Symmetries of the Helmholtz Equation

And Its Separable Coordinate Systems

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Abstract

Separation of variables is by far the most widely used method of solving linear partial differential equations. The Helmholtz equation ($\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \omega^2\Psi = 0$) is one of the simplest equations that can be solved by this method. This equation is widely used in physics, and so it has been studied for many years. It is a well-known fact that there are four orthogonal coordinate systems (up to a certain kind of equivalence) in which this equation separates. Recently, group-theoretical methods have been applied to differential equations to provide an explanation of their separable coordinate systems. One of the main references for this paper ([Miller 1977]) computes the first-order and second-order symmetry algebras for this equation, and shows how these give its four orthogonal separable coordinate systems. This reference, however, never defines what it means for a partial differential equation to separate in a coordinate system. In this paper, we provide some background Lie theory that is necessary for the symmetry analysis, we provide a definition of separation of variables that is applicable in a small but important set of types of equations[1] (a set that, of course, includes the Helmholtz equation in orthogonal coordinates), and we perform the first-order symmetry analysis, which explains two of the Helmholtz equation's separable coordinate systems.

[1] If we had been aware of the work of [Koornwinder 1980] earlier, we would not have needed to come up with our own definition; but we became aware of that too late to change the course of the thesis.

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Contents

Chapter 0

Preliminary Lie Theory

0.1 Lie Algebras

Lie groups and Lie algebras arose naturally out of Sophus Lie's study of the symmetries of differential equations. This paper examines such symmetries — in particular, the symmetries of the Helmholtz equation — and, as a result, much of the discussion in Chapter 2 will be framed in the language of Lie theory.

It is assumed that the reader is familiar with groups, rings, fields, and vector spaces. Another algebraic structure that will be important in later discussion is an algebra.

Definition 0.1 *Given a set $\mathcal{A}$ and a field $\mathcal{K}$, $\mathcal{A}$ is said to be an* algebra *over $\mathcal{K}$ if*
(i) : $\mathcal{A}$ is a vector space over $\mathcal{K}$.
(ii) : $\mathcal{A}$ is closed under a product $\cdot$ that distributes over $+$: Given $a, b, c \in \mathcal{A}$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$.
(iii) : If $\lambda \in \mathcal{K}$ and $a, b \in \mathcal{A}$, then $\lambda(a \cdot b) = (\lambda a) \cdot b = a \cdot (\lambda b)$.

When there is no ambiguity regarding which product we are referring to, we will often write ab instead of $a \cdot b$. Now, notice that there is no stipulation that the product $\cdot$ be commutative or associative. It is often useful to investigate whether two elements of $a, b \in \mathcal{A}$ commute with one another, and, if not, by how much ab differs from ba. We define the commutator $[-,-]$ operation as follows: $[a, b] = ab - ba$. The commutator (also called the "Lie commutator" or "Lie bracket," for

reasons that will be clear presently) can be thought of as measuring the degree of non-commutativity of a and b; if a and b commute, $[a, b] = 0$.

Theorem 0.1 *Let $\mathcal{A}$ be an associative algebra. Then, given $a, b, c \in \mathcal{A}$, and $k \in \mathcal{K}$, the commutator operation $[-, -]$ satisfies the following properties:*

$(i) : [a, b] = -[b, a]$

$(ii) : [a, kb] = [ka, b] = k[a, b]$

$(iii) : [a, b + c] = [a, b] + [a, c]$ *and* $[a + c, b] = [a, b] + [c, b]$

$(iv) :$ (Jacobi Identity) $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$

Each of these properties can be verified by a straightforward computation, and so we will omit the verifications.

Example 0.1

Let $\mathcal{K}$ be any field. An example of an algebra is $M_{n\times n}(\mathcal{K})$, the set of all n by n matrices whose entries are elements of $\mathcal{K}$, under ordinary matrix multiplication. We will not prove that this is an algebra, but the reader should recall that $M_{n\times n}(\mathcal{K})$ is a classic example of a vector space over $\mathcal{K}$, and that matrix multiplication distributes over addition. This algebra happens to be associative and noncommutative, but we will omit the proof of this claim.

Definition 0.2 *If $\mathcal{A}$ is an algebra that has a multiplicative identity I, then, $x \in \mathcal{A}$ is said to be a* unit *if there exists an element $y \in \mathcal{A}$ such that $xy = yx = I$.*

It is routine to check that the set $\mathcal{A}^\times$ of units of an associative algebra $\mathcal{A}^\times$ is a group under multiplication: (i) $I \in \mathcal{A}^\times$; (ii) every element of $\mathcal{A}^\times$ has an inverse; (iii) $\cdot$ is associative; and (iv) given $a, b \in \mathcal{A}^\times$, choose any $c, d \in \mathcal{A}^\times$ such that ($ac = ca = I$ and $bd = db = I$), and since $\cdot$ is associative it is clear that $(ab)(dc) = (dc)(ab) = I$.

Notice that since it turns out that $\mathcal{A}^\times$ is a group, the inverses had to be unique after all; and so, for the c and d of (iv), we may write $c = a^{-1}$, $d = b^{-1}$, and $(b^{-1}a^{-1}) = (ab)^{-1}$.

Definition 0.3 *The set $\mathcal{A}^\times$ of units of an associative algebra $\mathcal{A}$ is called the* group of units of $\mathcal{A}$.

Example 0.2

Let $\mathcal{K}$ be a field. When $M_{n\times n}(\mathcal{K})$ (also written $gl(n,\mathcal{K})$) is the associative algebra under consideration, the group of units is called the general linear group of $\mathcal{K}^n$ and is denoted $GL(n,\mathcal{K})$. If 1 is the identity of $\mathcal{K}$, then it is clear that

$$I = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & & & \\ \vdots & & \ddots & & \vdots \\ & & & 1 & 0 \\ 0 & & \cdots & 0 & 1 \end{pmatrix},$$

with 1 along the main diagonal and 0 everywhere else, is the identity of $GL(n,\mathcal{K})$. Note that $GL(n,\mathcal{K})$ is the group of invertible n by n matrices, which also happens to be the set of matrices with nonzero determinants.

Definition 0.4 *Let $\mathcal{K}$ be a field. A set $G \subseteq GL(n,\mathcal{K})$ is said to be a* matrix group *if it is a subgroup under with ordinary matrix multiplication. The* special linear group *$SL(n,\mathcal{K})$ is the set of $A \in GL(n,\mathcal{K})$ such that* $\det(A) = 1$.

Example 0.3

Consider if our field is $\mathbf{R}$ and $n = 2$. $SL(2,\mathbf{R})$ is then seen to be:

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbf{R} \text{ and } ad - bc = 1 \right\}.$$

Definition 0.5 *Given a set $\mathcal{A}$ and a field $\mathcal{K}$, $\mathcal{A}$ is said to be a* Lie algebra *over $\mathcal{K}$ if it is closed under a product operation $[-,-]$ that satisfies properties (i) through (iv) of Theorem 0.1. This product operation is referred to as the* Lie bracket.

The terminology might be somewhat confusing here. The product operation from Definition 0.5 is not required to be a commutator, as it is in the hypotheses of Theorem 0.1. If it *were*, there would have to be an additional product defined on the Lie algebra. But the definition of a Lie algebra only requires one product operation. Now, it turns out that every Lie algebra has an associated Universal Enveloping Algebra for which the Lie algebra's $[-,-]$ is a commutator; and this is why

the Lie bracket is often also called is often called the "Lie commutator." But to prove this or to discuss it further would take us too far afield. At this point, it will be useful to consider several examples of Lie algebras.

Example 0.4
The set $M_{n\times n}(\mathbf{R})$ from Example 0.1 is a Lie algebra if we define $[A,B] = AB - BA$, for all $A, B \in M_{n\times n}(\mathcal{K})$.

Definition 0.6 *We denote by* $\underline{sl(n,\mathcal{K})}$ *the set of all matrices* $A \in M_{n\times n}$ *such that* $\mathrm{tr}(A) = 0$, *where* $\mathrm{tr}(A)$, *the trace of* A, *is the sum of the diagonal entries of* A.

Example 0.5
It turns out that $sl(n,\mathbf{R})$ is a Lie algebra under commutator. The only property of Lie algebras that is in question is closure under Lie bracket. Recall that given two matrices A and $B \in M_{n\times n}$, $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. Recall also that $\mathrm{tr}(-)$ is a linear operator; that is, $\mathrm{tr}(A+c\cdot B) = \mathrm{tr}(A)+c\cdot\mathrm{tr}(B)$. Thus, $\mathrm{tr}([A,B]) = \mathrm{tr}(AB-BA) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = 0$. So, $sl(n,\mathbf{R})$ is closed under Lie bracket.

Notice that $sl(n,\mathbf{R})$ (for $n > 1$) is not closed under ordinary matrix multiplication. Thus, if we were in a world that contained only elements of $sl(n,\mathbf{R})$, it would not make sense to speak of $[-,-]$ as being a commutator (for, given $A, B \in sl(n,\mathbf{R})$, we could not in general make sense of AB). The reason we do speak of it as a commutator is that we think of $sl(n,\mathbf{R})$ as a subset of $M_{n\times n}(\mathbf{R})$, and the latter is an enveloping algebra for the former — that is, the Lie bracket of $sl(n,\mathbf{R})$ is a commutator in $M_{n\times n}(\mathbf{R})$.

0.2 The One-Parameter Subgroup

Here, we will give only a brief introduction to this concept, for we will revisit it in detail later. It should be assumed that all vector spaces that we discuss have metrics defined on them. In fact, for the most part we will deal only with vector spaces over $\mathbf{R}$, although vector spaces over $\mathbf{C}$ are in most respects completely analogous. In order to define a one-parameter subgroup, we must build up some machinery; so we will begin by introducing some necessary terms.

First, observe that a matrix group $G \subseteq M_{n\times n}(\mathbf{R}) \cong \mathbf{R}^{n^2}$ has a natural metric d_G induced by the usual metric in $\mathbf{R}^{n^2}$.

Definition 0.7 *If G and H are matrix groups with associated metrics d_G and d_H, and $\phi : G \longrightarrow H$ is a homomorphism such that for every $g_0 \in G$ and for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $d_G(g, g_0) < \delta$, $d_H(\phi(g), \phi(g_0)) < \epsilon$, then we say that ϕ is a* continuous homomorphism.

From now on, it will be assumed that the matrix groups that we speak of are in vector spaces that have metrics defined on them, and that the homomorphisms that we speak of are continuous.

Definition 0.8 *If V is a finite-dimensional vector space over $\mathbf{R}$, then $\gamma : (a, b) \longrightarrow V$ is said to be a* curve *in V (where (a, b) denotes an open interval in $\mathbf{R}$).*

Definition 0.9 *If γ is a curve in a vector space V, then γ is said to be* differentiable *if for all $t \in (a, b)$, $\lim_{h\to 0} \frac{\gamma(t+h)-\gamma(t)}{h}$ exists.*

Definition 0.10 *If G and H are matrix groups, and $\phi : G \longrightarrow H$ is a continuous homomorphism, then ϕ is said to be a* smooth homomorphism *if for every differentiable curve γ in G, $\phi \circ \gamma$ is a differentiable curve in H.*

Example 0.6
Let $G = H = GL(n, \mathbf{R})$. The identity map $A \longmapsto A$ for all A is a smooth homomorphism.

Definition 0.11 *Let $\gamma : \mathbf{R} \longrightarrow G$ be a homomorphism from $\mathbf{R}$ to some matrix group G, that is differentiable in the sense of Definition 0.9. Then we say that γ is a* one-parameter subgroup *in G.*

Since the identity map on $\mathbf{R}$, $f_I(x) = x$, is a differentiable curve in $\mathbf{R}$, the fact that γ is a smooth homomorphism implies that $\gamma \circ f_I$ is a differentiable curve in G. But $\gamma \circ f_I$ is just γ. So a one-parameter subgroup is a mapping that is a differentiable curve in G whose image is a

subgroup of G. This follows from the fact that a one-parameter subgroup γ has the property that $\gamma(t_1) \cdot \gamma(t_2) = \gamma(t_1 + t_2)$ (proof omitted[1]).

A useful example of a one-parameter subgroup has to do with the exponential of a matrix. The exponential of a square matrix is the natural generalization of the exponential of a real (or complex) number — using the MacLaurin series definition of exponential.

Definition 0.12 *Let $A \in M_{n\times n}(\mathcal{K})$, where $\mathcal{K} \in \{\mathbf{R}, \mathbf{C}\}$. We define the* exponential *of A ($\exp A$ or e^A) to equal $\sum_{n=0}^{\infty} \frac{1}{n!} A^n$.*

We will now state without proof several properties of the matrix exponential.

(i): The infinite sum defining e^A converges for every A.

(ii): If matrices A and B commute, then $e^{A+B} = e^A e^B = e^B e^A$.

(iii): Given $A \in M_{n\times n}(\mathbf{R})$ or $M_{n\times n}(\mathbf{C})$, $\det(e^A) = e^{\mathrm{tr}(A)} \neq 0$.

And now we are ready for an example of a one-parameter subgroup.

Example 0.7

We define $\gamma : \mathbf{R} \longrightarrow M_{n\times n}(\mathbf{R})$ as follows. Let $A \in M_{n\times n}(\mathbf{R})$. Then, for $t \in \mathbf{R}$, $\gamma(t) = \exp(tA)$. We then have that γ is a one-parameter subgroup in $GL(n, \mathbf{R})$. Why? Well, let f be a differentiable curve in $\mathbf{R}$. Then $(\gamma \circ f)(t) = \gamma(f(t)) = \exp(f(t)A)$, and this happens to be a differentiable curve in $GL(n, \mathbf{R})$ (proof omitted). That the image of γ lies in $GL(n, \mathbf{R})$ follows from property (iii) above: the determinant of a matrix exponential never vanishes.

Example 0.8

We define $\gamma : \mathbf{R} \longrightarrow SL(n, \mathbf{R})$ as follows. Let $A \in sl(n, \mathbf{R})$. Then, for $t \in \mathbf{R}$, $\gamma(t) = \exp(tA)$. It is easy to verify that γ is a one-parameter subgroup in $SL(n, \mathbf{R})$: Since $\mathrm{tr}(A) = 0$, $\det(e^{tA}) = e^{t\cdot \mathrm{tr}(A)} = e^0 = 1$; and thus, $e^{tA} \in SL(n, \mathbf{R})$.

[1] For a more complete discussion of matrix groups, see for instance [Curtis 1979].

0.3 The Tangent Space

In general, a tangent space (for instance, a tangent line to a curve, or a tangent plane to a surface) consists of the set of tangent vectors. The case of matrix groups is no exception.

Definition 0.13 *Let $\gamma : \mathbf{R} \longrightarrow V$ be a curve that is differentiable at x_0. We define $\gamma'(x_0) = \frac{d\gamma}{dx}(x_0)$ to equal $\lim_{h\to 0} \frac{\gamma(x_0+h)-\gamma(x_0)}{h}$, and we call $\gamma'(x_0)$ the* tangent vector *at $\gamma(x_0)$.*

A tangent space, properly speaking, is a tangent space at a point (or vector) to a *submanifold* (for our purposes, a submanifold of $\mathbf{R}^n$). So the idea of a submanifold is a logically prior concept. We will not define a submanifold of $\mathbf{R}^n$ carefully, but will instead rely on our intuition.[2] Roughly speaking, a d-dimensional submanifold of $\mathbf{R}^n$ is a subset of $\mathbf{R}^n$, that, in a small enough neighborhood of every point, resembles $\mathbf{R}^d$, for some d, $0 < d < n$. For instance, a smooth curve is a 1-dimenional submanifold in $\mathbf{R}^n$ for $n \geq 2$, because in a small enough neighborhood of every point, the curve resembles a line. Similarly, a well-behaved surface is a 2-dimensional submanifold in $\mathbf{R}^n$ for $n \geq 3$, because in a small enough neighborhood of every point, the surface resembles a plane.

Definition 0.14 *Let S be a submanifold in $\mathbf{R}^n$, and let $\mathbf{v}_0 \in S$. Let $\Gamma = \{\gamma : (-\epsilon, \epsilon) \longrightarrow S$ for some $\epsilon > 0$, where γ is differentiable and $\gamma(0) = \mathbf{v}_0\}$. We define the* tangent space *to S at $\mathbf{v}_0$ to be*

$$T_{\mathbf{v}_0}S = \{\gamma'(0) : \gamma \in \Gamma\}.$$

One can prove that $T_{\mathbf{v}_0}S$ is a subspace of $\mathbf{R}^n$ of dimension $d = \dim(S)$.

Example 0.9

A simple example is the tangent space to the unit circle parametrized by $\gamma(\theta) = (\cos(\theta+\pi/2), \sin(\theta+\pi/2))$, for $-\pi \leq \theta \leq \pi$, at $(0,1)$. Every curve in this submanifold that maps 0 to $(0,1)$ is of the form $(\cos f(\theta), \sin f(\theta))$, where f is differentiable and $f(0) = \pi/2$. The tangent vectors at $(0,1)$ are then of the form $(-\sin(\pi/2)f'(0), \cos(\pi/2)f'(0)) = (-f'(0), 0)$, and the tangent space is the set of points of the form $(-f'(0), 0)$ for all differentiable f, which is just the subspace $y = 0$.

