Numerical Methods - Assignment 1

[a]**5.** Determine the Taylor series for $\cosh x$ about zero. Evaluate $\cosh(0.7)$ by summing four terms. Compare with the actual value.

1. $f(x) = \cosh(x) = 1$ even/odd
2. $f'(x) = \sinh(x) = 0$ alternate
3. $f''(x) = \cosh(x) = 1$ 0/1
4. $f'''(x) = \sinh(x) = 0$

Taylor Series general formula

$$f(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} \dots$$

$$\cosh(x) = 1 + \cancel{\frac{0(x-0)}{1!}} + \frac{1(x-0)^2}{2!} + \cancel{\frac{0(x-0)^3}{3!}} + \frac{1(x-0)^4}{4!} \dots$$

going to use the first four non zero terms

$$\cosh(x) \approx 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!}$$

$$\cosh(0.7) \approx 1 + \frac{0.7^2}{2!} + \frac{0.7^4}{4!} + \frac{0.7^6}{6!}$$

$$\approx 1.25516756805556$$

Actual value of $\cosh(0.7)$ from google $=$ 1.25516900563

6. Determine the first two nonzero terms of the series expansion about zero for the following:

[a]**a.** $e^{\cos x}$ [a]**b.** $\sin(\cos x)$

c. $(\cos x)^2(\sin x)$

a) $f(x) = e^{\cos(x)}$ $f(0) = e^{\cos(0)} = e^1 = e$

$f'(x) = e^{\cos(x)} \cdot (-\sin(x))$ $f'(0) = e(-\sin(0)) = e \cdot 0 = 0$

$f''(x) = e^{\cos(x)}(-\cos(x)) + e^{\cos(x)}(-\sin(x))(-\sin(x))$

$f''(0) = e^{\cos(0)}(-\cos(0)) + e^{\cos(0)}(-\sin(0))(-\sin(0))$

$f''(0) = e^1(-1) + e^1 \cancel{(0)(0)}$

[illegible]

$f''(0) = -e$

$e^{\cos(x)} = e + \frac{(-e)x^2}{2!}$ ← first two non zero terms

b) $f(x) = \sin(\cos(x))$ $\quad f(0) = \sin(\cos(0)) = \sin(1)$

$f'(x) = \cos(\cos(x)) \cdot (-\sin(x))$ $\quad f'(0) = \cos(\cos(0))(-\sin(0))$

$= \cos(1)(0) = 0$

$f''(x) = \cos(\cos(x))(-\cos(x)) + (-\sin(\cos(x)))(-\sin(x))(-\sin(x))$

$f''(0) = \cos(\cos(0))(-\cos(0)) + \cancel{(-\sin(\cos(0)))(-\sin(0))(-\sin(0))}$

$f''(0) = \cos(1)(-1) = -\cos(1)$

$\sin(\cos(x)) = \sin(1) + \left(\frac{-\cos(1)x^2}{2!}\right)$

first two non-zero terms

c) $f(x) = \cos(x)^2 \sin(x)$ $\quad f(0) = 0$

$f'(x) = 2\cos(x)(\sin(x))(-\sin(x)) + \cos(x)^2(\cos(x))$

$f'(0) = \cancel{2\cos(0)(\sin(0))}(-\sin(0)) + \cos(0)^2 \cos(0)$

$f'(0) = 1^2 \cdot 1 = 1$

$f''(x) = -2\cos(x)(\sin(x))^2 + \cos^3(x)$

$= -2\left(\cos(x)2\sin(x)\cos(x) + \sin(x)^2(-\sin(x))\right)$

$+ 3\cos^2(x)(\sin(x))$

$f''(0) = -2(\cos(0)2\cancel{\sin(0)}\cos(0) + \sin(0)^2\cancel{(-\sin(0))}$

$+ 3\cos^2(0)\cancel{\sin(0)}$

$= 0$

$f'''(x) = -2(-\sin(x)^3 + 2\cos(x)^2\sin(x)) - 3\cos^2(x)(\sin(x))$

$f'''(x) = 2\sin(x)^3 - 4\cos(x)^2\sin(x) - 3\cos^2(x)(\sin(x))$

$f'''(x) = 2\sin(x)^3 - 7\cos(x)^2\sin(x)$

$= 6\sin(x)^2(\cos(x)) + 14\cos(x)(+\sin(x))\sin(x)$

$+ 7\cos(x)^2(-\cos(x))$

[illegible]

