Numerical Methods Assignment 4

Derrick Subnaik

[a] **1.** Show that the system of equations

$$\begin{cases} x_1 + 4x_2 + \alpha x_3 = 6 \\ 2x_1 - x_2 + 2\alpha x_3 = 3 \\ \alpha x_1 + 3x_2 + x_3 = 5 \end{cases}$$

possesses a unique solution when $\alpha = 0$, no solution when $\alpha = -1$, and infinitely many solutions when $\alpha = 1$. Also, investigate the corresponding situation when the right-hand side is replaced by 0's.

$\alpha = 0$

pivot row one

$$\left[\begin{array}{ccc|c} 1 & 4 & 0 & 6 \\ 2 & -1 & 0 & 3 \\ 0 & 3 & 1 & 5 \end{array}\right] \xrightarrow{r2=(-2)(r1)+r2} \left[\begin{array}{ccc|c} 1 & 4 & 0 & 6 \\ 0 & -9 & 0 & -9 \\ 0 & 3 & 1 & 5 \end{array}\right]$$

pivot now row 2

divide r2 by -9

$$\left[\begin{array}{ccc|c} 1 & 4 & 0 & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 3 & 1 & 5 \end{array}\right]$$

eliminate 3 in bottom row and 4 in row 1 by

$r3 + -3r2$

$r1 + -4r2$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \hline \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$x_1 = 2$

$x_2 = 1$

$x_3 = 2$

verifies unique solution when $\alpha = 0$

$\alpha = -1$

r1 pivot

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 6 \\ 2 & -1 & -2 & 3 \\ -1 & 3 & 1 & 5 \end{array}\right] \xrightarrow{(-2)r1+r2,\ r1+r3} \left[\begin{array}{ccc|c} 1 & 4 & -1 & 6 \\ 0 & -9 & 0 & -9 \\ 0 & 7 & 0 & 11 \end{array}\right]$$

$r2\left(\frac{7}{9}\right) + r3$

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ 1 & 4 & -1 & 6 \\ 0 & -9 & 0 & -9 \\ 0 & 0 & 0 & 4 \end{array}$$

x_3

$0=4$

bottom row of 0's verifies no solution when $\alpha=-1$

$a=1$

$$\left[\begin{array}{ccc|c} 1 & 4 & 1 & 6 \\ 2 & -1 & 2 & 3 \\ 1 & 3 & 1 & 5 \end{array}\right] \xrightarrow[r3-r1]{-2r1+r2} \left[\begin{array}{ccc|c} 1 & 4 & 1 & 6 \\ 0 & -9 & 0 & -9 \\ 0 & -1 & 0 & -1 \end{array}\right]$$

$r2(-9)+r3$

$$\left[\begin{array}{ccc|c} 1 & 4 & 1 & 6 \\ 0 & -9 & 0 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1=2-t$
$x_2=1$
$x_3=t$

$0=0$ so many solutions variable t

$x_1+4+t=6$
$x_1=2-t$

bottom row $0=0$ verifies that when $\alpha=1$ there are infinite many solution

when right hand side $=0$

$$\left[\begin{array}{ccc|c} 1 & 4 & \alpha & 0 \\ 2 & -1 & 2\alpha & 0 \\ \alpha & 3 & 1 & 0 \end{array}\right]$$

looking at my above matrices when the right hand side $=0$

when $\alpha=-1$ infinite many solution

$\alpha=1$ infinite many solution

$\alpha=0$ trivial solution

3. Apply naive Gaussian elimination to these examples and account for the failures. Solve the systems by other means if possible.

[a]**a.** $\begin{cases} 3x_1 + 2x_2 = 4 \\ -x_1 - \frac{2}{3}x_2 = 1 \end{cases}$ [a]**b.** $\begin{cases} 6x_1 - 3x_2 = 6 \\ -2x_1 + x_2 = -2 \end{cases}$

c. $\begin{cases} 0x_1 + 2x_2 = 4 \\ \ldots \end{cases}$ **d.** $\begin{cases} x_1 + x_2 + 2x_3 = 4 \\ x_1 + x_2 + 0x_3 = 2 \\ \ldots \end{cases}$

c. $\{ \; x_1 - \; x_2 = 5$ d. $\begin{cases} x_1 + x_2 + 0x_3 = 2 \\ 0x_1 + x_2 + \; x_3 = 0 \end{cases}$

