June 4th—Example 1

DEF Let I be an ideal in R (commutative ring).

$$\sqrt{I} = \{r \mid r^n \in I\}$$

EX

$$\sqrt{(x^2, y^3)},\ (x, y) \subseteq \sqrt{I}$$
$$I = (x^2, ^3)$$
$$\sqrt{I} \subseteq (x, y)$$

So now let's consider the ideal test. J is an ideal of R if:

1) J is closed under addition
2) $rJ \subseteq J$, $\forall r \in R$.
   Essentially says that it's closed under ANY multiplication.
   → NOTE: this gives also ADDITIVE INVERSES, so the subgroup test works!.

Sasha's work:

WTS: $rJ \subseteq J$, $\quad \&(Jr \subseteq R)$ (commutative ring), and the additive subgroup.

Proof: Let $r_1, r_2 \in J$. We know that $\exists n, m$ such that $r_1^n, r_2^m \in I$.

Let $a \in R$, WTS $ar_1 \in J$.

$(ar_1)^n = a^n r_1^n \qquad r_1^n \in I$

so $a^n r_1^n \in I_1$ so $(ar_1)^n \in I$.

Thus, $ar_1 \in J$. ∎

Marcus work:

We want to show that $(r_1 + r_2)^{n_1+n_2} \in I \Rightarrow r_1 + r_2 \in J$.

Proof: We have that:

$$(r_1 + r_2)^{n_1+n_2} = r_1^{n_1+n_2} + b_1 r_1^{n_1+n_2-1} + \cdots + bn_2 r_1^{n_1} r_2^{n_2} + \cdots + r_2^{n_1+n_2}$$

What's useful is that we can group our terms such that we can pull out a $r_1^{n_1}$ from all of our terms, which we know that $r_1^{n_1} \in I$.

Furthermore, the second group of terms will at least have a $r_2^{n_2}$ term, and we know that $r_2^{n_2} \in I$.

So that means every term in the binomial sum is in I, and because I is closed under addition, the sum must be in I, and therefore the sum must be in J.

Thus, J is an ideal. ∎