Consider $f(\bar{x})$

$\bar{x} = \langle x_1, x_2, \cdots, x_n \rangle$

And consider

$f(x,y) = xy^2+1$

$h(x,y) = xy+1.$

using our divison algorithm, we get that $f(x,y) = y \cdot h(x,y) + r$

$r = -y+1$

However, let's swap out our f & h for something harder.

$f = xy^2+1 \quad h = xy+y^2$

Let's first select a leading term that will establish a term order to help solve this problem.

Let's use lex order, where

$a > b > c > d > \cdots > x > y > z$

So when we compare:

a^2b vs. a^2b^2

$aab > aabb$

alphabetically

If we use graded lex order, we look at the degree first & then alphabetically

so, $a^2b^{\boxed{1}} < a^2b^{\boxed{2}}$

Now, let's look at Degree Choice.

Our first x has the highest degree

1 $\underline{x}^{\beta} > \underline{x}^{\alpha}$ if $|\beta| > |\alpha|$

2 $\underline{x}^{\beta} > \underline{x}^{\beta}$ if it comes first in lex order

Abbreviation

LT = leading term

LC = leading coeficcient

LM = leading monomial

So now let's use lex order in an example

$f = xy^2+1 \quad h = xy+y^2$

And we need f in terms of h. x is our leading term

$$
\begin{aligned}
xy^2+1 &= q(xy^2+y^2)+r \\
&= y(xy+y^2)+r \\
&= xy^2+y^3+r \\
r &= -y^3+1
\end{aligned}
$$

Let's try with y as our leading term

$$
\begin{aligned}
xy^2+1 &= x(xy+y^2)+r \\
&= x^2y+xy^2+r \\
r &= -x^2y+1
\end{aligned}
$$

So our remainder changes depending on our ordering.

Now, another problem arises. What happens if we want to divide $f(x,y)$ by $g(x,y)$ & $h(x,y)$ at the same time?

How do we choose what to divide first?

We get that $f(x,y) = q_1 g + r_1$

$r_1 = q_2 h + r_2$

$= q_1 g + q_2 h + r_2$

↳ a linear combination of g & h.

This can be problematic though.

Let $f = xy^2 + 1$

$g = xy + 1$

$h = y + 1$

$\frac{f}{g} \Rightarrow yg - y + 1$

↳ $-y+1 = -(y+1) + r_2$

$r_2 = 2$

so we have:

$xy^2 + 1 = y(xy+1) - (y+1) + 2$

What happens if we divide by h first?

$\frac{f}{h} \Rightarrow f = xy(y+1) - xy + 1$

$-xy + 1 = -(xy+1) + r_2$

$r_2 = 2$

We got different quotients, but the same remainder.

Now consider:

$f = x^2y + xy + y^2$

$f \div g \div h \Rightarrow f = (x+y)g + 1h \boxed{-x-1}$

$f \div h \div g \Rightarrow \boxed{r = x + 1 - x^2}$