3a) $\left|\begin{array}{cc|c} 3 & 2 & 4 \\ -1 & \frac{-2}{3} & 1 \end{array}\right|$ $r2 = \frac{1}{3}r1 + r2$ $\left|\begin{array}{cc|c} 3 & 2 & 4 \\ 0 & 0 & 2\frac{1}{3} \end{array}\right|$

row of 0's no solution

b) $\left|\begin{array}{cc|c} 6 & -3 & 6 \\ -2 & 1 & -2 \end{array}\right| \xrightarrow{\frac{1}{3}r1 + r2} \left|\begin{array}{cc|c} 6 & -3 & 6 \\ 0 & 0 & 0 \end{array}\right|$

$x_1 = \frac{1}{2}t + 1$ $0 = 0$ means infinite many solutions

$x_2 = t$

c) $\begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left|\begin{array}{cc|c} 0 & 2 & 4 \\ 1 & -1 & 5 \end{array}\right| \end{array}$ swap the rows to get triangle $\left|\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 2 & 4 \end{array}\right|$

$\frac{2x_1}{2} = \frac{4}{2}$

$x_0 - 2 = 5$ $\boxed{\begin{array}{l} x_1 = 2 \\ x_2 = 7 \end{array}}$

$x_0 = 7$

d) $\left|\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 \end{array}\right| \xrightarrow{r2 - r1} \left|\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 0 & -2 & -2 \\ 0 & 1 & 1 & 0 \end{array}\right|$

swap r2 and r3

$\begin{array}{c} \begin{array}{ccc} x_0 & x_1 & x_2 \end{array} \\ \left|\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & -2 \end{array}\right| \end{array}$

$-2x_2 = -2$ $x_2 = 1$

$x_1 = -1$

$x_1 + x_2 = 0$ $x_0 =$

$x_1 = -x_2$

$x_0 - 1 + 2 = 4$

$+1$

-2

$x_0 = 3$

$\boxed{\begin{array}{l} x_1 = 3 \\ x_2 = -1 \\ x_3 = 1 \end{array}}$

6\.

d. $\begin{cases} 3x_1 + 2x_2 - x_3 = 7 \\ 5x_1 + 3x_2 + 2x_3 = 4 \\ -x_1 + x_2 - 3x_3 = -1 \end{cases}$

$$-\tfrac{5}{3}r1 \begin{bmatrix} -5 & \frac{-10}{3} & \frac{5}{3} \end{bmatrix} \frac{-35}{3}$$
$$+\ r2 \begin{bmatrix} 5 & 3 & 2 \end{bmatrix} 4$$
$$0 \quad -\tfrac{1}{3} \quad \tfrac{11}{3} \quad -\tfrac{23}{3}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & 7 \\ 5 & 3 & 2 & 4 \\ -1 & 1 & -3 & -1 \end{array}\right]$$

$$\downarrow$$

$$\tfrac{1}{3}r1 \begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \frac{7}{3}$$
$$+\ r3 \begin{bmatrix} -1 & 1 & -3 \end{bmatrix} -1$$
$$0 \quad 5/3 \quad \tfrac{-10}{3} \quad 4/3$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & 7 \\ 0 & -1/3 & \frac{11}{3} & -23/3 \\ 0 & 5/3 & \frac{-10}{3} & 4/3 \end{array}\right]$$

$$5r2 \begin{bmatrix} 0 & \frac{-5}{3} & \frac{55}{3} \end{bmatrix} \frac{-115}{3}$$
$$\begin{bmatrix} 0 & 5/3 & \frac{-10}{3} \end{bmatrix} 4/3$$
$$0 \quad 0 \quad \tfrac{45}{3} \quad \tfrac{-111}{3}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & 7 \\ 0 & -1/3 & \frac{11}{3} & -23/3 \\ 0 & 0 & 15 & -37 \end{array}\right]$$

$$\frac{15x_2}{15} = \frac{-37}{15} \qquad \boxed{x_3 = \frac{-37}{15}}$$

$$-\frac{1}{3}x_1 + \frac{11}{3}\left(\frac{-37}{15}\right) = \frac{-23}{3}$$

$$-\frac{1}{3}x_1 - \frac{407}{45} = \frac{-23}{3} \qquad \frac{-345}{45} + \frac{407}{45}$$
$$+\frac{407}{45} \qquad +\frac{407}{45} \qquad \frac{62}{45}$$

