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PLANET EVOLUTION

Greg Novak[1], David S. Spiegel[2]
[1]IAP and
[2]Astrophysics Department, Institute for Advanced Study, Princeton, NJ 08540
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ABSTRACT

This is an abstract

Subject headings: planetary systems – radiative transfer

1. INTRODUCTION

One of the goals of this work is to present the community with an easy-to-use, publicly available, robust platform for experimentation with planet evolution.

Another goal is to provide the simplest complete model for planet evolution that includes a ideal gas EOS, a degenerate gas EOS, and an atmosphere model. This is for pedagogical purposes as well as to facilitate mapping of the large parameter space without running a "full blown" simulation, which potentially includes many free parameters that are very poorly constrained (such as the exact abundances of each element).

2. MIDDLE

2.1. *Equation of State*

2.1.1. *Gas Phase*

Roughly copy useful text and equations out of my own notes. Wish that I hadn't formulated anything in terms of sound speed. Should have just used energy. Will need to change this.

It is easy to write down an equation of state that covers all major regimes of the gas-phase equation of state: ideal gas, non-relativistic degenerate gas, relativistic degenerate gas, relativistic non-degenerate gas.

Basic changes in the physics occur when the Fermi energy and thermal energies are equal; when the Fermi velocity approaches c; and when the thermal velocity approaches c.

Relativistic effects become important when the thermal energy is greater than the rest mass energy

$$kT > \frac{2}{3} m_e c^2 \tag{1}$$

for electrons, and similarly for protons.

The Fermi momentum for spin 1/2 particles is

$$p_F = (3\pi^2)^{1/3} Y_e^{1/3} n_b^{1/3} \hbar \tag{2}$$

where Y_e is the number of electrons per baryon:

$$Y_e = \frac{n_e}{n_b} \tag{3}$$

Then the Fermi energy is $E_F = p_F^2/2m$ if $E_f \ll mc^2$ and $E_F = p_F c$ if $E_F \gg mc^2$.

Degeneracy effects become important when the Fermi energy is equal to the thermal energy, $3kT/2$. Using the non-relativistic expression for the Fermi energy

$$kT = \frac{\pi^{4/3} \hbar c Y_e^{2/3} n_b^{2/3} \bar{\lambda}_e}{3^{1/3}} \tag{4}$$

while for protons it is

$$kT = \frac{\pi^{4/3} \hbar c n_b^{2/3} \bar{\lambda}_p}{3^{1/3}} \tag{5}$$

Using the relativistic expression for the Fermi energy yields

$$\frac{2\pi^{2/3}}{3^{2/3}} \hbar c Y_e^{1/3} n_b^{1/3} \tag{6}$$

for electrons and

$$\frac{2\pi^{2/3}}{3^{2/3}} \hbar c n_b^{1/3} \tag{7}$$

for protons.

The relevant physics also changes when the Fermi energy becomes larger than the rest mass en-

greg.novak@obspm.fr, dave@ias.edu

ergy of the particles:

$$n_b = \frac{2^{3/2}\bar{\lambda}_e^{-3}}{3\pi^2 Y_e} \quad (8)$$

and, for protons

$$n_b = \frac{2^{3/2}\bar{\lambda}_p^{-3}}{3\pi^2} \quad (9)$$

For a single species, the expressions for pressure are: $E_F < E_T < E_R$, (ND, NR ideal gas)

$$p = nkT \quad (10)$$

$E_T < E_F < E_R$, (D, NR gas of spin-2 fermions)

$$p = \frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m}n^{5/3} \quad (11)$$

$E_T < E_R < E_F$, or $E_R < E_T < E_F$ (D, R gas of spin-2 fermions)

$$p = \frac{(3\pi^2)^{1/3}}{4}\hbar c n^{4/3} \quad (12)$$

$E_F < E_R < E_T$ or $E_R < E_F < E_T$ (ND, R gas, $g = 2$ bosons)

$$p = \frac{\pi^2}{45}\frac{1}{\hbar^3 c^3}(kT)^4 \quad (13)$$

or, for fermions in the same limit:

$$p = \frac{7}{8}p_{\mathrm{Boson}} \quad (14)$$

Now the entropy is: $E_F < E_T < E_R$, (ND, NR ideal gas)

$$\frac{\Sigma}{N} = \log\left(\frac{n_Q}{n}\right) + \frac{5}{2} \quad (15)$$

with

$$n_Q = \left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} \quad (16)$$

$E_T < E_F < E_R$, (D, NR gas of spin-2 fermions)

$$\frac{\Sigma}{N} = \left(\frac{\pi}{3}\right)^{2/3}\frac{m}{\hbar^2}\frac{kT}{n^{2/3}} \quad (17)$$

$E_T < E_R < E_F$, or $E_R < E_T < E_F$ (D, R gas of spin-2 fermions)

$$\frac{\Sigma}{N} = \frac{\pi^{4/3}}{3^{1/3}}\frac{1}{\hbar c}\frac{kT}{n^{1/3}} \quad (18)$$

$E_F < E_R < E_T$ or $E_R < E_F < E_T$ (ND, R gas, $g = 2$ bosons)

$$\frac{\Sigma}{N} = \frac{4\pi^2}{45}\frac{1}{\hbar^3 c^3}\frac{(kT)^3}{n} \quad (19)$$

or, for fermions in the same limit:

$$s = \frac{7}{8}s_{\mathrm{Boson}} \quad (20)$$

Finally, Kittel + Kroemer give the relation between their dimensionless entropy and the "conventional entropy." Landau + Lifshitz use the same convention as Kittel + Kroemer, where temperature is measured in energy and entropy is dimensionless.

$$S = k\sigma \quad (21)$$

Considering each region of 1 in turn, for the pressure we have the following.

In region A

$$p = n_b kT(1 + X) \quad (22)$$

In region B electrons are degenerate but protons are not. The condition for the protons to provide as much pressure is almost the same as the condition for the electrons to be degenerate, so as you become fully degenerate the protons become negligible. So neglect them in region B. In region C, both electrons and protons are degenerate, but the m_p or m_e that appears in the denominator ensures that the proton contribution is always negligible. In region F, electrons are degenerate and relativistic while protons are degenerate and non-relativistic. The condition for the protons to contribute is the same as for the proton Fermi energy to be such that the protons are relativistic. Therefore the protons are negligible in region F. Finally, in regions B, C, and F, the pressure is given by degenerate electrons.

$$p = \frac{3^{2/3}\pi^{4/3}}{40}\frac{\hbar^2 n_b^{5/3}}{m_e} \quad (23)$$

In region G, both electrons and protons are degenerate and relativistic. The expression doesn't contain their rest mass so they both contribute equally and you pick up a factor of 2.

$$p = \frac{3^{4/3}\pi^{2/3}}{4}\hbar c n_b^{4/3} \quad (24)$$

In region D protons are an ideal gas but the electrons are behaving like photons, except for a factor of 7/8 due to Fermi statistics instead of Bose-Einstein statistics. The condition for the protons to contribute is $n_b > (\pi^2/45)(kT/\hbar c)^3$ (there must be more than one baryon per mean photon wavelength), which falls in region H, so protons never contribute. Therefore in region D we have

$$p = \frac{\pi^2 k^4 T^4}{45 c^3 \hbar^3} \quad (25)$$

In region E, both protons and electrons are behaving like photons

$$p = \frac{\Gamma \pi^2 k^4 T^4}{51 c^3 \hbar^3} \quad (26)$$

where Γ refers to the number of degrees of freedom (I think this usage is standard) and is somewhere

