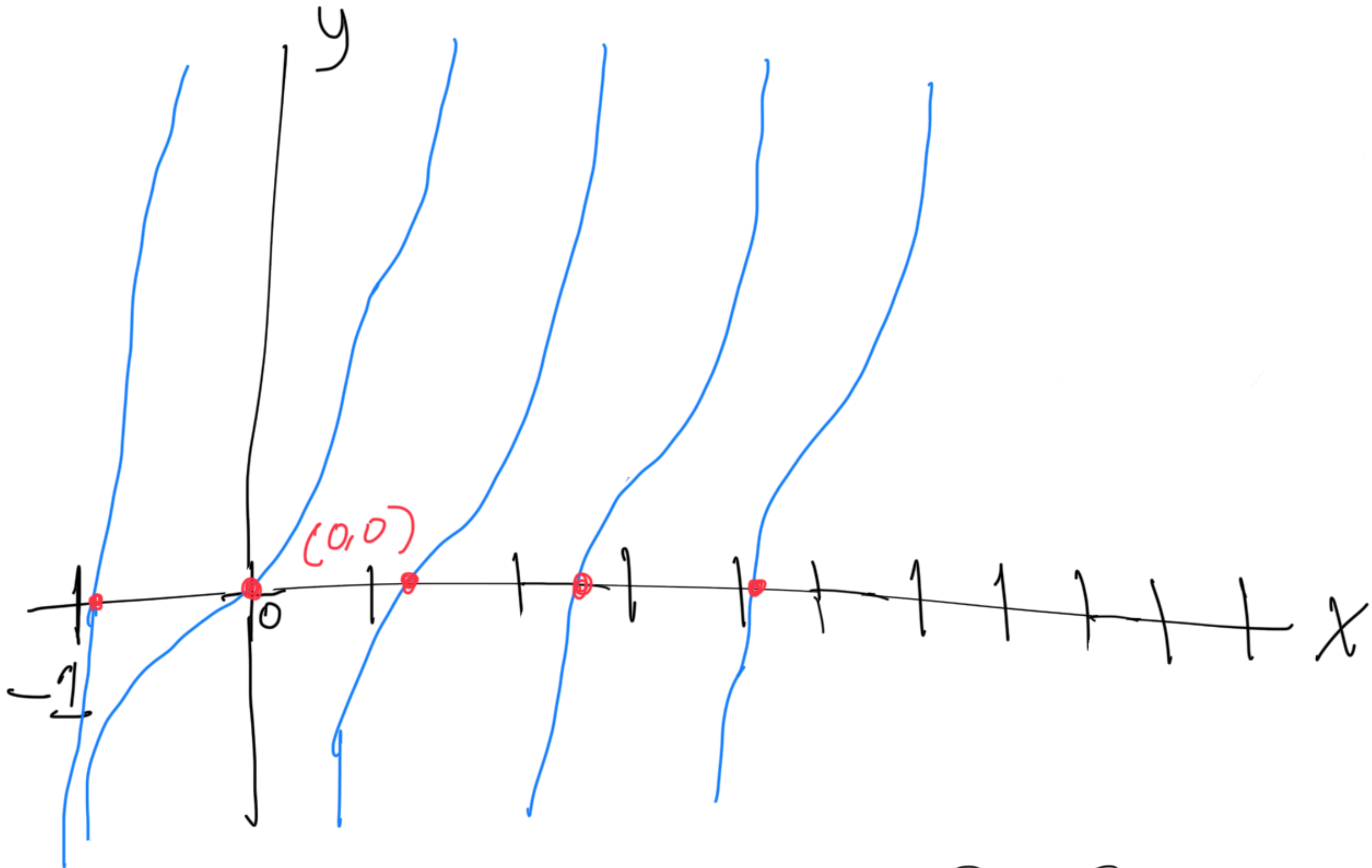
Numerical Methods Homework 2

[a]**4.** By graphical methods, locate approximations to all roots of the nonlinear equation $\ln(x+1) + \tan(2x) = 0$.



$x \approx -0.98$, $x \approx 0$, $x \approx 1.23$, $x \approx 2.68$, $x \approx 4.2$

Answer from back of book:

$\{-\frac{\pi}{4} - \delta, 0, \frac{\pi}{4} + \varepsilon, \frac{3\pi}{4} + \varepsilon, \frac{5\pi}{4} + \varepsilon, \dots\}$

where $\delta \approx 0.2$ and ε starts at approximately 0.4 and decreases

8. If $a = 0.1$ and $b = 1.0$, how many steps of the bisection method are needed to determine the root with an error of at most $\frac{1}{2} \times 10^{-8}$?