$$-3 \cdot -\frac{1}{3}x_1 = \frac{62}{45} \cdot -3$$

$$\boxed{x_2 = \frac{-62}{15}}$$

$$3x_0 + 2\left(\frac{-62}{15}\right) + \frac{37}{15} = 7$$

$$3x_0 + \frac{-87}{15} = 7 + \frac{37}{15} = \frac{105}{15} + \frac{87}{15}$$
$$+\frac{87}{15} \qquad = \frac{192}{15}$$

$$3x_0 = \frac{192}{15} \cdot \frac{1}{} = \boxed{64}$$

$3 \quad -15 \quad 3 \quad - \boxed{15}$

$$\boxed{\begin{aligned} x_1 &= \frac{64}{15} \\ x_2 &= \frac{-62}{15} \\ x_3 &= \frac{-37}{15} \end{aligned}}$$

7 **e.** $\begin{cases} x_1 + 3x_2 + 2x_3 + \ \ x_4 = -2 \\ 4x_1 + 2x_2 + \ \ x_3 + 2x_4 = \ \ 2 \\ 2x_1 + \ \ x_2 + 2x_3 + 3x_4 = \ \ 1 \\ x_1 + 2x_2 + 4x_3 + \ \ x_4 = -1 \end{cases}$

$$\left|\begin{array}{cccc|c} 1 & 3 & 2 & 1 & -2 \\ 4 & 2 & 1 & 2 & 2 \\ 2 & 1 & 2 & 3 & 1 \\ 1 & 2 & 4 & 1 & -1 \end{array}\right|$$

$-4(r1) = -4 \ -12 \ -8 \ -4 \ | \ 8$
$+ \ r2 \qquad 4 \ \ 2 \ \ 1 \ \ 2 \ | \ 2$
$0 \ \ -10 \ \ -7 \ \ -2 \ \ 10$

$-2(r1) \quad -2 \ -6 \ -4 \ -2 \ | \ 4$
$+r3 \qquad 2 \ \ 1 \ \ 2 \ \ 3 \ | \ 1$
$0 \ \ -5 \ \ -2 \ \ 1 \ | \ 5$

$r4 - r1 \quad | \ 1 \ 2 \ 4 \ 1 \ | \ -1$
$- \ 1 \ 3 \ 2 \ 1 \ | \ -2$
$0 \ -1 \ 2 \ 0 \ \ 1$

$\downarrow$

$$\left|\begin{array}{cccc|c} 1 & 3 & 2 & 1 & -2 \\ 0 & -10 & -7 & -2 & 10 \\ 0 & -5 & -2 & 1 & 5 \\ 0 & -1 & 2 & 0 & 1 \end{array}\right|$$

$r3 - 1/2 \, r2 \quad | \ 0 \ -5 \ -2 \ 1 \ | \ 5$
$0 \ -5 \ -7/2 \ -1 \ | \ 5$
$0 \ \ 0 \ \ 3/2 \ \ 2 \ | \ 0$

$r4 - \frac{1}{10} r2 \quad | \ 0 \ -1 \ 2 \ 0 \ | \ 1$
$0 \ -1 \ -7/10 \ -1/5 \ | \ 1$
$0 \ \ 0 \ \ 27/10 \ \ 1/5 \ | \ 0$

$\downarrow$

$$\left|\begin{array}{cccc|c} 1 & 3 & 2 & 1 & -2 \\ 0 & -10 & -7 & -2 & 10 \\ 0 & 0 & 3/2 & 2 & 0 \\ 0 & 0 & 27/10 & 1/5 & 0 \end{array}\right|$$

$\frac{2}{3}\left(\frac{27}{10}\right) = \frac{54}{30} = \frac{9}{5}$

$r4 - \frac{9}{5} r3 \qquad 0 \ \ 0 \ \ 27/10 \ \ 1/5 \ | \ 0$
$- \ 0 \ \ 0 \ \ 27/10 \ \ 18/5 \ | \ 0$
$0 \ \ 0 \ \ 0 \ \ -17/5 \ | \ 0$

