A short note on the discontinuous Galerkin discretization of the pressure projection operator in incompressible flow

D.T. Steinmoeller[a,*], M. Stastna[a], K.G. Lamb[a]

[a] *200 University Ave. W., Waterloo, ON, N2L 3G1*

Abstract

This note reports on an issue that may arise when the popular pressure projection method (PPM) is used for unsteady simulations of incompressible flow using the symmetric interior penalty discontinuous Galerkin (SIP-DG) method. Through a spectral analysis of the projection operator's DG discretization, we demonstrate that a global smoothing operator (e.g., viscosity) is required for stability and that, with insufficient smoothing, the PPM operator's eigenfunctions exhibit poor weak continuity properties and possess entropy-violating characteristics that can cause instabilities for inviscid, advection-dominated, and density stratified flow simulations, especially in marginally resolved situations or for long-time integrations. Some remedies to the problem are proposed and outlined.

Keywords: Incompressible Navier–Stokes Equations, Projection Methods, Discontinuous Galerkin Method, High-order element methods

*Corresponding author at: Department of Applied Mathematics, University of Waterloo, Waterloo, ON, Canada. N2L 3G1. Tel.: +1 519 888 4567 ext. 32588; fax: +1 519 746 4319.

Email address: `dsteinmo@uwaterloo.ca` (D.T. Steinmoeller)

1. Introduction

Recent interest in the possibility of using the DG method for numerical solutions to the unsteady incompressible Navier–Stokes (INS) equations has been sparked due its succesful employment in compressible flow simulations [3, 10] and its many attractive features such as geometric flexibility, stencil locality/compactness, upwind-biased fluxes for advection-dominated flows, and high-order accuracy [4, 6, 12]. The desire for DG solutions is further motivated by the Ladyzhenskaya-Babuska-Brezzi (LBB) stability problem that plagues continuous Galerkin (CG) formulations due to spurious pressure modes, resulting in stability for only certain spatially mixed-order velocity-pressure formulations [7, 11, 13, 14].

However, since the INS equations do not comprise a hyperbolic system it is unclear if the DG method can be used to recover consistent, stable, and accurate numerical solutions in general since DG methods only weakly impose continuity across elemental interfaces, typically by solving Riemann problems [12]. Indeed, for the spectral element ocean model (SEOM), Levin et al. [17] chose a DG formulation for scalar transport equations but not for the momentum equations, citing the ill-posedness of the Riemann problem for incompressible flow as a central difficulty. On the other hand, some studies have succesfully obtained DG solutions using the PPM to some standard test cases for the INS equations, such as the Taylor-Green Vortex [21, 12, 9], laminar Koznavay flow [12], and flow past square [21, 9] and circular [12] cylinders. These test cases however, typically only focus on short time integrations or highly viscous/damped situations, and the long-term stability properties of the methods remain unclear.

Here, we focus on difficulties encountered when using the standard PPM for the INS equations in velocity-pressure formulation for long time integrations or stratified flow simulations under the Boussinesq approximation. These difficulties are especially prominent in inviscid or low-viscosity situations. *[Already said in abstract – Remove?]* The main contribution of this work is a spectral analysis of the PPM operator that is furnished by the numerical calculation of the eigenvalues and eigenfunctions on a unit square domain with solid-wall boundary conditions. We argue that the results shown here are robust and apply to more general domains and meshes as well as to the three-dimensional INS equations.

2. Methods

2.1. Pressure Projection Method (PPM) and its weak form

The non-dimensionalized homogenous INS equations are [16]

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) = -\nabla p + \frac{1}{Re}\nabla^2 \mathbf{u} \, , \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \, , \tag{2}$$

corresponding to conservation of momentum and mass, respectively. Here, $\mathbf{u}$ is the dimensionless velocity field with components $(u(\mathbf{x}), v(\mathbf{x}))$ (and $\mathbf{x} = (x, y)$) in two dimensions, $p = p(\mathbf{x})$ is the dimensionless pressure and $Re = UL/\nu$ is the Reynolds number as a function of kinematic viscosity ν and typical velocity and length scales U and L, respectively. The inviscid incompressible Euler equations are recovered in the limit $Re \to \infty$. We are also interested in simulations of density stratified flow under the Boussinesq approximation. In that case, the equations are similar to (1)–(2) with the inclusion of a buoyancy source term in (1) and an additional scalar transport equation. We omit the Boussinesq approximation equations here for the sake of brevity and refer the interested reader to [16].

