General Relativity

Installment III.

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1 Vector Analysis Continued

3.4 Outer Product When working in three-dimensional Euclidean space we may define an outer, or cross, product of two vectors $\mathbf{a}$ and $\mathbf{b}$ written as $\mathbf{a} \times \mathbf{b}$ such that

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \cos\varphi \tag{1}$$

where φ is the angle between the two vectors when they are referred to the same origin. (Remember that we can slide vectors around at will in Euclidean space.) The product is taken as a vector in the line perpendicular to each of $\mathbf{a}$ and $\mathbf{b}$ and its direction is that given by the right hand screw rule when $\mathbf{a}$ is turned into $\mathbf{b}$. It follows that

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}\ . \tag{2}$$

If $\mathbf{a} = a^i \mathbf{e}_i$ and $\mathbf{b} = b^j \mathbf{e}_j$, then

$$\mathbf{a} \times \mathbf{b} = a^i b^j\, \mathbf{e}_i \times \mathbf{e}_j \tag{3}$$

But $\mathbf{e}_i \times \mathbf{e}_j$ is perpendicular to each of $\mathbf{e}_i$ and $\mathbf{e}_j$ and so is $\mathbf{e}^k$ when $k \neq j$ and $k \neq i$. Therefore we may express $\mathbf{a} \times \mathbf{b}$ as a linear combination of the reciprocal basis vectors $\mathbf{e}^k$ This is convenient but it does convey a feeling that there is something that distinguishes cross products from conventional vectors.

The coefficients in the linear combination of reciprocal vectors that gives us the cross product need to be determined but we may write the combination in an abstract way as

$$\mathbf{e}_i \times \mathbf{e}_j = A_{ijk} \mathbf{e}^k \tag{4}$$

where the coefficients A_{ijk} are as yet unknown. (We need the three indices since one is summed on and two are free to keep track of the two vectors in the product.) We dot $\mathbf{e}_\ell$ into this formula and recall that $\mathbf{e}_\ell \cdot \mathbf{e}^k = \delta_\ell^{\cdot k}$. Since $A_{ijk}\delta_\ell^{\cdot k} = A_{jk\ell}$ we find that

$$A_{ij\ell} = \mathbf{e}_\ell \cdot \mathbf{e}_i \times \mathbf{e}_j \tag{5}$$

When we put this into (4), we get

$$\mathbf{e}_i \times \mathbf{e}_j = [\mathbf{e}_k\, \mathbf{e}_i\, \mathbf{e}_j]\mathbf{e}^k \tag{6}$$

where

$$[\mathbf{e}_k\, \mathbf{e}_i\, \mathbf{e}_j] = \mathbf{e}_k \cdot \mathbf{e}_i \times \mathbf{e}_j \tag{7}$$

The three vectors $\mathbf{e}_i$, with $i = 1, 2, 3$, form a parallelpiped whose volume is $[\mathbf{e}_k\, \mathbf{e}_i\, \mathbf{e}_j]$ and the value of the triple product is unchanged by cyclic permutation of the indices. A permutation that is odd (noncyclic) merely changes the sign. (This sort of thing is discussed in older books on the mathematics of physics such as Margenau and Murphy.)

Let us introduce $v = [\mathbf{e}_k\, \mathbf{e}_i\, \mathbf{e}_j]$. Then we may write (6) out as

$$\mathbf{e}^1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{v}, \qquad \text{etc.} \tag{8}$$

where the other components are found by cyclic permutation. To write this in a convenient way we introduce the permutation symbol ϵ^{ijk} which equals zero if any two of the indices are the same, equals one if the order of the indices is a cyclic permutation of 123 and equals minus one if the indices are a cyclic permutation of 132. Then (8) is compressed into

$$\mathbf{e}^i = \epsilon^{ijk}\,(\mathbf{e}_j \times \mathbf{e}_k)/(2v)\ . \tag{9}$$

Let us now write (3) as

$$\mathbf{a} \times \mathbf{b} = a^2 b^3\, \mathbf{e}_2 \times \mathbf{e}_3 + a^3 b^2\, \mathbf{e}_3 \times \mathbf{e}_2 + ... = (a^2 b^3 - a^3 b^2)\, \mathbf{e}_2 \times \mathbf{e}_3 + ... \tag{10}$$