$\downarrow$

$1 \ \ 3 \ \ 2 \ \ 1 \ | \ -2$

$$\left[\begin{array}{cccc|c} 0 & -10 & -7 & -2 & 10 \\ 0 & 0 & 3/2 & 2 & 0 \\ 0 & 0 & 0 & -17/5 & 0 \end{array}\right]$$

$$\frac{-5}{17}\cdot -\frac{17}{5}(x_4) = 0\cdot\frac{-5}{17}$$
$$x_4 = 0$$
$$\frac{2}{3}\cdot\frac{3}{2}x_3 + \cancel{2(0)} = 0\cdot\frac{2}{3}$$
$$x_3 = 0$$
$$\frac{-10x_2}{-10} + \cancel{-7(0)} + \cancel{-2(0)} = \frac{10}{-10}$$
$$x_2 = -1$$
$$x_1 + 3(-1) + \cancel{2(0)} + \cancel{1(0)} = -2$$
$$+3 \qquad +3$$
$$x_1 = 1$$

$$\boxed{\begin{array}{l} x_1 = 1 \\ x_2 = -1 \\ x_3 = 0 \\ x_4 = 0 \end{array}}$$

7. If the Gaussian elimination algorithm with scaled partial pivoting is used on the example shown, which row is selected as the third pivot row?

$$\begin{bmatrix} 8 & -1 & 4 & 9 & 2 \\ 1 & 0 & 3 & 9 & 7 \\ -5 & 0 & 1 & 3 & 5 \\ 4 & 3 & 2 & 2 & 7 \\ 3 & 0 & 0 & 0 & 9 \end{bmatrix}$$

$$s = [9, 9, 5, 7, 9]$$

$$\begin{array}{c} 8/9 \\ 1/9 \\ -5/5 \\ 4/7 \\ 4/9 \end{array}\left[\begin{array}{ccccc} 8 & -1 & 4 & 9 & 2 \\ 1 & 0 & 3 & 9 & 7 \\ -5 & 0 & 1 & 3 & 5 \\ 4 & 3 & 2 & 2 & 7 \\ 3 & 0 & 0 & 0 & 9 \end{array}\right]$$

$$r1 \leftrightarrow r3$$

$$\left[\begin{array}{ccccc} -5 & 0 & 1 & 3 & 5 \\ 1 & 0 & 3 & 9 & 7 \\ 8 & -1 & 4 & 9 & 2 \\ 4 & 3 & 2 & 2 & 7 \\ 3 & 0 & 0 & 0 & 9 \end{array}\right] \qquad \begin{array}{l} \frac{1}{5}(r1) + r2 \\ \frac{8}{5}(r1) + r3 \\ \frac{4}{5}(r1) + r4 \\ \frac{3}{5}(r1) + r5 \end{array}$$

$$\begin{array}{c} 0 \\ 0 \\ -1/9 \\ 3/7 \\ 0 \end{array}\left[\begin{array}{ccccc} -5 & 0 & 1 & 3 & 5 \\ 0 & 0 & 16/5 & 48/5 & 8 \\ 0 & -1 & 28/5 & 69/5 & 10 \\ 0 & 3 & 14/5 & 22/5 & 11 \\ 0 & 0 & 3/5 & 9/5 & 12 \end{array}\right] \qquad s = (5, 99, 7, 9)$$

$$r2 \leftrightarrow r4$$

$$\left[\begin{array}{ccccc} -5 & 0 & 1 & 3 & 5 \\ 0 & 3 & 14/5 & 22/5 & 11 \\ 0 & -1 & 28/5 & 69/5 & 10 \\ 0 & 0 & 16/5 & 48/5 & 8 \\ 0 & 0 & 3/5 & 9/5 & 12 \end{array}\right] \qquad \frac{1}{3}r2 + r3$$

$$
\begin{array}{c|ccccc|}
1/5 & -5 & 0 & 1 & 3 & 5 \\
14/35 & 0 & 3 & 14/5 & 22/5 & 11 \\
98/135 & 0 & 0 & 98/15 & 229/15 & 41/3 \\
16/45 & 0 & 0 & 16/5 & 48/5 & 8 \\
3/45 & 0 & 0 & 3/5 & 9/5 & 12
\end{array}
\qquad s = [5, 7, 9, 9, 9]
$$

row 3 would be the third pivot

[a]**8.** Solve the system

$$
\begin{cases}
2x_1 + 4x_2 - 2x_3 = 6 \\
x_1 + 3x_2 + 4x_3 = -1 \\
5x_1 + 2x_2 = 2
\end{cases}
$$

using Gaussian elimination with scaled partial pivoting. Show intermediate results at each step; in particular, display the scale and index vectors.