The PPM time-stepping algorithm has been thoroughly explained in the literature, a very popular approach is the high-order stiffly-stable splitting algorithm due to Karniadakis *et al.* [15]. The method first forms a predicted velocity $\hat{\mathbf{u}}$ by explicitly evolving the advection term $\nabla \cdot (\mathbf{u} \otimes \mathbf{u})$ with a linear multi-step method. The projection step involves solving the following Poisson equation, along with suitable boundary conditions (see [15]), for the pressure p at time $t_{n+1} = t_n + \Delta t$:

$$\frac{1}{\Delta t}\left[\nabla \cdot \hat{\hat{\mathbf{u}}} - \nabla \cdot \hat{\mathbf{u}}\right] = -\nabla^2 p^{n+1} \, , \tag{3}$$

that is obtained by taking the divergence of the semi-discrete form of equation (1) after the advective terms have been evolved but before the viscous terms have been evolved. The constraint (2) is enforced by removing the $\nabla \cdot \mathbf{u}^{n+1}$ term in (3), thereby projecting the approximate solution onto the space of approximately non-divergent velocity fields. Once p^{n+1} has been computed, the pressure gradient is evolved in the semi-discrete form of (1) to recover the corrected velocity field,

$$\hat{\hat{\mathbf{u}}} = \hat{\mathbf{u}} - \Delta t \nabla p^{n+1} \, , \tag{4}$$

and the viscous terms can subsequently be evolved to recover $\mathbf{u}^{n+1}$. These terms are often evolved using implicit time-stepping, i.e.,

$$\left[1 - (\Delta t/Re)\nabla^2\right]\mathbf{u}^{n+1} = \hat{\hat{\mathbf{u}}} \, , \tag{5}$$

typically along with the no-slip boundary condition $\mathbf{u} = 0$ on solid boundaries. Clearly, this step is omitted in the inviscid case, $Re \to \infty$. In the DG framework, this last step is usually discretized using the SIP-DG formulation of the implicit viscosity operator $\left[1 - (\Delta t/Re)\nabla^2\right]$ [12, 21, 9].

In the context of DG methods, the difficulty highlighted in this work stems from the discretization of the left-hand side of (3), and we pre-suppose that the right-hand side is discretized via the nodal SIP-DG method (see [12] for an overview and a MATLAB implementation). The weak DG formulation of the left-hand side can be found by considering the local solution to (3) on a particular element (or sub-domain of Ω) D^k (where $k = 1, \cdots, K$), multiplying by a member of the space of test-functions $\{\phi_j^k\}_{j=1}^{N_p}$ and integrating by parts, yielding

$$\frac{1}{\Delta t}\left[\left(\int_{\partial D^k}\left(\phi_j\hat{\hat{\mathbf{u}}}\right)^* \cdot \hat{\mathbf{n}} - \int_{D^k}\hat{\hat{\mathbf{u}}}^k \cdot \nabla\phi_j^k \, d\mathbf{x}\right) - \left(\int_{\partial D^k}\left(\phi_j\hat{\mathbf{u}}\right)^* \cdot \hat{\mathbf{n}} - \int_{D^k}\hat{\mathbf{u}}^k \cdot \nabla\phi_j^k \, d\mathbf{x}\right)\right] , \tag{6}$$

where superscript $*$ denotes an appropriate numerical flux function chosen to impose weak continuity across element interfaces in a way that is consistent with the underlying dynamics of the INS equations.

We notice that the PPM method results in simply removing the first two terms in eqn. (6). Thus, despite the that fact weak continuity is imposed in p via the SIP-DG method, the PPM does not explicitly enforce weak continuity in the corrected velocity $\hat{\hat{\mathbf{u}}}$, and the overall impact on the resulting DG scheme remains unclear. Here, we attempt to understand the effects of using the PPM with DG via a numerical eigenvalue analysis as explained below.

2.2. Numerical Method

Beginning with the nodal DG implementation of the INS solver presented in [12], we have computed the eigenvalues of the PPM operator $\mathbb{P} : \hat{\mathbf{u}} \mapsto \hat{\hat{\mathbf{u}}}$ by grouping the set of linear operations corresponding to the DG discretization of (3)-(4) into a single MATLAB function and passing its corresponding function-handle into MATLAB's `eig` eigenvalue solver. In cases where viscosity was considered (finite *Re*), the viscous step (5) was included in $\mathbb{P}$.