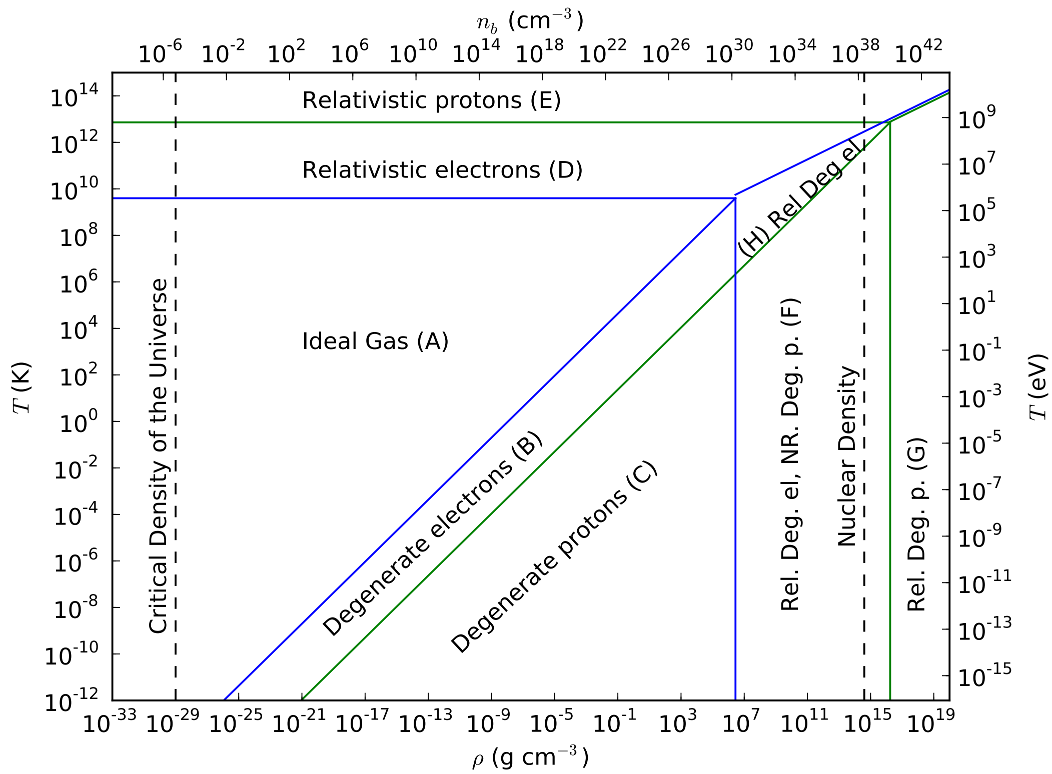

FIG. 1.— Phase diagram of the relevant regimes. *Top:* The sound speeds given just use the non-relativistic idea gas formula. There are no relativistic corrections (so it can be greater than c), I don't worry about the how the mean molecular weight changes with temperature, or about how the sound speed changes when, e.g. the electrons are degenerate. *Bottom:* The dashed lines are just reference points and do not divide physical regions.

between 10 and a hundred. In region I, the same thing is going on but the electron states are full, so it's the same expression with a value of Γ decremented by one.

In region H, the protons are behaving as an ideal gas while the electrons are relativistic and degenerate. The condition for the baryons to contribute is almost the same as that given in the previous paragraph, so for most/all of region H, the baryons dominate the pressure. Then we have:

$$p = n_b kT \tag{27}$$

The reason that in the absence of cooling the equation of state is proportional to $\rho^{5/3}$ in both the low density (adiabatic ideal gas) and high density (degenerate fermion gas) state is that both gases are dominated by the kinetic energy of the constituent particles, while the potential energies associated with inter-particle forces are negligible. (**?**, §57, p 168). At intermediate densities the interaction energies become comparable to the thermal energies, the situation is much more complicated. This happens to be the situation at typical densities encountered in everyday life.

Comparing to the detailed equation of state given by... (Mesa? Who?), the pressures given in this section are accurate to... (a factor of two?) over an enormous range of densities and temperatures.

Finally, consider the entropy in each of the regions given by Figure 1.

In region A, both electrons and protons are nonrelativisitc, so their individual contributions to the entropy are given by the Sakur-Tetrode equation:

$$\frac{\Sigma}{N_b} = \log \frac{n_q}{n} \tag{28}$$

where

$$n_q = \left(\frac{m_p kT}{2\pi\hbar^2} \right)^{3/2} \tag{29}$$

is the density where degeneracy effects become important, the quantum concentration. It is the same as the condition we have already used (thermal energy equal to the Fermi energy), up to a factor of order unity. The difference in definitiions is for convenience in writing the Sakur-Tetrode equation.

Taking the Sakur-Tetrode equation for the protons and electrons plus the entropy of mixing gives the total entropy, which is dominated by the protons.

$$\frac{\Sigma}{N_b} = (1+X)\log\frac{n_{qe}}{n_b} + -(1-X)\log(1-X) \\ -2X\log X + X\frac{5}{2} + \frac{3}{2}(\log\frac{m_p}{m_e} + \frac{5}{3}) \quad (30)$$

In region B, the electron contribution switches to $(n_q/n)^{2/3}$ so it's less than one and we have

$$\frac{\Sigma}{N_b} = \log\frac{n_{qe}}{n_b} - (1-X)\log(1-X) - X\log X + \frac{3}{2}(\log\frac{m_p}{m_e} + \frac{5}{3}) \quad (31)$$

In region C, the proton contribution switches to the degenerate case. In region F, the electrons are already negligible, so it doesn't matter if they become relativistic. In region G, it's not clear to me if it matters that the Fermi sea of protons is becoming relativistic. The entropy will be bounded above by one, but the expression may or may not change. Therefore in regions C, F, and maybe G, we have:

$$\frac{\Sigma}{N_b} = \frac{2\pi^{5/3}}{3^{2/3}}\left(\frac{n_{qp}}{n_b}\right)^{2/3} \quad (32)$$

In region D, the entropy per baryon provided by the electrons is of order the ratio of the photon density to the baryon density. Temperatures are so high in this region that this will generally be strongly dominated by the electrons. Thus in region D the entropy per baryon is

$$\Sigma = \frac{4\pi^2\Gamma k^3 T^3}{45\hbar^3 c^3 n_b} \quad (33)$$

Note that this is a bit confusing: When you have a mixture of high mass and low mass non-relativistic particles, the massive particles carry the entropy. However, if you let the mass of the lower-mass particle go to zero, then *it* carries the entropy. Letting the mass go to zero ensures that the particle becomes relativistic. In the *classical* expression for entropy (equation 28), letting the mass go to zero makes the quantum concentration zero, so there's no classical regime—it's always quantum. One would then conclude that gases of low-mass particles have very little entropy. However, the *relativistic* expression (equation 33) gives large entropy. So it must be that passing to the relativistic limit is what changes things so dramatically. This bears further thought.

