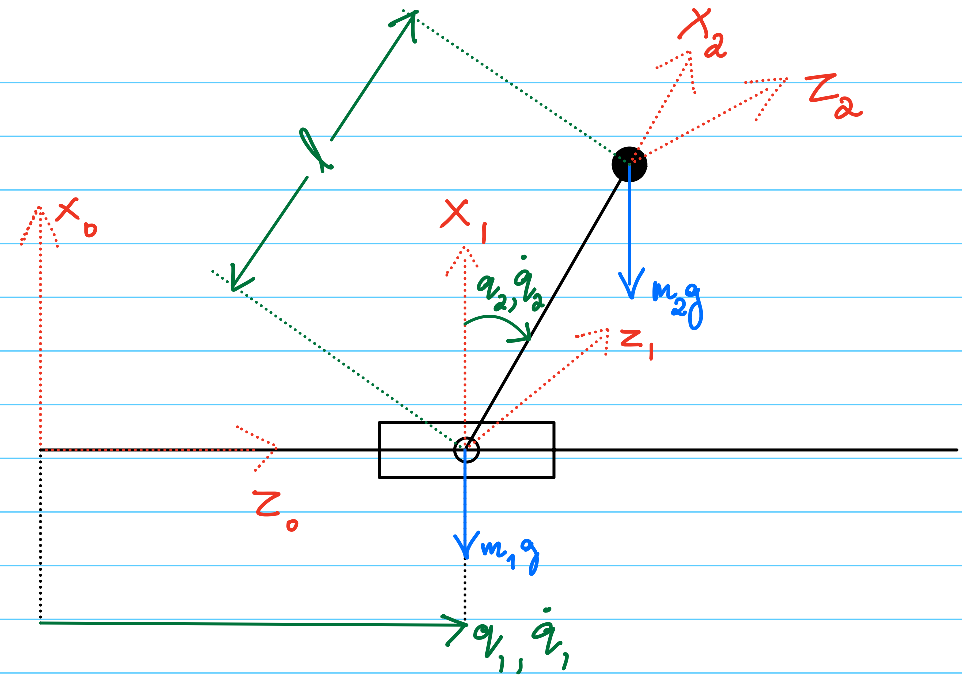

Newton Euler

Forward Pass

$${}^{0}R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

1

$${}^{1}r_{1_0,0}^{0} = [0, q_1, 0]^T \; ; \; {}^{1}\dot{r}_{1_0,0}^{0} = [0, \dot{q}_1, 0]^T$$

$${}^{1}\ddot{r}_{1_0,0}^{0} = [g, \ddot{q}_1, 0]^T$$

2

$${}^{1}R_2 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^{2}r_{2_0,0}^{0} = {}^{2}R_1 \, {}^{1}r_{1_0,0}^{0} + {}^{2}r_{2_0,0}^{1} = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ q_1 \\ 0 \end{bmatrix} + \begin{bmatrix} l \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} q_1 s_2 + l \\ q_1 c_2 \\ 0 \end{bmatrix}$$

$$
{}^{2}\dot{r}_{2_0,\,{}^{0}0} = {}^{2}R_1\,{}^{1}\dot{r}_{1_0,\,{}^{0}0} + {}^{2}\dot{r}_{2_0,\,{}^{1}0}
$$

$$
= \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \dot{q}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{q}_2 \end{bmatrix} \times \begin{bmatrix} l \\ 0 \\ 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} \dot{q}_1 s_2 \\ \dot{q}_1 c_2 + \dot{q}_2 l \\ 0 \end{bmatrix}
$$

$$
{}^{2}\ddot{r}_{2_0,\,{}^{0}0} = {}^{2}R_1\,{}^{1}\ddot{r}_{1_0,\,{}^{0}0} + {}^{2}\dot{\omega}_2 \times {}^{2}r_{2_0,\,{}^{1}0} + {}^{2}\omega_2 \times \left({}^{2}\omega_2 \times {}^{2}r_{2_0,\,{}^{1}0}\right)
$$

$$
= \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} g \\ \ddot{q}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{q}_2 \end{bmatrix} \times \begin{bmatrix} l \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{q}_2 \end{bmatrix} \times \left(\begin{bmatrix} 0 \\ 0 \\ \dot{q}_2 \end{bmatrix} \times \begin{bmatrix} l \\ 0 \\ 0 \end{bmatrix} \right)
$$

$$
= \begin{bmatrix} g c_2 + \ddot{q}_1 s_2 \\ -g s_2 + \ddot{q}_1 c_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \ddot{q}_2 l \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{q}_2 \end{bmatrix} \times \begin{bmatrix} 0 \\ \dot{q}_2 l \\ 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} g c_2 + \ddot{q}_1 s_2 - \dot{q}_2^2 l \\ -g s_2 + \ddot{q}_1 c_2 + \ddot{q}_2 l \\ 0 \end{bmatrix}
$$

