General Relativity

Installment IV.

September 17, 2003

1 Fictitious Forces Continued

1.1 An Expanding Medium

The universe is expanding (at least locally), so this is an especially interesting phenomenon for us. In this brief look, we shall study the classical version using Newton's gravitational force. A discussion of Newtonian cosmology can be found in H. Bondi's 'Cosmology.'

We select an origin of coordinates and consider an object — say a galaxy — at position $\mathbf{r}$ with respect to that origin. Assume that the expansion is uniform so that, apart from possible disturbances in that uniform expansion, all distances are continuously increased by a time-dependent scale factor $R(t)$. The position of the observed (or test) object is then

$$\mathbf{r} = R(t)\mathbf{x} \tag{1}$$

where $\mathbf{x}$ is the position of the galaxy in the expanding coordinate system.

We shall not proceed as formally as we did in the case of the rotating coordinate system since you have already seen how that goes. Let us just differentiate (1) with respect to time to obtain the velocity of the galaxy in the inertial frame as

$$\frac{d\mathbf{r}}{dt} = R(t)\frac{d\mathbf{x}}{dt} + \dot{R}\mathbf{x} \tag{2}$$

where the dot means time derivative and $\dot{R}$ is the rate of change of the scale factor. This can be rewritten as

$$\frac{d\mathbf{r}}{dt} = R(t)\frac{d\mathbf{x}}{dt} + H\mathbf{r} \tag{3}$$

where

$$H(t) = \frac{\dot{R}}{R} \; . \tag{4}$$

This relation between the velocities as seen in the two frames is the analogue of (21) in Installment III. In cosmology, the present value of $H(t)$ is called the Hubble constant. The quantity $H\mathbf{r}$ is the expansion velocity of the background medium at position $\mathbf{r}$ and it is called the Hubble velocity. The velocity of the galaxy relative to the expanding background medium, $R\dot{\mathbf{x}} = \dot{\mathbf{r}} - H\mathbf{r}$, is called the peculiar velocity. Relativists call $\dot{\mathbf{x}}$ the coordinate velocity to distinguish it from the (physical) peculiar velocity, $R\dot{\mathbf{x}}$. In that sense, an angular velocity may also be called a coordinate velocity.

When we differentiate (3), we get

$$\frac{d^2\mathbf{r}}{dt^2} = R\frac{d^2\mathbf{x}}{dt^2} + \dot{R}\frac{d\mathbf{x}}{dt} + \dot{H}\mathbf{r} + H\frac{d\mathbf{r}}{dt} \tag{5}$$

Now we use (3) and resort completely to the dot notation to obtain

$$\ddot{\mathbf{r}} = R\ddot{\mathbf{x}} + 2HR\dot{\mathbf{x}} + \dot{H}\mathbf{r} + H^2\mathbf{r} \tag{6}$$

The four terms on the right of this equation are (1) the acceleration in the expanding frame, (2) a drag term on the peculiar velocity analogous to the Coriolis term, (3) an acceleration like the Euler acceleration, and (4) something like the centrifugal term. However, in this case we have the simplification that

$$\dot{H} = \frac{\ddot{R}}{R} - H^2 \tag{7}$$

so that (6) becomes

$$\ddot{\mathbf{r}} = R\ddot{\mathbf{x}} + 2HR\dot{\mathbf{x}} + \frac{\ddot{R}}{R}\mathbf{r} \ . \tag{8}$$

A nice way to rewrite this is

$$\ddot{\mathbf{r}} = \frac{1}{R}\left(R^2\dot{\mathbf{x}}\right)^{\cdot} + \ddot{R}\mathbf{x} \ . \tag{9}$$

To complete this story, let specify $R(t)$ for the case of Newtonian cosmology. Having chosen a radius vector $\mathbf{r}$, we may say that the background outflow is opposed by the gravitational pull of the mass M interior to the sphere of radius $r = |\mathbf{r}|$. We express that as

$$\ddot{\mathbf{r}} = -\frac{GM\mathbf{r}}{r^3} \tag{10}$$

where $\mathbf{r}/r$ is the unit vector in the direction of $\mathbf{r}$. When we start off the motion described in this way with no peculiar velocity, the solution tells how the background expansion is proceeding. If the medium is uniform with density ρ, we have

$$M = \frac{4\pi}{3}\rho r^3 \ . \tag{11}$$

This interior mass is expanding since it is part of the background medium. So as, r grows, ρ dimishes so as to keep M constant. Since r grows like R, ρ decreases like R^{-3}. Then

$$\ddot{\mathbf{r}} = -\frac{4\pi}{3}G\rho\mathbf{r} = -\frac{4\pi}{3}G\rho R\mathbf{x} \tag{12}$$