In region E, similar considerations to the previous section lead us to the same expression for photons, with an additional factor for the number of degrees of freedom being excited. In region I, the same thing is going on with one or two fewer degrees of freedom.

$$\Sigma = \frac{4\pi^2\Gamma k^3 T^3 V}{45\hbar^3 c^3} \quad (34)$$

In region H, the electrons are supposed to be degenerate so they shouldn't contribute much entropy. I suppose then that we're back to the Sakur-Tetrode equation for just the baryons

$$\frac{\Sigma_p}{N_b} = \log\frac{n_{qp}}{n_b} + \frac{5}{2} - [(1-X)\log(1-X) + X\log X] \quad (35)$$

Comparing to the detailed entropy computed by... (Mesa? Who?), the entropies given in this section are accurate to... (a factor of two?) over an enormous range of densities and temperatures.

2.2. *Solid Phase*

Unfortunately laboratory densities are such that inter-particle forces are important. Furthermore, the above discussion does not account for the tendency of electrons to collect near the nucleus at low pressures and (what exactly? Electron exchange energy? What exactly is that?) that become important at low pressures.

At pressures below about 1 Mbar, these effects cause condensation of materials to solids where there is a finite density at zero pressure solution. Accounting for the tendency of electrons to collect around the nucleus is the basis of the Thomas-Fermi-Dirac (**?**) model of solids, and the electron exchange energy corrections given by **?** correct the density at zero temperature to close to observed values.

We implement the **?** equation of state to account for planetary cores.

2.3. *Convection and mixing length theory*

2.4. *Mixing length theory*

Following the discussion in **?**, define

$$W = \nabla_{\rm rad} - \nabla_{\rm ad} \quad (36)$$

which parameterizes how hard convection is being driven.

The temperature gradient is parameterized by

$$\nabla == \frac{d\log T}{d\log p} \quad (37)$$

The temperature gradient that would pertain if the luminosity is transported by radiation is:

$$\nabla_{\rm rad} := \frac{3\kappa l P}{16\pi a c G m T^4} \quad (38)$$

where l is the luminosity carried by the atmosphere. This will generally be constant for us but is a function of radius in the case of stars.

Specifying constant entropy gives the adiabatic temperature gradient

$$\nabla_{\rm ad} = \frac{\gamma - 1}{\gamma} \quad (39)$$

As convection becomes more effective at transporting energy through the atmosphere, then the actual gradient ∇ will approach the adiabatic gradient $\nabla_{\rm ad}$.

Define a parameter for the effectiveness of convection (how close is the actual gradient ∇ to the adiabatic gradient $\nabla_{\rm ad}$?) by

$$U = \frac{3acT^3\sqrt{8H_P}}{c_P\rho^2\kappa l_m^2\sqrt{g\delta}} \tag{40}$$

where $\delta = (d\log\rho/d\log T)_P$, $H_p = P/\rho g$, $c_P = (dq/dT)_P$ where q is the heat per unit mass, defined via $dq = du + Pdv$ where v is the specific volume per unit mass $v = 1/\rho$, and l_m is the famous mixing length, usually taken to be an order unity multiple of the pressure scale height of the atmosphere.

The parameter U is more accurately characterized as parameterizing the *in*effectivenes given that small U means effective convection. If $\log U \gg 0$, then convection is ineffective and you get $\nabla = \nabla_{\rm rad}$, even when $\nabla_{\rm rad} \gg \nabla_{\rm ad}$. $U \ll 1$ means convection is effective, and if you take $U \leftarrow 0$, then $\nabla = \nabla_{\rm ad}$.

Where mixing length theory makes a difference is when $U \ll 1$ (convection is doing something) and $\nabla >> \nabla_{\rm ad}$ (convection isn't successfully driving the gradient very close to $\nabla_{\rm ad}$. This is the region of the U, W plane bounded by $U \ll 1$ and $W \gg 1/U$.

Rewrite

$$U = \frac{9lp\sqrt{2H_p}}{8\pi c_P T\rho^2 l_m^2\sqrt{g\delta}Gm\nabla_{\rm rad}} \tag{41}$$

and define a convective length scale

$$l_c^2 = \frac{9lp\sqrt{2H_p}}{8\pi c_P T\rho^2\sqrt{g\delta}Gm} \tag{42}$$

so that

$$U = \frac{l_c^2}{l_m^2\nabla_{\rm rad}} \tag{43}$$

Two questions are interesting: For fixed driving (value of W), what is the range $l_{\rm min} < l_m < l_{\rm max}$ for which mixing length theory makes a difference, where l_m small means convection is ineffective and l_m big means that convection is efficient enough to drive the gradient to the adiabatic one?

Then we have that for fixed driving force W, ML theory makes a difference between

$$\frac{l_c}{\sqrt{\nabla_{\rm rad}}} < l_m < l_c\sqrt{\frac{\nabla_{\rm rad}-\nabla_{\rm ad}}{\nabla_{\rm rad}}} \tag{44}$$

The second interesting question is, for fixed mixing length, how hard do you need to drive convection to make it super-adiabatic?

$$W > \frac{l_c^2}{l_m^2} - \nabla_{\rm ad} \tag{45}$$

For the interesting case $\nabla_{\rm rad} \gg \nabla_{\rm ad}$ these reduce to

$$\frac{l_c}{\sqrt{\nabla_{\rm rad}}} < l_m < l_c \tag{46}$$

and

$$\nabla_{\rm rad} > \frac{l_c^2}{l_m^2} \tag{47}$$

Now it remains to estimate l_c. We can choose between g or R and between H and P. Then we may want either l_c or l_c/H. I have written out all combinations of these in my notes, we need only choose which one. I include one here.

Writing this out gives

$$l_c = \left(\frac{9l}{8\pi\tilde{c}_P N_A kT\rho R}\right)^{1/2}\left(\frac{2H^3}{\delta Gm}\right)^{1/4} \tag{48}$$

or, putting in the scalings:

$$l_c = 3.8\,{\rm cm}\left(\frac{L}{10^{-6}L_\odot}\right)^{1/2}\left(\frac{T}{10^5\,{\rm K}}\right)^{-1/2} \left(\frac{\delta}{7}\right)^{-1/4}\left(\frac{H}{10\,{\rm km}}\right)^{3/4}\left(\frac{\rho}{{\rm g\,cm^{-3}}}\right)^{-1/2} \left(\frac{R}{\rm Rjup}\right)^{-1/2}\left(\frac{m}{\rm Mjup}\right)^{-1/4} \tag{49}$$

That is to say that the details of the theory of convection will be important only if the mixing length is taken to be small compared to 3.8 cm. Otherwise the planet will achieve a nearly perfectly adiabatic temperature/pressure profile. A tiny excess of the gradient over a perfectly adiabatic one will be sufficient to push as much energy as needed through the planet atmosphere via convection.