[2] For a proper definition of a manifold and a submanifold, see [Munkres 1991].

0.4 Lie Groups

Lie groups, much like manifolds, and submanifolds, are objects whose general definition is more complicated than can be explored in this paper. As with the other two, we can give examples of Lie groups without knowing the general definition. The important aspect of Lie groups for our purpose is that they are intimately related to Lie algebras. If we have a Lie algebra $\mathcal{A}$ of matrices, there is an associated Lie group $\mathcal{G}$ of matrices that is obtained by exponentiating the elements of $\mathcal{A}$. The Lie algebra of a Lie group is the tangent space at the group's identity. Several examples of this follow.

Example 0.10

Recall from Example 0.2 that $gl(n, \mathbf{R})$ is another name for $M_{n\times n}(\mathbf{R})$. Now we see why: as we saw in Example 0.7, given an element $A \in gl(n, \mathbf{R})$, $\exp(A) \in GL(n, \mathbf{R})$, and, since for all $B \in GL(n, \mathbf{R})$, it turns out that there exists some $A \in gl(n, \mathbf{R})$ such that $B = \exp(A)$. Furthermore, one can prove that $gl(n, \mathbf{R})$ is the tangent space at the identity of $GL(n, \mathbf{R})$. The Lie algebra and the Lie group are similar entities, and so we name them similarly, differentiating the names only by capitalizing the Lie group.

Example 0.11

Consider the Lie algebra $\mathcal{E}(2)$ consisting of elements of the form

$$\begin{pmatrix} 0 & \theta & 0 \\ -\theta & 0 & 0 \\ a & b & 0 \end{pmatrix}.$$

A basis for this can be written as follows:

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{P_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{P_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

One can easily verify that the commutation relations are $[\mathbf{P_1}, \mathbf{P_2}] = [\mathbf{P_2}, \mathbf{P_1}] = 0$, $[\mathbf{M}, \mathbf{P_1}] = \mathbf{P_2}$, and $[\mathbf{M}, \mathbf{P_2}] = -\mathbf{P_1}$. (Clearly, as always, each element commutes with itself.)

This algebra is the tangent space at the identity of the Lie group $E(2)$, called the *Euclidean group in the plane*, which consists of matrices of the form $g(\theta, a, b)$ for $\theta, a, b \in \mathbf{R}$, where

$$g(\theta, a, b) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ a & b & 1 \end{pmatrix}.$$

The elements of $E(2)$ act on points in the plane as follows: if $\mathbf{x} = (x, y)$ and $g \in E(2)$, then $\mathbf{x}g = (x', y')$, where $x' = \hat{\imath} \cdot \Big((x, y, 1) \times g\Big)$ (with usual matrix multiplication and the ordinary dot-product) and $y' = \hat{\jmath} \cdot \Big((x, y, 1) \times g\Big)$. Note: $\hat{\imath}$ and $\hat{\jmath}$ are unit vectors in the x and y directions, respectively. Thus if $\mathbf{x} = (x, y)$, then

$$\mathbf{x}g(\theta, a, b) = (x\cos\theta + y\sin\theta + a, -x\sin\theta + y\cos\theta + b),$$

so that $\mathbf{x}g$ is a composition of a rotation and a translation.

The elements of $E(2)$ can also be taken to act on functions on the plane. If we are discussing the action of $g \in E(2)$ on a function $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$, we write T_g instead of g, so as to distinguish the notation from when we are discussing the action of g on a point. We then have $T_g f(\mathbf{x}) = f(\mathbf{x}g)$.

Note: since every element of $E(2)$ is the exponential of some element of $\mathcal{E}(2)$ (proof omitted), we always can speak of the action of T_g on f as the action of $e^{c_1\mathbf{P_1} + c_2\mathbf{P_2} + c_3\mathbf{M}}$ on f, for some $c_1, c_2, c_3 \in \mathbf{R}$.

The Lie group $E(2)$ and the Lie algebra $\mathcal{E}(2)$ will be very important to the later discussion.

Chapter 1

Separation of Variables

1.1 Solving the Helmholtz Equation by Separating the Variables

When a solution to a partial differential equation is the product of several functions, each of which is a function of just one of the independent variables, we say that the solution *separates*, and we may find this solution to the differential equation through the method of *separation of variables*. (We will actually revise this description of separable solutions later in the paper, but it is a good first approximation.) One of the simplest equations that is solvable via this method is the Helmholtz equation:

$$\mathcal{H}\Psi(x,y) = (\Delta_2 + \omega^2)\Psi(x,y) = 0, \tag{1.1}$$

where $\mathcal{H} = \Delta_2 + \omega^2$, with $\omega \in \mathbf{R} - \{0\}$, is the so-called *Helmholtz operator* (made precise below).

Let $\mathcal{F}$ denote the set of all $\mathbf{C}$-valued real-analytic functions on Ω, a non-empty open, connected subset of $\mathbf{R}^2$. In (1.1), Δ_2 is the two-dimensional Laplace operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, which we will often write $\partial_{xx} + \partial_{yy}$. Similarly, we will often write $\frac{\partial^3}{\partial x^3}$ as ∂_{xxx}. Sometimes, however, it will prove convenient to write multiple derivatives as iterated first derivatives: $\frac{\partial^n}{\partial x^n} \equiv \partial_x^n$.

Above, we mention the Helmholtz operator, for, when discussing differential equations, it is helpful to use the language of differential operators. We will now state explicitly what these are. Consider a (partial) differential equation of the form

$$\sum_{i=0}^{m}\sum_{j=0}^{n} C_{ij}\partial_x^i\partial_y^j f(\mathbf{x}) = 0 \qquad (C_{ij} \in \mathcal{F},\, \mathbf{x} \in \Omega).$$

In such a situation, we say that $Q = \sum_{i=0}^{m} \sum_{j=0}^{n} C_{ij} \partial_x^i \partial_y^j$ is the *differential operator* associated with this equation; and Q acts on a function f just as one would expect: $Q(f) = \sum_{i=0}^{m} \sum_{j=0}^{n} C_{ij} \partial_x^i \partial_y^j f$. Note: we will often write $Q(f)$ as Qf.

Now, to solve (1.1) by separating the variables, we assume that a solution Ψ_0 can be expressed as a product of a real-analytic function only of x and a real-analytic function only of y:

$$\Psi_0(x, y) = X(x)Y(y).$$

Since Ψ_0 satisfies (1.1), we have

$$X''(x)Y(y) + X(x)Y''(y) + \omega^2 X(x)Y(y) = 0. \tag{1.2}$$

It turns out that if $X(x)Y(y) \not\equiv 0$, we may divide through by XY and make sense of the resulting quotient functions. In Section 2 of this chapter, we will return to a justification for this statement, but for now we assume it is true and proceed to divide through by XY. After rearranging terms, we arrive at

$$\frac{X''}{X} = -\frac{Y''}{Y} - \omega^2. \tag{1.3}$$

In (1.3), the left side is a function only of x, and the right side is a function only of y. The only way for this equality to hold is if both sides are equal to a constant. If they were not, then, by holding constant either x or y and varying the other, we could force the equality not to hold. We therefore write $\frac{X''}{X} = -k^2$, for some $k \in \mathbf{C}$. We call $-k^2$ the *separation constant*.

Note that we have now reduced (1.1) to two ordinary differential equations:

$$X''(x) + k^2 X(x) = 0, \tag{1.4}$$

and

$$Y''(y) + (\omega^2 - k^2)Y(y) = 0. \tag{1.5}$$

Let $X_1(x) = e^{ikx}$ and $X_2(x) = e^{-ikx}$, $Y_1(y) = e^{i\sqrt{\omega^2 - k^2}y}$ and $Y_2(y) = e^{-i\sqrt{\omega^2 - k^2}y}$. Solutions to (1.4) are of the form $X(x) = A_1 X_1(x) + A_2 X_2(x)$, and solutions to (1.5) are of the form $Y(y) = B_1 Y_1(y) + B_2 Y_2(y)$, for A_i, $B_i \in \mathbf{C}$ with $i \in \{1, 2\}$.

Thus, we see that a typical separated solution Ψ_0 to (1.1) may be expressed as follows:

$$\Psi_0(x,y) = (A_1 e^{ikx} + A_2 e^{-ikx})(B_1 e^{i\sqrt{\omega^2-k^2}y} + B_2 e^{-i\sqrt{\omega^2-k^2}y}). \tag{1.6}$$

So far, we have not imposed any restrictions on k; we have merely required that both sides of (1.3) equal some constant. In practice, one often uses the method of separation of variables to solve so-called boundary-value problems, where a solution Ψ to (1.1) must take on certain values at the boundary of a region. In such cases, we have restrictions on the possible values of k. Since (1.1) is a linear partial differential equation, a well-behaved infinite sum of solutions is also a solution. In fact, all solutions to (1.1) can be expressed as (possibly infinite) linear combinations of separated (product) solutions. Since the separated solutions involve the complex exponential, it turns out that solutions to the boundary-value problem of (1.1) together with some boundary condition can often be expressed as Fourier series.[1]

1.2 The Field of Fractions

We now return to the issue, raised earlier, of why we may divide (1.2) through by XY.

Later in the discussion, the connectedness of Ω will prove to be important, so we now prove a lemma about open, connected sets.

Definition 1.1 *A topological space $(\mathbf{X}, \tau)$, where τ is the topology on $\mathbf{X}$, is said to be* connected *if and only if the only sets $V \in \tau$ that are closed are $\mathbf{X}$ itself and $\emptyset$.*[2] *We extend this definition to subsets $U \subseteq \mathbf{X}$ by saying that U is* connected *if and only if the only sets $V \subseteq \tau_U$ that are closed are U itself and $\emptyset$, where τ_U denotes the subspace topology on U.*

Definition 1.2 *A topological space $(\mathbf{X}, \tau)$ is said to be* path connected *if and only if given any two (not necessarily distinct) points P and P' in $\mathbf{X}$, there exists a continuous map γ from the interval*

[1] For a clear exposition of using separation of variables in practice, see [Hildebrand 1976].

[2] This can easily be shown to be equivalent to the perhaps more common definition that $(\mathbf{X}, \tau)$ is connected if and only if it is not disconnected, and it is disconnected if and only if there exist non-empty, disjoint sets $U, V \in \tau$ such that $\mathbf{X} = U \cup V$.

$[0,1]$ *to* $\mathbf{X}$ *such that* $\gamma(0) = P$ *and* $\gamma(1) = P'$. *Such a map is called a* path *connecting* P *and* P'. *We extend this definition to subsets* $U \subseteq \mathbf{X}$ *in the natural way:* U *is* path connected *if and only if given any two points in* U, *there is a path connecting them lying entirely in* U.

Lemma 1.1 *If some open subset* $U \subseteq \mathbf{R}^2$ *is connected, then it is path connected.*

Proof: If $U = \emptyset$, it is vacuously path connected. If $U \neq \emptyset$, let $\mathbf{x}_0 \in U$. Let $A = \{\mathbf{x} \in U$: there exists a path γ whose image lies entirely within U connecting $\mathbf{x}_0$ and $\mathbf{x}\}$. We will show that A is both open and closed in U.

Open: Choose $\mathbf{x} \in A$, with an associated path γ connecting it to $\mathbf{x}_0$. Choose δ small enough so that that $B(\mathbf{x}, \delta) \subseteq U$, where $B(\mathbf{x}, \delta) = \{\mathbf{y} \in \mathbf{R}^2 : \|\mathbf{y} - \mathbf{x}\| < \delta\}$. Choose $\mathbf{x}_1 \in B(\mathbf{x}, \delta)$. Define $\psi : [0,2] \to U$ as follows:

$$\psi(t) = \begin{cases} \gamma(t) & : \ 0 \leq t \leq 1 \\ \mathbf{x} + t(\mathbf{x}_1 - \mathbf{x}) & : \ 1 < t \leq 2 \end{cases}$$

and define $\gamma_1(t) = \psi(2t)$. Note that ψ is continuous, for $\lim_{t \to 1^-} \psi(t) = \lim_{t \to 1^+} \psi(t) = \psi(1) = \mathbf{x}_0$. Now $\gamma_1 : [0,1] \longrightarrow U$ is a path connecting $\mathbf{x}_0$ and $\mathbf{x}_1$ whose image clearly lies entirely within U. Thus, $\mathbf{x}_1 \in A$. Since $\mathbf{x}_1$ was arbitrary, $B(\mathbf{x}, \delta) \subseteq A$. Since $\mathbf{x}$ was arbitrary, A is open.

Closed: Choose $\mathbf{x}_1 \in \overline{A}$ (the closure of A in U). Since A is open in U, we may choose δ small enough so that $B(\mathbf{x}_1, \delta) \subseteq U$. Since $\mathbf{x}_1 \in \overline{A}$, we have that $B(\mathbf{x}_1, \delta)$ contains some point $\mathbf{x} \in A$. But now we use the same construction we used above. Given a path γ that connects $\mathbf{x}_0$ to $\mathbf{x}$, we define

$$\psi(t) = \begin{cases} \gamma(t) & : \ 0 \leq t \leq 1 \\ \mathbf{x} + t(\mathbf{x}_1 - \mathbf{x}) & : \ 1 < t \leq 2 \end{cases}$$

and $\gamma_1(t) = \psi(2t)$. Now γ_1 is a path connecting $\mathbf{x}_0$ and $\mathbf{x}_1$ whose image clearly lies entirely within U, and thus $\mathbf{x}_1$ meets the criterion for being in A. Since $\mathbf{x}_1 \in A$, and $\mathbf{x}_1$ was arbitrary, we have that $\overline{A} \subseteq A$, and thus A is closed.

Since U is connected, and $A \subseteq U$ is non-empty and both open and closed, Definition 1.1 implies that $A = U$. Since $\mathbf{x}_0$ was arbitrary, there exists a path connecting every pair of points in U. In other words, U is path connected. $\diamondsuit$

Lemma 1.2 *Let $f \in \mathcal{F}$, the set of real-analytic functions on Ω, and let there be an open set $U \subseteq \Omega$ such that for all $\mathbf{x} \in U$, $f(\mathbf{x}) = k$. Then for all $\mathbf{x} \in U$, $\nabla f(\mathbf{x}) = \mathbf{0}$, where ∇ denotes gradient, and $\mathbf{0}$ denotes the zero vector.*

Proof: Let $\mathbf{x} \in U$. $\partial_x f(\mathbf{x}) = \lim_{h\to 0} \frac{f(\mathbf{x}+h\hat{\imath})-f(\mathbf{x})}{h}$, where $\hat{\imath}$ is a unit vector in the x-direction. Since U is open, there exists a neighborhood of $\mathbf{x}$ contained entirely in U; and for small enough h, $\mathbf{x} + h\hat{\imath}$ will be in this neighborhood. Thus, for small enough h, $f(\mathbf{x} + h\hat{\imath}) = k = f(x)$. So, $\lim_{h\to 0} \frac{f(\mathbf{x}+h\hat{\imath})-f(\mathbf{x})}{h} = \lim_{h\to 0} \frac{k-k}{h} = 0 = \partial_x f(\mathbf{x})$. Similarly, $\partial_y f(\mathbf{x}) = 0$. This completes the proof. $\diamondsuit$

Lemma 1.3 *Let $f \in \mathcal{F}$. Suppose that there is an open set $U \subseteq \Omega$ such that, for all $\mathbf{x} \in U$, $f(\mathbf{x}) = 0$. Then*

$$\textit{For all } \mathbf{x} \in U \textit{ and for all } i, j \in \mathbf{N}, \ \left(\partial_x^i \partial_y^j f\right)(\mathbf{x}) = 0.$$

Proof: We will prove by induction on n that, for all $n \in \mathbf{N}$,

$$i, j \leq n \Longrightarrow \text{ for all } \mathbf{x} \in U \ \left(\partial_x^i \partial_y^j f\right)(\mathbf{x}) = 0. \tag{1.7}$$

Base Case: $n = 0$. Here, (1.7) holds by hypothesis.

Inductive Hypothesis: (1.7) *holds for* $n = N$. This means that for all $\mathbf{x} \in U$, $\left(\partial_x^N \partial_y^N f\right)(\mathbf{x}) = 0$. We must show that for all $\mathbf{x}$ in this same U, (i) $\left(\partial_x^{N+1} \partial_y^N f\right)(\mathbf{x}) = 0$, (ii) $\left(\partial_x^N \partial_y^{N+1} f\right)(\mathbf{x}) = 0$, and (iii) $\left(\partial_x^{N+1} \partial_y^{N+1} f\right)(\mathbf{x}) = 0$. The proof of (ii) is identical in form to the proof of (i), and so we will omit it.