$n > \log_2\left(\frac{b-a}{\varepsilon}\right)$ ← from google

$\log_2\left(\frac{1-0.1}{\frac{1}{2} \times 10^{-8}}\right) \approx 27.4234$

round up $\boxed{28}$

8. In Exercises 1.2.10–1.2.12, several methods are suggested for computing ln 2. Compare them with the use of Newton's method applied to the equation $e^x = 2$.

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ $\qquad x_1 = 1 - \frac{e^1 - 2}{e^1} \approx 0.735759$

$f'(x_n)$

$f(x) = e^x - 2$

$f'(x) = e^x$

$x_{n+1} = x_n - \frac{e^{x_n} - 2}{e^{x_n}}$

e^1

$x_2 = 0.73 - \frac{e^{0.73} - 2}{e^{.73}} \approx 0.694042$

$x_3 = 0.69 - \frac{e^{0.69} - 2}{e^{0.69}} \approx 0.69314$

$\ln(2) \approx 0.6931$

Compare – Newton's method converges faster than the methods in 1.2.10 – 1.2.12. In the back of the book it says it took at least 10 terms using those methods. Using Newton's Method it converged faster, because it took three times to converge.

20. Starting at $x = 3$, $x < 3$, or $x > 3$, analyze what happens when Newton's method is applied to the function $f(x) = 2x^3 - 9x^2 + 12x + 15$.

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ $\quad f'(x) = 6x^2 - 18x + 12$

$x = 3$

$f(3) = 2(3)^3 - 9(3)^2 + 12(3) + 15$

$= 54 - 81 + 36 + 15 = 24$

$f'(3) = 6(3)^2 - 18(3) + 12$

$54 - 54 + 12 = 12$

$x_1 = 3 - \frac{24}{12} = 3 - 2 = 1$

$\underline{x < 3}$

$x = 2$

$f(2) = 2(2)^3 - 9(2)^2 + 12(2) + 15$

$= 16 - 36 + 24 + 15 = 19$

$f'(2) = 6(2)^2 - 18(2) + 12$

$= 24 - 36 + 12 = 0$

$x_1 = 2 - \frac{19}{0}$ ← division by 0 method can fail

$x = 1$

$f(1) = 2(1)^3 - 9(1)^2 + 12(1) + 15$

$= 2 - 9 + 12 + 15 = 20$

$f'(1) = 6(1) - 18(1) + 12$

$= 6 - 18 + 12 = 0$

$x_1 = 1 - \frac{20}{0}$ ← division by $\emptyset$ again, method can fail

$x > 3$

$x = 4$

$f(4) = 2(4)^3 - 9(4)^2 + 12(4) + 15$

$= 128 - 144 + 48 + 15 = 47$

$f'(4) = 6(4)^2 - 18(4) + 12$

$= 96 - 72 + 12 = 36$

$x_1 = 4 - \frac{47}{36} \approx 2.69$

$x = 5$

$f(5) = 2(5)^3 - 9(5)^2 + 12(5) + 15 =$

$= 250 - 225 + 60 + 15 = 100$

$f'(5) = 6(5)^2 - 18(5) + 12$

$= 150 - 90 + 12 = 72$

$x_1 = 5 - \frac{100}{72} \approx 3.6$

at $x = 3$, method converges at $x_1 = 1$

$x < 3$, method may fail

$x > 3$, method converges at different roots depending on starting point

5. Using the bisection method, Newton's method, and the secant method, find the largest positive root correct to three decimal places of $x^3 - 5x + 3 = 0$. (All roots are in $[-3, +3]$.)