It is instructive to compare to the stellar case. Take the atmosphere for a main sequence star, scaling to solar values:

$$H = \frac{kT}{m_p g} = 200\,{\rm km}\left(\frac{T}{5000\,{\rm K}}\right)\left(\frac{g}{27g_\oplus}\right)^{-1} \tag{50}$$

and

$$l_c = 90\,{\rm km}\left(\frac{L}{L_\odot}\right)^{1/2}\left(\frac{T}{5000\,{\rm K}}\right)^{-1/2} \left(\frac{\delta}{7}\right)^{-1/4}\left(\frac{H}{200\,{\rm km}}\right)^{3/4}\left(\frac{\rho}{10^{-6}\,{\rm g\,cm^{-3}}}\right)^{-1/2} \left(\frac{R}{R_\odot}\right)^{-1/2}\left(\frac{m}{M_\odot}\right)^{-1/4} \tag{51}$$

owing mostly to the very small density at the photosphere.

(Rough sketch of thoughts, not sure how much is interesting or how much to keep): Using standard scalings (e.g. $L \propto M^3$) one finds that the scaling of the ratio l_c/H goes as a weak positive power of mass. This means that mixing length theory is expected to be slightly more important in the atmospheres of more massive stars, except that the atmospheres of massive stars become radiative so mixing length theory becomes irrelevant. Going to lower mass maintains convective atmospheres, but mixing length theory is expected to be somewhat less important. The atmosphere of the sun is thus at about the perfect mass to confound us with convection, albeit mildly.

Obviously the situation changes for giant stars, where the surface gravity can be 1000 times weaker. Then $H \simeq 200,000\,\mathrm{km}$ and $l_c \simeq 160,000\,\mathrm{km}$. The fact that H and l_c are in about the same ratio indicates that the overall scaling with R (involving corresponding scalings of surface temperature, surface gravity, and surface density) is very weak, which in turn means that I can simplify the previous expressions and get more insight into how things scale the mass and radius of the star. But it does correspond with what I've heard about where mixing length theory makes a difference: A bit in the solar atmosphere, and a lot in giant stars. If l_c were either much greater or much less than H, then mixing length theory wouldn't make a difference: you have either ineffective convection (hence $\nabla_{\rm rad}$) or very effective convection (hence $\nabla_{\rm ad}$) in those two limits. Mixing length theory makes a difference when they're about the same, which is what I'm finding.

The last interesting case is that where a star develops a convective core. Is mixing length theory important in that case? Reading off of a plot of one of Bahcall's solar model papers, I estimate that the about a third of the sun's mass is contained within 0.2 solar radii (outside the energy generation zone), so the local gravity is 10 times the surface gravity. Then

$$H = \frac{kT}{m_p g} = 80,000\,\mathrm{km}\left(\frac{T}{2\times10^7\,\mathrm{K}}\right)\left(\frac{g}{270 g_\oplus}\right)^{-1} \tag{52}$$

and

$$l_c = 37\,\mathrm{m}\left(\frac{L}{L_\odot}\right)^{1/2}\left(\frac{T}{2\times10^6\,\mathrm{K}}\right)^{-1/2} \left(\frac{\delta}{7}\right)^{-1/4}\left(\frac{H}{8000\,\mathrm{km}}\right)^{3/4}\left(\frac{\rho}{10^2\,\mathrm{g\,cm^{-3}}}\right)^{-1/2} \left(\frac{R}{0.2R_\odot}\right)^{-1/2}\left(\frac{m}{0.35M_\odot}\right)^{-1/4} \tag{53}$$

So we find that mixing length theory is irrelevant when stars have convective cores.

2.4.1. *Small rumination on scalings*

The previous section has convinced me that there's more to figure out here, so I will ramble for a bit.

How to estimate interior properties? Central temperatures are pinned to betweeen 1 and 10 million degrees owing to the steep dependence of the energy generation on temperature. This gives radiation pressure equal to gas pressure when

$$\rho = \frac{m_p a T^3}{k_B} = 0.1\left(\frac{T}{10^7\,\mathrm{K}}\right)^3 \tag{54}$$

which is a bit lower than I thought.

Write the mass, density, and pressure scalings as $M = 4\pi R^3\rho/3$ and $P/R = GM\rho/R^2$ so that

$$P = \frac{3GM^2}{4\pi R^4} \tag{55}$$

And we need a mass-radius relation which we will say is:

$$R = R_\odot\left(\frac{M}{M_\odot}\right)^n \tag{56}$$

with $0.5 < n < 1$ as I remember. So the scale pressure for the sun is roughly

$$P = 2.65\,\mathrm{Gbar}\left(\frac{M}{M_\odot}\right)^{2-4n} \tag{57}$$

This makes the scale density (via the EOS) $\rho = 1.63$ g/cc assuming a temperature of 20 million degrees. This is close to the mean density of the sun and about a factor of a hundred less than the central pressure. So keep that in mind, but seem to be on the right track.

Now write the ratio and get rid of the local gravity in favor of radius. Also use the equation of state to get rid of density in favor of pressure, so we can use the pressure scaling just above.

$$\frac{l_c}{H_P} = 2.40\times10^{-6}\left(\frac{L}{L_\odot}\right)^{1/2}\left(\frac{T}{2\times10^7\,\mathrm{K}}\right)^{-1/4}\left(\frac{\delta}{7}\right)^{-1/4} \tag{58}$$

$$\left(\frac{R}{R_\odot}\right)^{-1}\left(\frac{P}{2.65\,\mathrm{Gbar}}\right)^{-1/4} \tag{59}$$

Keep in mind the actual central pressure is higher than the scale pressure inserted above, but it goes as such a weak power it hardly makes a difference.

Putting in the scaling with mass: $LproptoM^3$, $P \propto M^{2-4n}$ and $T \propto M^b$ where b is small and positive, $R_* \propto M^n$, and introducing $f = r/R_*$ so that R_* is the radius of the entire object and f parameterizes where within the object you're looking gives:

$$\frac{l_c}{H_P} = 2.40\times10^{-6}\left(\frac{1}{f}\right)\left(\frac{M}{M_\odot}\right)^{1-b/4} \tag{60}$$

So the ratio of convective length to scale height goes as a power somewhat less than one in the mass of the star. Even if you look near the center (where f is small, which boosts the ratio, M has to be very large before you have to worry about mixing length theory in the convective core of a star.

Now what about the surface? Set conditions at photosphere:

$$P = \frac{g\tau}{\kappa} \tag{61}$$

near the surface, to take τ of order unity and κ of order unity (in CGS units) to get the pressure at the photosphere. Then the EOS gives the density and you have:

$$\frac{l_c}{H_P} = 0.060 \left(\frac{L}{L_\odot}\right)^{1/2} \left(\frac{M}{M_\odot}\right)^{-1/2} \left(\frac{T}{5000\,\mathrm{K}}\right)^{-1/4} \left(\frac{\delta}{7}\right)^{-1/4} \tag{62}$$

$$\left(\frac{\kappa}{\mathrm{cm}^2\,\mathrm{g}^{-1}}\right)^{1/2} \tag{63}$$

Well, one mystery is solved since the radius cancels, so it doesn't matter if you're talking about giants or main sequence stars. This surprises me. We also don't need to worry about the mass-radius relation. For constant opacity, the ratio goes as Mass to a bit less than the first power, since surface temp rises somewhat with mass. The opacity will be constant (electron scattering) for massive stars, but will rise for lower mass stars, partially compensating for the fact that luminosity is going down as mass goes down. So the ratio will be only a weak function of mass at the low-mass end and a bit less than linear in mass at the high mass end. So... surface convective layers depend on mixing length theory as you go up in mass, except that those stars become radiative...