Now we write out all the terms and use (8) to obtain

$$[\mathbf{e}_1\, \mathbf{e}_2\, \mathbf{e}_3]\, \mathbf{a} \times \mathbf{b} = (a^2 b^3 - a^3 b^2)\, \mathbf{e}^1 + (a^3 b^1 - a^1 b^3)\, \mathbf{e}^2 + (a^1 b^2 - a^2 b^1)\, \mathbf{e}^3, \tag{11}$$

which begins to resemble the familiar way of writing out the cross product. To complete the story we need to look a bit more at $[\mathbf{e}_k\, \mathbf{e}_i\, \mathbf{e}_j]$. This is a bit of manipulation that you do not really need to get into, but I'll leave it as an exercise for now and sketch it in a later installment for those who may be interested.

2 Fictitious Forces

2.1 Relativity

The statement by Galileo (Installment I) about the goings on in the hold of a ship being indifferent to the imposition of uniform motion of the ship itself may have surprised people in 1600. Nowadays, we are not so surprised by it since we encounter the phenomenon readily, as in planes moving in tranquil air. But the implication of Galileo's imagery are striking since it tells us that if two observers are moving with respect to each other at contsant velocity, neither one being accelerated, each of them has the right to consider heirself to be at rest. (As we shall see, there are developments in modern cosmology that alter that vision, but by the time we get to them, we shall have seen how to absorb their implication into our thinking.) Therefore, each of the relatively moving observers might wish to claim that the correct statement of the laws of mechanics would be with respect to, that is, *relative* to, heir own frame of reference. This form of relativity is called special since it presumes that the two observers move with respect to each other at constant velocity. (In other languages, the adjective used is more like *restricted*, but in English, the word special is used.) We shall be generalizing those notions and by way of preparation, let us begin to see how things look to accelerated observers.

2.2 Rotating Frames

Suppose that we are in a frame rotating with angular speed Ω, as indeed we are. The rotation may be described about an axis and this permits us to speak of an angular velocity $\mathbf{\Omega}$, where this vector is aligned with the axis of rotation. We would like then to set up a coordinate system rotating with the system. If we take the earth as our example, we should note that the earth's rotation rate is not constant, though it does vary only very slowly.

Let us use the term inertial frame to signify a frame that is not rotating. This in itself may raise questions in your mind, but let us postpone the issue of *precisely* what is meant by the inertial frame and settle for the frame of someone fixed in space. We write the basis vectors in the inertial frame as $\mathbf{e}_i$ and vector $\mathbf{a}$ is then written as usual as $a^i\mathbf{e}_i$.

Let us denote the basis vectors in the rotating frame as $\hat{\mathbf{e}}_\mathbf{i}$. (You may wonder why we do not use a prime to denote this second set of quantities to be consistent with the notation used earlier. We'll go into that later.) The vector $\mathbf{a}$ itself is expressed as $\hat{a}^i\,\hat{\mathbf{e}}_\mathbf{i}$ where $\hat{a}^i$ are the components in the rotating frame.

In the inertial frame, the basis vectors are fixed, so that

$$\frac{d\mathbf{a}}{dt} = \frac{da^i}{dt}\mathbf{e}_i \tag{12}$$

By contrast, the derivative of $\mathbf{a}$ in the rotating frame is

$$\frac{d\mathbf{a}}{dt} = \frac{d\hat{a}^i}{dt}\hat{\mathbf{e}}_i + \hat{a}^i\frac{d\hat{\mathbf{e}}_\mathbf{i}}{dt} \tag{13}$$

The qualitative difference between the two statements is that the basis vectors in the rotating frame, as seen in the inertial frame from which we are presumed to start out, are varying. We need to decide how the basis vectors in the rotating frame are varying.

