General Relativity

Installment XII.

October 13, 2003

1 Topics in Tensor Calculus

1.1 Parallel Transport

There seem to remain questions among you about the meaning of covariant differentiation. So let us look another facet of this process. Remember though that if you want to differentiate a vector field, you have in effect to look at the difference in the field at two points that are close by, separated by coordinate differentials, dx^i. To get that difference you need to move one of the vectors over to the other to make the comparison. This is no problem in flat space but it is more subtle in curved space. So we need to define what we mean by moving a vector from place to place in a manner parallel to itself. To express this in formulae, we need to be aware that components of a vector will change as we move it, not only because the space is curved but also because the basis vectors typically change. And even if you have a good definition of parallel transport, when you take a vector from one point to another in a curved space, keeping it parallel to itself as you go, you may wind up with a different end product depending on the path you take.

Recall that the absolute differential of a vector **a** is, in terms of components,

$$Da^i = da^i + \Gamma^{\,i}_{jk} a^j dx^k = a^i{}_{,k} dx^k + \Gamma^{\,i}_{jk} a^j dx^k \tag{1}$$

where the comma denotes partial differentiation as usual. The second term takes care of the differences in the basis vectors at two points. The derivative along a curve $x^i = x^i(s)$ is then

$$\frac{Da^i}{Ds} = a^i{}_{,k} \frac{dx^k}{ds} + \Gamma^{\,i}_{jk} a^j \frac{dx^k}{ds} = \frac{da^i}{ds} + \Gamma^{\,i}_{jk} a^j \frac{dx^k}{ds} \tag{2}$$

where Ds and ds are really the same thing, the difference in s between the neighboring points. When you move **a** along the curve so that

$$\frac{da^i}{ds} + \Gamma^{\,i}_{jk} a^j \frac{dx^k}{ds} = 0 \tag{3}$$

then you are keeping it parallel to itself as you go. To find what the vector is at any point once it has been transported in this way, you need to solve the indicated differential equations. This is doable since the problem is linear, but it may take some work.

Now suppose that the vector field that you want to transport along the curve is its tangent vector, $\mathbf{v}$. Then the condition of parallel transport is

$$\frac{dv^i}{ds} + \Gamma^{\ i}_{jk} v^j v^k \ = 0 \tag{4}$$

where we need to recall that $v^i = dx^i/ds$. But this is the equation for a geodesic and so we conclude that, along a geodesic, the tangent vector is parallel to itself. Look back at equation (11) of Installment XI. You should convince yourself that the geodesic equation can be written as

$$\nabla_{\mathbf{v}} \mathbf{v} = \mathbf{0} \tag{5}$$

Along a geodesic, the tangent vector moves parallel to itself. This is a property of straight lines in flat space and that is why the geodesics are considered the natural generalization to curved space of straight lines. If time permits, we shall see how these things can be used in some problems. But now we must go on to the next step that is needed to set up the Einstein equations.

1.2 Second Covariant Derivative

We saw that the metric is like a potential. In the classical theory of gravity, the potential is derived from the Poisson equation

$$\nabla^2 \phi = 4\pi\, G\rho \tag{6}$$

where ρ is the density of matter and G is Newton's gravitational constant. To extend this theory into relativity, we shall have to deal with second derivatives. So let us bite the bullet.

We have a four-vector a — no boldface for four-vectors in this game — with covariant components a_λ (covariant for ease of typing). What is its second derivative? Staying in spacetime with Greek indices, we write

$$a_{\lambda;\mu;\nu} = (a_{\lambda;\mu})_{;\nu} \tag{7}$$

Since $a_{\lambda;\mu}$ is a second rank covariant tensor, we recall the rule 'co below and minus.' Then

$$a_{\lambda;\mu;\nu} = (a_{\lambda;\mu})_{,\nu} - \Gamma^{\ \rho}_{\nu\mu} a_{\lambda;\rho} - \Gamma^{\ \rho}_{\nu\lambda} a_{\rho;\mu} \tag{8}$$

As we recall, a Christoffel symbol is needed for each index.

Slogging on, we get the second covariant derivative

$$a_{\lambda;\mu;\nu} = (a_{\lambda,\mu} - \Gamma^{\ \rho}_{\lambda\mu}\, a_\rho)_{,\nu} - \Gamma^{\ \rho}_{\nu\mu}(a_{\lambda,\rho} - \Gamma^{\ \sigma}_{\lambda\rho} a_\sigma) - \Gamma^{\ \rho}_{\nu\lambda}(a_{\rho,\mu} - \Gamma^{\ \sigma}_{\rho\mu} a_\sigma) \tag{9}$$

$$= a_{\lambda,\mu,\nu} - \Gamma^{\ \rho}_{\lambda\mu} a_{\rho,\nu} - \Gamma^{\ \rho}_{\mu\nu} a_{\lambda,\rho} - \Gamma^{\ \rho}_{\nu\lambda} a_{\rho,\mu} - \left[a_\rho \Gamma^{\ \rho}_{\lambda\mu,\nu} - \Gamma^{\ \rho}_{\nu\mu}\, \Gamma^{\ \sigma}_{\lambda\rho} a_\sigma - \Gamma^{\ \rho}_{\nu\lambda} \Gamma^{\ \sigma}_{\rho\mu} a_\sigma\right] \tag{10}$$