2.5. *Atmosphere*

2.5.1. *Robinson and Catling*

Robinson and Catling provide a simple but (complete, closed, what is it?) model atmosphere that has a radiative regime at small optical depths and a convective regime at large optical depths. This allows us to relate the bulk entropy of the planet (fixed in the convective regime) to the luminosity of the planet.

Furthermore, we can simply extend the RC model to compute the surface gravity.

Heating and Cooling - Simple Case— Robinson and Catling use the equation for energy conservation in steady state, that the derivative of the total flux is zero:

$$\frac{dF}{dz} = 0 \tag{64}$$

To get rid of the dimensions in the independent variable, we pass to the optical depth τ via

$$d\tau = \rho\kappa\, dz \tag{65}$$

Since their model is integrated over frequency, it's useful to split the unabsorbed stellar flux into a separate channel so that we have F_* and F_a, the flux being processed by the atmosphere (F_{net} in the notation used by **?**). Equation 64 stipulates that the derivative of a quantity is zero, so the equation can be trivially integrated to give

$$F_a(\tau) = -F_*(\tau) + F_{\mathrm{int}} \tag{66}$$

where F_{int} is an integration constant, taken to be the flux emitted from the interior of the planet as it cools.

The differential equations for the upward and downward thermal flux in the Eddington approximation (right? two stream?) are given by **?** (their equations 1 and 2, reproduced here)

$$\frac{dF^+}{d\tau} = D(F^+ - \sigma_{\mathrm{SB}}T^4) \tag{67}$$

$$\frac{dF^-}{d\tau} = -D(F^- - \sigma_{\mathrm{SB}}T^4) \tag{68}$$

where F^+ is the upwelling thermal flux and F^- is the down-welling thermal flux, and σ_{SB} is the Stefan-Boltzmann constant. The parameter D parameterizes the difference between the two-stream approximation and the true solution. It depends on the assumed form of the specific intensity and is generally taken to be between one and two. Together these determine the temperature gradient that the atmosphere must achieve in order to push the required flux through the atmosphere.

These equations are combined and differentiated once to give **?** equation 5 for the net flux in terms of the temperature gradient, reproduced here

$$\frac{d^2F_a}{d\tau^2} - D^2F_a = -2D\sigma_{\mathrm{SB}}\frac{dT^4}{d\tau} \tag{69}$$

where $F_a = F^+ - F^-$. The net flux processed by the atmosphere is known, so equation 69 can be integrated to give the temperature gradient. The boundary condition is set at the top of the atmosphere by subtracting equations 67 and 68 to give

$$\frac{dF_a}{d\tau} = D(F^+ - F^- - 2\pi B) \tag{70}$$

where B is the Planck function for a black body (modulo our understanding of where the π's come in. I think we understood that, right?) F_a is known everywhere, and $F^-[0] = 0$ so that $F^+[0] = F_a[0]$. This suffices to set the temperature at the top of the atmosphere.

From this it is clear that it is trivial to treat any heating or cooling mechanism that serves only to add or remove a fixed quantity vertical flux from the atmosphere independently of the thermodynamic state of the atmosphere. One must simply write down a new version of F_a and then use equation 69 to solve for the temperature profile as before. Fortunately a large class of heating/cooling function fall within this category, among them tidal dissipation and P_n style parameterizations of day-night energy transfer, ohmic dissipation as long as the temperature is high enough to induce sufficient ionization for the process to take place, or vertically propagating gravity waves (**?**). While nearly any physical heating or cooling mechanism will have some dependence on density and temperature, the important consideration to allow applying this method of solving for the temperature given the fluxes is that the mechanism be largely independent of the detailed thermodynamic state of the atmosphere.

Conceptually, one may think of adding an additional channel into which flux can go, the heating/cooling channel, F_{HC}. The $F = -F_* + F_a + F_{HC} + F_{\rm int}$, and equation 64 can be used to find F_a as a function of optical depth. Generally one will want to specify not F_{HC} but $dF_{HC}/d\tau$ so computing F_a will require a definite integral.

Heating and Cooling - General Case— We have treated the simple case in detail as a warm up for the general case, where the heating and cooling is allowed to depend on the local thermodynamic state of the atmosphere. This case is more complicated mostly because we no longer have a priori knowledge of the flux being processed by the atmosphere, nor even of the emergent flux from the top of the atmosphere.

We seek solutions for F^+, F^-, and T. In the case already considered, we had two differential equations (67 and 68) and an algebraic constraint on the net flux, providing sufficient information for a solution.

We can take the equation for energy conservation in steady state

$$\frac{dF}{d\tau} = 0 \tag{71}$$

and again divide the flux into the the flux being processed by the atmosphere, the unabsorbed stellar flux, the flux added/removed by heating/cooling, so that

$$\frac{dF_a}{d\tau} = \frac{dF_*}{d\tau} - \frac{dF_{a,HC}[\rho,T]}{d\tau} \tag{72}$$

where we have indicated the dependence of heating/cooling on density and temperature to emphasize that this function cannot be integrated directly as before.

We could solve for F^+ and F^- directly, but it remains true that there are stronger constraints on F_a compared to F^+ and F^- individually. It is useful to rewrite Equations 67 and 68 by adding and subtracting them to get:

$$\frac{dF_I}{d\tau} = DF_a \tag{73}$$

$$\frac{dF_a}{d\tau} = D(F - 2\sigma_{\rm SB}T^4) \tag{74}$$

where $F_I = F^+ + F^-$. It is proportional to the isotropic intensity of the radiation field, where the constant of proportionality depends on the form of the specific intensity of the radiation field.

Equation 72 gives the effect of heating/cooling on the net flux processed by the atmosphere. We must still write $\frac{dF_{a,HC}[\rho,T]}{d\tau}$ in terms of a more useful function, such as the volumetric cooling function. Equation 73 is a differential equation for something related to the intensity fo the radiation field and it will be modified by the addition/subtraction of flux. Equation 74 will be used to determine the temperature gradient required to push a *known* amount of flux through the atmosphere, where net flux processed by the atmosphere will be determined by equation 72.

Thus we add a term to equation 73 for heating/cooling and arrive at the three equations for our three unknown quantities.

$$\frac{dF_I}{d\tau} = DF_a - \frac{dF_{I,HC}[\rho,T]}{d\tau} \tag{75}$$

$$\frac{dF_a}{d\tau} = D(F - 2\sigma_{\rm SB}T^4) \tag{76}$$

$$\frac{dF_a}{d\tau} = \frac{dF_*}{d\tau} - \frac{dF_{a,HC}[\rho,T]}{d\tau} \tag{77}$$

Next we must specify $dF_{I,HC}[\rho,T]d\tau$ and $\frac{dF_{a,HC}[\rho,T]}{d\tau}$ in a more useful form. To do this, we must make a decision about what fraction of energy will be added/subtracted from the upward radiation stream versus the downward radiation stream. There are two interesting cases: where the amount added/diverted from each stream is proportional to the fraction of energy in that stream (expected for cooling) and where the amount added/diverted is independent of the radiation stream (expected for heating).