(i) : Note that since for all $\mathbf{x} \in U$, $\left(\partial_x^N \partial_y^N f\right)(\mathbf{x}) = 0$, by Lemma 1.2, we have that for all $\mathbf{x} \in U$,

$$\left(\partial_x \left[(\partial_x^N \partial_y^N f)\right]\right)(\mathbf{x}) = 0$$

Which means that, for all $\mathbf{x} \in U$,

$$\partial_x^{N+1} \partial_y^N f(\mathbf{x}) = 0.$$

(iii) : From (i) $\partial_{N+1} \partial_y^N f(\mathbf{x}) = 0$. Thus, by Lemma 1.2, we have that for all $\mathbf{x} \in U$,

$$\left(\partial_y \left[(\partial_x^{N+1} \partial_y^N f)\right]\right)(\mathbf{x}) = 0.$$

This means that, for all $\mathbf{x} \in U$,

$$\partial_x^{N+1} \partial_y^{N+1} f(\mathbf{x}) = 0.$$

This completes the proof by induction. $\diamondsuit$

Lemma 1.4 *Let $f \in \mathcal{F}$, the set of real-analytic functions on Ω, a non-empty, open, connected subset of $\mathbf{R}^2$. If f is zero everywhere on, a nonempty, open subset U of Ω, then f is zero on all of Ω.*

Proof: Let $A = \{\mathbf{y} \in \Omega :$ there exists a subset $B(\mathbf{y}, \delta) \subseteq \Omega$ for some $\delta > 0$ such that $f(\mathbf{x}) = 0$ for all $\mathbf{x} \in B(\mathbf{y}, \delta)\}$; that is, A is the set of all points in Ω in a neighborhood of which $f \equiv 0$. We now proceed with reasoning somewhat similar to that used in the proof of Lemma 1.1. We claim that A is both open and closed in Ω.

Open: Choose $\mathbf{y}_0 \in A$. Choose $\delta > 0$ such that $B(\mathbf{y}_0, \delta) \subseteq \Omega$ and $f \equiv 0$ in $B(\mathbf{y}_0, \delta)$. Then $B(\mathbf{y}_0, \frac{\delta}{2}) \subseteq A$, since it is clear that every point in $B(\mathbf{y}_0, \frac{\delta}{2})$ has a $\frac{\delta}{2}$-neighborhood of it in which $f \equiv 0$.

Closed: Choose $\mathbf{y}_0 \in \overline{A}$. Choose a sequence of points $\{\mathbf{y}_n\}_{n=1}^{\infty}$, each of which is in A, that converges to $\mathbf{y}_0$. By Lemma 1.3, for every n and for all i and j, $(\partial_x^i \partial_y^j f)(\mathbf{y}_n) = 0$. Since f is real-analytic on Ω, $(\partial_x^i \partial_y^j f)(\mathbf{y}_0)$ is continuous on Ω and hence equals 0 as well. Furthermore, since f is real-analytic on Ω, and $\mathbf{x}_0$ is a point in Ω, f has a Taylor series expansion around $\mathbf{y}_0$ valid in some δ-neighborhood of $\mathbf{y}_0$ that lies entirely within Ω. In other words, $\exists \delta > 0 \forall \mathbf{y} \in B(\mathbf{y}_0, \delta) \left(f(\mathbf{y}) = [\sum_{n=0}^{\infty} \frac{1}{n!} ((\mathbf{y} - \mathbf{y}_0) \cdot (\partial_x, \partial_y))^n](f)(\mathbf{y}_0) \right)$.[3] Notice that this series is just a sum of various factors multiplied by derivatives of f evaluated at $\mathbf{y}_0$ — but we showed above that all these derivatives must equal 0. Thus, for some appropriate δ, $\forall \mathbf{x} \in B(\mathbf{x}_0, \delta)\, (f(\mathbf{x}) = 0)$. As above, this means that there exists a δ-neighborhood of $\mathbf{y}_0$ in which $f \equiv 0$. This means that $\mathbf{y}_0 \in A$. Since $\mathbf{y}_0$ was arbitrary, we have that $\overline{A} \subseteq A$, and thus that A is closed in Ω.

[3] This notation might look unfamiliar. If $\mathbf{x}_0 = (x_0, y_0)$, and $\mathbf{x} = (x_0 + h, y_0 + k)$ for some $x_0, y_0, h, k \in \mathbf{R}$, then $(\mathbf{x} - \mathbf{x}_0) = (h, k)$, and $(h, k) \cdot (\partial_x, \partial_y) = h\partial_x + k\partial_y$. $(h\partial_x + k\partial_y)^n$ is then defined in the usual way, and equals $\sum_{i=0}^{n} \binom{n}{i} h^i k^{n-i} \partial_x^i \partial_y^{n-i}$.

Since Ω is connected, and $A \subseteq \Omega$ is non-empty and both open and closed, Definition 1.1 implies that $A = \Omega$. In other words, in all of Ω, $f \equiv 0$. $\diamondsuit$

Theorem 1.1 *$\mathcal{F}$, the set of real-analytic functions on Ω, is an integral domain.*

Proof: First, $\mathcal{F}$ is a ring, for it is clearly a group under $+$, and is clearly closed under $\times$. Further, since multiplication of real numbers is a commutative operation, $\mathcal{F}$ is commutative.

Now suppose $f, g \in \mathcal{F}$ and $fg \equiv 0$. We wish to prove that either $f \equiv 0$ or $g \equiv 0$. If $g \equiv 0$ we are done, so suppose that $g \not\equiv 0$. Thus, we may choose $\mathbf{x}_0$ with $g(\mathbf{x}_0) = c \neq 0$, for some $c \in \mathbf{C}$.

We will show that there exsits a neighborhood U of $\mathbf{x}_0$ such that for all $\mathbf{x} \in U$, $\left(\partial_x^0 \partial_y^0 f\right)(\mathbf{x}) = 0$, but this just means showing that there is a neighborhood U of $\mathbf{x}_0$ such that for all $\mathbf{x} \in U$, $f(\mathbf{x}) = 0$. Recall that $g(\mathbf{x}_0) \neq 0$. Since g is continuous, this implies that there is a neighborhood U of $\mathbf{x}_0$ in which g is never 0. Since $fg \equiv 0$, we have that for all $\mathbf{x} \in U$, $f(\mathbf{x})g(\mathbf{x}) = 0$; but since $g(\mathbf{x}) \neq 0$ for $\mathbf{x} \in U$, we may divide through by $g(\mathbf{x})$, and this forces $f(\mathbf{x}) = 0$ for $\mathbf{x} \in U$. Thus, $\forall \mathbf{x} \in U\,(f(\mathbf{x}) = 0)$. Now, by Lemma 1.4, we have that f is identically zero on Ω. $\diamondsuit$

It is possible to define a field of fractions on an integral domain. Let Φ denote the field of fractions for $\mathcal{F}$. Observe that a typical element $\varphi \in \Phi$ can be written several ways. For instance, $\frac{x}{y} = \frac{2x}{2y}$. Given $\varphi_1, \varphi_2 \in \Phi$, such that φ_1 can be written $\frac{f_1}{g_1}$ and φ_2 can be written $\frac{f_2}{g_2}$ for some $f_1, g_1, f_2, g_2 \in \mathcal{F}$, with $g_1, g_2 \not\equiv 0$, we say that $\varphi_1 = \varphi_2$ if $f_1 g_2 = f_2 g_1$, as elements of $\mathcal{F}$. It is clear that equality defined this way is an equivalence relation. We therefore say that $\varphi = \left[\frac{f}{g}\right]$ for some $f, g \in \mathcal{F}$ with $g \not\equiv 0$ such that φ can be written $\frac{f}{g}$, where $\left[\frac{f}{g}\right]$ denotes the equivalence class of $\frac{f}{g}$. We will often leave off the brackets, however, when dealing with elements of Φ. Addition, subtraction, multiplication, and division of fractions will be defined in the usual ways; for instance, $\frac{f_1}{g_1} + \frac{f_2}{g_2} = \frac{g_2 f_1 + g_1 f_2}{g_1 g_2}$. We define ∂_x and ∂_y in the usual ways, as well. It is a fact, whose proof we will omit, that if $\frac{f}{g}$ is defined at a point $\mathbf{x}$ (i.e., $g(\mathbf{x}) \neq 0$), then it is real-analytic in a neighborhood of $\mathbf{x}$.

Lemma 1.5 *Given $\frac{f}{g} \in \Phi$ such that $\nabla \frac{f}{g} \equiv 0$, there exists a complex number k such that $\frac{f}{g} = k$ in Φ.*

Proof: Pick a point $\mathbf{x}_0 \in \Omega$ such that $g(\mathbf{x}_0) \neq 0$. Since g is continuous, there exists a neighborhood U of $\mathbf{x}_0$ such that g is never zero on U. $\frac{f}{g}$, then, has a value at all points in U. Let $\frac{f}{g}(\mathbf{x}_0) = k$, for some $k \in \mathbf{C}$. We will first prove that $\frac{f}{g}(\mathbf{x}) = k$ for all $\mathbf{x} \in U$.

Since U is an open, connected subset of $\mathbf{R}^2$, Lemma 1.1 implies that it is path-connected. Choose $\mathbf{x} \in U$. Since U is path connected, there exists a continuous map γ: $[0,1] \to \Omega$ such that $\gamma(0) = \mathbf{x}_0$ and $\gamma(1) = \mathbf{x}$. According to the Fundamental Theorem of Calculus,

$$\int_\gamma \left(\nabla \frac{f}{g}\right)(\mathbf{y}) \cdot \vec{\mathbf{dr}} = \frac{f}{g}(\gamma(1)) - \frac{f}{g}(\gamma(0)) = \frac{f}{g}(\mathbf{x}) - k.$$

But, by hypothesis, $\nabla \frac{f}{g} \equiv 0$; so the integral equals 0. Thus, $\frac{f}{g}(\mathbf{x}) - k = 0$, which implies that $\frac{f}{g}(\mathbf{x}) = k$ for all $\mathbf{x}$ in U. So, the function $h = f - kg$ is clearly in $\mathcal{F}$, and is identically zero on U. Lemma 1.4 thus implies that $f - kg = 0$ on all of Ω. Thus, as an element of Φ, $\frac{f}{g} = \frac{k}{1}$. $\diamondsuit$

At this point, we are ready to return to the question that prompted this whole foray into the field of fractions. We have that, as elements of Φ,

$$\frac{X''}{X} = \frac{-Y'' - \omega^2 Y}{Y}.$$

We wish to conclude that both sides must equal a constant. Note that

$$\partial_y \left(\frac{X''}{X}\right) \equiv 0,$$

since X is a function of x alone, and

$$\partial_x \left(\frac{X''}{X}\right) = \partial_x \left(-\frac{Y''}{Y} - \omega^2\right) \equiv 0,$$

since Y is a function of y alone.

By Lemma 1.5, we conclude that $\frac{X''}{X} = k$, for some $k \in \mathbf{C}$.

1.3 Separation of Variables in Orthogonal Coordinate Systems

In Section 1 of this chapter, we saw that the Helmholtz equation separates in rectangular coordinates. It is natural to inquire in what other coordinate systems the Helmholtz equation separates.

We will examine only orthogonal curvilinear coordinates. Consider a new set of orthogonal coordinates $\{u, v\}$. If it is truly a change of coordinates, this means that (i) there are real-analytic functions u and $v : \Omega \longrightarrow \mathbf{R}$, where Ω is a non-empty, open, connected subset of $\mathbf{R}^2$, such that there is a bijection between pairs (x, y) and pairs $(u(x, y), v(x, y))$, and (ii) the Jacobian determinant $v_x u_y - u_x v_y$ never vanishes.

Now, in order to write the Helmholtz equation in another coordinate system $\{u, v\}$, we must transform the Helmholtz operator into the new coordinate system. If ψ is a real-analytic function on the plane ($\psi \in \mathcal{F}$) then the chain rule gives us $\partial_x \psi = \psi_u u_x + \psi_v v_x$. Continuing to apply the chain rule, we have

$$\begin{aligned}
\partial_{xx}\psi &= \partial_x \left(\psi_u u_x\right) + \partial_x \left(\psi_v v_x\right) \\
&= \left(\psi_{ux} u_x + \psi_u u_{xx}\right) + \left(\psi_{vx} v_x + \psi_v v_{xx}\right) \\
&= \left(\psi_{uu} u_x^2 + \psi_{uv} u_x v_x + \psi_u u_{xx}\right) + \left(\psi_{vv} v_x^2 + \psi_{uv} u_x v_x + \psi_v v_{xx}\right) \\
&= \psi_{uu} u_x^2 + \psi_u u_{xx} + 2\psi_{uv} u_x v_x + \psi_{vv} v_x^2 + \psi_v v_{xx} \\
&= \left(u_x^2 \partial_{uu} + u_{xx} \partial_u + 2 u_x v_x \partial_{uv} + v_x^2 \partial_{vv} + v_{xx} \partial_v\right) \psi.
\end{aligned}$$

The expression for $\partial_{yy}\psi$ is similar, and so we have that in the $\{u, v\}$ system,

$$\Delta_2 = \left(u_x^2 + u_y^2\right) \partial_{uu} + \left(u_{xx} + u_{yy}\right) \partial_u + 2\left(u_x v_x + u_y v_y\right) \partial_{uv} + \left(v_x^2 + v_y^2\right) \partial_{vv} + \left(v_{xx} + v_{yy}\right) \partial_v. \quad (1.8)$$

The condition that the system is orthogonal means that the unit vectors in the u and v directions ($\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$, respectively) are perpendicular at every point, which means that $\hat{\mathbf{u}}(x, y) \cdot \hat{\mathbf{v}}(x, y) \equiv 0$. Observe that $\hat{\mathbf{u}}$ is a unit vector in the direction of maximum increase of u — i.e., in the direction of $\nabla(u) = (u_x, u_y)$; and similarly $\hat{\mathbf{v}}$ is a unit vector in the direction of $\nabla(v) = (v_x, v_y)$. Since the dot-product of $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ is 0, the dot-product $\nabla(u) \cdot \nabla(v) = u_x v_x + u_y v_y$ must equal 0 as well. As a result, in an orthogonal coordinate system, the coefficient of the mixed second partial derivative vanishes:

$$\Delta_2 = \left(u_x^2 + u_y^2\right) \partial_{uu} + \left(u_{xx} + u_{yy}\right) \partial_u + \left(v_x^2 + v_y^2\right) \partial_{vv} + \left(v_{xx} + v_{yy}\right) \partial_v. \quad (1.9)$$

And so, since ω^2 is unchanged under the coordinate transformation, (1.1) is now written

$$\mathcal{H}\Psi = \left[\left(u_x^2 + u_y^2\right) \partial_{uu} + \left(u_{xx} + u_{yy}\right) \partial_u + \left(v_x^2 + v_y^2\right) \partial_{vv} + \left(v_{xx} + v_{yy}\right) \partial_v + \omega^2\right] \Psi = 0. \quad (1.10)$$

For convenience in viewing, we will write (1.10) as follows:

$$\mathcal{H}\Psi = \left(A_{11}\partial_{uu} + A_1\partial_u + A_{22}\partial_{vv} + A_2\partial_v + \omega^2\right)\Psi = 0, \tag{1.11}$$

where the functions A_{11}, A_1, A_{22}, and A_2 are equal to the corresponding coefficients in (1.10).

There are certain well-known "facts" concerning the form that the coefficient functions must assume if an equation of the form (1.11) is to be separable. For instance, [Miller 1977, p. 14] states that, assuming $\omega^2 \neq 0$, (1.11) is separable only if there exist functions $\mathcal{U}, \mathcal{V}, \mathcal{U}_1$, and $\mathcal{V}_1$ such that

$$(a) \quad A_{11} = \frac{\mathcal{U}(u)}{\mathcal{U}_1(u) + \mathcal{V}_1(v)} \qquad \text{and} \qquad (b) \quad A_{22} = \frac{\mathcal{V}(v)}{\mathcal{U}_1(u) + \mathcal{V}_1(v)}. \tag{1.12}$$

Also, [Robertson 1928] asserts that if (1.11) is separable, then there exist functions $B(u)$ and $C(v)$ such that

$$(c) \quad \frac{A_1(u,v)}{A_{11}(u,v)} = B(u) \qquad \text{and} \qquad (d) \quad \frac{A_2(u,v)}{A_{22}(u,v)} = C(v). \tag{1.13}$$

These "facts" prove very useful when characterizing the separable coordinate systems of (1.11), and they hold true in all separable coordinate systems that are commonly used in practice. But, if one adopts the naive, intuitive definition of separability that we present at the begining of this chapter, (1.12) and (1.13) are not always true.[4] The following counterexample demonstrates this.

Example 1.1

Let $\Psi_0(u,v) = e^u e^v$, and let the coefficient functions of the operator $(A_{11}\partial_{uu} + A_1\partial_u + A_{22}\partial_{vv} + A_2\partial_v + \omega^2)$ be defined by

$$\begin{array}{lllll} A_{11} & = & u - v & \qquad A_1 & = & v \\ A_{22} & = & -u - v & \qquad A_2 & = & -\omega^2 + v. \end{array}$$

One can easily verify that Ψ_0 is a solution. Since Ψ_0 is a product of a function only of u and a function only of v, it is a separated solution according to the naive definition. Yet, it is clear that none of (a) — (d) in (1.12) and (1.13) hold for the above coefficients.

Since (1.12) and (1.13) are so useful, we wish to define separation so that they do hold.

[4] In [Hildebrand 1976, p. 505], we encounter another characterization of the coefficients in a separable equation that is useful but not true in general if one adopts the naive definition of separability.

