Numerical Methods Assignment 3

4. Verify that the polynomials

$$p(x) = 5x^3 - 27x^2 + 45x - 21$$
$$q(x) = x^4 - 5x^3 + 8x^2 - 5x + 3$$

interpolate the data

x	1	2	3	4
y	2	1	6	47

and explain why this does not violate the uniqueness part of the theorem on existence of polynomial interpolation.

$$p(1) = 5(1)^3 - 27(1)^2 + 45(1) - 21$$
$$= 5 - 27 + 45 - 21 = 2 \checkmark$$

$$p(2) = 5(2)^3 - 27(2)^2 + 45(2) - 21$$
$$= 5(8) - 27(4) + 45(2) - 21$$
$$= 40 - 108 + 90 - 21 = 1 \checkmark$$

p(x) works for all data points

$$p(3) = 5(3)^3 - 27(3)^2 + 45(3) - 21$$
$$= 5(27) - 27(9) + 135 - 21$$
$$= 135 - 243 + 135 - 21 = 6 \checkmark$$

$$p(4) = 5(4)^3 - 27(4)^2 + 45(4) - 21$$
$$= 5(64) - 27(16) + 180 - 21$$
$$= 320 - 432 + 180 - 21 = 47 \checkmark$$

$$q(1) = 1^4 - 5(1)^3 + 8(1)^2 - 5(1) + 3$$
$$= 1 - 5 + 8 - 5 + 3 = 2 \checkmark$$

$$q(2) = 2^4 - 5(2)^3 + 8(2)^2 - 5(2) + 3$$
$$= 16 - 40 + 32 - 10 + 3 = 1 \checkmark$$

$$q(3) = 3^4 - 5(3)^3 + 8(3)^2 - 5(3) + 3$$
$$= 81 - 5(27) + 8(9) - 15 + 3$$
$$= 81 - 135 + 72 - 15 + 3 = 6 \checkmark$$

$$q(4) = 4^4 - 5(4)^3 + 8(4)^2 - 5(4) + 3$$
$$= 256 - 5(64) + 8(16) - 20 + 3$$

$= 256 - 320 + 128 - 20 + 3 = 47$ ✓

It does not violate the uniqueness part of the theorem because both polynomials are equal to the value in the table and both polynomials have different degree.

23. Form a divided-difference table for the following and explain what happened.

x	1	2	3	1
y	3	5	5	7

x	$f(x)$	$(x-x_0)$	$(x-x_1)$	$(x-x_2)$
1	3			
		2		
2	5		-1	
		0		$\frac{2}{0}$ ← undefined
3	5		1	
		-1		
1	7			

The repeated $x=1$ led to an undefined answer because of division by zero.

26. Use inverse interpolation to find an approximate value of x such that $f(x) = 0$ given the following table of values for f. Look into what happens and draw a conclusion.

x	-2	-1	1	2	3
$f(x)$	-31	5	1	11	61

$$= \frac{(0-5)(0-1)(0-11)(0-61)}{(-31-5)(-31-1)(-31-11)(-31-61)}(-2)$$

$$+ \frac{(0+31)(0-1)(0-11)(0-61)}{(5+31)(5-1)(5-11)(5-61)}(-1)$$

$$+ \frac{(0+31)(0-5)(0-11)(0-61)}{(1+31)(1-5)(1-11)(1-61)}(1)$$

$$+ \frac{(0+31)(0-5)(0-1)(0-61)}{(11+31)(11-5)(11-1)(11-61)}(2)$$

$$+\frac{(6+31)(6-5)(6-1)(6-11)\ldots(6)}{(61+31)(61-5)(61-1)(61-11)}$$

after plugging into calculator

$x \approx 1.93526$

Conclusion: We can conclude that $f(x)$ crosses zero at around 1.93

18. Consider $\int_1^2 dx/x^3$. What is the result of using the composite trapezoid rule with the partition points 1, $\frac{3}{2}$, and 2?

$h = \frac{1}{2}$

$$\frac{h}{2}\left[f(x_i) + f(x_{i+1})\right]$$

$$\frac{1}{2}\left(\frac{1}{2}\right)\left[f(1) + 2f(3/2) + f(2)\right]$$

$f(1) = 1$

$f(3/2) = \frac{1}{(3/2)^3} = \frac{1}{\frac{27}{8}} = \frac{8}{27}$

$f(2) = \frac{1}{2^3} = \frac{1}{8}$

$$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left[1 + 2\left(\frac{8}{27}\right) + \frac{1}{8}\right]$$

$$\frac{216}{216} + \frac{128}{216} + \frac{27}{216} = \frac{371}{216}$$

$$\left(\frac{1}{4}\right)\left(\frac{371}{216}\right) = \boxed{\frac{371}{864}}$$

[a]**19.** If the composite trapezoid rule is used with $h = 0.01$ to compute $\int_2^5 \sin x\, dx$, what numerical value will the error not exceed? (Use the absolute value of error.) Give the best answer based on the error formula.

