fix that as we discussed in the previous paragraphs. For this we make the substitution $t \to ct$. In the new units, $c = 1$ and we may write the Lorentz transformation as

$$t' = \frac{t - vx}{\sqrt{1-v^2}} \qquad\qquad x' = \frac{x - vt}{\sqrt{1-v^2}} \tag{17}$$

In matrix form, the transformation is

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \tag{18}$$

where $\gamma = \sqrt{1 - v^2}$.

A suggestive change of variables that is often made is

$$\cosh\chi = \gamma \qquad\qquad \sinh\chi = \gamma v. \tag{19}$$

The transformation then looks like this:

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\chi & -\sinh\chi \\ -\sinh\chi & \cosh\chi \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \tag{20}$$

This version makes it very clear that the determinant of the transformation matrix is unity. A consequence of this (which you might like to verify) is that

$$t'^2 - x'^2 = t^2 - x^2 \tag{21}$$

That is, $t^2 - x^2$ is a scalar under transformation to a new inertial frame. This quantity looks a lot like the one in the theorem of pythagoras, except for the sign. In Minkowski spacetime, the distance from the origin of a point with spacetime coordinates t and x is $\sqrt{t^2 - x^2}$. The locus of points equidistant from the origin is a hyperbola. So spacetime is a world in which hyperbolic trignonmetry reigns.

One interesting property of spacetime that follows from this definition of distance is that the sum of the lengths of two sides of a triangle is *l*ess than the length of the third side. This feature of the geometry of spacetime gives rise to some results that may seem paradoxical at first glance. (The so-called twin paradox is a popular example of this.)

We should also be aware that, when we travel, in normal circumstances the spatial distance covered is measured by an odometer. But the spatio-temporal distance is measured by a chronometer, or clock. That is, in our own frames, we are simply moving forward in our own or *p*roper times designated as τ (though the less pretentious t' will do). As we travel through spacetime, the distances covered along our paths, or worldlnes, are recorded by our clocks. In your own frame, you have no spatial velocity and your speed in your time direction is $dt'/dt' = 1$ (remember though that $c = 1$). In the unprimed frame your velocity has components $dt/d\tau$ and $dx/d\tau$. These are the components of the vector tangent to your world line and its magnitude (your speed in spacetime) is always unity. You cannot stop moving through spacetime but if you move through space at the speed of light, you will stand still in time because your total speed must be unity. All this makes for an interesting dynamics.

3 Energy, Momentum and All That

Consider a particle moving through spacetime. As usual, we denote Its spatial coordinates as x^i. The time, t, we call x^0. In four-dimensional spacetime, we shall use Greek indices that run from 0 to 3. The spacetime coordinates of an event are thus x^μ with $\mu = 0, 1, 2, 3$. In special relativity (no gravity yet) we shall call the metric tensor in spacetime $\eta_{\mu]nu}$ and the line element is written as

$$d\tau^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{22}$$

From the considerations of the previous section, we have

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{23}$$

The spacetime displacement corresponding to the change in dx^μ is the change in the proper time of the displace object. The μ component of the corresponding velocity is

$$u^\mu = \frac{dx^\mu}{d\tau} \tag{24}$$

This is called the four velocity. According to (22),

$$\eta_{\mu\nu} u^\mu u^\nu = 1 \tag{25}$$

If the moving object has mass (or inertia) m, the momentum is

$$p^\mu = m u^\mu = m \frac{dx^\mu}{d\tau} \tag{26}$$

Then

$$\eta_{\mu\nu} p^\mu p^\nu = (p^0)^2 - \mathbf{p}^2 = m^2 \tag{27}$$

where $\mathbf{p}$ is the spatial momentum, or three-momentum, by contrast with p^μ, the four-momentum. The zeroth component of the four-momentum, p^0, is interpreted as the energy, E, of the object. Thus,

$$E^2 = \mathbf{p}^2 + m^2 \tag{28}$$

or

$$E = \pm\sqrt{m^2 + \mathbf{p}^2} \tag{29}$$

The two possible signs of the energy has great ramifications for quantum mechanics; it also has significance for classical (continuum) mechanics that has not been fully exploited as yet.

When the momentum is not large, as in ordinary life, we may approximate the energy as

$$E = m\left(1 + \frac{\mathbf{p}^2}{m^2}\right)^{\frac{1}{2}} = m\left(1 + \frac{\mathbf{p}^2}{2m^2} + ...\right) \tag{30}$$

(The minus sign has been suppressed for simplicity.) Since $\mathbf{p}^2/(2m^2)$ is the classical kinetic energy, we see that the energy of the object has two terms, the standard kinetic energy and m. With our scaling of t by c, we have a situation where energy and mass have the same units. But to give the mass–energy its normal look, we recall that a factor of c^2 $(= 1)$ is needed. Then, we find that the extra term in the energy (from the classical point of view) is the famous mc^2.