Definition 1.3 *Consider a second-order partial differential equation that involves two independent variables and has no mixed partial derivatives. That is, our partial differential equation is of the form*

$$(A_{11}\partial_{uu} + A_1\partial_u + A_{22}\partial_{vv} + A_2\partial_v + A)\,\Psi = 0,$$

where A_{11}, A_1, A_{22}, A_2, and A are real-analytic functions of u and v. We say that this equation is weakly solvable by separation of variables *or* weakly separable *if there exists a solution $\Psi_0(u,v)$ that can be written $U_0(u)V_0(v)$, for some functions U_0 and V_0, and any such solution is said to be* weakly separated. *We say that this equation is* strongly solvable by separation of variables *or* strongly separable *if $\frac{A_1}{A_{11}}$ is a function only of u, $\frac{A_2}{A_{22}}$ is a function only of v, and the equation has at least five linearly independent weakly separated solutions.*

It is not immediately clear that the above definition implies (1.12) in the case of the Helmholtz equation, but we will show that indeed it does.

Lemma 1.6 *Let a partial differential equation of the form*

$$(A_{11}\partial_{uu} + A_1\partial_u + A_{22}\partial_{vv} + A_2\partial_v + A)\,\Psi = 0 \tag{1.14}$$

be strongly solvable by separation of variables, and assume that A_{11} and A_{22} are always positive. Then

$$\frac{A_{11}(u,v)}{A_{22}(u,v)} = \frac{\mathcal{U}(u)}{\mathcal{V}(v)},$$

for some $\mathcal{U} > 0$ that is a function only of u and some $\mathcal{V} > 0$ that is a function only of v.

Proof: Let us write our differential equation as $(A_{11}\partial_{uu} + A_1\partial_u + A_{22}\partial_{vv} + A_2\partial_v + A)\,UV = 0$, where $\Psi = UV$ represents a weakly separated, nonzero solution. This implies that

$$A_{11}U''V + A_1U'V + A_{22}UV'' + A_2UV' + AUV = 0. \tag{1.15}$$

We divide through by $A_{11}V$ and rearrange terms to arrive at

$$U'' + \frac{A_1}{A_{11}}U' + \left(\frac{A_{22}}{A_{11}}\frac{V''}{V} + \frac{A_2}{A_{11}}\frac{V'}{V} + \frac{A}{A_{11}}\right)U = 0. \tag{1.16}$$

We also re-group terms to arrive at

$$V'' + \frac{A_2}{A_{22}}V' + \left(\frac{A_{11}}{A_{22}}\frac{U''}{U} + \frac{A_1}{A_{22}}\frac{V'}{V} + \frac{A}{A_{22}}\right)V = 0. \tag{1.17}$$

The crucial aspect of (1.16) is that, since U'', $\frac{A_1}{A_{11}}$, U', and U are functions only of u, so must be $\frac{A_{22}}{A_{11}}\frac{V''}{V} + \frac{A_2}{A_{11}}\frac{V'}{V} + \frac{A}{A_{11}}$. Write

$$\mathcal{U}_0(u) = \frac{A_{22}}{A_{11}}\frac{V''}{V} + \frac{A_2}{A_{11}}\frac{V'}{V} + \frac{A}{A_{11}}, \tag{1.18}$$

for some $\mathcal{U}_0$ that is a function only of u. Similarly, from (1.17), we see that there exists some $\mathcal{V}_0$ that is a function only of v such that

$$\mathcal{V}_0(v) = \frac{A_{11}}{A_{22}}\frac{U''}{U} + \frac{A_1}{A_{22}}\frac{V'}{V} + \frac{A}{A_{22}}. \tag{1.19}$$

We multiply (1.16) by $A_{11}V$ and (1.17) by $A_{22}U$, and we add the results:

$$A_{11}U''V + A_1U'V + A_{22}UV'' + A_2UV' + (A_{11}\mathcal{U}_0 + A_{22}\mathcal{V}_0)\,UV = 0. \tag{1.20}$$

Comparing this to (1.15), we have $(A - (A_{11}\mathcal{U}_0 + A_{22}\mathcal{V}_0))UV = 0$, and so, since $\mathcal{F}$ is an integral domain and $UV \not\equiv 0$, we see that A must equal $A_{11}\mathcal{U}_0 + A_{22}\mathcal{V}_0$:

$$A \equiv A_{11}\mathcal{U}_0 + A_{22}\mathcal{V}_0. \tag{1.21}$$

Since, for a particular choice of $\mathcal{U}_0$ and $\mathcal{V}_0$, both (1.16) and (1.17) are linear, homogeneous, second-order ordinary differential equations, each has at most two linearly independent solutions. We will label these U_{01}, U_{02}, and V_{01}, V_{02}. Thus, for a particular choice of $\mathcal{U}_0$ and $\mathcal{V}_0$, there are at most four linearly independent solutions to (1.15): $U_{01}V_{01}$, $U_{01}V_{02}$, $U_{02}V_{01}$, and $U_{02}V_{02}$. Since (1.14) is strongly separable, Definition 1.3 requires that there be at least five linearly independent weakly separated solutions. By the preceding argument, the existence of a fifth linearly independent weakly separated solution necessitates the existence of a pair of solutions $(\mathcal{U}_1, \mathcal{V}_1) \neq (\mathcal{U}_0, \mathcal{V}_0)$ that satisfies (1.18) and (1.19). Suppose, without loss of generality, that $\mathcal{V}_1 \neq \mathcal{V}_0$. We claim that $\mathcal{U}_1 \neq \mathcal{U}_0$.

Assume for the moment that $\mathcal{U}_1 = \mathcal{U}_0$. If this is so, then by following the same procedure that led to (1.20), we see that

$$A \equiv A_{11}\mathcal{U}_0 + A_{22}\mathcal{V}_1. \tag{1.22}$$

Taking the difference between (1.22) and (1.21) yields

$$0 = A_{11}\left(\mathcal{U}_0 - \mathcal{U}_0\right) + A_{22}\left(\mathcal{V}_1 - \mathcal{V}_0\right),$$

and hence

$$0 = A_{22}\left(\mathcal{V}_1 - \mathcal{V}_0\right).$$

Since Φ, the field of fractions of $\mathcal{F}$, is a field, it is clearly an integral domain. Since we know that $\mathcal{V}_1 - \mathcal{V}_0 \not\equiv 0$, this forces $A_{22} \equiv 0$. $\Longrightarrow\Longleftarrow$! This contradicts the hypothesis that A_{22} is everywhere positive. Thus, we must have $\mathcal{U}_1 \neq \mathcal{U}_0$. We can therefore correctly rewrite (1.22) as $A \equiv A_{11}\mathcal{U}_1 + A_{22}\mathcal{V}_1$. Subtracting (1.21) from this gives us

$$0 = A_{11}\left(\mathcal{U}_1 - \mathcal{U}_0\right) + A_{22}\left(\mathcal{V}_1 - \mathcal{V}_0\right). \tag{1.23}$$

For no value of u does $\mathcal{U}_1(u) = \mathcal{U}_0(u)$; and for no value of v does $\mathcal{V}_1(v) = \mathcal{V}_0(v)$. Why? Well, suppose for instance that for $v = v_0$, we have $\mathcal{V}_1(v_0) = \mathcal{V}_0(v_0)$. Then, we see from (1.23) that for all u, $A_{11}\left(u, v_0\right)\left(\mathcal{U}_1(u) - \mathcal{U}_0(u)\right) = 0$, but then, since Φ is an integral domain and $A_{11} > 0$, this forces $\mathcal{U}_1 \equiv \mathcal{U}_0$. $\Longrightarrow\Longleftarrow$! Therefore, we may rearrange the terms in (1.23) to get

$$\frac{A_{11}}{A_{22}} = \frac{\mathcal{V}_1 - \mathcal{V}_0}{\mathcal{U}_0 - \mathcal{U}_1} = \frac{\frac{1}{\mathcal{U}_0 - \mathcal{U}_1}}{\frac{1}{\mathcal{V}_1 - \mathcal{V}_0}}, \tag{1.24}$$

which is a quotient of a function only of u and a function only of v, as required. And, note that (i) both $\mathcal{U}(u) \equiv \frac{1}{\mathcal{U}_0(u) - \mathcal{U}_1(u)}$ and $\mathcal{V}(v) \equiv \frac{1}{\mathcal{V}_1(v) - \mathcal{V}_0(v)}$ are defined everywhere, for their denominators never vanish; and (ii) neither $\mathcal{U}$ nor $\mathcal{V}$ is ever 0, since neither A_{11} nor A_{22} ever vanishes. Since A_{11} and A_{22} are both everywhere positive, their quotient is too. Thus, if it happens that $\mathcal{U} < 0$, we will have that $\mathcal{V} < 0$ too. In this case, replace both $\mathcal{U}$ and $\mathcal{V}$ with their negatives — this will make them both positive, and will preserve their quotient. $\diamondsuit$

Lemma 1.7 *Let A_{11}, A_1, A_{22}, and A_2 be real-analytic functions of both u and v that satisfy the following conditions:*

(i) $\quad \frac{A_1(u,v)}{A_{11}(u,v)} = B(u)$ *for some B that is a function only of u,*

(*ii*) $\frac{A_2(u,v)}{A_{22}(u,v)} = C(v)$ *for some* C *that is a function only of* v,

and

(*iii*) $\frac{A_{11}(u,v)}{A_{22}(u,v)} = \frac{\mathcal{U}(u)}{\mathcal{V}(v)}$ *for some never-zero* $\mathcal{U}$ *that is a function only of* u *and some never-zero* $\mathcal{V}$ *that is a function only of* v.

Consider a differential equation that has the following form:

$$A_{11}\frac{U''}{U} + A_1\frac{U'}{U} + A_{22}\frac{V''}{V} + A_2\frac{V''}{V} + \omega^2 = 0,$$

where U *is a real-analytic function only of* u *and* V *is a real-analytic function only of* v. *Then, if* $\omega^2 \neq 0$, *there exist functions* $\mathcal{U}_1$ *of* u *alone and* $\mathcal{V}_1$ *of* v *alone such that*

$$(a) \quad A_{11} = \frac{\mathcal{U}(u)}{\mathcal{U}_1(u) + \mathcal{V}_1(v)} \qquad \textit{and} \qquad (b) \quad A_{22} = \frac{\mathcal{V}(v)}{\mathcal{U}_1(u) + \mathcal{V}_1(v)}.$$

Proof: From hypotheses (*i*) and (*ii*) of the lemma, $A_1 = A_{11}B$ and $A_2 = A_{22}C$. Thus, we may rewrite the equation as follows:

$$A_{11}\left(\frac{U''}{U} + B\frac{U'}{U}\right) + A_{22}\left(\frac{V''}{V} + C\frac{V'}{V}\right) + \omega^2 = 0.$$

Notice that $\frac{U''}{U} + B\frac{U'}{U}$ is a function only of u, so we may write it $b(u)$; and $\frac{V''}{V} + C\frac{V'}{V}$ is a function only of v, so we may write it $c(v)$. From hypothesis (*iii*) above, $A_{22} = A_{11}\frac{\mathcal{V}}{\mathcal{U}}$. Thus, we again rewrite the equation as follows:

$$A_{11}b + A_{11}\frac{\mathcal{V}}{\mathcal{U}}c + \omega^2 = 0,$$

which implies that

$$A_{11} = \frac{-\omega^2}{b + c\frac{\mathcal{V}}{\mathcal{U}}} = \frac{-\omega^2\mathcal{U}}{b\mathcal{U} + c\mathcal{V}}.$$

Now, since $\omega^2 \neq 0$, we may divide numerator and denominator by $-\omega^2$, to arrive at

$$A_{11} = \frac{\mathcal{V}}{\left(-\frac{b\mathcal{U}}{\omega^2}\right) + \left(-\frac{c\mathcal{V}}{\omega^2}\right)}, \tag{1.25}$$

indicating that our choices of $\mathcal{U}_1$ and $\mathcal{V}$ ought to be $\mathcal{U}_1 = -\frac{b\mathcal{U}}{\omega^2}$ and $\mathcal{V}_1 = -\frac{c\mathcal{V}}{\omega^2}$. Note that $\mathcal{U}_1$ and $\mathcal{V}_1$ are clearly functions only of u and only of v, respectively, as required. This verifies result (a) of the lemma, and it is clear that (1.25) in conjunction with hypothesis (*iii*) now implies result (b) of the lemma. $\diamondsuit$

1.4 Orthogonal Coordinate Systems In Which the Helmholtz Equation is Strongly Separable

We are seeking orthogonal coordinate systems in which (1.1) is strongly separable, so we will assert (1.10) to be strongly separable and try to characterize u and v.[5] First, we note that the condition that the Jacobian determinant never vanishes implies that $u_x^2 + u_y^2 = A_{11}$ and $v_x^2 + v_y^2 = A_{22}$ never vanish either, and hence are everywhere positive. Now, we see immediately that Lemma 1.6 implies that $\frac{A_{11}}{A_{22}}$ is of the form required by hypothesis (*iii*) of Lemma 1.7, and, since (1.11) is strongly separable, Definition 1.3 implies that hypotheses (*i*) and (*ii*) of Lemma 1.7 are satisfied as well. Thus, Lemma 1.7 implies that

$$u_x^2 + u_y^2 = \frac{\mathcal{U}(u)}{\mathcal{U}_1(u) + \mathcal{V}_1(v)}, \quad \text{and} \quad v_x^2 + v_y^2 = \frac{\mathcal{V}(v)}{\mathcal{U}_1(u) + \mathcal{V}_1(v)}. \tag{1.26}$$

Note: in the above equation we replaced A_{11} and A_{22} in accordance with (1.10).

Recall that we are working with orthogonal coordinate systems, and so $u_x v_x + u_y v_y = 0$. As a result, $-\frac{v_x}{u_y} = \frac{v_y}{u_x}$. Define the function $\mathcal{R} \equiv -\frac{v_x}{u_y} = \frac{v_y}{u_x}$. Then,

$$v_y = \mathcal{R} u_x \tag{1.27}$$

and

$$v_x = -\mathcal{R} u_y. \tag{1.28}$$

Now, since $v_x^2 + v_y^2 = \mathcal{R}^2(u_x^2 + u_y^2)$, (1.26) implies that

$$\mathcal{R}^2 = \frac{\mathcal{V}}{\mathcal{U}}. \tag{1.29}$$

In order to proceed, we will need to define some new terminology.

Definition 1.4 *Let $\{u, v\}$ be a coordinate system on $\Omega \subseteq \mathbf{R}^2$. We will say that the image of a curve is a* coordinate-curve *if it is defined by an equation of the form $u(x, y) = c$ or $v(x, y) = c$, for a number $c \in \mathbf{R}$. It is a coordinate-curve of u if it is defined by $u = c$ for some $c \in \mathbf{R}$, and a coordinate-curve of v if it is defined by $v = c$ for some $c \in \mathbf{R}$.*

[5] Both [Miller 1977] and [Morse and Feshbach 1953] describe how to determine the separable coordinate systems; the line of reasoning here is a combination of those two descriptions.

Definition 1.5 *A matrix $M \in GL(n, \mathbf{R})$ is said to be* orthogonal *if the linear transformation that it represents preserves lengths and angles.*[6] *The set of all orthogonal, real $n \times n$ matrices is denoted $\mathcal{O}(n)$.*

Definition 1.6 *A map $\mathcal{L} : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is said to be* affine *if there exist a matrix $M \in GL(n, \mathbf{R})$ and a vector $\mathbf{v}_0 \in \mathbf{R}^n$ such that for every vector $\mathbf{v} \in \mathbf{R}^n$, $\mathcal{L}(\mathbf{v}) = \mathbf{v}_0 + M\mathbf{v}$. An affine transformation $\mathcal{L}(\mathbf{v}) = \mathbf{v}_0 + M\mathbf{v}$ is said to be an* affine similitude *if $M = cM'$, where $c \in \mathbf{R}$ and $M' \in \mathcal{O}(n)$.*

It turns out that an affine similitude is the composition of a dilation, a translation, a rotation, and possibly a reflection.

Definition 1.7 *We will say that the coordinate system $\{u, v\}$ is* shapewise equivalent *to the coordinate system $\{\tilde{u}, \tilde{v}\}$ if there exist invertible real-analytic functions $\eta(u)$ and $\phi(v)$, where $\eta'(u)$ and $\phi'(v)$ never vanish, such that $\tilde{u}(x, y) = \eta\Big(u(\mathcal{L}(x, y))\Big)$ and $\tilde{v}(x, y) = \phi\Big(v(\mathcal{L}(x, y))\Big)$, where $\mathcal{L}$ is an affine similitude.*

We summarize the relationship that holds among shapewise equivalent coordinate systems in the schematic diagram below.

$$\begin{array}{ccc} \mathbf{R}^2_{\{x,y\}} & \xleftarrow{\mathcal{L}} & \mathbf{R}^2_{\{x,y\}} \\ \downarrow & & \downarrow \\ \mathbf{R}^2_{\{u,v\}} & \xrightarrow[\eta,\phi]{} & \mathbf{R}^2_{\{\tilde{u},\tilde{v}\}} \end{array} \tag{1.30}$$

The vertical maps are given by $(x, y) \mapsto (u(x, y), v(x, y))$ on the left, and $(x, y) \mapsto (\tilde{u}(x, y), \tilde{v}(x, y))$ on the right. We say that the diagram is schematic because in order to be rigorous, we would need to replace each $\mathbf{R}^2$ with an appropriate open set. Still, the diagram makes it clear that one gets the same result going from $\{x, y\}$ coordinates to $\{\tilde{u}, \tilde{v}\}$ regardless of whether one uses the direct functions $\tilde{u}(x, y)$ and $\tilde{v}(x, y)$ or one follows the path around the other way, using the affine similitude $\mathcal{L}$, the functions $u(x, y)$ and $v(x, y)$, and the reparametrizations $\eta(u)$ and $\phi(v)$. The diagram is suggestive of the fact (which we will not prove) that shapewise equivalence is an equivalence relation.