1.3 The Commutator of Covariant Derivatives

When you look at the expression for the second covariant derivative, you may not be surprised to learn that, unlike partial derivatives, covariant derivatives do not commute: the order in which you take the derivatives matters. When you repeat the differentiations in the other order, you get

$$a_{\lambda;\nu;\mu} = a_{\lambda,\nu,\mu} - \Gamma^{\ \rho}_{\lambda\nu} a_{\rho,\mu} - \Gamma^{\ \rho}_{\nu\mu} a_{\lambda,\rho} - \Gamma^{\ \rho}_{\mu\lambda} a_{\rho,\nu} - \left[a_\rho \Gamma^{\ \rho}_{\lambda\nu,\mu} - \Gamma^{\ \rho}_{\mu\nu} \Gamma^{\ \sigma}_{\lambda\rho} a_\sigma - \Gamma^{\ \rho}_{\mu\lambda} \Gamma^{\ \sigma}_{\rho\nu} a_\sigma\right] \tag{11}$$

The effect of changing the order of differentiation is seen in the evaluation of $a_{\lambda;\mu;\nu} - a_{\lambda;\nu;\mu}$. This is the difference of two third rank covariant tensors and, amazingly enough, it is expressible as a linear combination of the components of a. The intermediate steps are easy enough to work out but too difficult to type, so here is the result:

$$a_{\lambda;\mu;\nu} - a_{\lambda;\nu;\mu} = R^{\rho}_{\cdot\lambda\mu\nu} a_\rho \tag{12}$$

where

$$R^{\rho}_{\cdot\lambda\mu\nu} = -\Gamma^{\ \rho}_{\lambda\mu,\nu} + \Gamma^{\ \rho}_{\lambda\nu,\mu} + \Gamma^{\ \sigma}_{\nu\lambda} \Gamma^{\ \rho}_{\sigma\mu} - \Gamma^{\ \sigma}_{\mu\lambda} \Gamma^{\ \rho}_{\nu\sigma} \tag{13}$$

is called the Riemann tensor or the curvature tensor.

1.4 The Transition from Newton

We saw that the orbits produced by an external force — at least one coming from a potential — could be duplicated by geodesics in a suitably curved space. In the Newtonian limit, the appropriate geometry was given a metric with $g_{00} = 1 + 2\phi$ where ϕ is the potential. When we studied the motion of a particle in a freely falling elevator, we also saw that the spatial variations in the force are significant. In the elevator, the displacement, ξ^i, of a particle from the center satisfied the equation

$$\ddot{\xi}^i = -E_{ij} \xi^j \tag{14}$$

where

$$E_{ij} = \frac{\partial \phi}{\partial x^i \partial x^j} \tag{15}$$

We also discussed the similarity of this equation to the equation for geodesic deviation on $\mathcal{S}^2$. To see how this idea really works, we need the formula for geodesic deviation in a curved space. The derivation of the appropriate formula is too long for this discussion, so we shall simply state the results. For a nice discussion of these matters, I would recommend the book 'General Relativity and Cosmology' by Jayant Narlikar. (I assume that you are reading in Kenyon as well and have looked at Fig. 3.8 on p. 33. You ought to aim to have read the first six chapters by the midterm since much of this is covered in the notes as well.)

Consider geodesics diverging from a point P. Along each of these geodesics, the distance from P is s. For s not too large, consider a pair of geodesics not too far from each other.

Suppose that Q_1 on one of the geodesics is at the same distance from P as is point Q_2 on the other. Let Q_1 be at x^i and Q_2 be at $x^i + \xi^i$. Then it is possible to establish that

$$\frac{d^2\xi^i}{ds^2} = R^i_{jk\ell} v^j v^k \xi^\ell \tag{16}$$

where $\mathbf{v}$ with components v^i is the tangent vector on the first geodesic. What Einstein did was to use the curvature tensor to construct a suitable second rank tensor to serve the same purpose as E_{ij}. That is, he replaced E^i_j by $R^i_{k\ell j} v^j v^k$. Thus he captured not only the basic effect of the equivalence between uniform accelerations and uniform gravity fields, but also could describe the basic differential gravitational forces. Then he needed to give a method for determining the curvature tensor.

Now

$$\mathrm{Trace}\,\mathbb{E} = E^i_i = \triangle\phi = 0, \tag{17}$$

where $\triangle = \nabla^2$ is the Laplacian. Hence the trace of the surrogate for E_{ij} is going to vanish and this statement replaces Laplace's equation for the gravitational potential. That is a good place to start with the new equation that is to replace Poisson's equation.

Queries

What is a null geodesic?

What is there that we have discussed so far that suggests that nothing can go faster than the speed of light?

When you see the astronauts in orbit on the tv, you observe that objects in the space ship seem to float in the air. Why is that?

Do covariant derivatives commute in flat space?