Consider the cooling case first. Define a box with horizontal area dA and vertical height dl, along with the volumetric cooling function C. If a fraction f of all photons traversing the box are absorbed, then the flux lost to the radiation streams is:

$$dF^+ = fF^+ \quad dF^- = fF^- \tag{78}$$

Requiring that the total energy lost match the volumetric cooling function gives

$$C = \frac{f(F^+ + F^-)dA}{dA\,dl} \tag{79}$$

so that

$$f = \frac{C dl}{F^{+} + F^{-}} \tag{80}$$

Substituting for f in equation 78 gives the derivatives dF^{+}/dl and dF^{-}/dl. Adding and subtracting these to get dF_I/dl and dF_a/dl, and finally dividing by $\rho\kappa$ to give derivatives in terms of optical depths yields:

$$\frac{dF_{I,HC}[\rho,T]}{d\tau} = -\frac{C}{\rho\kappa} \tag{81}$$

$$\frac{dF_{a,HC}[\rho,T]}{d\tau} = -\frac{CF_a}{\rho\kappa F_I} \tag{82}$$

The other interesting case is a heating mechanism that adds energy isotropically, ie, it adds equally to the upward and downward streams. In this case we have

$$dF^{+} = K \quad dF^{-} = K \tag{83}$$

where K is some constant. Requiring that the total energy lost match the volumetric heating function gives

$$H = \frac{2K dA}{dA\, dl} \tag{84}$$

so that

$$K = \frac{1}{2} H dl \tag{85}$$

Substituting for K in equation 83 and again adding, subtracting, and dividing by $\rho\kappa$ gives

$$\frac{dF_{I,HC}[\rho,T]}{d\tau} = -\frac{H}{\rho\kappa} \tag{86}$$

$$\frac{dF_{a,HC}[\rho,T]}{d\tau} = 0 \tag{87}$$

This at last allows us to write down the set of equations governing the atmosphere in the presence of heating and cooling.

$$\frac{dF_I}{d\tau} = DF_a + \frac{H-C}{\rho\kappa} \tag{88}$$

$$\frac{dF_a}{d\tau} = D(F - 2\sigma_{\rm SB}T^4) \tag{89}$$

$$\frac{dF_a}{d\tau} = \frac{dF_*}{d\tau} + \frac{CF_a}{\rho\kappa F_I} \tag{90}$$

Equations 89 and 90 can be set equal to give an algebraic equation for the temperature if the fluxes are given. This can be solved numerically at every step while while equations 88 and either 89 or 90 are integrated to give the fluxes.

In the presence of general heating and cooling, it is no longer the case that energy added at some depth must eventually find its way out at the surface of the planet. Energy removed by cooling is being transferred to another form that exits the planet without being processed by the atmosphere. This could be, for example, a low opacity "window" that's narrower in frequency than the width of the black body spectrum at the atmospheric temperature. Photons will escape through the window from deeper in the atmosphere than they would if the window didn't exist.

This fact makes it more difficult to set the boundary condition, since we cannot compute the net flux at the surface of the planet beforehand. Define $F_{\rm out}$ to be the flux exiting the planet's atmosphere. We have $F^{+} = F_{\rm out}$, $F^{-} = 0$ so $F_I = F_{\rm out}$ and $F_a = F_{\rm out}$.

Energy conservation still holds, so that $F_{\rm out} = F_{\rm int} + F_* + F_{HC}$. However, we cannot compute F_{HC} until the temperature and density profiles are known, and thus must resort to an iterative method to find the solution. The energy emerging from the surface of the planet as thermal radiation is

$$F_{\rm out} = F_{\rm int} + F_* + \epsilon \int_0^\infty \frac{(H-C)\, d\tau}{\rho\kappa} \tag{91}$$

The factor of ϵ has been added to facilitate a solution since we cannot compute the integral until the temperature and density profiles are known. Therefore we initially take ϵ to be zero and increase it to one in a number of steps to arrive at the self-consistent atmosphere profile with heating and cooling that depends on the detailed thermodynamic state of the atmosphere.

One issue that deserves some thought is what happens if energy is added or removed within the convective region. Any flux removed within the convective region will simply be replaced by more vigorous convection due to the very steep dependence of the convective energy flux on the extent to which the temperature gradient is super-adiabatic for the conditions that prevail in planets. Therefore the internal flux will be larger to compensate. However, the extra internal flux will be used up in keeping the temperature profile nearly adiabatic, and therefore it will not emerge at the top of the atmosphere. The planet will, however, lose its energy more rapidly.

If there is heating within the convective region, it will tend to make the region radiative. If this results in a detached radiative zone, the **?** model is not equipped to handle it. If it simply pushes the radiative-convective boundary deeper, then we find that we in fact deposited the energy in the radiative zone, though the region would have been convective if the heating hadn't been there.

2.5.2. *Robinson + Catling model with gravity*

The **?** model assumes that the planet internal flux and irradiation fluxes are known, allowing one to solve for the temperature profile as a function of optical depth. However, the interesting situation for planet evolution is where the planet mass and radius, and therefore the surface gravity, are known and we would like to *solve* for the internal flux to compute the planet luminosity and hence evolution.

Gravity does not enter into the **?** model, but we may straightforwardly extend it to include gravity.

The equation of hydrostatic equilibrium is

$$\frac{dp}{dr} = \rho g \tag{92}$$

where g is some constant, the surface gravity.

To connect to the **?** model, we write in the derivative in terms of optical depth

$$\frac{dp}{d\tau} = \frac{g}{\kappa(p, T(\tau))} \tag{93}$$

where $T(\tau)$ is given by the **?** model. We seek to integrate this equation and then deduce the dependence of p and τ necessary to ensure that g is constant. It is true that g is not precisely constant, but $dg/d\tau$ is of order the photo mean free path over the planet radius, so it is quite close to constant over the region of interest.

Assume that the opacity has a power law dependence on pressure and temperature:

$$\kappa(p, T) = \kappa_0 \left(\frac{p}{p_\kappa}\right)^a \left(\frac{T}{T_\kappa}\right)^b \tag{94}$$

where κ_0, a, b, p_κ and T_κ are constants, the latter two being kept separate from p_0 and T_0 to keep specification of the opacity separate from the atmosphere structure.

Working in terms of optical depth relieves the **?** model from speaking of pressure in the radiative region. In the convective region, the pressure profile is used to define the temperature profile, where a power law dependence of pressure on optical depth is assumed.

We then have a choice about whether to carry out this integral "low" in the atmosphere (within the convective zone) or "high" in the atmosphere (within the radiative zone).