[6] One can prove that M is orthogonal iff $M^t = M^{-1}$.

The definition of shapewise equivalence has to do with the analytical, not the geometrical, properties of shapewise equivalent coordinate systems. We define shapewise equivalence this way because the analytical properties are what are useful for proving that a change of coordinates that preserves shapewise equivalence also preserves strong separability of the Helmholtz equation (i.e., if (1.1) is strongly separable in one orthogonal coordinate system, it is strongly separable in any shapewise equivalent coordinate system). It is the geometrical similarities among coordinate systems that are shapewise equivalent, however, that lead to the term "shapewise equivalent". We will therefore prove a lemma about the geometrical relationships among shapewise equivalent coordinate systems.

Lemma 1.8 *Let two coordinate systems $\{u, v\}$ and $\{\tilde{u}, \tilde{v}\}$ be shapewise equivalent. Let $\mathcal{L}$ be the affine similitude that relates $\{\tilde{u}, \tilde{v}\}$ to $\{u, v\}$, as in Definition 1.7. Then, every non-empty coordinate-curve C_u of u is equal to $\mathcal{L}(C_{\tilde{u}})$ for some non-empty coordinate-curve $C_{\tilde{u}}$ of $\tilde{u}$.*

Proof: We can prove this most efficiently with the following chain of equivalences.

$$
\begin{aligned}
(x,y) \in \mathcal{L}(C_{\tilde{u}}) \quad &\Longleftrightarrow \quad (x,y) = \mathcal{L}(\tilde{x},\tilde{y}) \quad \text{and} \quad (\tilde{x},\tilde{y}) \in C_{\tilde{u}} \\
&\Longleftrightarrow \quad (x,y) = \mathcal{L}(\tilde{x},\tilde{y}) \quad \text{and} \quad \tilde{u}(\tilde{x},\tilde{y}) = \tilde{c}, \text{ for some } \tilde{c} \in \mathbf{R} \\
&\Longleftrightarrow \quad (x,y) = \mathcal{L}(\tilde{x},\tilde{y}) \quad \text{and} \quad \eta\Big(u(\mathcal{L}(\tilde{x},\tilde{y}))\Big) = \tilde{c} \\
&\Longleftrightarrow \quad \eta\Big(u(x,y)\Big) = \tilde{c} \\
&\Longleftrightarrow \quad u(x,y) = c, \text{ where } \eta(c) = \tilde{c} \\
&\Longleftrightarrow \quad (x,y) \in C_u \qquad \diamondsuit
\end{aligned}
$$

Example 1.2

Frequently, when people graph a data set that includes both some very small and some very large values, they use so-called log graph-paper. Log graph-paper is based on a coordinate system $\{\tilde{x} = x, \tilde{y} = \log y\}$ that is, by the above argument, shapewise equivalent to rectangular coordinates $\{x, y\}$. The reason it is said to be shapewise equivalent should be clear: a rectangle that is parallel to the coordinate axes in ordinary rectangular coordinates is a rectangle on log graph-paper, as well.

Theorem 1.2 *Let two coordinate systems $\{u, v\}$ and $\{\tilde{u}, \tilde{v}\}$ be shapewise equivalent. Then if the Helmholtz equation is strongly separable in $\{u, v\}$, it is also strongly separable in $\{\tilde{u}, \tilde{v}\}$, although the transformation of $\mathcal{H}$ from $\{u, v\}$ into $\{\tilde{u}, \tilde{v}\}$ might change ω^2 into $\tilde{\omega}^2$, a different positive quantity.*

Proof: Suppose, without loss of generality, that the Helmholtz equation is strongly separable in $\{u, v\}$. We will first verify that a reparametrization of u and v leaves the equation strongly separable; then, we will verify that under an affine similitude the equation remains strongly separable (although possibly changing ω^2). It will therefore follow that the composition of a reparametrization and an affine similitude leaves the equation strongly separable.

Let there be reparametrizations $\tilde{u} = \eta(u)$ and $\tilde{v} = \phi(v)$. Now, $A_{11}(u, v) = u_x^2 + u_y^2$. Note that

$$\tilde{u}_x = \frac{\partial \tilde{u}}{\partial x} = \frac{d\tilde{u}}{du}\frac{\partial u}{\partial x} = \eta'(u)u_x.$$

Similarly, $\tilde{u}_y = \eta'(u)u_y$. Thus, we have that the corresponding coefficient

$$\tilde{A}_{11}(\tilde{u}, \tilde{v}) = \tilde{u}_x^2 + \tilde{v}_y^2 = (\eta'(u))^2 \left(u_x^2 + u_y^2\right) = (\eta'(u))^2 A_{11}(u, v).$$

Note that $\eta'(u)$ can be expressed as a function of $\tilde{u}$ using the inverse function of η. Similarly, one can determine that, if $A_1(u, v) = u_{xx} + u_{yy}$, then

$$\tilde{A}_1(\tilde{u}, \tilde{v}) = \eta''(u)(u_{xx} + u_{yy}) = \eta''(u)A_1(u, v).$$

Note, as above, that $\eta''(u)$ can be expressed as a function of $\tilde{u}$ using the inverse function of η. Thus, the ratio $\frac{\tilde{A}_1}{\tilde{A}_{11}}$ equals a function of $\tilde{u}$ multiplied by $\frac{A_1}{A_{11}}$. The latter is a function only of u. Again, we can use the inverse function of η to express this as a function of $\tilde{u}$, thus showing that the ratio $\frac{\tilde{A}_1}{\tilde{A}_{11}}$ is a function only of $\tilde{u}$, as required. Similarly, $\frac{\tilde{A}_2}{\tilde{A}_{22}}$ is a function only of v. We must also show that there are five linearly independent solutions. If $\Psi(u, v) = U(u)V(v)$ is a solution in $\{u, v\}$ coordinates, then a straightforward calculation reveals that $\tilde{\Psi}(\tilde{u}, \tilde{v}) = \Psi(\eta^{-1}(\tilde{u}), \phi^{-1}(\tilde{v})) = U(\eta^{-1}(\tilde{u}))V(\phi^{-1}(\tilde{v}))$ is a solution in $\{\tilde{u}, \tilde{v}\}$ coordinates, where η^{-1} and ϕ^{-1} are the inverse functions of η and ϕ, respectively. We omit the proof that the transformations of the five linearly independent solutions form a linearly independent set of solutions, but this should seem plausible.[7]

[7]Each $\tilde{\Psi}$ takes on the same value at a given (x, y) point as the corresponding Ψ, and thus the functions $\tilde{\Psi}$ are

When we see in Chapter 2, Section 2 that $\mathcal{H}$ is invariant under the one-parameter subgroups of its first-order symmetry operators, this will show that $\mathcal{H}$ is invariant under rotations and translations. We first observe that dilations serve only to multiply ω^2 by the square of the dilation factor. For reflections, we may consider only reflections over the y-axis, for all other reflections are the composition of reflections over the y-axis and translations and rotations. If we reflect our space over the y-axis, ∂_x turns into $-\partial_{\tilde{x}}$, but then taking a second derivative yields $\partial_{xx} = -(-\partial_{\tilde{x}\tilde{x}}) = \partial_{\tilde{x}\tilde{x}}$. That the ω^2 term will remain unchanged follows from the fact that the ∂_{xx} and ∂_{yy} terms maintain the same form (only ∂_{xx} is re-labeled as $\partial_{\tilde{x}\tilde{x}}$). $\diamondsuit$

We now return to the Helmholtz equation's separable coordinate systems. If it is not already the case that $\mathcal{U} \equiv 1$ and $\mathcal{V} \equiv 1$ in (1.26), then there exist real-analytic functions $\tilde{u}(u)$ and $\tilde{v}(v)$ such that

$$\tilde{u}'(u) = \mathcal{U}^{-\frac{1}{2}} \quad \text{and} \quad \tilde{v}'(v) = \mathcal{V}^{-\frac{1}{2}}.$$

Since $\mathcal{U}$ and $\mathcal{V}$ are both always positive (from Lemma 1.6), $\tilde{u}'(u)$ and $\tilde{v}'(v)$ are everywhere defined. It is clear that $\tilde{u}'(u)$ and $\tilde{v}'(v)$ are never 0, for there are no real values of $\mathcal{U}$ and $\mathcal{V}$ that would lead to this. As a result, these functions describe a coordinate system that is shapewise equivalent to $\{u, v\}$. We now wish to come up with expressions for $\tilde{u}_x^2 + \tilde{u}_y^2$ and for $\tilde{v}_x^2 + \tilde{v}_y^2$. Note that

$$\frac{\partial \tilde{u}}{\partial x} = \frac{d\tilde{u}}{du}\frac{\partial u}{\partial x} = \mathcal{U}^{-\frac{1}{2}} u_x$$

and so

$$\tilde{u}_x^2 = \frac{1}{\mathcal{U}} u_x^2.$$

Expressions for $\tilde{u}_y^2, \tilde{v}_x^2$ and $\tilde{v}_y^2$ are analogous, and so, (1.26) now implies that

$$\tilde{u}_x^2 + \tilde{u}_y^2 = \frac{1}{\mathcal{U}}\left(u_x^2 + u_y^2\right) = \frac{1}{\mathcal{U}_1 + \mathcal{V}_1} \quad \text{and} \quad \tilde{v}_x^2 + \tilde{v}_y^2 = \frac{1}{\mathcal{V}}\left(v_x^2 + v_y^2\right) = \frac{1}{\mathcal{U}_1 + \mathcal{V}_1}. \tag{1.31}$$

At this point it is important to make sure that we understand what we have just derived. Our discovery is that if $\{u, v\}$ is a coordinate system in which (1.11) is strongly separable, and thus in

essentially the same as the functions Ψ. As a result, the linear independence of one set can be shown to imply the linear independence of the other.

which (1.26) holds, then there is a shapewise equivalent coordinate system $\{\tilde{u}, \tilde{v}\}$ in which (1.11) is strongly separable such that (1.31) holds. For the remainder of the discussion, we will consider only coordinate systems $\{u, v\}$ such that (1.31) holds.

Since, for these coordinate systems, $\mathcal{U}(u) \equiv 1 \equiv \mathcal{V}(v)$, this in conjunction with (1.29) forces $\mathcal{R} \equiv \pm 1$. If we assume $\mathcal{R} \equiv 1$, then

$$u_x = v_y \quad \text{and} \quad u_y = -v_x. \tag{1.32}$$

This means that u and v satisfy the Cauchy-Riemann equations. Thus, if we let $z = x + iy$ and $w = u + iv$, then the mapping $z \mapsto w$ is conformal. If $\mathcal{R}$ in fact equals -1, we may achieve the same relation between w and z merely by interchanging x and y, and so we assume that $\mathcal{R} \equiv 1$. We define

$$\frac{df}{dz} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad \text{and} \quad \frac{df}{d\bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right), \tag{1.33}$$

in accordance with [Greene and Krantz 1997, pp. 12-17]. By the Cauchy-Riemann equations,

$$f \text{ complex-analytic} \iff \frac{df}{d\bar{z}} = 0 \tag{1.34}$$

Since w is an analytic function of z, we may take its derivative. This gives

$$\frac{dw}{dz} = \frac{1}{2}\left(u_x + iv_x - i(u_y + iv_y)\right) = \frac{1}{2}\left(u_x + v_y + i(v_x - u_y)\right) = \text{ (from (1.32)) } u_x - iu_y,$$

and thus

$$\left|\frac{dw}{dz}\right|^2 = \frac{dw}{dz}\frac{d\bar{w}}{d\bar{z}} = (u_x - iu_y)(u_x + iu_y) = u_x^2 + u_y^2 = \frac{1}{\mathcal{U}_1 + \mathcal{V}_1}. \tag{1.35}$$

Hence,

$$\left|\frac{dz}{dw}\right|^2 = \mathcal{U}_1 + \mathcal{V}_1, \tag{1.36}$$

which forces

$$\partial_{uv}\left(\left|\frac{dz}{dw}\right|^2\right) = 0, \tag{1.37}$$

since $\mathcal{U}_1$ is a function only of u and $\mathcal{V}_1$ is a function only of v.

We now wish to express ∂_{uv} in terms of $w = u + iv$ and $\bar{w} = u - iv$. Notice that $\frac{d}{dw}$ and $\frac{d}{d\bar{w}}$ can be defined in terms of ∂_u and ∂_v, in analogy with (1.33). Using this, we obtain that, given a differentiable function f,

$$\frac{\partial^2 f}{\partial u \partial v} = i f_{ww} - i f_{\bar{w}\bar{w}}.$$

So, $\partial_{uv} = i\partial_{ww} - i\partial_{\bar{w}\bar{w}}$. Thus, (1.37) now implies that $\partial_{uv}\left(\left|\frac{dz}{dw}\right|^2\right) = (i\partial_{ww} - i\partial_{\bar{w}\bar{w}})\left(\frac{dz}{dw}\frac{d\bar{z}}{d\bar{w}}\right) = 0$. Now, since $\frac{dz}{d\bar{w}} = \frac{d\bar{z}}{dw} = 0$ (from (1.33), performing the indicated differentiation and simplifying yields

$$\left(\frac{d\bar{z}}{d\bar{w}}\right)\frac{d^2}{dw^2}\left(\frac{dz}{dw}\right) - \left(\frac{dz}{dw}\right)\frac{d^2}{d\bar{w}^2}\left(\frac{d\bar{z}}{d\bar{w}}\right) = 0,$$

which in turn leads to

$$\left(\frac{dz}{dw}\right)^{-1}\frac{d^2}{dw^2}\left(\frac{dz}{dw}\right) = \left(\frac{d\bar{z}}{d\bar{w}}\right)^{-1}\frac{d^2}{d\bar{w}^2}\left(\frac{d\bar{z}}{d\bar{w}}\right).$$

The left side is the complex conjugate of the right side. The only way for this to happen is for both sides to be real. Since the left side is an analytic function of w and is always real, it must be constant. We conclude that each side is equal to a constant $\lambda \in \mathbf{R}$. Some rearranging yields

$$\frac{d^2}{dw^2}\left(\frac{dz}{dw}\right) = \lambda\left(\frac{dz}{dw}\right) \quad \text{and} \quad \frac{d^2}{d\bar{w}^2}\left(\frac{d\bar{z}}{d\bar{w}}\right) = \lambda\left(\frac{d\bar{z}}{d\bar{w}}\right). \tag{1.38}$$

Over these past few pages, we have shown in great detail how to arrive at this differential equation. We will present in some detail how to arrive at the solutions in the case where $\lambda = 0$, but will only state the solutions in the case where $\lambda \neq 0$. Both [Miller 1977] and [Morse and Feshbach 1953] show how to arrive at these solutions in greater detail.

Case $\lambda = 0$: Our differential equation is $\frac{d^2}{dw^2}(\frac{dz}{dw}) = 0$. Thus $\frac{dz}{dw} = \beta + \gamma w$, for some $\beta, \gamma \in \mathbf{C}$.

When $\gamma = 0$, we have that $z = \alpha + \beta w$ ($\alpha \in \mathbf{C}$). Let $\alpha = a + ib$ and $\beta = c + id$. Then

$$\begin{aligned} z &= (a + ib) + (c + id)(u + iv) \\ &= (a + ib) + ((cu - dv) + i(du + cv)) \\ &= acu - dv + i(b + du + cv) \\ &= x + iy. \end{aligned}$$

Thus, we see that $x = a + cu - dv$ for $a, c, d \in \mathbf{R}$, and $y = b + du + cv$, for $b \in \mathbf{R}$. This corresponds to a translation, rotation, and dilation of the coordinates, leaving the coordinate axes still rectangular. The transformed Helmholtz equation (1.10) is now the same as (1.1) except with ω^2 replaced by $\omega^2(\sqrt{c^2 + d^2})$. So, if we wish to preserve the value of ω^2 under the change of coordinates, we must choose c, d so that $c^2 + d^2 = 1$, which restricts us to translations and rotations.