$= 6\sin(x)^2(\cos(x)) + 14\cos(x)(\sin(x)^2)$
$-7\cos(x)^3$

$f'''(x) = 20\sin(x)^2\cos(x) - 7\cos(x)^3$

$f'''(0) = \cancel{20\sin(0)^2\cos(0)} - 7\cos(0)^3$
$= -7(1)^3$
$= -7$

$\cos(x)^2\sin(x) = x + \frac{-7x^3}{3!}$

[a]**7.** Find the smallest nonnegative integer m such that the Taylor series about m for $(x-1)^{1/2}$ exists. Determine the coefficients in the series.

$x > 0$

$f(x) = (x-1)^{1/2}$

$f'(x) = \frac{1}{2}(x-1)^{-1/2} = \frac{1}{2(x-1)^{1/2}} = \frac{1}{2\sqrt{x-1}}$

$f'(1) = \frac{1}{2\sqrt{1-1}} = \frac{1}{2\sqrt{0}} = \frac{1}{0}$ divison by zero is undefined so this does not exist

$f'(2) = \frac{1}{2\sqrt{2-1}} = \frac{1}{2\sqrt{1}} = = \frac{1}{2}$ ✓exists

$f(2) = (x-1)^{1/2} = (2-1)^{1/2} = (1)^{1/2} = 1$

$f''(x) = \frac{1}{2}(x-1)^{-1/2} = \frac{-1}{4}(x-1)^{-3/2}$

$f''(2) = \frac{-1}{4}(2-1)^{-3/2} = (-1/4)(1) = \frac{-1}{4}$

$f'''(x) = \frac{-1}{4}(x-1)^{-3/2} = \frac{3}{8}(x-1)^{-5/2}$

$f'''(2) = \frac{3}{8}(2-1)^{-5/2} = \frac{3}{8}(1) = \frac{3}{8}$

$f''''(x) = \frac{3}{8}(x-1)^{-5/2} = \frac{-15}{16}(x-1)^{-7/2}$

$f''''(2) = \frac{-15}{16}(2-1)^{-7/2} = \frac{-15}{16}$

Taylor Series

$f(x) = 1 + \frac{1}{2}(x-2) - \frac{1}{4}\left(\frac{(x-2)^2}{2!}\right) + \frac{3}{8}\left(\frac{(x-2)^3}{3!}\right) - \frac{15}{16}\left(\frac{(x-2)^4}{4!}\right)$

$1 \qquad 1/2 \qquad -\left(\frac{1}{4}\right)\left(\frac{1}{2!}\right) = -\frac{1}{8} \qquad \left(\frac{3}{8}\right)\left(\frac{1}{3!}\right) = \frac{3}{48} = \frac{1}{16} \qquad -\frac{15}{16}\left(\frac{1}{4!}\right) = -\frac{15}{384}$

$16 \cdot 24$

$\frac{-5}{128}$

Coefficients of first five terms

$c_0 = 1$

$C_1 = \frac{1}{2}$

$C_2 = -1/8$

$C_3 = 1/16$

$C_4 = -5/128$

[a]**23.** What is the second term in the Taylor series of $\sqrt[4]{4x-1}$ about 4.25?

$f(x) = \sqrt[4]{4x-1} = (4x-1)^{1/4}$ $f(4.25) = (16)^{1/4} = 2$

$f'(x) = (4x-1)^{1/4}$

$= \frac{1}{4}(4x-1)^{-3/4} \cdot 4 = (4x-1)^{-3/4}$

$f'(4.25) = (4(4.25)-1)^{-3/4} = 16^{-3/4} = \frac{1}{8}$

$2 + \frac{1}{8}(x-4.25)$

2^{nd} term: $\boxed{\frac{1}{8}(x-4.25)}$

36. Using the Taylor series expansion in terms of h, determine the first three terms in the series for $e^{\sin(x+h)}$. Evaluate $e^{\sin 90.01^\circ}$ accurately to ten decimal places as Ce for constant C.