The domain under consideration is the closed unit square $\Omega = [0, 1] \times [0, 1]$ subject to no normal flow, and when viscosity is included, no-slip boundary conditions, i.e., $\mathbf{u} = 0$ on $\partial\Omega$. For a domain with solid walls only, the Poisson problem (3) is subject to Neumann conditions only ([15]), and there is no unique solution. To avoid this issue, we have used the approach of [22] and introduced an auxiliary additive scalar variable to the discretized version of equation (3) in order to impose the additional constraint

$$\int_{\Omega} p^{n+1} \, d\mathbf{x} = 0 , \tag{7}$$

i.e., zero-mean pressure *[Suggestion: omit the equation?]*. Since only the gradient of p is physical in incompressible flow, this approach is quite reasonable, although alternatives exist (cite null singular vector guys?).

3. Results

3.1. DG Simulations using the PPM

We have carried out long-time integrations of (1)-(2) using the nodal DG implemention in [12] and found spikes in the solution to form at element interfaces that lead to numerical instability and blow up. Modal filtering [12] alleviates the issue somewhat allowing for longer times, but does not prevent the instability in general. Plotting the divergence $\nabla \cdot \mathbf{u}$ suggests the issue is related to spurious compressibility artifacts near element interfaces that are not zero to numerical precision. This behaviour is a consequence of the discontinuous nature of the DG method at interfaces that causes the numerical solution to be inconsistent with the INS equations.

For stratified flow simulations under the Boussinesq approximation, the situation is worse since those equations are subject to additional types of physical instabilities [16]. We found that spurious compressions caused regions of high density to artifically emerge over regions of low density at certain element interfaces, resulting in unphysical grid-scale Rayleigh-Taylor instabilities that destroyed the numerical solution. Using the PPM with DG, we were unable to obtain virtually any useful results for stratified flow unless large amounts of viscosity were used. *[Not sure about where this discussion/subsection goes, if anywhere...]*

3.2. Spectral analysis of the DG PPM operator

The unit square domain Ω was partioned into 8 uniform triangular elements, and the eigenvalues and eigenfunctions of the PPM operator $\mathbb{P}$ were computed for a variety of *Re* as well as the inviscid case. To assess the continuity properties of the eigenmodes, in Figure 1 we plot the maximum absolute jump in either of the eigenfunction's u or v component for polynomial order $N = 8$ corresponding to $N_p = 45$ nodal points on each triangle [12], or 360 grid points in total. Each u and v component was scaled to have a maximum absolute value of 1.

Perhaps the most revealing result in Figure 1 is the inviscid case shown in panel **(d)**. Since there are no damping terms in this case, we find a collection of modes with eigenvalue $\lambda = 0$ forming a basis of compressible velocity fields that belong to the null space of $\mathbb{P}$ and a collection of incompressible modes with $\lambda = 1$. The main feature to notice here is that for the (inviscid) incompressible modes, the maximum jump across element interfaces is $O(1)$, pointing to the fact that weak continuity is not being correctly imposed by the inviscid $\mathbb{P}$ operator. The projection onto such modes represents a source of numerical instability, since it is not clear if these rather large jumps will yield entropy-satisfying contributions at interfaces when the advection terms are evolved with an approximate Riemann solver. For example, the popular Rusanov/Lax-Friedrichs flux presumes that the signed jump in the