$a = 2$

$b = 5$

$f(x) = \sin(x)$

$f'(x) = \cos(x)$

$f''(x) = -\sin(x)$

$$\frac{b-a}{12} h^2 f''(c) \leftarrow \text{max so } f''(5)$$

$$\frac{5-2}{12}(0.01)^2(-\sin(5)) \approx \boxed{0.000025}$$

[a]**23.** We want to approximate $\int_1^2 f(x)\, dx$ given the table of values

x	1	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{7}{4}$	2
$f(x)$	10	8	7	6	5

$h = \frac{1}{4}$

Compute an estimate by the composite trapezoid rule.

$= \frac{h}{2}\left[f(x_i) + f(x_{i+2})\right]$

$\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\left[(10) + (2)(8) + (2)(7) + (2)(6) + (5)\right]$

$= \left(\frac{1}{8}\right)\left[10 + 16 + 14 + 12 + 5\right]$

$= \left(\frac{1}{8}\right)(57) \approx \boxed{\frac{57}{8} \approx 7.125}$

For the following equation calculate the numerical derivative around x = 3 using h = 0.01.

$$f(x) = x^3 + 4\ \sin x$$

a- Estimate the derivative using the first formula $f'(x) = (f(x+h) - f(x))/h$
b- Estimate the derivative using the second formula $f'(x) = (f(x-h) - f(x+h))/2h$
c- Estimate the derivative using Richerdson extrapolation.
d- Compute the exact derivate and measure the absolute error, relative error, and relative error percentage for the different methods.

$f(x+h) = f(3+0.01) = f(3.01)$

$f(3.01) = (3.01)^3 + 4\sin(3.01) \approx 27.79$

$f(3) = 3^3 + 4\sin(3) \approx 27.56$

$f(x-h) = f(3-0.01) = f(2.99)$

$f(2.99) = 2.99^3 + 4\sin(2.99) = 27.39$

a) $f'(x) = \dfrac{f(x+h) - f(x)}{h} = \dfrac{f(3.01) - f(3)}{h}$

$= \dfrac{27.79 - 27.56}{0.01} \approx \boxed{23.1274}$

b) $f'(x) = \dfrac{f(x+h) - f(x-h)}{2h} = \dfrac{f(3.01) - f(2.99)}{2(0.01)}$

$= \dfrac{27.79 - 27.39}{.02} \approx \boxed{23.0402}$

c) $f'_R = \dfrac{4D(h/2) - D(h)}{3}$

$D(h/2) = D(0.005) = \dfrac{f(3.005) - f(2.995)}{2(0.005)}$

$f(3.005) = 3.005^3 + 4\sin(3.005) \approx 27.67$

$\dots \approx 27.44$

$f(2.995) = 2.995^3 + 4\sin(2.995)$

$D(0.005) = \dfrac{27.67 - 27.44}{.01} \approx 23.0401$

$$f'_R = \frac{4(23.0401) - (23.0402)}{3}$$

$$\approx \boxed{23.04}$$

D) $f'(x) = 3x^2 + 4\cos(x)$

$f'(3) = 3(3)^2 + 4\cos(3) \approx 23.04$

first formula:

$23.12 - 23.04 \approx 0.087 \leftarrow$ Abs Error

$\dfrac{.087}{23.04} \approx .003784 \leftarrow$ Rel Error

$.003784 \times 100 \approx 0.38\% \leftarrow$ Rel Error Percent

second formula:

$23.0402 - 23.04 \approx .00016 \leftarrow$ Abs Error

$\dfrac{.00016}{23.04} \approx 0.000007 \leftarrow$ Rel Error

$.000007 \times 100 = 0.00072\% \leftarrow$ Rel Error %

Richardson formula:

$23.0401 - 23.04 \approx 2e{-12} \leftarrow$ Abs Error

$\dfrac{2e{-12}}{23.04} \approx 8.68054e{-14} \leftarrow$ Rel Error

$8.68054E{-14} \times 100 \approx 8.68054e{-12}\% \leftarrow$ Rel Error %