When we introduce this into (9), and choose the background dynamics according to

$$\ddot{R} = -\frac{4\pi}{3} G\rho R \tag{13}$$

we get

$$\left(R^2\dot{\mathbf{x}}\right)^{\cdot} = 0 \ . \tag{14}$$

Therefore

$$R^2\dot{\mathbf{x}} = \mathbf{constant}. \tag{15}$$

The magnitude of the peculiar velocity of a galaxy decreases as R^{-1}. This is a result of the fictitious drag force in an expanding medium. When you let the air out of a tire, the air expands and the peculiar velocities of the constituent molecules diminish. Since the temperature is a measure of the mean peculiar velocity, you expect the gas to cool. Of course, it does, and we shall later discuss how to estimate the cooling rate. We merely state for now that the universe must have been hotter in the past.

When we study the relativistic case, the expansion will be seen as an expansion of space itself. Waves traveling through space will have their wavelengths stretched like $R(t)$. Quantum mechanics tells us that, as the wavelength associated with a particle grows with R, its momentum diminishes like R^{-1}, in agreement with our present result. Photons too have their wavelengths stretched and this gives rise to the famous cosmological redshift. However you look at it, the acceleration produces an effective force but, in our discussion so far, we have thrown in gravity as if it were a real force. This is what we want to change.

2 Equivalence of Acceleration and Gravity

Suppose that you are in a box, or elevator, and cannot look out. Figure 1 shows two possible situations. In (a) the elevator is on the ground and subject to the downward acceleration of the earth's gravity, $\mathbf{g}$. In (b), the elevator is in space, far from any sources of gravity, and being towed from its ceiling with an acceleration $\mathbf{a} = \mathbf{g}$ in a direction that we shall call upward. Can you tell which situation you are in without looking outside? You may ask for any equipment you need for this determination, except perhaps for a mobile phone.

You could ask for a laser and send a beam across the elevator from one wall to the one opposite. In case (b), the beam would hit the opposite wall at a lower height than the one at which it started. Einstein's theory of gravity started from the premise that the two situations are equivalent and that a downward deflection would be observed in case (a) also. He had already demonstrated the equivalence of matter and energy, so it was natural to suppose that the light beam would be bent downward by gravity acting on its weight. This prediction was in fact later confirmed by the study of the deflection of light by the sun during an eclipse. However, there is a way that the distinction between cases (a) and (b) could be made. No real gravitational field is constant in space and the difference in gravitational force from place to place produces tidal distortions. This is what we shall look into next.

2.1 Force and Potential

** The material in this section can be skipped and in any case need not be worried over. **

In equation (I-11) we referred to the notion of a force derivable from a potential on the assumption that you knew a little about such things. I assume that you know that a body like the earth has a gravitational potential $\phi(\mathbf{x})$ where $\mathbf{x}$ is the position vector with respect to the earth's center. In principle, ϕ could depend on time, but let us leave out such refinements for now. An equation like $\phi(\mathbf{x}) = constant$ defines surfaces in space and these have dimension one less than the dimension of the space itself. The surfaces are then said to have codimension one.

Consider two points $\mathbf{x}$ and $\mathbf{x} + \delta\mathbf{x}$ close to each other. The difference in potential between the two points is

$$\delta\phi = \phi(\mathbf{x} + \delta\mathbf{x}) - \phi(\mathbf{x}) = \frac{\partial\phi}{\partial x^i}\delta x^i + ... \tag{16}$$

where x^i are the components of $\mathbf{x}$ (and likewise for the δx^i). Let us introduce a quantity

$$d\phi = \frac{\partial\phi}{\partial x^i}dx^i \tag{17}$$

that is not to be thought of as infinitesimal, though its form may have been inspired by that of $\delta\phi$, which is infinitesimal. Many people have a bit of trouble with this concept, so don't worry if you are worried. Now the differential forms make up an abstract vector space in which the dx^i are treated as basis vectors. Then the $\partial\phi/\partial x^i$ are considered components of what is called the gradient of ϕ. If you already know what a gradient is from your vector analysis (and I hope you do), then you need not worry. I include this to give some idea of why the components of gradients have their indices down. Also, I wanted you to have a sense of *deja vu* when we see some similar things later on. There is a book by Bill Burke called 'Spacetime, Geometry and Cosmology' (not to be confused with his geometry book) that tries to tell these things like they are in terms aimed at freshpersons. It is pleasant reading but it is pretty incomprehensible, but have a look at it.

