General Relativity

Installment XI.

October 13, 2003

1 The Differential

This is a subject that leads into some advanced topics that we shall not really get into, but as some people are not comfortable with differentials, it may be worth saying something about them here.

Consider a function f on the real line with coordinate x. The value of f at $x + dx$, where dx is a small increment in coordinate, is

$$f(x + dx) = f(x) + f'(x)\,dx + \frac{1}{2} f''(x)(dx)^2 + ... \tag{1}$$

where $f' = df/dx$ and so on. In the limit of small displacement, dx, we may approximate this as

$$df = f(x + dx) - f(x) = \frac{df}{dx}\,dx \tag{2}$$

In a space of dimension n with coordinates x^i, this generalizes to

$$df = \frac{\partial f}{\partial x^i}\,dx^i \tag{3}$$

One way to look at this is to think of the dx^i as a set of (reciprocal) basis vectors and of the $\partial f/\partial x^i$ as components of a (covariant) vector called the gradient of f. This image proved so appealing to mathematicians that they promoted df from its lowly status as an approximation with an infinitesimal character to being a finite object in its own right. It is a pity that a more compelling notation was not introduced with this elevation, though in some variants on the notation various doodads are put on the d, as in $\tilde{d}$. What matters most about this object though is that when you see something like $\tilde{d}f$, this thing is not necessarily integrable in the way that $(\partial f/\partial x)dx$ integrates up to f. That is, in the expression $a_i dx^i$, a_i

is not necessarily the derivative of anything. This feature of the generalization of differentials to forms is of fundamental importance in thermodynamics and it also plays an important role in some aspects of classical mechanics. In a less mathematical way, we might write the gradient operator in terms of more commonplace looking basis vectors as $\nabla = \mathbf{e}^i(\partial/\partial x^i)$. When applied to a function f, this is

$$\nabla f = \mathbf{e}^i \frac{\partial f}{\partial x^i} \tag{4}$$

Recall that we have already said that the derivative along a curve parameterized by s is

$$\frac{d}{ds} = \frac{dx^i}{ds}\frac{\partial}{\partial x^i} \tag{5}$$

This is also called the derivative along the tangent vector $v^i = dx^i/ds$. As we have seen, the operator d/ds is a vector with components v^i and basis vectors $\partial/\partial x^j$. If we think of the latter as $\mathbf{e}_i$ we may write

$$\mathbf{e}^i \cdot \mathbf{e}_j = dx^i[\frac{\partial}{\partial x^j}] = \delta^i_{\cdot j} \tag{6}$$

This is what underlies the distinction between covariant and contravariant, but we are not going into these more advanced topics. What we do need to appreciate though is that one may speak of the derivative along any given vector, **a**. This is

$$\mathbf{a} \cdot \nabla = (a^i \mathbf{e}_i) \cdot (\mathbf{e}^j \frac{\partial}{\partial x^j}) = a^i \frac{\partial}{\partial x^i} \tag{7}$$

This will also work for the directional derivative of a vector.

The derivative of **b** along **a** is

$$\nabla_{\mathbf{a}} \mathbf{b} = \nabla_{(a^i \mathbf{e}_i)}(b^j \mathbf{e}^j) = a^i \nabla_{\mathbf{e}_i}(b^j \mathbf{e}_j) \tag{8}$$

As before, the derivative along $\mathbf{e}_i$ is the derivative in the x^i direction. Taking the derivative of the product, we obtain

$$\nabla_{\mathbf{a}} \mathbf{b} = a^i(\mathbf{e}_j b^j_{\ ,i} + b^j \nabla_{\mathbf{e}_i} \mathbf{e}_j) \tag{9}$$

(The typesetting is not so successful here — $\mathbf{e}_i$ is a subscript on ∇ but the $\mathbf{e}_j$ is not.) From past experience, we surmise that with a little work we would find that

$$\nabla_{\mathbf{e}_i} \mathbf{e}_j = \Gamma^{\ k}_{ij} \mathbf{e}_k \tag{10}$$

where the Γ will turn out to be just what the notation suggests. Thus,

$$\nabla_{\mathbf{a}} \mathbf{b} = a^i b^k_{\ ;i} \mathbf{e}_k \tag{11}$$

2 The Metric Tensor as Potential

2.1 Four-Velocity

Now let us go into an inertial frame with time coordinate $x^0 = t$ and the space coordinates x^i, $i = 1, 2, 3$, as usual. The distance between two neighboring points is $d\tau$ where $d\tau^2 = \eta_{\mu\nu} dx^\mu dx^\nu$. ($d\tau^2$ means $(d\tau)^2$.)

