General Relativity

Installment V.

September 22, 2003

1 Moving on the Earth's Surface

1.1 Relative Motion

What we saw in the last installment is that, in the box falling toward the earth, the horizontal motion of a particle not at the center of the box is toward the center of the box. If the horizontal displacement of the particle from box center is ξ, the equation for the horizontal motion with respect to the box is

$$\ddot{\xi} = -\frac{GM}{r^3}\xi \; . \tag{1}$$

The quantity GM/r^3 is a frequency squared and is a measure of the strength of the tidal force. This gathering in the horizontal is reminscent of motion on the earth.

Consider two travelers on earth starting out at the same time from the equator, but at slightly different longitudes. Each heads due north in heir own frame. If their northward velocities are the same, they will also be moving toward each other unintentionally till they meet at the north pole. This is a similar sort of motion to what we saw in the tidal case. Let us see how to treat this.

Suppose that the earth's surface is a perfect sphere with radius R. This geometrical object is called $\mathcal{S}^2$ and it will prepare us for thinking about Einstein's model of the universe, $\mathcal{S}^3$. As in the tidal problem, let us consider that the two travelers are near to each other and that they are moving along their respective meridians. Let their longitudes be ϕ and $\phi + d\phi$ and their (common) latitude be θ. The distance between them is $R\sin\theta d\phi$. This is measured on the earth's surface and perpendicularly to the meridians. This distance is small; let us express it as $\xi = R\sin\theta d\phi$. How does ξ vary as the travelers move north?

Along a meridian, the distance moved in a displacement $d\theta$ in latitude is $ds = Rd\theta$. So we calculate that

$$\frac{d^2\xi}{ds^2} = \frac{1}{R^2}\frac{d^2\xi}{d\theta^2} = \frac{1}{R^2}\frac{d^2(R\sin\theta d\phi)}{d\theta^2} = -\frac{1}{R}d\phi\sin\theta \ . \tag{2}$$

Hence, we have (essentially) what is called Jacobi's formula

$$\frac{d^2\xi}{ds^2} = -\frac{1}{R^2}\xi \ . \tag{3}$$

which is of the same form as (**??**). However, in this case, the coefficient which determines the way the two travelers approach one another is the radius of the earth, a (manifestly) geometric property, related to its curvature. This is a special case of a formula for what is called geodesic deviation that we shall derive in a higher dimensional example later on.

1.2 Geometry of $\mathcal{S}^2$

We can study the geometry of a space either by acting as someone living in the space or by being someone in a space of higher dimension looking at the space in question. Let us look at distances on the surface of the two-dimensional sphere, a.k.a. $\mathcal{S}^2$, from our standpoint of the three-dimensional Euclidean space, $\mathbb{E}^3$. Suppose two points on $\mathcal{S}^2$, one at x, y, z and the other at $x + dx, y + dy, z + dz$, are separated by a very small distance ds. We have

$$ds^2 = dx^2 + dy^2 + dz^2 \ . \tag{4}$$

But the points are both on the sphere. So we require that

$$x^2 + y^2 + z^2 = R^2 \tag{5}$$

where R is the radius of the sphere. From this, we find that $xdx + ydy + zdz = 0$ so that

$$dz^2 = \frac{(xdx + ydy)^2}{z^2} = \frac{x^2dx^2 + y^2dy^2 + 2xydxdy}{R^2 - (x^2 + y^2)} \tag{6}$$

The formula for ds^2 becomes

$$ds^2 = \frac{(R^2 - y^2)dx^2 + (R^2 - x^2)dy^2 + 2xydxdy}{R^2 - (x^2 + y^2)} \tag{7}$$

And so we have a distance formula for two neighboring points on $\mathcal{S}^2$. But a resident of $\mathcal{S}^2$ might not like using x and y and would prefer some more convenient coordinates.

Nice coordinates on $\mathcal{S}^2$ are latitude θ and longitude ϕ defined by

$$x = R\sin\theta\cos\phi \qquad\qquad y = R\sin\theta\sin\phi \tag{8}$$

From this we find that

$$dx = R\cos\theta\cos\phi d\theta - R\sin\theta\sin\phi d\phi \tag{9}$$

$$dy = R\cos\theta\sin\phi d\theta - R\sin\theta\cos\phi d\phi \tag{10}$$

Now we may calculate $(dx)^2, (dy)^2$ and $dxdy$ and substitute and rearrange to obtain

$$ds^2 = R^2 d\theta^2 + sin^2\theta d\phi^2, \tag{11}$$

as we might have found directly. However, when we come to look at $\mathcal{S}^3$, we shall find the present approach more relaxing.

We have found the distance on $\mathcal{S}^2$ between two points separated in three dimensional Euclidean space by the infinitesimal vector $d\mathbf{x}$. In the latter space, we may write $d\mathbf{x} = dx^i\mathbf{e}_i$ where the i here is $1, 2, 3$. In the Euclidean case, we have that the distance is the square root of $d\mathbf{x} \cdot d\mathbf{x} = dx^i dx^j \mathbf{e}_i \cdot \mathbf{e}_j = dx^i dx^j g_{ij}$ where $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$. Though we do not have the same situation in $\mathcal{S}^2$ — the vectors at a point are in the space tangent to $\mathcal{S}^2$ at that point, not in $\mathcal{S}^2$ itself — we see that the formula for distance in $\mathcal{S}^2$ looks like the one in Eucldean space. So, instead of calculating g_{ij} from an inner product as before, we specify the nature of the geometry by simply specifying g_{ij}. That is, we give what is called a metric on our space.