Reverse Pass

#2

$$ {}^2F_{12} = m_2\, {}^2\ddot{r}_{20,0} = m_2 \begin{bmatrix} g c_2 + \ddot{q}_1 s_2 - \dot{q}_2^2 l \\ -g s_2 + \ddot{q}_1 c_2 + \ddot{q}_2 l \\ 0 \end{bmatrix} $$

$$ {}^2\mu_{12} = -{}^2F_{12} \times {}^2r_{20,10} = -m_2 \begin{bmatrix} g c_2 + \ddot{q}_1 s_2 - \dot{q}_2^2 l \\ -g s_2 + \ddot{q}_1 c_2 + \ddot{q}_2 l \\ 0 \end{bmatrix} \times \begin{bmatrix} l \\ 0 \\ 0 \end{bmatrix} $$

$$ = \begin{bmatrix} 0 \\ 0 \\ m_2 l\left(-g s_2 + \ddot{q}_1 c_2 + \ddot{q}_2 l\right) \end{bmatrix} $$

$$ {}^1F_{10} = {}^1R_2\, {}^2F_{12} + m_1\, {}^1\ddot{r}_{10,0} $$

$$ = m_2 \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} g c_2 + \ddot{q}_1 s_2 - \dot{q}_2^2 l \\ -g s_2 + \ddot{q}_1 c_2 + \ddot{q}_2 l \\ 0 \end{bmatrix} $$

$$ + m_1 \begin{bmatrix} -g \\ \ddot{q}_1 \\ 0 \end{bmatrix} $$

$$ = m_2 \begin{bmatrix} g c_2^2 + \ddot{q}_1 s_2 c_2 - \dot{q}_2^2 l c_2 + g s_2^2 - \ddot{q}_1 c_2 s_2 - \ddot{q}_2 l s_2 \\ g c_2 s_2 + \ddot{q}_1 s_2^2 - \dot{q}_2^2 l s_2 - g s_2 c_2 + \ddot{q}_1 c_2^2 + \ddot{q}_2 l c_2 \\ 0 \end{bmatrix} + m_1 \begin{bmatrix} -g \\ \ddot{q}_1 \\ 0 \end{bmatrix} $$

$$= m_2 \begin{bmatrix} g - \dot{q}_2^2 l c_2 - \ddot{q}_2 l s_2 \\ \ddot{q}_1 - \dot{q}_2^2 l s_2 + \ddot{q}_2 l c_2 \\ 0 \end{bmatrix} + m_1 \begin{bmatrix} -g \\ \ddot{q}_1 \\ 0 \end{bmatrix}$$

Finally, incorporating friction (only viscous for now)

Inverse Dynamics

$$\tau = \begin{bmatrix} m_2(\ddot{q}_1 - \dot{q}_2^2 l s_2 + \ddot{q}_2 l c_2) + m_1 \ddot{q}_1 \\ m_2 l(-g s_2 + \ddot{q}_1 c_2 + \ddot{q}_2 l) \end{bmatrix} + \begin{bmatrix} f_p \dot{q}_1 \\ f_r \dot{q}_2 \end{bmatrix} \quad \text{--- (1)}$$

from (1), Joint Space inertia Matrix M

$$= \begin{bmatrix} (m_1 + m_2) & m_2 l c_2 \\ m_2 l c_2 & m_2 l^2 \end{bmatrix}$$

Vector of Coriolis & Gravity forces CG

$$= \begin{bmatrix} -m_2 \dot{q}_2^2 l s_2 \\ -m_2 g l s_2 \end{bmatrix}$$

Friction forces $f = \begin{bmatrix} f_p \dot{q}_1 \\ f_r \dot{q}_2 \end{bmatrix}$

Forward Dynamics

$$\ddot{q} = M^{-1}(\tau - f - CG)$$