Consider the "high" case first. The boundary condition specified in the **?** model results in a constant temperature as the optical depth goes to zero, given by

$$\sigma_{\rm SB} T^4 = \frac{1}{2}(F_1(1 + \frac{k_1}{D}) + F_2(1 + \frac{k_2}{D}) + F_{\rm int}) \tag{95}$$

We carry out the integral over the top of the atmosphere, from 0 to $d\tau$, where the temperature may be considered to be constant. In this equation 93 may be integrated using 94 to obtain

$$g = \frac{p_0 \kappa(p_0, T_h)}{(1+a)\tau_0} \tag{96}$$

where T_h has been left as the temperature high in the atmosphere to accommodate cases where atmospheric heating/cooling is considered.

Therefore in this case it is reasonable to use $n = 1 + a$ where a is determined by the specific physical origin of the opacity in question. However, note that in this case we require knowledge of the opacity in the radiative region, but the relation between pressure and optical depth is required only in the deeper convective region. Therefore it is not necessarily required that $n = 1 + a$.

Next consider the "low" case. Here we extend the integral determining the surface gravity into the convective region. Break up the integral of equation 93 into the part over the radiative region and the part over the convective region to obtain

$$g\tau = \int_0^{p_{rc}} \kappa(p, T(p))\, dp + \int_{p_{rc}}^{p} \kappa(p, T(p))\, dp \tag{97}$$

This results in:

$$g = \frac{p_0 \kappa(p_0, T_0)}{\tau_0(a + b\beta + 1)} \left(\frac{\tau}{\tau_0}\right)^{(a+b\beta+1-n)/n} + \frac{1}{\tau}\left[\int_0^{p_{rc}} \kappa(p, T(p)\, dp - \frac{p_0 \kappa(p_0, T_0)}{a + b\beta + 1}\left(\frac{p_{rc}}{p_0}\right)^{a+b\beta+1}\right] \tag{98}$$

where we have used $p/p_0 = (\tau/\tau_0)^{1/n}$, as assumed in the **?** model to pertain in the convective region. The term in brackets will be zero if $p/p_0 = (\tau/\tau_0)^{1/n}$ holds in the radiative region as well. Otherwise, the term in brackets will be a positive constant. However, it is multipled by $1/\tau$ and thus may be made as small as desired by carrying the integral to larger optical depths. Assuming the term in brackets is small, we have

$$g = \frac{p_0 \kappa(p_0, T_0)}{\tau_0(a + b\beta + 1)} \tag{99}$$

where $n = a + b\beta + 1$. In this case, the integral is being carried into the convective region and therefore the relation for n *must* hold in order for the **?** model to be self consistent.

2.5.3. *Characteristics of the Robinson + Catling model*

If one of the stellar irradiation channels deposits energy low in the atmosphere (below the radiative/convective boundary), it raises the possibility of creating a deep radiative zone. The relationship between the atmosphere and the interior of the planet is determined by the *deepest* transition from radiative region to convective region, below which the planet is entirely convective.

Figure 2 shows The quantity $4\beta/n = 4\alpha(\gamma - 1)/n\gamma$, characterizes the atmosphere and is large for atmospheres that are undergoing dry convection, have γ large compared to 1, and have opacities with little dependence on temperature. The parameter α characterizes the deviation of the temperature profile from adiabatic and is driven to values below unity by moist convection.

Figure 3 shows the dependence on the optical depth of the radiative convective transition on the structure of the atmosphere (along the x axis) and the ratio of external flux to internal flux for two cases, where one deposits the energy high in the atmosphere and low in the atmosphere.

Figure 4 shows what happens to the optical depth of the rad/conv boundary if you hold the ratio of external to internal flux fixed and vary the atmosphere (x axis) and depth of energy deposit (y axis).

Figure 5 shows what happens to the optical depth of the rad/conv boundary as a function of external to internal flux ratio and depth of energy deposit.

Figure 6 shows what happens to the internal flux as you change the irradiation temperature for fixed surface gravity. At low Text, the internal temp is whatever it will be, and then as the external flux increases the internal flux drops.

Figure 7 shows the same thing as the previous figure, except that the overall temperature profile is shallower and Tint as a function of Text at fixed surface gravity becomes double-values for certain choices of Text. It is not clear to me what this means.

2.5.4. *Via Tables*

An atmospheric boundary condition can be provided via any function that takes surface gravity and bulk entropy of the planet and computes the luminosity. This can be via a table lookup for such a table computed by a program such as TLUSTY.

3. CONCLUSIONS

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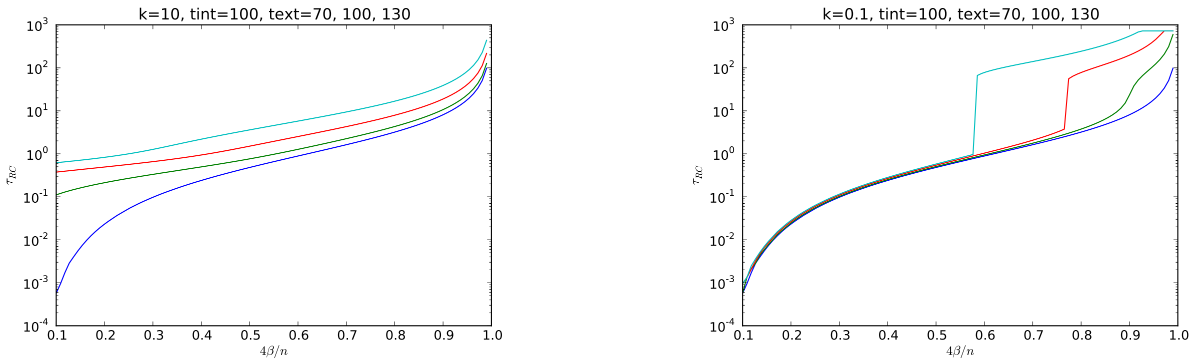

FIG. 2.— Left, optical depth of radiative/convective boundary for a single channel of stellar irradiation, deposited high in the atmosphere ($\tau \simeq 0.1$). The x axis, characterizes the temperature profile of the atmosphere, with shallow profiles on near zero and steep profiles near 1. On the right, the optical depth of the radiative-convective boundary for the case where the stellar irradiation is deposited low in the atmosphere. The jumps in the value of τ_{rc} indication places where a deep radiative layer is formed.