When $\gamma \neq 0$, we have $z = \alpha + \beta w + \frac{1}{2}\gamma w^2$, for some $\alpha \in \mathbf{C}$. The α and β terms translate and rotate, and β is also a dilation if $|\beta| \neq 1$. What is the effect of the γ term? Consider, for the moment, when α and β both equal 0 and $\gamma = 1$. Then

$$z = \frac{1}{2}w^2 = \frac{1}{2}\left(u^2 - v^2\right) + iuv = x + iy.$$

Thus, $x = \frac{1}{2}\left(u^2 - v^2\right)$ and $y = uv$. These $\{u, v\}$ coordinates are called *parabolic*, because the coordinate-curves are confocal parabolas, with focus at the origin and directrix parallel to the y-axis. The coordinate-curves take the form

$$u = \sqrt{\sqrt{x^2 + y^2} + x} = c \quad \text{and} \quad v = \pm\sqrt{\sqrt{x^2 + y^2} - x} = d \quad \text{for } c, d \in \mathbf{R}.$$

It is possible to prove (by completing the square) that all coordinate systems that arise from this case are shapewise equivalent to the specific case we worked out, where $\alpha = \beta = 0$ and $\gamma = 1$.

Case $\lambda \neq 0$: For convenience, we set $\lambda = 1$, because varying λ does not change the shapewise equivalence class of coordinate systems, which is what we are primarily interested in. The transformation is then given by the solution to (1.38) with $\lambda = 1$. Since $\frac{d^2}{dw^2}\phi = \phi \Longrightarrow \phi$ is a linear combination of e^w and e^{-w}, we have $\frac{dz}{dw} = \beta e^w - \gamma e^{-w}$, where $\beta, \gamma \in \mathbf{C}$. We write $\frac{dz}{dw}$ this way, subtracting the γ-term, because this makes the expression for z a little nicer:

$$z = \alpha + \beta e^w + \gamma e^{-w} \quad (\alpha \in \mathbf{C}).$$

We ignore the α term, for this just moves the origin (thus leaving the coordinate system shapewise equivalent to the one defined by the above equation with $\alpha = 0$).

When $\gamma = 0$, let $\beta = a + ib$ for some $a, b \in \mathbf{R}$. We see that

$$z = \beta e^{u+iv} = e^{u+a}e^{i(v+b)} \Longrightarrow x = e^{u+a}\cos(v + b) \quad \text{and} \quad y = e^{u+a}\sin(v + b).$$

If we define $r \equiv e^{u+a} = \sqrt{x^2+y^2}$ and $\theta \equiv v + b$ (which clearly preserves shapewise equivalence, since we are merely replacing u and v with the reparametrizations $r(u)$ and $\theta(v)$), then we see that we are dealing with polar coordinates, where the coordinate-curves of r are concentric circles and the coordinate-curves of θ are rays from the origin.

When $\gamma \neq 0$ and $\beta \neq 0$ as well, we will perform a rotation and translation of coordinates to end up with a shapewise equivalent coordinate system that is easier to describe. First, the rotation: let $\tilde{z} = e^{i\theta} z$, where the choice of θ will be made clear in a moment. This leads to

$$\tilde{z} = \tilde{\beta} e^{w} + \tilde{\gamma} e^{-w},$$

where $\tilde{\beta} = e^{i\theta}\beta$ and $\tilde{\gamma} = e^{i\theta}\gamma$. We choose θ so that $\tilde{\beta}\tilde{\gamma} = e^{2i\theta}\beta\gamma = \frac{1}{2}e^{R} > 0$, where $R = \ln(2\,|\beta\gamma|)$. Now the translation: let $\widetilde{w} = w - w_0$, where w_0 is chosen so that $e^{-w_0} = 2e^{R}\tilde{\beta}$. Replacing $\widetilde{w} + w_0$ for w, we have

$$\begin{aligned} \tilde{z} &= \tilde{\beta} e^{\widetilde{w}+w_0} + \tilde{\gamma} e^{-\widetilde{w}-w_0} \\ &= \left(\tilde{\beta} e^{w_0}\right) e^{\widetilde{w}} + \left(\tilde{\gamma} e^{-w_0}\right) e^{-\widetilde{w}} \\ &= \frac{1}{2} e^{-R} e^{\widetilde{w}} + \frac{1}{2} e^{R} e^{-\widetilde{w}}. \end{aligned}$$

The coordinate system described by $\tilde{z}$ and $\widetilde{w}$ is shapewise equivalent to the one described by z and w, for the former involves just a rotation and a translation of the latter (which is described by an affine similitude). We therefore drop the tildes in the notation:

$$z = \frac{e^{w-R} + e^{-(w-R)}}{2} = \cosh(w - R).$$

Replacing $u + iv$ for w yields

$$z = \cosh\left((u-R) + iv\right) = \cosh(u-R)\cos v + i(\,\sinh(u-R)\sin v).$$

From this, we see that $x = \cosh(u-R)\cos v$ and $y = \sinh(u-R)\sin v$; and these equations describe *elliptic coordinates*. In the x-y plane, the coordinate-curves of u are confocal ellipses, and the coordinate-curves of v are confocal hyperbolas.

To summarize, we have discovered that every orthogonal separable coordinate system of the Helmholtz equation is shapewise equivalent to rectangular, parabolic, polar, or elliptic coordinates. For reasons that will become clear in Chapter 2, we will now briefly give the solution in polar coordinates.

In polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, and $r^2 = x^2 + y^2$. As a result, the chain-rule gives us that for a differentiable function f, $\partial_r f = \partial_x f \partial_r x + \partial_y f \partial_r y$. Clearly, then,

$$\partial_r f = \cos\theta \partial_x f + \sin\theta \partial_y f = \left(\frac{x}{r}\partial_x + \frac{y}{r}\partial_y\right) f = \left(\frac{x}{\sqrt{x^2+y^2}}\partial_x + \frac{y}{\sqrt{x^2+y^2}}\partial_y\right) f,$$

which indicates that

$$\partial_r = \frac{x}{r}\partial_x + \frac{y}{r}\partial_y = \frac{x}{\sqrt{x^2+y^2}}\partial_x + \frac{y}{\sqrt{x^2+y^2}}\partial_y. \tag{1.39}$$

We also have that $\partial_\theta f = \partial_x f \partial_\theta x + \partial_y f \partial_\theta y$. Thus,

$$\partial_\theta = -r\sin\theta\partial_x f + r\cos\theta\partial_y f = (-y\partial_x + x\partial_y) f,$$

and this shows that

$$\partial_\theta = -y\partial_x + x\partial_y. \tag{1.40}$$

Using these two expressions, we can verify that in polar coordinates the Helmholtz equation is written

$$\mathcal{H}\Psi = \left(\partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta} + \omega^2\right)\Psi(r,\theta) = 0. \tag{1.41}$$

If we look for solutions of the form $\Psi_0 = R(r)\Theta(\theta)$, then, just as in Section 1, we find ordinary differential equations for R and Θ that have solutions of the form $R(r) = J_{\pm k}(\omega r)$ and $\Theta(\theta) = e^{\pm ik\theta}$, where $J_p(x)$ is the Bessel function defined by the power series

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+p}}{n!(n+p)!}.$$

Chapter 2

Symmetry Operators

2.1 The First-Order Symmetry Algebra for the Helmholtz Equation

As in Chapter 1, let $\mathcal{F}$ denote the set of all $\mathbf{C}$-valued real-analytic functions on Ω, an open, connected subset of $\mathbf{R}^2$. When we are referring to some specific differential operator Q, we refer to its solution set as $\mathcal{F}_0$. That is, $\mathcal{F}_0 = \{f \in \mathcal{F} : Q(f) = 0\}$. We define the order of a differential operator to be the order of the highest-order derivative in the operator, where $\partial_x^i \partial_y^j$ has order $i+j$. Let L be a first-order differential operator: $L = A(x,y)\partial_x + B(x,y)\partial_y + C(x,y)$ (with $A, B, C \in \mathcal{F}$). We can make sense of the composition of two diffferential operators (for instance, L and Q) as follows: $L \circ Q$ (often written LQ) acts on f by $LQ(f) = L(Q(f))$. It is possible to show (via a tedious computation) that the composition of two differential operators is itself a differential operator, although we will omit the proof. Notice that composition of differential operators is clearly associative. Given two differential operators Q_1 and Q_2, and a function $f \in \mathcal{F}$, we have $(Q_1 Q_2)(f)$ unambiguously means $Q_1(Q_2(f))$ and so, given three differential operators Q_1, Q_2, and Q_3, we see that $((Q_1 Q_2) Q_3)(f) = (Q_1 (Q_2 Q_3))(f) = Q_1(Q_2(Q_3(f)))$.

Recall the commutator $[-,-]$, introduced in Chapter 0. Note that the set of all differential operators is a Lie algebra under the commutator. We say that L is a *first-order symmetry operator* for Q if $[L, Q] = R(x,y) \circ Q$ $(= R(x,y)Q)$ for some function $R \in \mathcal{F}$.

We now prove several properties of first-order symmetry operators.

Theorem 2.1 *First-order symmetry operators map solutions to solutions.*

Proof: Let $f \in \mathcal{F}_0$, so that $Qf = 0$. We have that $[L, Q]f = RQf$, and so $LQf - QLf = R(0) = 0$. Thus, $L(0) - Q(Lf) = 0$, which implies that $Q(Lf) = 0$, and hence $Lf \in \mathcal{F}_0$ as well. $\diamondsuit$

Lemma 2.1 *Let a first-order differential operator L be of the form $L = A(x,y)\partial_x + B(x,y)\partial_y$. Then L obeys the product rule for differentiation. That is, $L(f(x,y)g(x,y)) = f(x,y)(Lg(x,y)) + (Lf(x,y))g(x,y)$.*

Proof: In order to show this, we calculate the action of L on a product of two functions f and g.

$$\begin{aligned}
L(f(x,y)g(x,y)) &= (A(x,y)\partial_x + B(x,y)\partial_y)(f(x,y)g(x,y)) \\
&= A(x,y)\partial_x(f(x,y)g(x,y)) + B(x,y)\partial_y(f(x,y)g(x,y)) \\
&= A(x,y)(f(x,y)\partial_x g(x,y)) + A(x,y)(\partial_x f(x,y)g(x,y)) \\
&\quad + B(x,y)(f(x,y)\partial_y g(x,y)) + B(x,y)(\partial_y f(x,y)g(x,y)) \\
&= f(x,y)(A(x,y)\partial_x g(x,y) + B(x,y)\partial_y g(x,y)) \\
&\quad + g(x,y)(A(x,y)\partial_x f(x,y) + B(x,y)\partial_y f(x,y)) \\
&= f(x,y)(Lg(x,y)) + g(x,y)(Lf(x,y)) \qquad \diamondsuit
\end{aligned}$$

Theorem 2.2 *The set $\mathcal{S}$ of first-order symmetry operators of a given differential operator Q is a complex Lie algebra. That is, (i) $\mathcal{S}$ is closed under Lie bracket, and (ii) $\mathcal{S}$ is closed under $\mathbf{C}$-linear combinations.*

Proof: Let $L_1, L_2 \in \mathcal{S}$. We first verify that $[L_1, L_2]$ is a first-order differential operator:

$$\begin{aligned}
L_1 L_2 f &= (A_1\partial_x + B_1\partial_y + C_1)(A_2\partial_x + B_2\partial_y + C_2)f \\
&= (A_1\partial_x + B_1\partial_y + C_1)(A_2 f_x + B_2 f_y + C_2 f) \\
&= A_1(A_{2x}f_x + A_2 f_{xx} + B_{2x}f_y + B_2 f_{xy} + C_{2x}f + C_2 f_x) \\
&\quad + B_1(A_{2y}f_x + A_2 f_{xy} + B_{2y}f_y + B_2 f_{yy} + C_{2y}f + C_2 f_y) \\
&\quad + C_1(A_2 f_x + B_2 f_y + C_2 f),
\end{aligned}$$

where $A_{2x} = \partial_x(A_2)$, etc.

Similarly,

$$\begin{aligned} L_2L_1f &= A_2(A_{1x}f_x + A_1f_{xx} + B_{1x}f_y + B_1f_{xy} + C_{1x}f + C_1f_x) \\ &\quad + B_2(A_{1y}f_x + A_1f_{xy} + B_{1y}f_y + B_1f_{yy} + C_{1y}f + C_1f_y) \\ &\quad + C_2(A_1f_x + B_1f_y + C_1f). \end{aligned}$$

Notice that, in the above expressions, wherever f appears twice differentiated, its coefficient is not differentiated at all. As a result, the coefficients of f_{xx}, f_{xy}, and f_{yy} are the same in the expressions for L_1L_2f and for L_2L_1f. Thus, in the commutator, all second-derivatives of f drop out, and we are left with

$$\begin{aligned} [L_1, L_2]f &= (A_1A_{2x} - A_2A_{1x} + B_1A_{2y} - B_2A_{1y})\partial_x f \\ &\quad + (A_1B_{2x} - A_2B_{1x} + B_1B_{2y} - B_2B_{1y})\partial_y f \\ &\quad + (A_1C_{2x} - A_2C_{1x} + B_1C_{2y} - B_2C_{1y})f. \end{aligned}$$

The above is clearly a first-order differential operator acting on f.

(i) From the Jacobi identity, we have that

$$[[L_1, L_2], Q] + [[L_2, Q], L_1] + [[Q, L_1], L_2] = 0. \tag{2.1}$$

Thus, since $[L_1, Q] = R_1Q$ and $[L_2, Q] = R_2Q$, we have $[[L_1, L_2], Q] + [R_2Q, L_1] + [-R_1Q, L_2] = 0$. Thus, (2.1) now implies that

$$[[L_1, L_2], Q] = [R_1Q, L_2] - [R_2Q, L_1]. \tag{2.2}$$

We define the following function on first-order operators: Given $L = A(x,y)\partial_x + B(x,y)\partial_y + C(x,y)$, we set $\widetilde{L} = L - C(x,y) = A(x,y)\partial_x + B(x,y)\partial_y$.

Note that

$$\begin{aligned} [R_1Q, L_2]f &= R_1QL_2f - L_2R_1Qf \\ &= R_1\left(L_2Q - [L_2, Q]\right)f - \left(\widetilde{L}_2 + C_2\right)R_1Qf \end{aligned}$$

$$
\begin{aligned}
&= R_1\left((\widetilde{L}_2 + C_2)Q - R_2Q\right)f - \left(\widetilde{L}_2 + C_2\right)R_1Qf \\
&= \left(R_1\widetilde{L}_2Q + R_1C_2Q - R_1R_2Q\right)f - \left(\widetilde{L}_2R_1Q + R_1C_2Q\right)f \\
&= R_1\widetilde{L}_2Qf - \widetilde{L}_2(R_1(Qf)) - R_1R_2Qf \\
&= R_1\widetilde{L}_2Qf - \left(R_1\widetilde{L}_2(Qf) + (\widetilde{L}_2(R_1))(Qf)\right) - R_1R_2Qf \quad \text{(by Lemma 2.1)} \\
&= -\left(\widetilde{L}_2(R_1)\right)Qf - R_1R_2Qf.
\end{aligned}
$$

Similarly, $[R_2Q, L_1]f = -(\widetilde{L}_1(R_2))Qf - R_1R_2Qf$. Thus,

$$[[L_1, L_2], Q] = (\widetilde{L}_1(R_2) - \widetilde{L}_2(R_1))Q.$$

(*ii*) Let $a_1, a_2 \in \mathbf{C}$. We consider $[a_1L_1 + a_2L_2, Q]$. It is a property of Lie bracket that $[a_1L_1 + a_2L_2, Q] = [a_1L_1, Q] + [a_2L_2, Q]$. Furthermore, $[a_1L_1, Q] = a_1[L_1, Q] = a_1R_1Q$, and similarly $[a_2L_2, Q] = a_2R_2Q$. So,

$$[a_1L_1 + a_2L_2, Q] = (a_1R_1 + a_2R_2)Q. \qquad \diamondsuit$$

Theorem 2.3 *Let Q be a differential operator of the form $Q = k + \sum_{i=1}^{m}\sum_{j=1}^{n}(c_{ij}\partial_x^i\partial_y^j)$ with $k \in \mathbf{R} - \{0\}$ and $\forall i, j \in \mathbf{Z}^+(C_{ij} \in \mathcal{F})$, and let L be a first-order symmetry operator for Q of the form $L = A\partial_x + B\partial_y + c$, with $A, B \in \mathcal{F}$ and $c \in \mathbf{R}$. Then $[L, Q] = 0$.*

Proof: Since L is a symmetry operator for Q, there exists some $R \in \mathcal{F}$ such that for all $f \in \mathcal{F}$, $[L, Q]f = RQf$. Choose $f(x, y) = 1$. Then, since the derivative of a constant is zero, $[L, Q]1 = LQ(1) - QL(1) = L(k) - Q(c) = ck - kc = 0$. So, we have $0 = RQ(1) = Rk$, which, since $k \neq 0$, forces $R(x, y) = 0$. Thus, $[L, Q] = 0Q = 0$. $\diamondsuit$

Example 2.1

Consider for example the symmetry algebra $\mathcal{A}$ of first-order symmetry operators of the Helmholtz equation (1.1). In this case, $Q = \mathcal{H} = \partial_{xx} + \partial_{yy} + \omega^2$, where $\omega^2 \neq 0$. Let $L \in \mathcal{A}$. We will characterize the possible forms that L can assume and thereby describe a basis for $\mathcal{A}$.

We write $L = A\partial_x + B\partial_y + C$, with $A, B, C \in \mathcal{F}$, and $[L, \mathcal{H}] = R\mathcal{H}$ for some $R \in \mathcal{F}$. Now, $[L, \mathcal{H}]$ acts on a function as follows: Given $f \in \mathcal{F}$, $[L, \mathcal{H}]f = L\mathcal{H}f - \mathcal{H}Lf$.