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k + E_{n+1}$$

$f(x) = e^{\sin(x)}$

$f'(x) = e^{\sin(x)} \cos(x)$

$f''(x) = e^{\sin(x)} \cos^2(x) - e^{\sin(x)} \sin(x)$

$f'''(x) = e^{\sin(x)} \cos^3(x) - e^{\sin(x)}(2\cos(x)\sin(x)) - \left(e^{\sin(x)}\cos(x)\sin(x) + e^{\sin(x)}\cos(x)\right)$

$f'''(x) = e^{\sin(x)} \cos^3(x) - 3e^{\sin(x)} \cos(x)\sin(x) - e^{\sin(x)} \cos(x)$

$e^{\sin(x+h)} = e^{\sin(x)} + he^{\sin(x)}\cos(x)$

$+ \frac{h^2}{2}\left(e^{\sin(x)} \cos^2(x) - e^{\sin(x)} \sin(x)\right)$

$e^{\sin(x+h)} = e^{\sin(x)}\left(1 + h\cos(x) + \frac{h^2}{2}\cos^2(x) - \sin(x)\right)$

$x=90 \quad h=0.01$

$$e^{\sin(90.01)} = e^{\sin(90)}\left(1+ \cancel{.01(\cos(90))} + \frac{(.01^\circ)^2}{2}(\cos^2(90)-\sin(90))\right)$$

$\frac{.01\pi}{180}$

$$= e^{\sin(90)}(1)$$

$$e^{\sin(90.01)} \approx 2.7181459144$$

[a] **38.** Determine a Taylor series to represent $\cos(\pi/3+h)$. Evaluate $\cos(60.001^\circ)$ to eight decimal places (rounded). *Hint*: π radians equal 180 degrees.

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k + E_{n+1}$$

$f(x) = \cos(x)$

$f'(x) = -\sin(x)$

$f''(x) = -\cos(x)$

$f'''(x) = \sin(x)$

$$\cos(x+h) = \cos(x) - h\sin(x) - \frac{h^2}{2}\cos(x) + \frac{h^3}{3!}\sin(x)$$

$$\cos\left(\tfrac{\pi}{3}+h\right) = \underbrace{\cos\left(\tfrac{\pi}{3}\right)}_{1/2} - h\underbrace{\sin\left(\tfrac{\pi}{3}\right)}_{\frac{\sqrt{3}}{2}} - \frac{h^2}{2}\underbrace{\cos\left(\tfrac{\pi}{3}\right)}_{\frac{1}{2}} + \frac{h^3}{3!}\underbrace{\sin\left(\tfrac{\pi}{3}\right)}_{\frac{\sqrt{3}}{2}}$$

$$\cos\left(\tfrac{\pi}{3}+h\right) = \frac{1}{2} - h\frac{\sqrt{3}}{2} - \frac{1}{4}h^2 + \frac{\sqrt{3}}{12}h^3$$

$$h = \frac{.001\pi}{180} \approx 1.745\times 10^{-5}$$

$$\cos(60.001) = \frac{1}{2} - \left(\frac{.001\pi}{180}\right)\frac{\sqrt{3}}{2} - \frac{1}{4}\left(\frac{.001\pi}{180}\right)^2 + \frac{\sqrt{3}}{12}\left(\frac{.001\pi}{180}\right)^3$$

$$\cos(60.001) \approx 0.49998488$$

44. How many terms are needed in the series

$$\operatorname{arccot} x = \frac{\pi}{2} - x + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \cdots$$

to compute $\operatorname{arccot} x$ for $x^2 < 1$ accurate to 12 decimal places (rounded)?

$$\frac{1}{2n+1} \qquad \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \left|\frac{x^{2n+1}}{2n+1}\right| < 10^{-12}$$

$$|x| \approx 1 \qquad \left|\frac{1^{2n+1}}{2n+1}\right| < 10^{-12}$$

$$\frac{1}{} < 10^{-12}$$

$$\overline{2n+1}$$

$$2n+1 > 10^{12}$$

$$-1 \qquad\qquad -1$$

$$\frac{2n}{2} > \frac{10^{12}-1}{2}$$

$$\approx \; n > 5\times 10^{11}$$

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