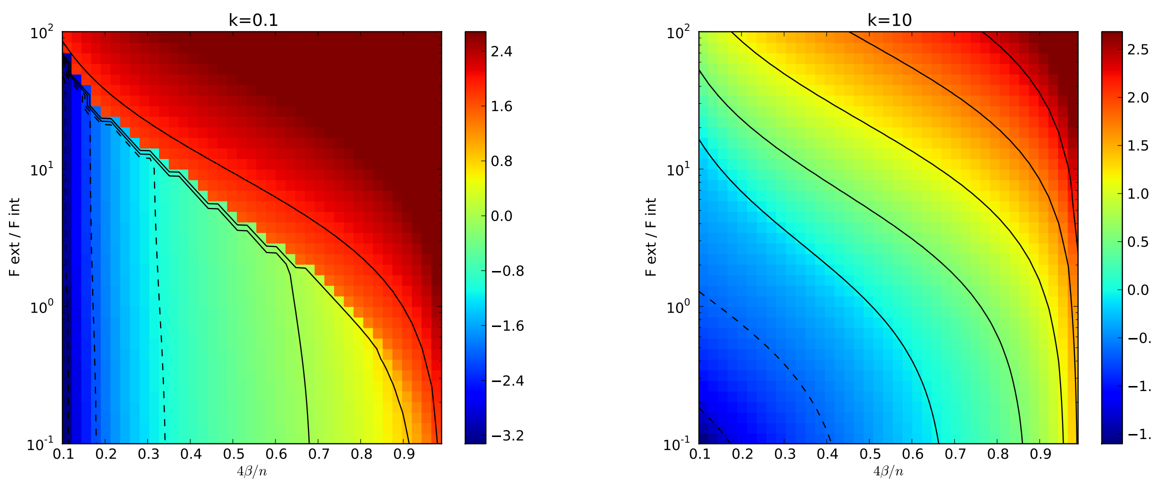

FIG. 3.— Contours of radiative/convective boundary vs. structure of atmosphere (along the x axis) and ratio of external flux to internal flux (y axis). On the left, depositing energy low in the atmosphere, on the right, depositing energy high. The behavior is relatively smooth when depositing energy high in the atmosphere. A discontinuity develops when depositing energy low in the atmosphere, and a fixed amount of energy has a larger effect.

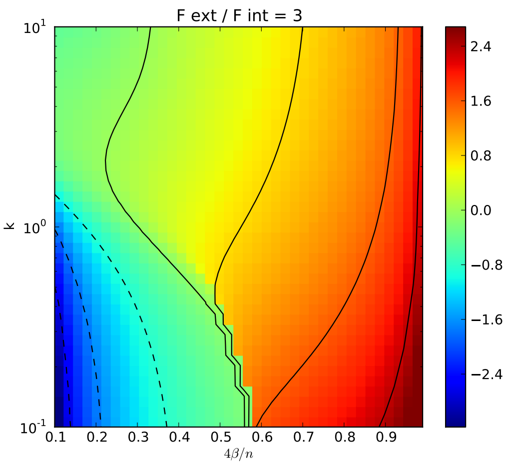

FIG. 4.— If we hold the ratio of external flux to internal flux constant and ask what happens to the rad/conv boundary as the structure of the atmosphere (x axis) and depth of energy deposit (y axis, high is high, low is low), then we can see the "fold" develop as the energy is deposited deeper and deeper in the atmosphere.

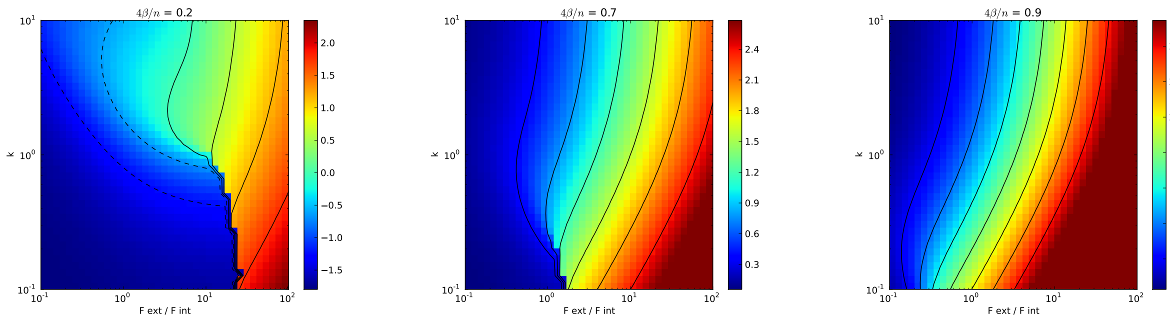

FIG. 5.— Optical depth of the rad/conv transition for three different atmosphere structures, from shallow temp profile (left) to steep temp profile (right). The x axis gives the ratio of external flux to internal flux (small ratio on the left, large ratio on the right) while the y axis gives the depth at which the energy is deposited (low for low, high for high). One sees that depositing the energy deeper has a larger effect, and a "fold" structure develops for atmospheres with shallow temperature profiles.

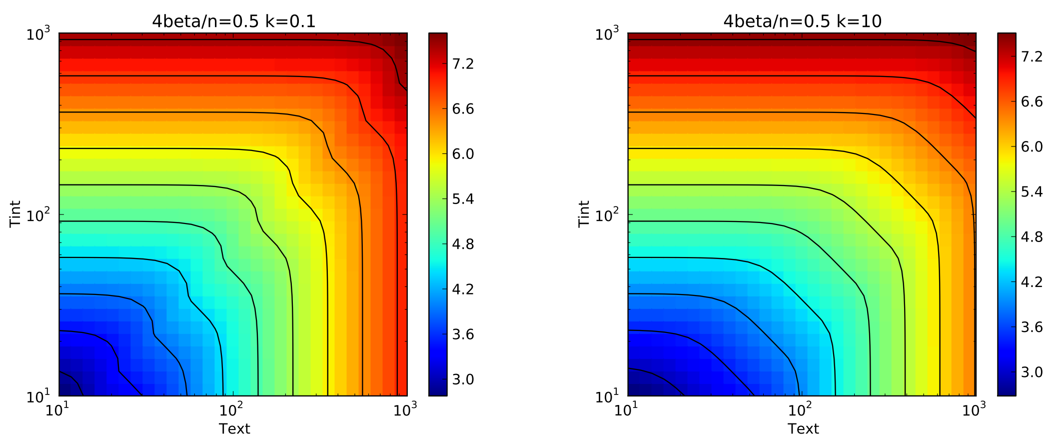

FIG. 6.— Contours of surface gravity as a function of effective temperature of external radiation (x axis) and effective temperature of of planet's cooling radiation (y axis). On the left, for energy deposited deep in the atmosphere, on the right, high in the atmosphere. One can see that at fixed surface gravity (along a contour), increasing Text causes Tint to fall in a well-defined fashion, with somewhat interesting behavior around $Text \simeq Tint$ when depositing energy deep in the atmosphere.

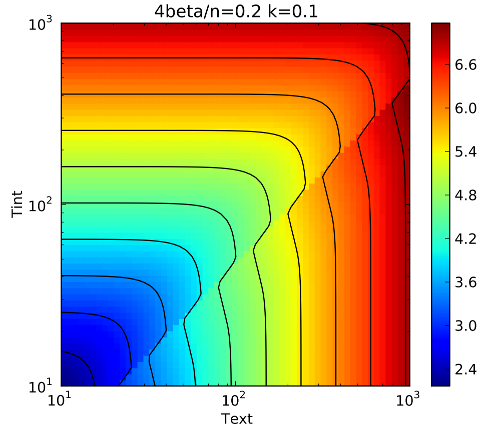

FIG. 7.— Contours of surface gravity as a function of effective temperature of external radiation (x axis) and effective temperature of of planet's cooling radiation (y axis), when depositing energy deep in the atmosphere. The same as the previous figure, except that the structure of the atmosphere is different, the temperature profile being shallower. One sees that the contours of surface gravity develop an "overbite" as Text approaches Tint. This would indicate that for fixed surface gravity, Tint is no longer a single-valued function of Text. I'm not sure what this means. Does stability determine which solution is selected?