When a short time δt goes by, the system rotates by a small amount $\Omega\delta t$. This causes each of the basis vectors to turn by a small amount without any change in length. The change in each basis vector is represented by the addition of $\delta\hat{\mathbf{e}}_\mathbf{i}$. Since a rotation does not change the length of a vector, this slight change in basis vector must be perpendicular to the basis vector itself and to $\mathbf{\Omega}$. We then conclude that the direction of $\delta\hat{\mathbf{e}}_\mathbf{i}$ is in the direction of $\mathbf{\Omega}\times\hat{\mathbf{e}}_\mathbf{i}$. The sign of this correction is positive in the direction obtained by turning $\mathbf{\Omega}$ into $\hat{\mathbf{e}}_\mathbf{i}$; that is in fact how the direction of Ω is chosen. The amount of the correction is proportional to $\Omega\delta t$. It is also proportional to the component of $\hat{\mathbf{e}}_\mathbf{i}$ perpendicular to $\mathbf{\Omega}$, which varies with the sine of the angle between $\hat{\mathbf{e}}_\mathbf{i}$ and $\mathbf{\Omega}$ — just what we get from the cross product between the two vectors. Hence we have

$$\frac{d\hat{\mathbf{e}}_\mathbf{i}}{dt} = \mathbf{\Omega}\times\hat{\mathbf{e}}_\mathbf{i} \tag{14}$$

and so

$$\frac{d\mathbf{a}}{dt} = \frac{d\hat{a}^i}{dt}\hat{\mathbf{e}}_i + \mathbf{\Omega}\times\mathbf{a} \tag{15}$$

To clarify these things one often writes $D_\mathbf{I}$ to denote the time derivative in the inertial frame. That is, $D_\mathbf{I}$ is the time derivative holding the $\mathbf{e}_\mathbf{i}$ fixed:

$$D_\mathbf{I} = \frac{da^i}{dt}\mathbf{e}_i \tag{16}$$

Then, we write $D_\mathbf{R}$ for the time derivative in the rotating frame, that is, the derivative holding $\hat{\mathbf{e}}_\mathbf{i}$ fixed:

$$D_\mathbf{R} = \frac{d\hat{a}^i}{dt}\hat{\mathbf{e}}_\mathbf{i} \tag{17}$$

As we see from (15), we have

$$D_\mathbf{I}\mathbf{a} = D_\mathbf{R}\mathbf{a} + \mathbf{\Omega}\times\mathbf{a} \tag{18}$$

Another way that people have of saying this is that when you transform into a rotating reference frame you should transform the time derivative as in

$$\frac{d}{dt} \to \frac{d}{dt} + \mathbf{\Omega}\times \tag{19}$$

Either way you look at this, the first interesting question to ask is what happens to the derivative of the position vector, $\mathbf{x}$? We find that

$$D_\mathbf{I}\mathbf{x} = D_\mathbf{R}\mathbf{x} + \mathbf{\Omega}\times\mathbf{x} \tag{20}$$

This says that

$$\mathbf{v}_\mathbf{I} = \mathbf{v}_\mathbf{R} + \mathbf{\Omega}\times\mathbf{x} \tag{21}$$

where $\mathbf{v}_\mathbf{I}$ and $\mathbf{v}_\mathbf{R}$ are the velocities relative to the two frames. So this shows how the velocity transforms when we go to a rotating frame. Even if you are stationary in the rotating frame, you will have a velocity $\mathbf{\Omega}\times\mathbf{x}$ with respect to the inertial frame. Since that velocity is not a constant in time, you will feel an acceleration by virtue of being in the (accelerating!)

rotating frame. If you happen to be on the surface of the earth, this motion will carry you in a direction tangent to your latitude circle along with the rest of us.