$$
\begin{array}{c|ccc|c}
2/4 & 2 & 4 & -2 & 6 \\
1/4 & 1 & 3 & 4 & -1 \\
5/5 & 5 & 2 & 0 & 2
\end{array}
\qquad s = [4, 4, 5]
$$

$r3 \leftrightarrow r1$

$$
\left|\begin{array}{ccc|c}
5 & 2 & 0 & 2 \\
1 & 3 & 4 & -1 \\
2 & 4 & -2 & 6
\end{array}\right.
$$

$r2 - \frac{1}{5}(r1)$

$r3 - 2/5(r1)$

$$
\begin{array}{c|ccc|c}
 & 5 & 2 & 0 & 2 \\
\frac{13}{20} & 0 & \frac{13}{5} & 4 & -7/5 \\
\frac{16}{20} & 0 & 16/5 & -2 & 26/5
\end{array}
\qquad s = [5, 4, 4]
$$

$r3 \leftrightarrow r2$

$$
\left|\begin{array}{ccc|c}
5 & 2 & 0 & 2 \\
0 & 16/5 & -2 & 26/5 \\
0 & 13/5 & 4 & -7/5
\end{array}\right.
\qquad r3 - \frac{13}{16} r2
$$

| 5 2 0 | 2 $\qquad 45 \, x_3 = -45$

$$\left|\begin{array}{ccc|c} 0 & 16/5 & -2 & 26/5 \\ 0 & 0 & 45/8 & -45/8 \end{array}\right.$$

$x_3 = -1$

$\frac{16}{5}x_2 - 2(-1) = \frac{26}{5} - 2$

-2

$\frac{16}{5}x_2 = \frac{16}{5}$ $\quad x_2 = 1$

$5x_1 + 2(1) = 2$

$-2 \quad -2$

$\frac{5x_1}{5} = \frac{0}{5}$ $\quad x_1 = 0$

$$\boxed{\begin{array}{l} x_1 = 0 \\ x_2 = 1 \\ x_3 = -1 \end{array}}$$

9. Consider the linear system

$$\begin{cases} 2x_1 + 3x_2 & = 8 \\ -x_1 + 2x_2 - x_3 & = 0 \\ 3x_1 + \quad 2x_3 & = 9 \end{cases}$$

Solve for x_1, x_2, and x_3 using Gaussian elimination with scaled partial pivoting. Show intermediate matrices and vectors.

$$\begin{array}{c} 2/3 \\ 1/2 \\ 3/3 \end{array}\left|\begin{array}{ccc|c} 2 & 3 & 0 & 8 \\ -1 & 2 & -1 & 0 \\ 3 & 0 & 2 & 9 \end{array}\right| \qquad s = [3, 2, 3]$$

$r3 \leftrightarrow r1$

$$\left|\begin{array}{ccc|c} 3 & 0 & 2 & 9 \\ -1 & 2 & -1 & 0 \\ 2 & 3 & 0 & 8 \end{array}\right| \qquad \begin{array}{l} r2 + \frac{1}{3}(r1) \\ r3 - \frac{2}{3}r1 \end{array}$$

$$\begin{array}{c} \\ 2/2 \\ 3/3 \end{array}\left|\begin{array}{ccc|c} 3 & 0 & 2 & 9 \\ 0 & 2 & -1/3 & 3 \\ 0 & 3 & -4/3 & 2 \end{array}\right| \qquad s = [3, 2, 3]$$

$r3 - \frac{3}{2}r1$

same so pick one

$$\left[\begin{array}{ccc|c} 3 & 0 & 2 & 9 \\ 0 & 2 & -1/3 & 3 \\ 0 & 0 & -5/6 & -5/2 \end{array}\right]$$

