General Relativity

Installment VIII.

October 7, 2003

1 Homework

1.1 The Covariant Derivative

What we started to do in the last installment was to look at the introduction of the covariant derivative by way of the local flatness of the space. Suppose then that in an infinitesimal region around point P we may think in terms of the simple concepts of vectors and their bases. So a vector $\mathbf{a}$ can be written as in daily life as

$$\mathbf{a} = a^i \, \mathbf{e}_i \tag{1}$$

The coordinates at P are x^i and the coordinates at neighboring Q are $x^i + dx^i$. The change in $\mathbf{a}$ in going from P to Q is

$$d\mathbf{a} = a^i{}_{,j} \, dx^j \, \mathbf{e}_i + a^i d\mathbf{e}_i \tag{2}$$

Remember that

$$a_{i,j} = \frac{\partial a^i}{\partial x^j} \tag{3}$$

In (**??**) there are two terms. One represents the actual change in the vector as we go from P to Q. The other expresses the change in the component representation that results from possible changes in the basis vectors in going from P to Q. The latter term is not there when we are using Cartesian coordinates in a flat space. As we shall see, it is also not there in the flat space time of special relativity. (You may have read that cosmologists think that the universe is flat, but that is a rough statement and does not allow for the bumps and local warps in spacetime that make our lives so interesting.)

The issue is then that we need to say what $d\mathbf{e}_i$ is. Let us represent it as a linear combination of the basis vectors. The coefficients in this linear combination will depend on which $d\mathbf{e}_i$ we are dealing with; that is, the coefficient will carry the index i. We also expect that $d\mathbf{e}_i$ will be proportional to dx^j to good approximation. (More precisely, there will be higher order terms — quadratic and cubic and so on in dx^j — and for a more global study we need to go to the tangent space. But this local, linear treatment is adequate for our purposes. So we may write

$$d\mathbf{e}_i = \Gamma_{ij}^{\,k}\,\mathbf{e}_k\,dx^j \tag{4}$$

The coefficients Γ naturally have three indices, one for the sum over the dx^j, one for the combination of the $\mathbf{e}_k$ and one to tell us which $\mathbf{e}_i$ we are referring to. On using (**??**) we have

$$d\mathbf{a} = a^i_{\ ,j}\,dx^j\,\mathbf{e}_i + a^i\Gamma_{ij}^{\,k}\,\mathbf{e}_k\,dx^j \tag{5}$$

With the renaming of a dummy index this is written as

$$d\mathbf{a} = \left(a^k_{\ ,j} + \Gamma_{ij}^{\,k}\,a^i\right)\mathbf{e}_k\,dx^j \tag{6}$$

The covariant derivative is then defined as

$$a^k_{\ ;j} = a^k_{\ ,j} + \Gamma_{ij}^{\,k}\,a^i \tag{7}$$

For homework: Figure out what the $\Gamma_{ij}^{\,k}$ are within the context of what we have in this handout. Make use of the fact that

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j \tag{8}$$

The manipulations that you need are exemplified in Installment VII, equations (32)-(37).

For homework in 4000: (a) Using results from the 3000 homework and the definition of the covariant derivative of a second rank covariant tensor given in class, show that $g_{ij:k} = 0$. (b) Try to show that the covariant derivative of a vector, say $a^i_{\ ;j}$, is a mixed tensor of second rank. Do not worry if you don't get this one. Indeed look it up if you want to. But do learn how it is done.