General Relativity

Installment VI.

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1 Spherical Coordinates

In the last Installment we saw how to describe motion along (the generalization of) "straight lines" in curved spaces. But we should be aware that, even in Euclidean space, this description could look complicated in other than Cartesian coordinates. Consider the case of spherical coordinates $x^1 = r, x^2 = \theta, x^3 = \phi$. It is not hard to see that the distance between two points separated by coordinate differentials $dr, d\theta, d\phi$ is given by

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \ . \tag{1}$$

This may be written as

$$ds^2 = g_{ij} dx^i \, dx^j \tag{2}$$

where

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \ . \tag{3}$$

If we repeat the calculation that we did for $\mathcal{S}^2$ to find the equation for paths with stationary lengths, we obtain the same equation as before, namely

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0 \tag{4}$$

where

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\ell} (g_{j\ell,k} + g_{k\ell,j} - g_{jk,\ell}) \tag{5}$$

You may (perhaps should) calculate the Christoffel symbols. They are

$$\Gamma^1_{22} = -r; \qquad \Gamma^1_{33} = -r \sin^2\theta; \qquad \Gamma^2_{12} = \Gamma^2_{12} = \frac{1}{r}; \qquad \Gamma^2_{33} = -\sin\theta\cos\theta \tag{6}$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}; \qquad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta \ . \tag{7}$$

1.1 Lagrangian Mechanics

Another way to think about this result is to go back to mechanics. Along the trajectory followed by (say) a particle, we can measure distance, s. But we may also put down the time, t, along the path, if we could but find it out. In any case, we may presume a relation between distance traveled and time, $s(t)$. The speed of the particle is then ds/dt and the components of the velocity vector are dx^i/dt, where it is understood that the basis vectors are along the directions of the spherical coordinates. (We are allowed to presume this as we are in Euclidean space in this example.) Then (2) tells us how to compute the square of the speed and thus the kinetic energy. Since there is no potential energy here, this latter is the same as the Lagrangian,

$$L = \frac{m}{2} g_{ij} \frac{dx^i}{dt}\frac{dx^j}{dt} \tag{8}$$

where m is the mass. The action for a trip from P to Q is

$$\mathcal{A} = \int_{t_P}^{t_Q} L dt \tag{9}$$

and it should be stationary on the physically chosen path. That is,

$$\delta\mathcal{A} = \delta \int_{t_P}^{t_Q} L dt = 0 \ . \tag{10}$$

This gives us the equation (see the end of Installment I, which may now seem more readable to you),

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0 \tag{11}$$

where $\dot{x}^i = dx^i/dt$

Now,

$$\frac{\partial L}{\partial \dot{x}^i} = m g_{ij} \dot{x}^j \tag{12}$$

This is basically the mass times the velocity and it is called the momentum (or canonical momentum) and indeed it does look like a momentum. But there is this g_{ij} in there. What it does is to lower the index on the mv^i. The momentum comes in with its index downstairs and so it is perhaps natural that in mechanics (both classical and quantum mechanical) it is said to be conjugate to the coordinate.

In explicit spherical coordinates,

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \dot{\phi}^2\right) \tag{13}$$

Hence

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}; \qquad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}; \qquad \frac{\partial L}{\partial \dot{\phi}} = m\dot{r}^2 \sin^2\theta\phi \tag{14}$$

Indeed, the first of these is a momentum and the other two are angular momenta. We find the equations of motion

$$\ddot{r} - r\,\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi} \tag{15}$$

$$\ddot{\theta} + \frac{2}{r}\,\dot{r}\,\dot{\theta} - \sin\theta\ \cos\theta\,\dot{\phi}^2 \tag{16}$$

$$\ddot{\phi} + \frac{2}{r}\,\dot{r}\,\dot{\phi} + 2\cot\theta\,\dot{\phi}\,\dot{\theta} = 0\ . \tag{17}$$

What a way to describe a straight line! But, as we see, this set of equations is none other than the geodesic equation (4). We can then read off the Christoffel symbols by comparing with the general form and we get (6)–(7). (This is the secret easy way to do things.)