A particle moves through the spacetime at constant spatial velocity $\mathbf{v}$ and in its frame its proper (or personal) coordinates are $x^{0'} = t'$ and $x^{i'} = 0$. That is, $x^{\mu'} = (t', \mathbf{0})$ where $\mu = 0, 1, 2, 3$. The Lorentz transformation (with $c = 1$) is

$$t = \gamma(t' + vx') \qquad x = \gamma(x' + vt') \tag{12}$$

The particle is assumed to move in the x-direction with $x^1 = x$. The time-component of the velocity of the particle is

$$\frac{dt}{dt'} = \gamma \tag{13}$$

since t' is the proper time of the particle. Here, $\gamma = 1/\sqrt{1 - \mathbf{v}^2}$and the components of $\mathbf{v}$ are dx^i/dt so that

$$\frac{dx^i}{dt'} = \gamma v^i = u^i \tag{14}$$

We can also write this as

$$u^\mu = \left(\frac{1}{\sqrt{1 - \mathbf{v}^2}}, \frac{\mathbf{v}}{\sqrt{1 - \mathbf{v}^2}} \right) \tag{15}$$

As we can see,

$$\eta_{\mu\nu} u^\mu u^\nu = 1 \tag{16}$$

Another way to say this is provided by the introduction of

$$u_\mu = \eta_{\mu\nu} u^\nu = (u^0, -\mathbf{v}) \tag{17}$$

so that

$$u_\mu u^\mu = 1 \tag{18}$$

2.2 Geodesic Motion

We need not repeat the derivation of the equation of motion along geodesics since the derivation is as before except for some notational changes: we replace Latin indices by Greek ones, and now use τ, the spacetime distance along the path, instead of the former s. Then we have

$$\ddot{x}^\mu = -\Gamma^{\ \mu}_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma \tag{19}$$

where $\dot{x} = dx/d\tau$. (τ and t' are being used interchangeably in this example.) We may write (19) out as

$$\ddot{x}^\mu = -\Gamma^{\,\mu}_{00}\, u^0 u^0 - 2\Gamma^{\,\mu}_{0i}\, u^0 u^i - \Gamma^{\,\mu}_{ij}\, u^i u^j \tag{20}$$

But since $u^0 = \gamma$ and $u^i = \gamma v^i$,

$$\ddot{x}^\mu = -(u^0)^2 \left(\Gamma^{\,\mu}_{00} - 2\Gamma^{\,\mu}_{0i} v^i - \Gamma^{\,\mu}_{ij} v^i v^j\right) \tag{21}$$

Since c is the unit of speed, we may assume that $|\mathbf{v}| << 1$ in the case of the mild motions we are used to, so that

$$\ddot{x}^\mu \approx -(u^0)^2\, \Gamma^{\,\mu}_{00} \approx -\Gamma^{\,\mu}_{00} \tag{22}$$

We need to compute $\ddot{x}^i$ to compare with Newton's second law to get an idea of what replaces it.

We know that

$$\Gamma^{\,\mu}_{00} = \frac{1}{2} g^{\mu\beta} \left(g_{0\beta,0} + g_{0\beta,0} - g_{00,\beta}\right) \tag{23}$$

In the limit where the space is not very warped, we expect the Minkowski metric to be a reasonable approximation so we write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{24}$$

where $h_{\mu\nu}$ is a small perturbation on the flatspace Minkowski metric. The inverse metric should be of the form

$$g^{\mu\nu} = \eta^{\mu\nu} + f^{\mu\nu} \tag{25}$$

since $\eta_{\mu\rho}\eta^{\rho\nu} = \delta_\mu^{\ \nu}$. We then require that

$$g_{\mu\rho} g^{\rho\nu} = (\eta_{\mu\rho} + h_{\mu\rho})(\eta^{\rho\nu} + f^{\rho\nu}) = \delta_\mu^{\ \nu} \tag{26}$$

When we write out the indicated product, we get

$$g_{\mu\rho} g^{\rho\nu} = \delta_\mu^{\ \nu} + f_\mu^{\ \nu} + h_\mu^{\ \nu} + ... = \delta_\mu^{\ \nu} \tag{27}$$

Then we conclude that

$$f_\mu^{\ \nu} = -h_\mu^{\ \nu} \tag{28}$$

We neglect terms quadratic in h and, since $\eta_{\mu\nu}$ is a constant,

$$\Gamma^{\,i}_{00} = \frac{1}{2}\eta^{i\beta}(2h_{0\beta,0} - h_{00,\beta}) = \frac{1}{2}\delta^{ij} h_{00,j} \tag{29}$$

And so,

$$\ddot{x}^i = -\frac{1}{2}\frac{\partial h_{00}}{\partial x^j}\delta^{ij} \tag{30}$$

On comparison of this with the Newton's second law,

$$\ddot{x}^i = -\frac{\partial \phi}{\partial x^j}\delta^{ij} \tag{31}$$

we conclude that in the classical limit

$$h_{00} = 2\phi \tag{32}$$

where ϕ is the potential of an imposed force. Then, since $g_{00} = \eta_{00} + h_{00}$, we have that

$$g_{00} = 1 + 2\phi \tag{33}$$

We had already seen that in Galilean spacetime, a suitable line element is dt. Now we find that the effect of an external potential is to produce a slight warping through g_{00}. More importantly, we have the clue that the metric tensor replaces the gravitational potential when we ascribe the influence of gravity to an effect of geometry rather than to a force.