$-5/6\, x_3 = -5/2$

$x_3 = \frac{-5}{2} \cdot \frac{-6}{5}$

$x_3 = \frac{30}{10} = 3$

$x_2(2) - 1 = 3$
$\quad\;\; +1 \quad\; +1$

$\frac{x_2(2)}{2} = \frac{4}{2} \quad x_2 = 2$

$3x_1 + 2(3) = 9$
$\qquad\quad -6 \quad -6$

$\frac{3x_1}{3} = \frac{3}{3}$

$x_1 = 1$

$$\boxed{\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 3 \end{aligned}}$$

14. Using scaled partial pivoting, show how a computer would solve the following system of equations. Show the scale array, tell how the *pivot* rows are selected, and carry out the computations. Include the index array for each step. There are no fractions in the correct solution, except for certain ratios that must be looked at to select pivots. You should follow exactly the scaled-partial-pivoting code, except that you can include the right-hand side of the system in your calculations as you go along.

$$\begin{cases} 2x_1 - x_2 + 3x_3 + 7x_4 = 15 \\ 4x_1 + 4x_2 \qquad\quad + 7x_4 = 11 \\ 2x_1 + x_2 + x_3 + 3x_4 = 7 \\ 6x_1 + 5x_2 + 4x_3 + 17x_4 = 31 \end{cases}$$

$$\begin{array}{c} 2/7 \\ 4/7 \\ 2/3 \\ 6/17 \end{array}\left[\begin{array}{cccc|c} 2 & -1 & 3 & 7 & 15 \\ 4 & 4 & 0 & 7 & 11 \\ 2 & 1 & 1 & 3 & 7 \\ 6 & 5 & 4 & 17 & 31 \end{array}\right]$$

$S = [7, 7, 3, 17]$

$IV = [1, 2, 3, 4]$

swap r1 and r3

$$\left[\begin{array}{cccc|c} 2 & 1 & 1 & 3 & 7 \\ 4 & 4 & 0 & 7 & 11 \\ 2 & -1 & 3 & 7 & 15 \\ 6 & 5 & 4 & 17 & 31 \end{array}\right]$$

$S = [3, 7, 7, 17]$

$IV = [3, 2, 1, 4]$

$r2 - 2(r1)$
$r3 - r1$
$r4 - 3(r1)$

$$\left[\begin{array}{cccc|c} 2 & 1 & 1 & 3 & 7 \end{array}\right.$$

$$\begin{array}{c|cccc|c} 2/7 & 0 & 2 & -2 & 1 & -3 \\ -2/7 & 0 & -2 & 2 & 4 & 8 \\ 2/17 & 0 & 2 & 1 & 8 & 10 \end{array}$$

$r3 + r2$
$r2 - r4$

$$\begin{array}{c|cccc|c} & 2 & 1 & 1 & 3 & 7 \\ & 0 & 2 & -2 & 1 & -3 \\ 0/7 & 0 & 0 & 0 & 5 & 5 \\ 1/17 & 0 & 0 & 3 & 7 & 13 \end{array}$$

swap r4 and r3

$$\left|\begin{array}{cccc|c} 2 & 1 & 1 & 3 & 7 \\ 0 & 2 & -2 & 1 & -3 \\ 0 & 0 & 3 & 7 & 13 \\ 0 & 0 & 0 & 5 & 5 \end{array}\right|$$

$\frac{5x_4}{5} = \frac{5}{5}$

$x_4 = 1$

$3x_3 + 7 = 13$
$\quad -7 \quad -7$

$\frac{3x_3}{3} = \frac{6}{3} \quad x_3 = 2$

$2x_2 - 2(2) + 1 = -3$
$\quad +4 \quad -1 \quad -1$
$\quad\quad\quad +4$

$\frac{2x_2}{2} = \frac{0}{2} \quad x_2 = 0$

$2x_1 + 0 + 2 + 3 = 7$
$\quad - 5 \quad -5$

$\frac{2x_1}{2} = \frac{2}{2} \quad x_1 = 1$

$$\boxed{\begin{array}{l} x_1 = 1 \\ x_2 = 0 \\ x_3 = 2 \\ x_4 = 1 \end{array}}$$

1. Using naive Gaussian elimination, factor the following matrices in the form $A = LU$, where L is a unit lower triangular matrix and U is an upper triangular matrix.