Observe that

$$\begin{aligned}\partial_{xx}(fg) &= \partial_x(\partial_x(fg)) \\ &= \partial_x(fg_x + f_x g) \\ &= fg_{xx} + f_x g_x + f_x g_x + f_{xx} g \\ &= fg_{xx} + 2f_x g_x + f_{xx} g. \end{aligned} \tag{2.3}$$

We are now prepared to compute an expression for $[L, \mathcal{H}]$.

$$\begin{aligned}L\mathcal{H}f &= (A\partial_x + B\partial_y + C)(\partial_{xx} f + \partial_{yy} f + \omega^2 f) \\ &= A(f_{xxx} + f_{xyy} + \omega^2 f_x) + B(f_{xxy} + f_{yyy} + \omega^2 f_y) \\ &\quad + C(f_{xx} + f_{yy} + \omega^2 f) \end{aligned} \tag{2.4}$$

And, using (2.3),

$$\begin{aligned}\mathcal{H}Lf &= (\partial_{xx} + \partial_{yy} + \omega^2)(A\partial_x f + B\partial_y f + Cf) \\ &= Af_{xxx} + 2A_x f_{xx} + A_{xx} f_x + Bf_{xxy} + 2B_x f_{xy} + B_{xx} f_y \\ &\quad + Cf_{xx} + 2C_x f_x + C_{xx} f \\ &\quad + Af_{xyy} + 2A_y f_{xy} + A_{yy} f_x + Bf_{yyy} + 2B_y f_{yy} + B_{yy} f_y \\ &\quad + Cf_{yy} + 2C_y f_y + C_{yy} f \\ &\quad + \omega^2 Af_x + \omega^2 Bf_y + \omega^2 Cf \\ &= A(f_{xxx} + f_{xyy} + \omega^2 f_x) + B(f_{xxy} + f_{yyy} + \omega^2 f_y) \\ &\quad + C(f_{xx} + f_{yy} + \omega^2 f) \\ &\quad + 2A_x f_{xx} + (2A_y + 2B_x) f_{xy} + 2B_y f_{yy} \\ &\quad + (A_{xx} + A_{yy} + 2C_x) f_x \\ &\quad + (B_{xx} + B_{yy} + 2C_y) f_y \\ &\quad + (C_{xx} + C_{yy}) f. \end{aligned} \tag{2.5}$$

Putting (2.4) and (2.5) together, we have $[L, \mathcal{H}]f = L\mathcal{H}f - \mathcal{H}Lf$, and so,

$$-[L, \mathcal{H}]f = \mathcal{H}Lf - L\mathcal{H}f$$

$$
\begin{aligned}
&= 2A_x f_{xx} + (2A_y + 2B_x) f_{xy} + 2B_y f_{yy} \\
&\quad + (A_{xx} + A_{yy} + 2C_x) f_x \\
&\quad + (B_{xx} + B_{yy} + 2C_y) f_y \\
&\quad + (C_{xx} + C_{yy}) f \\
&= 2A_x \partial_{xx} f + (2A_y + 2B_x) \partial_{xy} f + 2B_y \partial_{yy} f \\
&\quad + (A_{xx} + A_{yy} + 2C_x) \partial_x f \\
&\quad + (B_{xx} + B_{yy} + 2C_y) \partial_y f \\
&\quad + (C_{xx} + C_{yy}) f.
\end{aligned}
$$

Recall that $[L, \mathcal{H}] = R\mathcal{H}$. Thus,

$$-[L, \mathcal{H}] f = -R\partial_{xx} f - R\partial_{yy} f - \omega^2 R f.$$

Since this must hold true for all $f \in \mathcal{F}$, we are left with several conditions that must be met by A, B, and C. From matching up the coefficients of ∂_{xx} and ∂_{yy}, for instance, we see that

$$2A_x = -R = 2B_y. \tag{2.6}$$

From matching up the coefficients of ∂_{xy}, we see that

$$2A_y + 2B_x = 0. \tag{2.7}$$

In like manner, we match up the other corresponding coefficients and see that

$$A_{xx} + A_{yy} + 2C_x = 0 \tag{2.8}$$

$$B_{xx} + B_{yy} + 2C_y = 0 \tag{2.9}$$

$$C_{xx} + C_{yy} = -\omega^2 R. \tag{2.10}$$

From (2.6) and (2.7), we have $A_x = B_y$ and $A_y = -B_x$. These are the Cauchy-Riemann equations, and thus $A + iB$ is complex-analytic. This implies that A and B are harmonic functions of x and y, and so $A_{xx} + A_{yy} = 0$ and $B_{xx} + B_{yy} = 0$. Thus, (2.8) and (2.9) imply $C_x = 0$ and $C_y = 0$.

Thus, Lemma 1.5 forces $C(x,y) = k$, a $\mathbf{C}$-valued constant. Since $\omega^2 \neq 0$, this, in turn, forces $R = 0$ (2.10). This result in conjunction with (2.6) means that $A_x = 0$, so A is a function only of y: $A(x,y) = g(y)$ for some g; and $B_y = 0$, so B is a function only of x: $B(x,y) = f(x)$ for some f. From (2.7), we now have that $f'(x) = -g'(y)$. The only way for this to happen, as we saw in the separation of variables solution of the Helmholtz equation, is for $f'(x)$ and $-g'(y)$ both to equal some constant $-a \in \mathbf{C}$. Thus, $f(x) = -ax + c$ and $g(y) = ay + b$ for some $a, b, c \in \mathbf{C}$. Putting it all together, we have

$$L = (ay + b)\partial_x + (-ax + c)\partial_y + k$$

If we let $P_x = \partial_x$, $P_y = \partial_y$, $M = y\partial_x - x\partial_y$, and $E = 1$, then the set $\mathcal{A}$ of first-order symmety operators is clearly spanned by the linearly independent set $\mathcal{S} = \{P_x, P_y, M, E\}$, and so $\mathcal{S}$ is a basis (over $\mathbf{C}$) for $\mathcal{A}$, the symmetry algebra of the Helmholtz equation (1.1). We say that E is a *trivial* symmetry operator, for it just maps solutions to themselves. Furthermore, it will be useful to consider only the real symmetry operators. Therefore, we often refer to the Lie algebra $\mathcal{A}'$, the span of the basis $\{P_x, P_y, M\}$ over $\mathbf{R}$, as the symmetry algebra of the Helmholtz equation.

Let us determine the commutation relations. We have that every element commutes with itself, and furthermore P_x commutes with P_y, so that $[P_x, P_y] = [P_y, P_x] = 0$. Finally, $[M, P_x] = P_y$ and $[M, P_y] = -P_x$. Notice that these commutation relations are the same as we saw for $\mathcal{E}(2)$ in Example 0.11. Thus, $\mathcal{A}'$ is isomorphic to $\mathcal{E}(2)$, and so we often refer to $\mathcal{E}(2)$ as the symmetry algebra of the Helmholtz equation. Because of the analogy to Example 0.11, we say that P_x and P_y correspond to translations and M corresponds to a rotation. We often refer to $E(2)$ as the symmetry group of the Helmholtz equation, although, properly speaking, (*i*) the symmetry group of the Helmholtz equation really contains the set of exponentials[1] of the elements of $\mathcal{A}'$, not the elements of $E(2)$ (which are matrices, not operators), and (*ii*) the symmetry group also contains reflections, which are *not* exponentials.

[1]The exponential of an operator will be described in the next section.

2.2 The One-parameter Subgroup Revisited

We first remark that in order to make much of the discussion that follows fully rigorous, we would need to investigate topics from functional analysis that are beyond the scope of this paper. For instance, we will implicitly assume (without proof) that certain infinite sums of operators converge, and that certain expressions correctly denote derivatives.

Now, if L is a differential operator, there is a one-parameter subgroup associated with it that acts on functions. We define the one-parameter subgroup T_{aL} (for $a \in \mathbf{R}$) associated with L in much the same manner as in Example 0.7 — namely, with the exponential. The exponential of an operator is just what we would expect:

$$T_{aL}f = \exp(aL)f = \sum_{n=0}^{\infty} \left(\frac{1}{n!} a^n L^n f \right) \text{ for all } f \in \mathcal{F}.$$

Often, we drop the f from the above expression, and write

$$T_{aL} = \sum_{n=0}^{\infty} \left(\frac{1}{n!} a^n L^n \right).$$

Theorem 2.4 *Let Q be a differential operator, and let L be a first order differential operator such that $L = \widetilde{L}$ (i.e., $C = 0$). The following two conditions are equivalent:*
(i) : L is a first-order symmetry operator for Q such that $[L, Q] = 0$.
(ii) : Q is invariant under T_{aL}, the one-parameter subgroup associated with L, meaning that $QT_{aL}f = T_{aL}Qf$, for all $f \in cF$.

Proof: $(i) \Longrightarrow (ii)$ By (i), we have that

$$LQ = QL.$$

Therefore, if we operate on the left by L, we have

$$L^2Q = L(QL) = (LQ)L = (QL)L = QL^2.$$

Similarly, if (for $n > 1$) $L^{n-1}Q = QL^{n-1}$, then, operating on the left by L leaves us

$$L^nQ = L(QL^{n-1}) = (LQ)L^{n-1} = (QL)L^{n-1} = QL^n.$$

So, we have shown by induction on n that for all $n \in \mathbf{N}$, $L^n Q = QL^n$. Thus, for $f \in \mathcal{F}$,

$$\begin{aligned} T_{aL}Qf &= \sum_{n=0}^{\infty} \left(\frac{1}{n!} a^n L^n Q f \right) \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{n!} a^n Q L^n f \right) \\ &= \sum_{n=0}^{\infty} \left(Q \frac{1}{n!} a^n L^n f \right) \\ &= Q \sum_{n=0}^{\infty} \left(\frac{1}{n!} a^n L^n f \right) \\ &= QT_{aL}f. \end{aligned} \tag{2.11}$$

$(ii) \Longrightarrow (i)$ By (ii), we have that for all $a \in \mathbf{R}$ and for all $f \in \mathcal{F}$, $T_{aL}Qf = QT_{aL}f$. So $T_{aL}Qf - QT_{aL}f = 0$. When we expand this, we have

$$0 = \sum_{n=0}^{\infty} \left(\frac{1}{n!} a^n L^n Q f \right) - Q \sum_{n=0}^{\infty} \left(\frac{1}{n!} a^n L^n f \right).$$

And so, for all nonzero a, we have

$$\left(\frac{1}{a} \right) \left(\sum_{n=0}^{\infty} \left(\frac{1}{n!} a^n L^n Q f \right) - Q \sum_{n=0}^{\infty} \left(\frac{1}{n!} a^n L^n f \right) \right) = 0,$$

which means that

$$\begin{aligned} 0 &= \lim_{a \to 0} \left(\frac{1}{a} \right) \left(\sum_{n=0}^{\infty} \left(\frac{1}{n!} a^n L^n Q f \right) - Q \sum_{n=0}^{\infty} \left(\frac{1}{n!} a^n L^n f \right) \right) \\ &= \lim_{a \to 0} \left(\frac{1}{a} \right) \left(\left(Qf + aLQf + \frac{a^2}{2!} L^2 Qf + \cdots \right) - Q \left(f + aLf + \frac{a^2}{2!} L^2 f + \cdots \right) \right) \\ &= \lim_{a \to 0} \left(\frac{1}{a} \right) \left(\left(Qf + aLQf + \frac{a^2}{2!} L^2 Qf + \cdots \right) - \left(Qf + aQLf + \frac{a^2}{2!} QL^2 f + \cdots \right) \right) \\ &= \lim_{a \to 0} \left(\frac{1}{a} \right) \left(aLQf - aQLf + a^2 \left(\frac{1}{2!} \left(L^2 Qf - QL^2 f \right) \right) + a^3 (\cdots) + \cdots \right) \\ &= \lim_{a \to 0} \left(LQf - QLf + a \left(\frac{1}{2!} \left(L^2 Qf - QL^2 f \right) \right) + a^2 (\cdots) + \cdots \right) \\ &= LQf - QLf + \lim_{a \to 0} \left(a \left(\frac{1}{2!} \left(L^2 Qf - QL^2 f \right) \right) + a^2 (\cdots) + \cdots \right) \\ &= LQf - QLf + 0 \\ &= [L, Q] f. \end{aligned}$$

We have demonstrated that $[L, Q] = 0 = 0Q$, and thus that L is a symmetry operator for Q. $\diamondsuit$

It turns out that the set of operators $\{\exp(L) : L \in \mathcal{A}'\}$ is closed under composition, and, as a group is isomorphic to $E(2)$. If L is a first-order symmetry operator for the Helmholtz equation, so that $L = aP_x + bP_y + \theta M$, then L corresponds to the matrix $\mathbf{A} = a\mathbf{P_1} + b\mathbf{P_2} + \theta\mathbf{M} \in \mathcal{E}(2)$. The exponential T_L, then, corresponds to $g_{_L} = e^{\mathbf{A}} \in E(2)$. Now, in accordance with Example 0.11, the action of $T_{g_{_L}}$ on a function can be written as follows: $T_{g_{_L}} f(\mathbf{x}) = f(\mathbf{x}g_{_L})$. A useful fact, whose proof we omit, is that the action of $T_{g_{_L}}$ on f is the same as the action of the operator T_L on f:

$$T_L f(\mathbf{x}) = T_{g_{_L}} f(\mathbf{x}) = f(\mathbf{x}g_{_L}). \tag{2.12}$$

2.3 First-Order Symmetry and Separation of Variables

In rectangular coordinates, regardless of the choice of k, there exists a separated solution Ψ_k of (1.1) that is a simultaneous eigenfunction of P_x and P_y. There actually exist four linearly independent such separated solutions, but the existence of just one is sufficient for our purposes. Consider $\Psi_k = X_1 Y_1$, where $X_1 = e^{ikx}$ and $Y_1 = e^{i\sqrt{\omega^2 - k^2}y}$:

$$P_x(\Psi_k) = ik\Psi_k \quad \text{and} \quad P_y(\Psi_k) = i\sqrt{\omega^2 - k^2}\Psi_k. \tag{2.13}$$

Thus Ψ_k is an eigenfunction of P_x and P_y.

In polar coordinates, a similar comment applies. Here, regardless of the choice of k, there exists a separated solution Ψ_k (again, actually four such solutions) that is an eigenfunction of M. To see why, compare to (1.40), and notice that, by its definition, $M = y\partial_x - x\partial_y = -\partial_\theta$. Thus, if we let $\Psi_k = e^{ik\theta} J_k(\omega r)$, we have that

$$M(\Psi_k) = -ik\Psi_k. \tag{2.14}$$

It turns out that, given an arbitrary first-order symmetry operator for the Helmholtz equation $L \in \mathcal{E}(2)$, there exists a coordinate system $\{u, v\}$ in which (1.1) is strongly separable, such that at least one separated solution is an eigenfunction of L and $L = \frac{\partial}{\partial u}$. To see this, we need the following theorems.

Theorem 2.5 *Let $L = A(x,y)\partial_x + B(x,y)\partial_y$ be a first-order operator (so that $A, B \in \mathcal{F}$) that is defined on an open, connected set $\Omega \in \mathbf{R}^2$. Then, given $\mathbf{x}_0 = (x_0, y_0) \in \Omega$, there exist*

(i) : an open set U with $\mathbf{x}_0 \in U \subseteq \Omega$

(ii) : an $\epsilon > 0$

(iii) : a real-analytic map $X : (-\epsilon, \epsilon) \times U \longrightarrow \Omega$ which we will write as $X_t(a,b) = (x(t,a,b), y(t,a,b))$ such that

(1) : $X_0(a,b) = (a,b)$ — that is, $x(0,a,b) = a$ and $y(0,a,b) = b$;

(2) : if s, t and $s+t$ lie in $(-\epsilon, \epsilon)$, then there exists an open set V with $\mathbf{x}_0 \in V \subseteq U$ such that $X_t(V) \subseteq U$ and $X_{s+t} = X_s \circ X_t$ on V;

(3) : for $(a,b) \in U$ fixed, $x(t) = x(t,a,b)$ and $y(t) = y(t,a,b)$ is the unique solution to the initial value problem

$$\frac{dx}{dt} = A(x,y), \quad \frac{dy}{dt} = B(x,y),$$
$$x(0) = a, \quad y(0) = b.$$

This theorem follows from the existence and uniqueness theorems for first-order ordinary differential equations. We will not prove it; we refer the curious reader to [Hurewicz 1958, pp. 28-29] and [Warner 1971, pp. 37-38] (they prove this in the case where L and X are C^∞, but it is true in the real-analytic case as well).

For (a,b) fixed, note that $X_t(a,b)$ is a curve passing through (a,b) at $t = 0$ (by (1)) and the tangent vector $X_t'(a,b)$ is given by the vector field L. If we vary $t \in (-\epsilon, \epsilon)$, the curve $X_t(a,b)$ is called an *integral curve* of the vector field. Furthermore, the function $X(t,a,b)$ is called the *flow* of the vector field. Now the important result is the following theorem.

