General Relativity

Installment XVI

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1 Orbiting a Schwarzschild Object

1.1 The Newtonian Approach

When I say Newtonian, I really mean classical, though we even see that through Einsteinian lenses. This Newtonian subsection is for those of you who seem shaky on the provenance of Kepler's laws, so some of you can skip this.

Let us accept that the motion of a test body around a spherically symmetric mass is in the Euclidean plane. If there were no forces, the motion would be in straight lines. This is geodesic motion and it is described in Cartesian coordinates, x, y, by

$$\ddot{x} = 0; \qquad \ddot{y} = 0 \tag{1}$$

Since the Newtonian force of gravity is radial, it is advisable to use polar coordinates r, φ where

$$x = r\cos\varphi; \qquad y = r\sin\varphi \tag{2}$$

This decision is made on the basis of past experience in working with this problem. But you should note that it is always wise to select your coordinate system to suit the problem at hand. To accommodate our present choice, we note that

$$\dot{x} = \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi \tag{3}$$

$$\dot{y} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi \tag{4}$$

and

$$\ddot{x} = \ddot{r}\cos\varphi - 2\dot{r}\dot{\varphi}\sin\varphi - r\dot{\varphi}^2\cos\varphi - r\ddot{\varphi}\sin\varphi \tag{5}$$

$$\ddot{y} = \ddot{r}\sin\varphi + 2\dot{r}\dot{\varphi}\cos\varphi - r\dot{\varphi}^2\sin\varphi + r\ddot{\varphi}\cos\varphi \tag{6}$$

Then $\ddot{x}\cos\varphi + \ddot{y}\sin\varphi = 0$ gives us

$$\ddot{r} - r\dot{\varphi}^2 = 0 \tag{7}$$

and $\ddot{x}\sin\varphi + \ddot{y}\cos\varphi = 0$ is

$$r\ddot{\varphi} + 2\dot{r}\dot{\varphi} = 0 \tag{8}$$

These are the geodesic equations in polar coordinates and, from them, you can read off the Christoffel symbols

$$\Gamma^{\,r}_{rr} = 0; \qquad \Gamma^{\,r}_{r\varphi} = \Gamma^{\,r}_{\varphi r} = 0; \qquad \Gamma^{\,r}_{\varphi\varphi} = -r \tag{9}$$

$$\Gamma^{\,\varphi}_{rr} = 0; \qquad \Gamma^{\,\varphi}_{\varphi r} = \Gamma^{\,\varphi}_{r\varphi} = \frac{2}{r}; \qquad \Gamma^{\,\varphi}_{\varphi\varphi} = 0 \tag{10}$$

If there is a central mass, we need to modify (7) to include the central force

$$\ddot{r} - r\dot{\varphi}^2 = -\frac{Gm}{r^2} \tag{11}$$

while (8) remains the same. As usual, we may integrate (8) to obtain

$$r^2\dot{\varphi} = h \tag{12}$$

where h is the orbital angular momentum of the test particle. Then

$$\ddot{r} = \frac{h^2}{r^3} - \frac{Gm}{r^2} \tag{13}$$

When we multiply this equation by $\dot{r}$ we have something that is easily integrated and we find that

$$\frac{1}{2}\dot{r}^2 = -\frac{h^2}{2r^2} + \frac{Gm}{r} + E \tag{14}$$

where E is a constant of integration whose meaning you will readily perceive.

Our first interest is in finding the orbit, that is in $r(\varphi)$, and so we note that

$$\dot{r} = \frac{dr}{d\varphi}\,\dot{\varphi} = \frac{h}{r^2}\frac{dr}{d\varphi} \tag{15}$$

We then obtain

$$\frac{h^2}{2r^4}\left(\frac{dr}{d\varphi}\right)^2 = E - \frac{h^2}{2r^2} + \frac{Gm}{r} \tag{16}$$

Experience has shown that the variable $u = 1/r$ is a convenient one in this problem, so we introduce it and obtain (after some minor rearrangements) that

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 = K + \frac{2Gm}{h^2}u \tag{17}$$

where $K = 2E/h^2$. Surprisingly, perhaps, it is easier to solve this equation by first differentiating it. On doing that, we get

$$\frac{d^2u}{d\varphi^2} + u = \frac{Gm}{h^2} \tag{18}$$

This is an inhomogeneous, linear differential equation with constant coefficients. A general solution will be of the form of a particular solution, such as $u = Gm/h^2$, plus the solution of the homogeneous equation, $u = A\cos(\varphi - \varphi_0)$ where A and φ_0 are constants of integration. We make things look a bit nicer by introducing the new constant $e = Ah^2/(Gm)$. Then the orbit is given by

$$r = \frac{h^2/(Gm)}{1 + e\cos(\varphi - \varphi_0)} \tag{19}$$

If you know your conic sections, you will observe that e is the eccentricity and when $e < 1$ the curve is an ellipse. What has the cone got to do with the motion of a planet? One way to think about this is to say that there is no force but that the sun (in the case of the solar system) has deformed the two-dimensional space we are talking about into a conical shape that we can look at from our three-dimensional space. But in the cone, the intrinsic geometry is flat. If you make a cone out of paper and draw a closed orbit around it, you get an ellipse. Unwrap the cone and lie it flat and see what that orbit really is. Of course, in spacetime, the worldline has a helical character as it winds around the time axis