If we choose our coordinate system well, we can have the particle moving with constant ϕ. Then (15)–(16) become

$$\ddot{r} - r\,\dot{\theta}^2 = 0 \tag{18}$$

$$\ddot{\theta} + \frac{2}{r}\,\dot{r}\,\dot{\theta} = 0 \tag{19}$$

The second equation integrates to

$$r^2\dot{\theta} = \text{constant.} \tag{20}$$

This is an expression of the conservation of angular momentum in a system with no torques. (If you do not know how to find this result, please see me.) Call the constant ℓ and (18) becomes

$$\ddot{r} - \frac{\ell^2}{r^3} = 0\ . \tag{21}$$

This equation is of second order, so its solutions will have two arbitrary constants. A first integral is gotten by multiplying by $\dot{r}$ and integrating. We get

$$\frac{1}{2}\dot{r}^2 + \frac{\ell^2}{2\,r^2} = \text{constant} = E\ . \tag{22}$$

Since the origin of time is arbitrary we may write the solution as

$$r = \sqrt{\ell^2/(2E) + 2Et^2} \tag{23}$$

We can then solve (20) to obtain

$$t = \frac{\ell}{2E}\tan(\theta + \theta_0) \tag{24}$$

where θ_0 is an integration constant. And so we find that

$$r\ \cos(\theta + \theta_0) = \frac{\ell}{\sqrt{2E}} \tag{25}$$

So we have found that a free particle moving in Euclidean space with no forces acting on it moves in a straight line. This reminds me of what Voltaire wrote to Maupertuis when he went to Lapland to see whether the earth was oblate as Newton had said. Voltaire wrote (more or less):

Vous avez mesuré dans des lieux pleine d'ennui

ce que Newton savait sans sortir de chez lui.

Here a significant issue is raised. Someone coming in at the middle of this discussion could well imagine that we were talking about complicated motion in a curved space when in fact all we are doing is looking at the motion of a free particle. The compications arise only because we have injudiciously chosen curvilinear coordinates, which are not optimal for talking about straight lines. So how can we tell when this has happened and that we are really in a flat space? That involves what is called a test for curvature that we shall learn later.

1.2 Arbitrary Coordinates

As we see, the price of doing things in a general way is that we may overlook how simple things really are in our attempt to do it all at once. But once we are aware of that, we can throw caution to the winds and go into any coordinate system we feel like, knowing that there is a simplest version somewhere.

So suppose we have a coordinate system x^i, in whatever the space is and we go into another system. This time, let us see how we like using the notation that some people use for the other system of coordinates, that is $x^{i'}$. We then look for a transformation

$$x^i = x^i(x^{i'}). \tag{26}$$

If we have a particle moving on a particular trajectory, the tangent vector to the trajectory is

$$\frac{dx^i}{ds} = \frac{\partial x^i}{\partial x^{k'}}\frac{dx^{k'}}{ds} \ . \tag{27}$$

This is the transformation of the tangent vector from one coordinate system to the other and it is effected by the transformation matrix $\partial x^i/\partial x^{k'}$. Let us suppose that the transformation has an inverse. That is, that it is possible to solve for $x^{k'}(x^i)$. Then the transformation matrix taking vectors back the other way is $\partial x^{k'}/\partial x^i$ with

$$\partial x^i/\partial x^{k'}\partial x^{k'}/\partial x^j = \delta^i_{\cdot j} \tag{28}$$

The second derivative is

$$\frac{d^2x^i}{ds^2} = \frac{\partial^2 x^i}{\partial x^{k'}\partial x^{\ell'}}\frac{dx^{k'}}{ds}\frac{dx^{\ell'}}{ds} + \frac{\partial x^i}{\partial x^{k'}}\frac{d^2x^{k'}}{ds^2} \tag{29}$$

For a motion in a straight line in rectangular coordinates, we have $d^2x^i/ds^2 = 0$. If we impose this condition, and multiply by $\partial x^{i'}/\partial x^i$ we get

$$\delta^{i'}_{\cdot k'}\frac{d^2x^{k'}}{ds^2} + \frac{\partial x^{i'}}{\partial x^i}\frac{\partial^2 x^i}{\partial x^{k'}\partial x^{\ell'}}\frac{dx^{k'}}{ds}\frac{dx^{\ell'}}{ds} = 0 \ . \tag{30}$$

This may be written as

$$\frac{d^2x^{i'}}{ds^2} + \Gamma^{i'}_{k'\ell'}\frac{dx^{k'}}{ds}\frac{dx^{\ell'}}{ds} = 0 \tag{31}$$

where

$$\Gamma^{i'}_{k'\ell'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial^2 x^i}{\partial x^{k'}\partial x^{\ell'}}\ . \tag{32}$$

Notice that for a linear transformation, the nonacceleration property is invariant. More generally, the condition of no acceleration taken to arbitrary coordinates seems to tell us that the motion is geodesic; at least it will have once we shall have established the presumption that identification of the new expression for Γ is equivalent to the old one.