Now we consider what happens to the acceleration felt in the inertial frame when we go into the rotating frame. The inertial acceleration is $D_{\mathbf{I}}\mathbf{v_I}$. If we put $\mathbf{a} = \mathbf{v_I}$ into (18) we have

$$D_{\mathbf{I}}\mathbf{v_I} = D_{\mathbf{R}}\mathbf{v_I} + \boldsymbol{\Omega} \times \mathbf{a} \tag{22}$$

When we put (21) into this, we get

$$D_{\mathbf{I}}\mathbf{v_I} = D_{\mathbf{R}}(\mathbf{v_R} + \boldsymbol{\Omega} \times \mathbf{x}) + \boldsymbol{\Omega} \times (\mathbf{v_R} + \boldsymbol{\Omega} \times \mathbf{x}) \tag{23}$$

The acceleration with respect to the rotating frame is $\mathbf{a_R} = D_{\mathbf{R}}\mathbf{v_R}$ so we may write that the inertial acceleration $\mathbf{a_I}$ is given by

$$\mathbf{a_I} = \mathbf{a_R} + 2\boldsymbol{\Omega} \times \mathbf{v_R} + \dot{\boldsymbol{\Omega}} \times \mathbf{x} + \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \mathbf{x} \tag{24}$$

where we have written $\dot{\boldsymbol{\Omega}}$ for $D_{\mathbf{R}}\boldsymbol{\Omega}$. In the triple product we simplify by noting that $\boldsymbol{\Omega} \times \mathbf{x}$ is perpendicular to $\boldsymbol{\Omega}$ and that it has magnitude $\Omega|\mathbf{x}|\sin\varphi$ where φ is the angle between $\boldsymbol{\Omega}$ and $\mathbf{x}$. (If you are at position $\mathbf{x}$ on earth, then φ is your colatitude.) Also, since $\boldsymbol{\Omega} \times \mathbf{x}$ is perpendicular to $\boldsymbol{\Omega}$, the triple product has magnitude $\Omega^2\varpi$ where $|\varpi|\,|\mathbf{x}|\sin\varphi$ in a notation that some astronomers like. If you chase down the right-hand rule twice you see that the triple product is a vector perpendicular to $\boldsymbol{\Omega}$ pointing away from where $\mathbf{x}$ is. So if the vector ϖ is perpendicular to the rotation axis and pointing toward $\mathbf{x}$, the acceleration formula is

$$\mathbf{a_I} = \mathbf{a_R} + 2\boldsymbol{\Omega} \times \mathbf{v_R} + \dot{\boldsymbol{\Omega}} \times \mathbf{x} - \Omega^2\varpi \tag{25}$$

(The ϖ in this formula should be in boldface, but I cannot make it happen.)

Suppose that the inertial observer feels no force; heir acceleration is then zero. Nevertheless, the observer in the rotating frame will feel an acceleration given by

$$\mathbf{a_R} = -2\boldsymbol{\Omega} \times \mathbf{v_R} - \dot{\boldsymbol{\Omega}} \times \mathbf{x} + \Omega^2\varpi \tag{26}$$

Judging from the form of Newton's second law, these terms on the right might be called forces. However since they arise as a result of going into an accelerating frame of reference, the are sometimes called fictitious forces, even though you will definitely feel them in a rotating frame, as you may know from having been a whirling platform of some sort. The third term on the right is called the centrifugal force and it comes from the inertial tendency to go in a straight line when this tendency is frustrated by whatever is keeping you in the rotating frame. (This was the message of the parable of the bead on the wire.)

The middle term on the right of (26) is called the Euler force. Since the earth's rotation is being slowed down as a result of tidal dissipation, there is such a force and it drives a very weak circulation in the earth's interior, according to some calculations. The first term on the right is called the Coriolis force and it plays a great role in meteorology and oceanography.

If you move over the surface of the earth, the Coriolis force is perpendicular to your velocity. This is like the Lorentz force felt by an electron moving through a magnetic field. When you look at weather maps you see that the contours of pressure are quite wavy. The Coriolis force has a lot to do with those waves. (Coriolis, by the way, was very interested in rotation and even wrote a book on billiards.)

For our purposes, the most striking thing in (26) is that if you want to think of it as a form of Newton's second law, you should multiply by the mass of the moving body. When you do that., all the terms on the right are also multiplied by the mass. So the mass cancels from this form of Newton's laws because fictitious forces are always proportional to the mass of the object suffering acceleration.

The acceleration produced by a fictitious force is independent of the mass of the object acted upon. It is therefore worth recalling that Galileo found that falling bodies had accelerations independent of their masses. Once you do that, you see what Einstein's notion of gravity was. All we need to do now is to see how he implemented it.