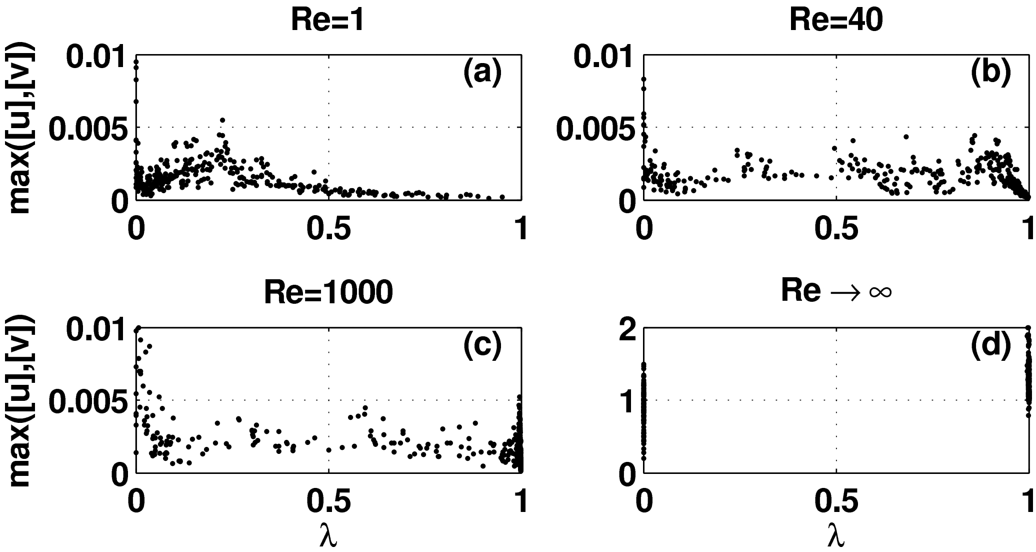

Figure 1: Maximum absolute jump $[\cdot]$ in either u or v component of velocity eigenfunction vs. corresponding eigenvalue λ at selected Re and in the inviscid case $Re \to \infty$. For finite Re, the time-step was set to $\Delta t = 10^{-3}$.

prognostic variable is the opposite of the sign of the flux across the interface in order to achieve a non-positive contribution to the global energy balance [12].

An inspection of panels **(a)–(c)** shows that viscosity appears to smooth out the eigenfunctions. However, it is unclear how much viscosity is required to obtain long-time numerically stable solutions in general. For instance, in the $Re = 1000$ case, there are a number of modes with $O(1)$ maximum jumps that do not have eigenvalue $\lambda = 0$ to numerical precision, and it is believed that these modes are a source of difficulty for long time integrations. *(Insert 3D plots of some ugly of the ugly modes for each Re in figure 1).*

3.3. Proposed remedies to the problem

In the present literature, approaches adopted to circumvent the problem highlighted here avoid the PPM altogether and ensure that the weak form of the divergence-free constraint explicitly appears in the scheme along with a suitable numerical flux function ([1, 2, 5]). Cockburn *et al.* [5] use a pressure stabilization term in their numerical flux choice for the weak DG form of (2) along with a post-processing procedure to obtain exactly non-divergent approximate velocity fields to solve the steady INS equations. The method of [1, 2] recovers a well-posed Riemann problem in the imposition of (2) by considering a numerical flux function from the artificial compressibility equations (that can be recovered by adding a $\frac{1}{c^2}\frac{\partial p}{\partial t}$ term to the left-hand side of (2), where c is an artificial sound speed. However, the method appears somewhat costly since all terms are discretized implicitly in time and exact Riemann problems must be solved numerically by nonlinear Newton iterations at each time-step. Other possibilities lay with the very recent explosion of interest in the class of hybridizable DG (HDG) methods [20, 19] or by adopting a streamfunction-vorticity formulation [18] that is limited to two-dimensional calculations.

In present work, we have followed up on the success of the (implicit-explicit) IMEX schemes for compressible flow [10], where the pressure terms (sound waves) are evolved implicitly and the advective terms are evolved explicitly, and we have adopted the same methodology for the artificial compressibility equations. The advective and pressure terms are each discretized in the DG framework with inexpensive approximate Riemann solvers. We have encountered no difficulties using this method, aside from the relative increase in size of the linear system that results from the discretization of the implicit sound-wave operator when compared to the PPM. We have succesfully validated the method for the stratified incompressible Euler equations under the Boussinesq approximation against the spectral method benchmark solutions to the Dubreil-Jacotin-Long equation [8]. The details of the method and its validation will be presented in a forthcoming publication.

[Show and discuss plot of well-behaved modes for the sound-evolution operator.]

4. Conclusions

Through an a numerical eigenvalue analysis, we conclude that the PPM is, in general, not viable for use in advection-dominated (weakly damped) DG simulations of the INS equations or the incompressible (inviscid) Euler equations. Simple remedies to the problem involve considering the artificial compressibility perturbation [1, 2] to the governing equations, thereby recovering hyperbolicity and a well-posed Riemann problem. Other remedies may be possible as well such as considering a vorticity-streamfunction formulation [18], using velocity fluxes with pressure stabilization terms along with a post-processing procedure [5], or by pursuing an HDG spatial discretization method [20, 19].

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