For $\mathcal{S}^2$, the result about the distance between two points that we found may be summarized as

$$ds^2 = g_{ij} dx^i dx^j \tag{12}$$

where $x^1 = \theta$, $x^2 = \phi$ and

$$g_{ij} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix} . \tag{13}$$

This g_{ij} does what our old one did. For instance, it can be used to lower an index as in $dx_i = g_{ij}dx^j$ so that $ds^2 = dx_i dx^i$. And we define the inverse metric by

$$g^{ik}g_{kj} = \delta^i_{\cdot j} \tag{14}$$

as before. (What is g^{ij} for $\mathcal{S}^2$?)

1.3 A Trip

Let us think about taking a trip over the surface of the earth from point P to point Q. The distance covered is

$$s(P - Q) = \int_P^Q ds \tag{15}$$

which evidently depends on the path taken. But

$$ds = \sqrt{g_{ij} dx^i\, dx^j} \tag{16}$$

so the path with a stationary length is the one for which

$$\delta s(P - Q) = \delta \int_P^Q \sqrt{g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}} ds = 0\ . \tag{17}$$

Since

$$\sqrt{g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}} = 1, \tag{18}$$

we may replace this quantity in the integrand by any power of it. All these choices in fact lead to same result. (If you want to pursue this, look in 'General Relativity' by H. Stephani, a Cambridge Press publication suitable for a slightly more advanced course than ours. The equivalence of the different approaches is shown there.) It should not be surprising that, if the integral of a quantity equal to one is stationary, the integral of its square is also stationary. So let us look instead for the path such that

$$\delta \int_P^Q \left(g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} \right) ds = 0 \ . \tag{19}$$

In fact, there is a reason in relativity to consider this case rather than the square root, as we shall see when we get there. Then

$$\int_P^Q \left[\frac{\partial g_{ij}}{\partial x^r}\delta x^r \dot{x}^i \dot{x}^j + g_{ij}\dot{x}^i \delta \dot{x}^j + g_{ij}\dot{x}^j \delta \dot{x}^i \right] = 0 \ . \tag{20}$$

where $dx^i/ds = \dot{x}^i$. This may also be written as

$$\int_P^Q \left[\frac{\partial g_{ij}}{\partial x^r}\delta x^r \dot{x}^i \dot{x}^j + 2 g_{ij}\dot{x}^i \delta \dot{x}^j \right] = 0 \ . \tag{21}$$

We integrate by parts to get rid of the s-derivatives on the δx^ℓ and we have

$$\int_P^Q \left[\frac{\partial g_{ij}}{\partial x^r}\delta x^r \dot{x}^i \dot{x}^j - 2\frac{d}{ds}\left(g_{ij}\dot{x}^i \right) \delta x^j \right] = 0 \ . \tag{22}$$

Now we may rename some dummy indices to obtain

$$\int_P^Q \left[\frac{\partial g_{ij}}{\partial x^r}\dot{x}^i \dot{x}^j - 2\frac{d}{ds}\left(g_{ir}\dot{x}^i \right) \right] \delta x^r = 0 \ . \tag{23}$$

The perturbation δx^r represents the difference in position between the route being tested and a neighboring one that we are examining for its promise as a shorter alternative. Therefore, δx^r is arbitrary. And, since the perturbed integral must vanish for any (small) δx^r if we are to obtain a true extremum, the integrand must vanish. Hence

$$-2\frac{d}{ds}\left(g_{kr}\dot{x}^k \right) + \frac{\partial g_{k\ell}}{\partial x^r}\dot{x}^k \dot{x}^\ell = 0. \tag{24}$$

On carrying out the derivative of the product, we get

$$-2\dot{x}^i \frac{dg_{ij}}{ds} - 2g_{ij}\ddot{x}^i + \frac{\partial g_{ik}}{\partial x^j}\dot{x}^i \dot{x}^k = 0 \tag{25}$$

Then, since $dg_{ij}/ds = (\partial g_{ij}/\partial x\ell)\dot{x}^\ell$, we may write the equation for the optimal path as

$$2g_{ij}\ddot{x}^i = -2g_{ji,\ell}\dot{x}^\ell\dot{x}^i + g_{ik,j}\dot{x}^i\dot{x}^k \tag{26}$$

where we recall the notation from a previous installment, $[\cdot]_{,i} = \partial[\cdot]/\partial x^i$, where $[\cdot]$ is whatever you like.

Now we multiply by g^{jr}. Since $g^{jr}g_{ij} = \delta^r_{.i}$, we see that

$$\ddot{x}^r = \frac{1}{2}g^{jr}\left[g_{ik,j} - g_{ji,k} - g_{jk,i}\right]\dot{x}^i\dot{x}^k \tag{27}$$

Finally, we may introduce the three-index quantity

$$\Gamma^k_{ij} = \frac{1}{2}g^{rk}\left[g_{ir,j} + g_{jr,i} - g_{ij,r}\right] \tag{28}$$

to give us

$$\ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0\ . \tag{29}$$

The solution to this equation gives us stationary paths in the sense that a slight deviation from such a path makes a very small change in the distance traveled.

If you followed the section on Lagrange's equations, you may recognize the similarity of this derivation to the one there. Even if you skipped that part, you may see a resmblance of this equation to Newton's second law. Of course, the dots here refer to spatial derivatives, but that distinction will no longer be made once we get into spacetime.