$f(-3) = (-3)^3 - 5(-3) + 3 = -27 + 15 + 3 = -9$

$f(-2) = (-2)^3 - 5(-2) + 3 = -8 + 10 + 3 = 5$

$f(-1) = (-1)^3 - 5(-1) + 3 = 7$

$f(0) = 3$

$f(1) = 1^3 - 5(1) + 3 = 1 - 5 + 3 = -1$

$f(2) = 2^3 - 5(2) + 3 = 8 - 10 + 3 = 1$

$f(3) = 3^3 - 5(3) + 3 = 27 - 15 + 3 = 15$

largest where sign chages so we will use

(1, 2)

Bisection Method

(1, 2)

$x_m = \frac{1+2}{2} = 1.5$

$f(1.5) = (1.5)^3 - 5(1.5) + 3 = -1.125$ $f(1.5) < 0$

(1.5, 2)

$x_m = \frac{1.5+2}{2} = 1.75$

$f(1.75) = (1.75)^3 - 5(1.75) + 3 \approx -0.3906$

$f(1.75) < 0$

(1.75, 2)

$x_m = \frac{1.75+2}{2} = 1.875$

$f(1.875) = (1.875)^3 - 5(1.875) + 3 \approx 0.1268$ $f(1.875) > 0$

(1.75, 1.875)

$x_m = \frac{1.75+1.875}{2} = 1.8125$

$f(1.8125) = (1.8125)^3 - 5(1.8125) + 3 \approx -0.108$ $f(1.8125) < 0$

(1.8125, 1.875)

$x_m = \frac{1.8125+1.875}{2} = 1.84375$

$f(1.84375) = (1.84375)^3 - 5(1.84375) + 3 \approx 0.04892$

$f(1.84375) > 0$

(1.8125, 1.84375)

$x_m = \frac{1.8125+1.84375}{2} \approx 1.82813$ $f(1.82813) < 0$

$f(1.82813) = (1.82813)^3 - 5(1.82813) + 3 \approx -0.03$

(1.82813, 1.83594)

$x_m = \frac{1.82813+1.84375}{2} \approx 1.83594$

$f(1.83594) = 1.83594^3 - 5(1.83594) + 3 \approx 0.00865$ $f(1.83) > 0$

by bisection ≈ 1.835

Newtons Method

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ $f(x) = x^3 - 5(x) + 3$

$f'(x) = 3x^2 - 5$

$x = 2$ $f(2) = 2^3 - 5(2) + 3 = 1$

$f'(2) = 3(2)^2 - 5 = 7$

$f(2) = 3(2)^2 - 5 = 7$

$x_1 = 2 - \frac{1}{7} \approx 1.857$

$f(x_1) = (x_1)^3 - 5(x_1) + 3 \approx 0.119$

$f'(x_1) = 3(x_1)^2 - 5 \approx 5.345$

$$x_2 = 1.857 - \frac{.119}{5.345} \approx 1.83479$$

$f(x_2) = (x_2)^3 - 5(x_2) + 3 = .002773$

$f'(x_3) = 3(x_2)^2 - 5 = 3.09933$

$$x_3 = 1.83479 - \frac{.002773}{3.09933} \approx 1.83424$$

by Newton Method $\boxed{1.834}$

Secant method

$$x_{n+1} = x_n - f(x_n)\left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right)$$

$$x_2 = 2 - f(2)\left(\frac{2-1}{f(2) - f(1)}\right)$$

$$x_2 = 2 - (1)\left(\frac{1}{1-(-1)}\right) = 2 - \frac{1}{2} = 1.5$$

$$x_3 = 1.5 - f(1.5)\frac{(1.5-2)}{f(1.5) - f(2)}$$

$f(1.5) = 1.5^3 - 5(1.5) + 3 = -1.125$

$$x_3 = 1.5 - (-1.125)\frac{(1.5-2)}{(-1.125) - 1} \approx 1.76471$$

$$x_4 = 1.76 - f(1.76)\left(\frac{1.76 - 1.5}{f(1.76) - f(1.5)}\right)$$

$f(1.76) = 1.76^3 - 5(1.76) + 3 \approx -0.348$

$$x_4 = 1.76 - (-0.348)\left(\frac{1.76 - 1.5}{-0.348 - (-1.125)}\right) \approx 1.88337$$

$f(1.88) = 1.88^3 - 5(1.88) + 3 \approx 0.26363$

$$x_5 = 1.88 - f(1.88)\left(\frac{1.88 - 1.76}{f(1.88) - f(1.76)}\right)$$

$$= 1.88 - 0.26\left(\frac{1.88 - 1.76}{0.26 - (-0.348)}\right) \approx 1.83224$$

by secant method ≈ 1.832

bisection: 1.835
Newton: 1.834 $\quad \approx 1.834$
secant: 1.832