2.2 Dynamics in a Falling Elevator

Suppose you are in an elevator well above the earth's surface and falling freely. Imagine that it is not tumbling — it it sufficiently symmetric laterally that the force of gravity from the earth does not exert a torque. We want to look at the dynamics of this falling.

So you are in a box falling in a potential *per unit mass* $\phi(\mathbf{x})$. Your position vector wherever you are in the box has components x^i. Your equation of motion in component form is

$$\ddot{x}^i = \delta^{ij}\frac{\partial\phi}{\partial x^j} \tag{18}$$

where δ^{ij} follows the usual rules of the Kronecker delta: it is one if i and j are the same and zero if they are not.

You may wonder why I have written Newton's second law in this seemingly odd way. The real reason is that I am being unnecessarily pedantic The fact is that it is more natural in some sense to express gradients in terms of the reciprocal basis because, when you take a derivative of a function with respect to x^i, the index winds up downstairs. We emphasize this by using a notation that people use in relativity (and some other subjects), namely

$$\frac{\partial\phi}{\partial x^i} = \phi_{,i} \tag{19}$$

In both forms of this partial differential equation, the index is a lower one. Then the gradient of ϕ is written as

$$\nabla\phi = \phi_{,i}\mathbf{e}^i \tag{20}$$

where $\mathbf{e}^i$ are the reciprocal basis vectors, of which we shall speak farther down the line. On the other hand, we have been writing the coordinate components as x^i, so I used the Kronecker delta in (18) to make everything look grammatical. I apologize for this unnecessary fussiness. If you like, just replace (18) by

$$\ddot{\mathbf{x}} = -\nabla\phi \tag{21}$$

Let $\mathbf{x}_{(0)}$ be the position vector of the center of the box with components $x^i_{(0)}$. The Taylor series for ϕ around the box center is

$$\phi(\mathbf{x}) = \phi(\mathbf{x}_{(0)}) + \phi_{,i}(\mathbf{x}_{(0)})\xi^i + \frac{1}{2}\phi_{,i,j}(\mathbf{x}_{(0)})\xi^i\xi^j + ... \tag{22}$$

where

$$\xi^i = x^i - x^i_{(0)} \tag{23}$$

When apply (21) at $\mathbf{x}_{(0)}$ and subtract it from (21) itself, we obtain

$$\ddot{\xi}^i = -\xi^k\delta^{ij}\phi_{,j,k}(\mathbf{x}_{(0)}) \tag{24}$$

This equation tells how the coordinates with respect to the center of the box are changing.

If we take as the potential

$$\phi(\mathbf{x}) = -\frac{GM}{r}\ , \tag{25}$$

we need to do some differentiating of this to make (24) explicit. With

$$r = \sqrt{\delta_{ij}x^ix^j} \tag{26}$$

we find

$$\left(\frac{1}{r}\right)_{,k} = -\frac{1}{2r^3}\left(\delta_{ij}x^ix^j\right)_{,k} \tag{27}$$

Since $x^i_{\cdot,j} = \delta^i_{\cdot j}$, we find that

$$\phi_{,i} = \frac{GM}{r^3} x_i \tag{28}$$

Taking the second derivative proceeds in a similar way and it is left for homework. We get

$$\phi_{,i,j} = \frac{GM}{r^3}\left(\delta_{ij} - \frac{3x_i x_j}{r^2}\right). \tag{29}$$

We then conclude that

$$\ddot{\xi}^i = -\frac{GM}{r^3_{(0)}}\left[\delta^i_{\cdot j} - \frac{3x_j x^i}{r^2}\right]_{(0)} \xi^j \ . \tag{30}$$

Suppose we work in spherical coordinates, which is the natural choice for this spherical problem. The box falls down a radius vector and the coordinates of its center are $x^1 = r_{(0)}$, $x^2 = 0$, $x^3 = 0$. The equation of motion of a displacement from the center of the box becomes

$$\ddot{\xi}^i = -E_{ij}\xi^j \tag{31}$$

where

$$E_{ij} = -\frac{GM}{r_0^3}\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{32}$$

This equation describes how the differential gravitational force deforms things. That effect is called a tide and, as you see, the tidal interaction varies as the inverse cube of the distance to the object.