Finally, the point of closest approach to the sun, or perihelion, occurs at $\varphi = \varphi_0$ and of greatest distance, or aphelion, at $\varphi = \varphi_0 + \pi$ so that we have

$$r_{min} = \frac{h^2/(Gm)}{1+e}; \qquad r_{max} = \frac{h^2/(Gm)}{1-e} \tag{20}$$

The major diameter of the ellipse is

$$2a = r_{min} + r_{max} \tag{21}$$

so that the semi-major axis is

$$a = \frac{h^2}{Gm(1-e^2)} \tag{22}$$

The perihelion distance then turns out to be $q = r_{min} = a(1-e)$ which I ask you to work out for yourselves. All these things have been well measured for all the planets.

1.2 Back to Schwarzschild

The conical model space is singular at the apex. A more satisfactory model is a curved two-space, which, for our purposes, is provided by Schwarzschild's exterior metric. As we saw, that is

$$d\tau^2 = (1 - \frac{2m}{r})dt^2 - (1 - \frac{2m}{r})^{-1}\, dr^2 - r^2\, d\Omega \tag{23}$$

with

$$d\Omega = d\theta^2 + \sin^2\theta\, d\phi^2 \tag{24}$$

We may try to draw the same kind of closed orbit on the stretched rubber sheet that this leads to when we limit ourselves to the planar orbit of the theory. Then we again find something very close to an ellipse. However, the ellipse does not quite close on itself. in terms of the ellipse we found in the Newtonian case — equation (19) — it is as if φ_0, is slowly drifting. The rate of drift, known as the precession of the perihelion, may be calculated from the orbital equation we found in the last installment:

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = K + \frac{2m}{h^2}u + 2mu^3 \tag{25}$$

where K is the renamed constant, $(C^2 - 1)/h^2$. Again, $u = 1/r$ and h is the angular momentum over the mass, but now we have units with $G = 1$ and $c = 1$.

When we differentiate (25), we obtain

$$\frac{d^2u}{d\varphi^2} + u = \frac{m}{h^2} + 3mu^2 \tag{26}$$

We improve the appearance of this equation with the substitution $v = h^2u/m$. Then we have

$$\frac{d^2v}{d\varphi^2} + v = 1 + \epsilon v^2 \tag{27}$$

where

$$\epsilon = \frac{3m^2}{h^2} \tag{28}$$

Roughly speaking, ϵ measures the square of the ratio of the Schwarzschild radius to the radius (or semi-major axis) of the orbit. This equation is a little more difficult to solve than the classical one because of the nonlinear term v^2 but it can be done. As in many such cases, an approximate solution is more directly informative, at least when the approximation is of the right kind. Here we use what is called (singular) pertubation theory.

When $\epsilon = 0$, the equation reduces to the classical one whose solution — call it v_0 — we found readily. In fact, ϵ is very small for the planets, so we look for a solution close to the classical one by assuming an expansion of the form

$$v = v_0 + \epsilon v_1 + \epsilon^2 v_2 + ... \tag{29}$$

When we introduce this expression into the equation for v, we get

$$\frac{d^2v_0}{d\varphi^2} + v_0 - 1 + \epsilon[\frac{d^2v_1}{d\varphi^2} + v_1 - v_0^2] + \epsilon\{...\} + ... = 0 \tag{30}$$

When we let $\epsilon \to 0$, the only part of this equation that survives must be zero, that is,

$$\frac{d^2v_0}{d\varphi^2} + v_0 - 1 = 0. \tag{31}$$

But this is essentially the equation we solved in the classical problem so we know that the solution is

$$v_0 = 1 + e\cos(\varphi - \varphi_0) \tag{32}$$

Equation (31) does not involve ϵ and it remains true even in studying (30). Hence, when we introduce it into (30), we are left with

$$\epsilon[\frac{d^2v_1}{d\varphi^2} + v_1 - v_0^2] + \epsilon\{...\} + ... = 0 \tag{33}$$

We can cancel a factor of ϵ from this equation and again let $\epsilon \to 0$. This forces the result

$$\frac{d^2v_1}{d\varphi^2} + v_1 - v_0^2 = 0 \tag{34}$$

And now the plot thickens.