$$\begin{bmatrix} 3 & 0 & 3 \end{bmatrix}$$

[a]**a.** $A = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 3 & 0 \end{bmatrix}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad U = A = \begin{bmatrix} 3 & 0 & 3 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{bmatrix}$

$\frac{1}{3}r1 - r2$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 0 & 1 \end{bmatrix} \qquad = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & 3 & -1 \end{bmatrix}$

$-3r3 - r1$

$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & -3 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 3 & 0 & 3 \\ 0 & -1 & 3 \\ 0 & 0 & 8 \end{bmatrix}}$$

b. $A = \begin{bmatrix} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 3 & -1 \\ 3 & -3 & 0 & 6 \\ 0 & 2 & 4 & -6 \end{bmatrix}$

$L = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \qquad U = \begin{vmatrix} 1 & 0 & 1/3 & 0 \\ 0 & 1 & 3 & -1 \\ 3 & -3 & 0 & 6 \\ 0 & 2 & 4 & -6 \end{vmatrix}$

$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \qquad \begin{vmatrix} 1 & 0 & 1/3 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & -3 & -1 & 6 \\ 0 & 2 & 4 & -6 \end{vmatrix}$

$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & -3 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{vmatrix} \qquad \begin{vmatrix} 1 & 0 & 1/3 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 8 & 3 \\ 0 & 0 & -2 & -4 \end{vmatrix}$

$$\boxed{L = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & -3 & 1 & 0 \\ 0 & 2 & -1/4 & 1 \end{vmatrix} \quad U = \begin{vmatrix} 1 & 0 & 1/3 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 8 & 3 \\ 0 & 0 & 0 & -7/4 \end{vmatrix}}$$

c. $A = \begin{bmatrix} -20 & -15 & -10 & -5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$$L = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \qquad U = \begin{vmatrix} -20 & -15 & -10 & -5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ -1/20 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \qquad \begin{vmatrix} -20 & -15 & -10 & -5 \\ 0 & -3/4 & -1/2 & -1/4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ -1/20 & 1 & 0 & 0 \\ 0 & -4/3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \qquad \begin{vmatrix} -20 & -15 & -10 & -5 \\ 0 & -3/4 & -1/2 & 1/4 \\ 0 & 0 & 2/3 & 1/3 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$

$$L = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -1/20 & 1 & 0 & 0 \\ 0 & -4/3 & 1 & 0 \\ 0 & 0 & 3/2 & 1 \end{vmatrix} \qquad U = \begin{vmatrix} -20 & -15 & -10 & -5 \\ 0 & -3/4 & -1/2 & -1/4 \\ 0 & 0 & 2/3 & 1/3 \\ 0 & 0 & 0 & -1/2 \end{vmatrix}$$

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 \\ 0 & 9 & 4 & 0 \\ 5 & 0 & 8 & 10 \end{bmatrix}$$

[a] **a.** Determine a unit lower triangular matrix $\boldsymbol{M}$ and an upper triangular matrix $\boldsymbol{U}$ such that $\boldsymbol{MA} = \boldsymbol{U}$.

b. Determine a unit lower triangular matrix $\boldsymbol{L}$ and an upper triangular matrix $\boldsymbol{U}$ such that $\boldsymbol{A} = \boldsymbol{LU}$. Show that $\boldsymbol{ML} = \boldsymbol{I}$ so that $\boldsymbol{L} = \boldsymbol{M}^{-1}$.

a)

$$M = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \qquad U = \begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 \\ 0 & 9 & 4 & 0 \\ 5 & 0 & 8 & 10 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -5 & 0 & 0 & 1 \end{vmatrix} \qquad \begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 \\ 0 & 9 & 4 & 0 \\ 0 & 0 & 8 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ -5 & 6 & 0 & 1 \end{vmatrix} \qquad \begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 8 & 0 \end{vmatrix}$$

$$\boxed{M = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ -5 & 6 & -2 & 1 \end{vmatrix} \qquad U = \begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}}$$

b) using M from part b

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ -5 & 6 & -2 & 1 \end{vmatrix}$$

$$L = M^{-1} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 5 & 0 & 2 & 1 \end{vmatrix}$$

U same as part a

$$\begin{vmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$