Theorem 2.6 *Let $L = A(x,y)\partial_x + B(x,y)\partial_y$ be a real-analytic first-order operator defined on an open, connected set $\Omega \subseteq \mathbf{R}^2$ and $\mathbf{x}_0 = (x_0, y_0)$ be a point in Ω such that $(A(\mathbf{x}_0), B(\mathbf{x}_0)) \neq (0,0)$. Then there is a change of coordinates $X : U \longrightarrow V$ such that*

(1) : U is a neighborhood of $(0,0)$ and V is a neighborhood of $\mathbf{x}_0$

(2) : If $X(u,v) = (x(u,v), y(u,v))$, then $\partial_u = L$.

Proof: Pick a real-analytic curve γ such that $\gamma(0) = \mathbf{x}_0$ and $\gamma'(0)$ is linearly independent from $(A(\mathbf{x}_0), B(\mathbf{x}_0))$. We may do this since, by hypothesis, $(A(\mathbf{x}_0), B(\mathbf{x}_0)) \neq (0,0)$. Let $X_t(a,b)$ be the flow of the operator L, from Theorem 2.5. Then we define $X(u,v) = X_u(\gamma(v)) = (x(u,v), y(u,v))$.

Claim 1

$$\frac{\partial x}{\partial u}(u,v) = A(x(u,v), y(u,v))$$
$$\frac{\partial y}{\partial u}(u,v) = B(x(u,v), y(u,v))$$

This follows by fixing v and letting $(a,b) = \gamma(v)$. Then $(x(u,v), y(u,v)) = X_u(a,b)$ Claim 1 now follows from (3) of Theorem 2.5.

Claim 2

$$\det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \neq 0 \quad \text{at } (u,v) = (0,0)$$

Proof of Claim 2:

Note that $X(0,0) = X_0(\gamma(0)) = X_0(\mathbf{x}_0) = \mathbf{x}_0 = (x_0, y_0)$. Then, Claim 1 implies that $\frac{\partial x}{\partial u}(0,0) = A(\mathbf{x}_0)$ and $\frac{\partial y}{\partial u}(0,0) = B(\mathbf{x}_0)$. Furthermore, note that

$$\begin{aligned} \left(\frac{\partial x}{\partial v}(0,0), \frac{\partial y}{\partial v}(0,0)\right) &= \frac{d}{dv} X(0,v)|_{v=0} \\ &= \frac{d}{dv} X_0(\gamma(v))|_{v=0} \quad \text{from the definition of } X \\ &= \frac{d}{dv} \gamma(v)|_{v=0} \quad \text{since } X_0(a,b) = (a,b) \\ &= \gamma'(0). \end{aligned}$$

Since $\gamma'(0)$ and $(A(\mathbf{x}_0), B(\mathbf{x}_0))$ are linearly independent, the rows of the Jacobian matrix are linearly independent. Thus, the determinant is nonzero, as claimed.

Now, Claim 2 implies that the map $(u,v) \mapsto X(u,v) = (x,y)$ is a local change of coordinates in the neighborhoods of $(0,0)$ and (x_0, y_0) (by the Inverse Function Theorem). Then, Claim 1 implies that $L = \partial_u$, because the chain rule implies that $\partial_u = \frac{\partial x}{\partial u}\partial_x + \frac{\partial y}{\partial u}\partial_y = A\partial_x + B\partial_y = L$. $\diamondsuit$

We have not required that the new coordinates $\{u,v\}$ be orthogonal, so the Laplace operator is written in full generality as in (1.8). More simply, we write the Helmholtz operator as follows:

$$\mathcal{H} = A_{11}\partial_{uu} + A_{12}\partial_{uv} + A_{22}\partial_{vv} + A_1\partial_u + A_2\partial_v + \omega^2, \tag{2.15}$$

where the A_{ij} and A_i equal the corresponding coefficient functions in (1.8). Let L be a first-order symmetry operator. We have that $[L, \mathcal{H}] = 0$. By Theorem 2.6, we have that $L = \partial_u$. We insert this into the commutator, and find that $\partial_u \mathcal{H} - \mathcal{H}\partial_u = 0$. After some simplification, this yields

$$A_{11_u}\partial_{uu} + A_{12_u}\partial_{uv} + A_{22_u}\partial_{vv} + A_{1_u}\partial_u + A_{2_u}\partial_v = 0.$$

This means that the above operator yields 0 when operated on *any* function $f \in \mathcal{F}$. It follows that each of the above coefficient functions is identically 0. Therefore, A_{ij} and A_i are functions only of v. As a result, one finds that the function $\Psi_k = e^{iku}V(v)$ is a solution to $\mathcal{H}\Psi = 0$, as long as V satisfies

$$A_{22}V'' + (ikA_{12} + A_2)\,V' + \left(-k^2A_{11} + ikA_1 + \omega^2\right)V = 0.$$

Clearly, then, $L\Psi_k = ik\Psi_k$, and so Ψ_k is an eigenfunction of L.

Although Ψ_k is a weakly separated solution, the coordinates $\{u, v\}$ might not be strongly separable in the sense of Chapter 1. But, it is possible to change coordinates so that we (i) preserve $\partial_u = L$, and (ii) achieve a coordinate system in which the Helmholtz equation is strongly separable. If we perform this change of coordinates, the new coordinate system will be shapewise equivalent to either rectangular or polar coordinates, as we will discover below.

Below, we will define an "orbit" of a first-order symmetry operator operator. In brief, two operators K and L will be said to lie on the same orbit if $K = cL^g$, where $g \in E(2)$ and the expression L^g will be made clear presently. We will need some fairly involved discussion before we see exactly why this definition of an orbit is useful, but the idea will be that operators that lie on the same orbit correspond to shapewise equivalent coordinate systems.

Recall that the set of exponentials of symmetry operators of the Helmholtz equation is isomorphic to $E(2)$. Let $g = g_{L'} \in E(2)$, where $g_{L'}$ is the element of $E(2)$ corresponding to $T_{L'} = e^{L'}$, for some symmetry operator L'. Let $L^g = T_g L T_{g^{-1}}$, where T_g and $T_{g^{-1}}$ operate on functions as described in Example 0.11. As we saw in (2.11), $T_{L'}$ and $T_{L'}^{-1}$ both commute with $\mathcal{H}$. Clearly, L commutes with $\mathcal{H}$, since all first-order symmetry operators do. Thus, in accordance with (2.12), we may write L^g as a product of three operators that commute with $\mathcal{H}$ (namely, $L^g = T_g L T_{g^{-1}} = T_{L'} L T_{-L'}$), and hence it commutes with $\mathcal{H}$ and is a symmetry operator. This follows, since L^g commuting with

$\mathcal{H}$ means that $[L^g, \mathcal{H}] = 0 = 0\mathcal{H}$. Write $L = A(\mathbf{x})\partial_x + B(\mathbf{x})\partial_y$; $\mathbf{x} = (x, y)$; $\mathbf{x}' = (x', y') = \mathbf{x}g$; and $\tilde{\mathbf{x}} = (\tilde{x}, \tilde{y}) = \mathbf{x}g^{-1}$. Note also that L^g is a first-order operator, for L involves only first-order derivatives, and T_g and $T_{g^{-1}}$ only change coordinates. Finally, notice that

$$\begin{aligned} L^{g_1 g_2} &= T_{g_1 g_2} L T_{g_2^{-1} g_1^{-1}} \\ &= T_{g_1} \left(L^{g_2}\right) T_{g_1^{-1}} \\ &= \left(L^{g_2}\right)^{g_1}. \end{aligned}$$

Let $\Psi_k = U(u)V(v)$ be a separated solution in the strongly separable coordinates $\{u, v\}$. Since $g \in E(2)$ implies that g is an affine similitude and hence preserves shapewise equivalence, we have that $\mathbf{u}' = \mathbf{u}g$ is a strongly separable coordinate system (Theorem 1.2). Since T_g acts on Ψ_k via g's action on $\mathbf{u} = (u, v)$, we have that $T_g\Psi_k$ is a separated solution in $\mathbf{u}'$ coordinates, for $T_g\Psi_k(\mathbf{u}) = \Psi_k(\mathbf{u}') = U(u')V(v')$. If Ψ_k is an eigenfunction of L with eigenvalue ik, then so is $T_g\Psi$, for $LT_g\Psi = T_gL\Psi$ (this follows from Theorem 2.4) which equals $T_g(ik)\Psi_k = (ik)T_g\Psi_k$. Clearly, for $c \neq 0$, cL will have the same eigenfunctions as L (the eigenvalues will simply be multiplied by c): $cL\Psi_k = cik\Psi_k$. Furthermore, the coordinates $\{\tilde{u}, v\}$ such that $cL = \partial_{\tilde{u}}$ are just a reparametrization of the coordinates $\{u, v\}$ given by $L = \partial_u$, and hence are shapewise equivalent to $\{u, v\}$. So, for $g \in E(2)$, we have

$$\begin{aligned} cL^g\Psi_k &= c\left(T_gLT_{g^{-1}}\right)\Psi_k \\ &= cT_gL\left(T_{g^{-1}}\Psi_k\right) \\ &= cT_g(ik)T_{g^{-1}}\Psi_k \\ &= cik\Psi_k. \end{aligned}$$

And, of course, the coordinates given by cL^g are a combination of a reparametrization of $\{u, v\}$ and an affine similitude, and hence are shapewise equivalent to $\{u, v\}$. So, as stated above, if $K \in \mathcal{E}(2)$, K is said to lie on the same *orbit* as $L \in \mathcal{E}(2)$ if $K = cL^g$ for some $c \in \mathbf{R} - \{0\}$ and some $g \in E(2)$, for the coordinates that K gives are shapewise equivalent to those that L gives.

We therefore want to find all equivalence classes of coordinate systems that we can obtain by requiring at least one separated solution to be an eigenfunction of L for some $L \in \mathcal{E}(2)$. Define

g_1, g_2, and g_3 so that $T_{g_1} = \exp(aP_x)$ and $T_{g_2} = \exp(bP_y)$, and $T_{g_3} = \exp(\theta M)$, where we will define a, b, θ below. It is proven in [Hausner and Schwartz 1968, p. 66] that $e^K L e^{-K} = e^{\mathrm{Ad}K} L$, where $\mathrm{Ad}K(L) = [K, L]$. A straightforward computation then leads to

$$P_x^{g_3} = \cos\theta P_x + \sin\theta P_y, \quad P_y^{g_3} = -\sin\theta P_x + \cos\theta P_y, \quad M^{g_3} = M. \tag{2.16}$$

Therefore, if a first-order symmetry operator L is of the form $c_1P_x + c_2P_y + c_3M$ where $c_3 \neq 0$, then we claim that $L^{g_1g_2} = c_3M$ if $a = \frac{c_2}{c_3}$ and $b = -\frac{c_1}{c_3}$. This follows since $g_1g_2 = e^{aP_x + bP_y}$, which we will write as e^P, where $P = aP_x + bP_y$; and then we have

$$\begin{aligned}
L^{g_1g_2} &= e^P L e^{-P} = e^{\mathrm{Ad}P} L = L + [P, L] + \tfrac{1}{2}[P, [P, L]] + \tfrac{1}{6}[P, [P, [P, L]]] + \cdots \\
&= \text{(since } [P, [P, L]] = 0) \; L + [P, L] = L + [P, c_3M] = L + c_3a[P_x, M] + c_3b[P_y, M] \\
&= L + c_2(-P_y) - c_1(P_x) \\
&= (c_1P_x + c_2P_y + c_3M) - c_2P_y - c_1P_x = c_3M.
\end{aligned}$$

If, however, $L = c_1P_x + c_2P_y \neq 0$, then setting $\theta = \begin{cases} \tan^{-1}(\frac{c_1}{c_2}) \text{ if } c_2 \neq 0 \\ \cot^{-1}(\frac{c_2}{c_1}) \text{ otherwise} \end{cases}$ yields $L^{g_3} = \sqrt{c_1^2 + c_2^2}P_y$:

$$\begin{aligned}
L^{g_3} &= e^{\theta M} L e^{-\theta M} = e^{\theta \mathrm{Ad}M} L \\
&= \text{(from (2.16))} \; c_1(\cos\theta P_x + \sin\theta P_y) + c_2(-\sin\theta P_x + \cos\theta P_y) \\
&= (c_1\cos\theta - c_2\sin\theta)P_x + (c_1\sin\theta + c_2\cos\theta)P_y \\
&= 0P_x + \sqrt{c_1^2 + c_2^2}P_y.
\end{aligned}$$

In the first case, where $c_3 \neq 0$, L lies on the same orbit as M; in the second case, where $c_3 = 0$, L lies on the same orbit as P_y. Thus, rectangular and polar are the two shapewise inequivalent, orthogonal, strongly separable coordinate systems that first-order symmetry analysis explains.

2.4 Second-Order Symmetry

In our conception, the main goal of this paper is to clarify the terminology in the first two sections of [Miller 1977], and to provide enough background material and theoretical justification to allow a reader who is familiar with the typical undergraduate math curriculum to read those sections of

Miller's book and hence to understand the group-theoretical explanation for the four coordinate systems in which the Helmholtz equation is strongly separable. Because of (i) the similarity between the way one obtains the second-order symmetry operators and the way one obtains the first-order symmetry operators, and (ii) the similarity of the orbit-analysis of second-order symmetry operators to that of first-order symmetry operators, it seems that the above goal has now been accomplished. In order to accomplish this goal, it has been necessary, in many places, to follow a line of reasoning similar to Miller's, with added detail or other clarifications. We stress, however, that the goal of this paper is *not* to recapitulate Miller's work. Thus, although we will present the important results from the second-order symmetry analysis for the sake of completeness, we refer the reader to [Miller 1977, pp. 17-22] for all details.

A second-order symmetry operator S for (1.1) is a real-analytic second-order differential operator $A_{11}\partial_{xx} + A_{12}\partial_{xy} + A_{22}\partial_{yy} + A_1\partial_x + A_2\partial_y + A$ (real-analytic means that $A_{ij}, A_i, A \in \mathcal{F}$) such that $[S, \mathcal{H}] = Q \circ \mathcal{H}$, where Q is a real-analytic first-order differential operator. Let $\mathcal{S}$ denote the second-order symmetry algebra of the Helmholtz operator $\mathcal{H}$. Note that $[\mathcal{H}, \mathcal{H}] = 0 = 0\mathcal{H}$, so $\mathcal{H}$ is a second-order symmetry operator. One can show that, in fact, the product of any real-analytic function and $\mathcal{H}$ is also a second-order symmetry operator. Thus, $\mathcal{Z} = \{S : S = R\mathcal{H} \text{ where } R \in \mathcal{F}\}$ lies in $\mathcal{S}$. Since (i) $\mathcal{Z}$ is infinite dimensional, and (ii) all symmetry operators in $\mathcal{Z}$ map solutions of $\mathcal{H}$ to 0 and hence are said to be *trivial*, it is useful to construct the set of non-trivial symmetries $\mathcal{S}' = \mathcal{S}/\mathcal{Z}$. We compute $\mathcal{S}'$ in much the same way as we compute $\mathcal{A}$ in the first-order case; and one can prove that a basis over $\mathbf{C}$ for $\mathcal{S}'$ can be expressed as

$$\left\{P_x, P_y, M, E, P_x^2, P_xP_y, M^2, \{M, P_x\}, \{M, P_y\}\right\},$$

where $\{-,-\}$ denotes the *anti-commutator*, so that $\{Q_1, Q_2\} = Q_1Q_2 + Q_2Q_1$.

As in the first-order case, we restrict our attention to the real Lie algebra, and furthermore, we consider only purely second-order symmetry operators:

$$\mathcal{S}^{(2)} = \left\{P_x^2, P_xP_y, M^2, \{M, P_x\}, \{M, P_y\}\right\}.$$

The important result of the second-order orbit analysis is Theorem 2.7 below, which we will present after defining an orbit of a second-order symmetry operator.

First, note that the set of affine similitudes on $\mathbf{R}^2$ forms a group, which we will denote $AS(2)$. Just as with $E(2)$, we will say that given $g \in AS(2)$, T_g acts on functions by $T_g f(\mathbf{x}) = f(\mathbf{x}g)$.

Definition 2.1 *Let S and S' be second-order symmetry operators of the Helmholtz equation, and let $g \in AS(2)$. The construction S^g is defined to equal $T_g S T_{g^{-1}}$. We say that S lies on the same orbit as S' if $S = c(S')^g$, for some $c \in \mathbf{R} - \{0\}$ and some $g \in E(2)$.*

Theorem 2.7 *Let $S_p = \{M, P_y\}$ and $S_e = M^2 + P_y^2$. Then,*
(1) : *To every coordinate system that is shapewise equivalent to parabolic coordinates, there corresponds some second-order symmetry operator $S_{p'} \in \mathcal{S}^{(2)}$ of which at least one separated solution is an eigenfunction, where $S_{p'}$ lies on the same orbit as S_p.*
(2) : *To every coordinate system that is shapewise equivalent to elliptic coordinates, there corresponds some second-order symmetry operator $S_{e'} \in \mathcal{S}^{(2)}$ of which at least one separated solution is an eigenfunction, where $S_{e'}$ lies on the same orbit as S_e.*

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