We find that

$$v_0^2 = 1 + 2e\cos(\varphi - \varphi_0) + e^2\cos^2(\varphi - \varphi_0) \tag{35}$$

which we may write as

$$v_0^2 = 1 + 2e\cos(\varphi - \varphi_0) + \frac{e^2}{2}\left(1 + \cos[2(\varphi - \varphi_0)]\right) \tag{36}$$

We then have the equation

$$\frac{d^2v_1}{d\varphi^2} + v_1 = 1 + \frac{e^2}{2} + 2e\cos(\varphi - \varphi_0) + \frac{e^2}{2}\cos[2(\varphi - \varphi_0)] \tag{37}$$

This is another inhomogeneous differential equation, but a particular solution is a bit harder to find than for the last inhomogeneous equation we encountered. It is not so hard to see that if we take as our particular solution

$$v_1^{(p)} = 1 + \frac{e^2}{2} + V_1^{(p)} \tag{38}$$

we get

$$\frac{d^2V_1^{(p)}}{d\varphi^2} + V_1^{(p)} = 2e\cos(\varphi - \varphi_0) + \frac{e^2}{2}\cos[2(\varphi - \varphi_0)] \tag{39}$$

And if we let

$$V_1^{(p)} = -\frac{e^2}{6}\cos[2(\varphi - \varphi_0)] + U_1^{(p)} \tag{40}$$

we obtain

$$\frac{d^2U_1^{(p)}}{d\varphi^2} + U_1^{(p)} = 2e\cos(\varphi - \varphi_0) \tag{41}$$

The solution of this equation is the nub of the problem: the term on the right of this equation is resonant. What this means is that the solution to this problem grows without bound as a function of φ, thus as a function of time as well. One says that the perturbation theory is singular for this reason. This sort of situation arises in classical celestial mechanics and in a large number of problems of a practical kind. We shall not go into the methods for dealing with these difficulties but shall settle for the growing solution on the understanding that if we do not use it too extensively we can find out what we need to know. Thus the morality of physics (according to G.E. Uhlenbeck): "Do it! If it blows up in your face, then you worry."

In this instance it does no good to try a solution to (41) that is proportional to $2e\cos(\varphi-\varphi_0)$ because, when you insert that on the left of (41), you get zero and you cannot balance the right hand side. In such cases, you are advised to try next something like $\varphi\cos(\varphi-\varphi_0)$. In fact, as you may verify by substitution,

$$U_1^{(p)} = e\varphi\sin(\varphi-\varphi_0) \tag{42}$$

is a solution of (41). A particular solution of the equation for v_1 is then

$$v_1^{(p)} = 1 + \frac{e^2}{2} - \frac{e^2}{6}\cos[2(\varphi-\varphi_0)] + e\varphi\sin(\varphi-\varphi_0) \tag{43}$$

To this, we can (and should) add a solution to the homogeneous equation as before. Such a solution may be useful in fitting to initial conditions but we need not concern ourselves with it here. What want is to see whether the additional terms from Einstein's theory makes for any interesting effects. The one we need to focus on is the resonant term since such terms always tend to change the character of the results and books have been written on this kind of problem. We shall settle here for a down-to-earth simplified version but, if ever you learn the so-called two-time method, go back and look at this problem.

As we are thinking about planetary orbits for now, we have the simplification that, for them, $e^2 << 1$. Then we may write an approximate solution to first order in e as

$$v = 1 + e\cos(\varphi-\varphi_0) + \epsilon[1 + e\varphi\sin(\varphi-\varphi_0)] + ... + \text{homogenous solution} \tag{44}$$

The terms represented by dots do not play much of a role and we shall not worry about them.

We may make the approximations

$$\cos(\epsilon\varphi) = 1 + \mathcal{O}(\epsilon^2); \qquad \sin(\epsilon\varphi) = \epsilon\varphi + \mathcal{O}(\epsilon^3) \tag{45}$$

Then we note that we may write that

$$\begin{aligned}\cos(\varphi-\varphi_0) + \epsilon\varphi\sin(\varphi-\varphi_0) &\approx \cos(\varphi-\varphi_0)\cos(\epsilon\varphi) - \sin(\varphi-\varphi_0)\sin(\epsilon\varphi) \\ &= \cos\left[\varphi - (\varphi_0 + \epsilon\varphi)\right] \end{aligned} \tag{46}$$

Then we write that

$$v = 1 + e\cos(\varphi-\hat{\varphi}_0) + \text{nonsingular terms} \tag{47}$$

This is basically the classical solution except that, instead of φ_0, we have

$$\hat{\varphi}_0 = \varphi_0 + \epsilon\varphi \tag{48}$$

Every time one orbit is completed and φ increases by 2π, $\hat{\varphi}_0$ increases by $4\pi\epsilon/3$. Therefore, for a planet whose orbital period is P, the rate at which the perihelion advances is

$$\frac{6\pi m^2}{h^2 P} \tag{49}$$

If we introduce the semi-major axis, and retreat to conventional units, this is

$$\frac{6\pi G m}{a c^2 P} \tag{50}$$

where $e^2 << 1$ and so is neglected.

The application of this formula is not so simple since there are other factors that may influence the position of the perihelia of planets, most importantly the influence of the other planets. But all those effects had been pretty well worked out before the end of the nineteenth century and the discrepancy for Mercury was known quite accurately. So it was a triumph for relativity when the answer came out right and that had been the major empirical support of general relativity, at least until it became clear that black holes do exist. Still, we may expect that there is more to come in the theory of gravity.